PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 67, Number 1, November 1977

TRANSFORMATIONS INTO BAIRE 1 FUNCTIONS

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ABSTRACT. A measurable f from $I = [0, 1]$ to R is equivalent to a Baire 2 function but may not be equivalent to any Baire 1 function. Gorman has obtained the following interesting contrasting facts. If f assumes finitely many values there is a homeomorphism h of I such that $f \circ h$ is equivalent to a Baire 1 function, but there is a measurable f which assumes countably many values which does not have this property. However, the example of Gorman is such that for some homeomorphisms h the function $f \circ h$ is not measurable. It is shown here that if $f \circ h$ is measurable, for every homeomorphism h, then there is an h for which $f \circ h$ is equivalent to a Baire 1 function.

1. A measurable f from $I = [0, 1]$ to R is equivalent to a Baire 2 function but may not be equivalent to any Baire 1 function. Gorman [1] has obtained the following interesting contrasting facts. If f assumes finitely many values there is a homeomorphism h of I such that $f \circ h$ is equivalent to a Baire 1 function, but there is a measurable f which assumes countably many values which does not have this property. However, the example of Gorman is such that for some homeomorphisms h the function $f \circ h$ is not measurable.

A function f is said to be *absolutely measurable* if for every homeomorphism h of I the function $f \circ h$ is measurable. This is tantamount to saying that f is measurable with respect to every Lebesgue-Stieltjes measure derived from a strictly increasing continuous distribution function. We prove the following result.

THEOREM. *If $f\colon I \to R$ is absolutely measurable there is a homeomorphism h of I such that $f \circ h$ is equivalent to a Baire* 1 *function.*

2. We give some preliminary definitions and lemmas. A set $E \subset I$ is of *absolute measure zero* if for every homeomorphism h the set $h(E)$ is of measure zero. A point $x \in I$ is a *c-point* of a set E if for every neighborhood N of x the set $N \cap E$ has cardinality c and x is a *perfect c-point* of E if for every neighborhood N of x the set $N \cap E$ contains a nonempty perfect set. E is c-dense (perfectly dense) in a set D if every point of D is a c-point (perfect c-point) of E. Gorman [2] has obtained the following lemma.

LEMMA 1. *If $E \subset I$ is of the first category there is a homeomorphism h of I*

Received by the editors March 18, 1977.
AMS (MOS) subject classifications (1970). Primary 26A21.
[1]Supported in part by NSF Grant No. MCS-76-06573.

such that $h(E)$ is of measure zero. If $E \subset I$ is an F_σ set and is c-dense in I there is a homeomorphism h of I such that $h(E)$ has measure one.

For every $E \subset I$ let $P(E)$ be the set of points in I which are perfect c-points of E.

LEMMA 2. *If $E \subset I$ is absolutely measurable the set $E \setminus P(E)$ is of absolute measure* 0.

PROOF. Since $P(E)$ is closed, $E \setminus P(E)$ is absolutely measurable. Since $E \setminus P(E)$ has no perfect c-points neither does $h(E \setminus P(E))$ for any homeomorphism h of I. So $h(E \setminus P(E))$ has measure 0.

Let E be absolutely measurable and let $G(E) = P(E)^0$. Partition E into 3 sets, $E^* = E \cap G(E)$, $E^{**} = E \cap [P(E) \setminus G(E)]$, and $E^{***} = E \setminus P(E)$.

LEMMA 3. *E^* is perfectly dense in $G(E)$, E^{**} is nowhere dense, and E^{***} is of absolute measure* 0.

PROOF. By Lemma 2.

LEMMA 4. *Let $J = \bigcup_1^\infty E_n$, where $J \subset I$ is an open interval and each is absolutely measurable. There is an n such that $G(E_n)$ is nonempty.*

PROOF. Otherwise $J = (\bigcup_1^\infty E_n^{**}) \cup (\bigcup_1^\infty E_n^{***})$. The set $\bigcup_1^\infty E_n^{**}$ is of the first category so that by Lemma 1 there is a homeomorphism h such that $h(\bigcup E_n^{**})$ is of measure 0. But $\bigcup E_n^{***}$ is of absolute measure 0 so that $h(\bigcup E_n^{***})$ is also of measure 0.

3. We now prove the main lemma.

LEMMA 5. *If f: $I \to R$ is absolutely measurable and takes only countably many values, there is a homeomorphism h of I such that $f \circ h$ is equivalent to a Baire 1 function.*

PROOF. Let $\{a_n\}$ be the sequence of values assumed by f and let $E_n = f^{-1}(a_n)$, $n = 1, 2, \ldots$. The sets E_n are absolutely measurable and their union is I. Denote the open components of $P(E_1)^0$ by $I_{11}, I_{12}, \ldots$ and the open components of $I \setminus P(E_1)$ by $J_{11}, J_{12}, \ldots,$ and for each $n = 2, 3, \ldots$ the open components of $P(E_n)^0 \cap (\bigcup_m J_{n-1,m})$ by $I_{n1}, I_{n2}, \ldots$ and the open components of $(\bigcup_m J_{n-1,m}) \setminus P(E_n)$ by $J_{n1}, J_{n2}, \ldots$. It follows by Lemma 4 that the union G of all the I_{nm} is a dense open subset of I. By Lemma 1, there is a homeomorphism h_1 of I such that $h_1^{-1}(G)$ has measure one. Now $E_n \cap I_{nm}$ contains an F_σ set which is c dense in I_{nm}. So by Lemma 1, the homeomorphism h_1 may be modified on each $h_1^{-1}(I_{nm})$ to a homeomorphism h so that $h^{-1}(E_n \cap I_{nm})$ is of full measure in $h^{-1}(I_{nm})$. Define

$$g(x) = \begin{cases} a_n & \text{if } x \in h^{-1}(I_{nm}) \textit{ for some } n, m, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is Baire 1 and $f \circ h$ is equal almost everywhere to g.

4. The following lemma serves as a bridge for carrying our result from countable valued functions to arbitrary absolutely measurable functions.

LEMMA 6. *If f: $I \to R$ is absolutely measurable, and $P \subset I$ is perfect there is a perfect $Q \subset P$ such that $f|Q$ is continuous.*

PROOF. There is a homeomorphism h of I such that $h^{-1}(P)$ has positive measure. By Lusin's theorem $f \circ h$ is continuous on a perfect subset R of $h^{-1}(P)$ of positive measure. Set $Q = h(R)$.

We also need the following two lemmas.

LEMMA 7. *Let f: $I \to R$. There is a homeomorphism h of I such that $f \circ h$ equals a Baire 1 function almost everywhere if and only if there is a c-dense F_σ set E and a Baire 1 function g such that $f(x) = g(x)$ for every $x \in E$.*

PROOF. This follows from Lemma 1 and the fact that a set of full measure in I contains an F_σ set which is a c-dense in I.

LEMMA 8. *If $E \subset I$ is of type F_σ and is c-dense in an open set H and S is the union of a nowhere dense set and a set of absolute measure 0, then $E \setminus S$ contains an F_σ set F which is c-dense in H.*

PROOF. There is a homeomorphism h such that $h(H)$ and $h(E)$ have the same measure. But $h(S)$ is the union of a nowhere dense set and a set of measure 0. Then $h(E \setminus S)$ contains an F_σ set K which is c-dense in $h(H)$, and $h^{-1}(K) \subset E \setminus S$ is c-dense in H.

5. We are now ready to prove the theorem.

THEOREM 1. *Every absolutely measurable function on $I = [0, 1]$ can be transformed by a homeomorphic change of variable into a function which is equal almost everywhere to a Baire 1 function.*

PROOF. Let f be absolutely measurable. By Lemma 7, it suffices to show that there is a g in Baire 1 such that $f(x) = g(x)$ on a set containing a c-dense F_σ. Let $\{f_k\}$ take on only values of the form $m/2^{k-1}$ such that

$$\text{(i)} \qquad f_k(x) \leqslant f(x) < f_k(x) + \frac{1}{2^{k-1}} \quad \text{for every } x \in I.$$

Then each f_k is absolutely measurable. By Lemmas 5 and 7 there is a Baire 1 function b_1 such that the set of points for which $b_1(x) = f_1(x)$ contains a c-dense set E of type F_σ.

Let $A = \{f_1(x) = f_2(x)\} \cap E$. Then A is absolutely measurable so that, by Lemma 3, there are pairwise disjoint sets A^*, A^{**}, and A^{***} such that $A = A^* \cup A^{**} \cup A^{***}$ and A^* is perfectly dense on an open set G and $A^* = A \cap G$, A^{**} is nowhere dense, and A^{***} is of absolute measure 0.

Let H be the interior of the complement of G. Then $H \cap A^*$ is empty. So, $H \cap A$ is the union of a nowhere dense set and a set of absolute measure zero. If $H \neq \varnothing$, $E \cap H$ is a c-dense F_σ subset of H since E is a c-dense F_σ

subset of I. So, by Lemma 8, $(E \cap H) \setminus A$ contains an F_σ set which is c-dense in H. Call this set F.

Now, on the complement of A we have

$$f_2(x) = f_1(x) + \tfrac{1}{2}.$$

On E, $b_1(x) = f_1(x)$. Since $F \subset E \setminus A$,

$$f_2(x) = b_1(x) + \tfrac{1}{2} \quad \text{on } F.$$

Let $b_2 = b_1 + \frac{1}{2}\chi_H$. Since H is open χ_H is Baire 1 so that b_2 is Baire one. Moreover, $\|b_2 - b_1\| \leqslant \frac{1}{2}$ where $\|\phi\|$ is the $\sup[|\phi(x)|: x \in I]$ for any function ϕ on I.

Finally, we note that $b_2(x) = f_2(x)$ on a c-dense set of type F_σ. First, F is an F_σ set which is c-dense in H and on F,

$$b_2(x) = b_1(x) + \tfrac{1}{2} = f_1(x) + \tfrac{1}{2} = f_2(x).$$

Next, on A^*,

$$b_2(x) = b_1(x) = f_1(x) = f_2(x).$$

But A^* is perfectly dense in G so that it contains an F_σ set K which is dense in G. Now, $b_2(x) = f_2(x)$ on $F \cup K$ which is a c-dense F_σ in $G \cup H$, a dense open set in I. Thus $b_2(x) = f_2(x)$ on a dense F_σ in I.

Proceeding by induction, obtain a sequence of Baire 1 functions $\{b_k\}$ such that

$$\|b_{k+1} - b_k\| \leqslant 1/2^k$$

where $b_{k+1} = f_{k+1}$ on a c-dense F_σ.

We now modify $\{b_k\}$ to obtain $\{g_k\}$, a sequence of Baire one functions which converges uniformly to a Baire 1 function g such that $g(x) = f(x)$ on a c-dense F_σ.

For this purpose, let $\{I_k\}$ be an enumeration of the rational intervals in [0, 1]. For each k, choose $P_k \subset I_k$ so that

(a) P_k is perfect,

(b) $P_i \cap P_j = \varnothing$ if $i \neq j$,

(c) $b_k(x) = f_k(x)$ on P_k.

By Lemma 6, there is a perfect set $Q_k \subset P_k$ such that $f|Q_k$ is continuous.

Let

$$g_k(x) = \begin{cases} f(x) & \text{for } x \in Q_1 \cup Q_2 \cup \cdots \cup Q_k, \\ b_k(x) & \text{elsewhere.} \end{cases}$$

Since $Q_1 \cup \cdots \cup Q_k$ is closed, b_k is Baire 1, and $f|Q_1 \cup \cdots \cup Q_k$ is continuous, g_k is Baire 1. Now, $|b_{k+1}(x) - b_k(x)| \leqslant 1/2^k$ for every $x \in I$, $g_k(x) = b_k(x)$ on $I \setminus (Q_1 \cup \cdots \cup Q_k)$ and $g_{k+1}(x) = b_{k+1}(x)$ on $I \setminus (Q_1 \cup \cdots \cup Q_k \cup Q_{k+1})$ and $g_k(x) = g_{k+1}(x) = f(x)$ on $Q_1 \cup \cdots \cup Q_k$. Hence

$$\sup\left[|g_{k+1}(x) - g_k(x)|: x \in I \setminus Q_{k+1}\right] \leqslant 1/2^k.$$

For $x \in Q_{k+1}$, $g_{k+1}(x) = f(x)$ and so

$$\begin{aligned} |g_k(x) - g_{k+1}(x)| &= |g_k(x) - f(x)| = |h_k(x) - f(x)| \\ &\leqslant |h_k(x) - h_{k+1}(x)| + |h_{k+1}(x) - f_{k+1}(x)| + |f_{k+1}(x) - f(x)| \\ &\leqslant \frac{1}{2^k} + 0 + \frac{1}{2^k} = \frac{1}{2^{k-1}} \end{aligned}$$

by (c) and (i). So $\| g_{k+1} - g_k \| \leqslant 1/2^{k+1}$.

Since each g_k is Baire 1 and the sequence $\{ g_k \}$ converges uniformly, the limit g is also Baire 1 and is equal to f on $\bigcup_k Q_k$, a c-dense F_σ set. This proves the theorem.

References

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