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# FINITE HOMOLOGICAL DIMENSION OF $BP_*(X)$ FOR INFINITE COMPLEXES

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ABSTRACT. The main result proved here is that $BP_*(EG \times_G X)$ has finite homological dimension when $G = \mathbf{Z}_p$ and $X$ is a finite $G$-CW-complex. The argument uses $BP_*BP$-comodules.

It is well known that the Brown-Peterson homology $BP_*(X)$ of a finite CW-complex $X$ has finite homological dimension (= projective dimension) as a $BP_*$-module (cf. L. Smith [**10**], D. C. Johnson and W. S. Wilson [**3**]). On the other hand, for infinite complexes and spectra, hom dim $BP_*(X)$ may be infinite [**4**].

The modules $BP_*(B\mathbf{Z}_p)$ are well understood in terms of a standard resolution, and have homological dimension one (BP denotes Brown-Peterson homology for the same prime $p$). A recent unpublished study by D. C. Johnson and W. S. Wilson includes the result that $BP_*((B\mathbf{Z}_p)^n)$ has homological dimension $n$ for $p$ odd and all $n$; this has been an intractable problem for ten years, having been verified previously only for $n = 2$. In this note I shall obtain qualitative information on $BP_*(B\mathbf{Z}_p \times X)$ for finite complexes $X$; this approach suffices in the case of BP and MU *cohomology* to calculate $BP^*(BA)$ and $MU^*(BA)$ for all compact abelian Lie groups $A$ (see [**6**] and Stretch [**11**]).

The results to be proved here depend on a theorem about $BP_*BP$-comodules (Theorem A below) proved in [**9**], which I refer to as the coherence theorem. The results, and the key lemma on which they depend, are as follows.

THEOREM 1. hom dim $BP_*(B\mathbf{Z}_p \times X) < \infty$ *for each finite complex X.*

The proof will of course begin with the common knowledge that

$$\text{hom dim } BP_*(B\mathbf{Z}_p) = 1,$$

and make use of the standard projective resolution.

THEOREM 2. *If* hom dim $BP_*(Y) < \infty$, *this remains true after attaching a finite number of cells to Y.*

More generally, one may ask if $BP_*(EG \times_G X)$ has finite homological dimension for all compact Lie groups $G$ and finite $G$-complexes $X$.

THEOREM 3. *If* $G = \mathbf{Z}_p$, *then* hom dim $BP_*(EG \times_G X) < \infty$, *for each finite G-complex.*

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These results depend essentially on one argument, based on the following lemma.

KEY LEMMA. *Let* $\phi$: $A \to B$ *be a homomorphism of* $BP_*$*-modules, where* $A$ *is finitely-presented and* $B$ *is a* $BP_*BP$*-comodule with* hom dim $B < \infty$. *Then* $\operatorname{Im}\phi$ *and* $\operatorname{Ker}\phi$ *are finitely-presented, and* hom dim $\operatorname{Coker}\phi < \infty$.

We remark that a (graded) $BP_*$-module $M$ is finitely-presented if and only if it is *coherent* (it is finitely-generated, and each of its finitely-generated submodules is finitely-presented). Since $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$, it is equivalent to assert that $M$ is isomorphic to

$$BP_* \underset{\mathbf{Z}_{(p)}[v_1,\ldots,v_n]}{\otimes} N$$

for some finitely-generated module $N$ over $\mathbf{Z}_{(p)}[v_1,\ldots,v_n]$, with $n < \infty$; then

$$\text{hom dim } M \leqslant n + 1$$

by the Hilbert syzygy theorem [**2**, §1].

COROLLARY. *If*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \nwarrow \quad \swarrow & \\ & C & \end{array}$$

*is an exact triangle of* $BP_*BP$*-comodules, with* $A$ *coherent and* hom dim $B < \infty$, *then* hom dim $C < \infty$.

The corollary follows at once from the Key Lemma, applied to the exact sequence

$$0 \to \operatorname{Coker}\phi \to C \to \operatorname{Ker}\phi \to 0.$$

Notice further that Theorem 2 follows immediately from the corollary; in particular, if hom dim $BP_*(Y) < \infty$ and $\alpha \in BP_*(Y)$, then the annihilator ideal $\operatorname{Ann}(\alpha)$ is finitely-generated.

The Key Lemma follows from the coherence theorem (Theorem 4) of [**9**].

THEOREM A. *Let* $M$ *be a* $BP_*BP$*-comodule, which is finitely-generated as a* $BP_*$*-module. Then* $M$ *is coherent if and only if* $v_n$: $M \to M$ *is injective for some* $n \geqslant 0$ (*by convention*, $v_0 = p$).

PROOF OF THE KEY LEMMA. The assumptions imply that $\operatorname{Im}\phi$ is a finitely-generated $BP_*$-module; $\operatorname{Im}\phi$ may not be a subcomodule of the $BP_*BP$-comodule $B$, but it generates a subcomodule $B_0$ which is finitely-generated as a $BP_*$-module. If hom dim $B = n$, then $v_n$: $B \to B$ is injective [**9**, Lemma 3.4], hence also $v_n$: $B_0 \to B_0$ is injective, and so $B_0$ is coherent by Theorem A. Then its finitely-generated subcomodule $\operatorname{Im}\phi$ is finitely-presented. The conclusions about $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ now follow from the exact sequences

$$0 \to \operatorname{Ker}\phi \to A \to \operatorname{Im}\phi \to 0, \quad 0 \to \operatorname{Im}\phi \to B \to \operatorname{Coker}\phi \to 0$$

(cf. [**2**, §1]).

We turn now to $BP_*(B\mathbf{Z}_p \times X)$, and shall make use of the Künneth formula [5],

$$0 \to BP_*(B\mathbf{Z}_p) \underset{BP_*}{\otimes} BP_*(X) \to BP_*(B\mathbf{Z}_p \times X)$$
$$\to \mathrm{Tor}_1^{BP_*}\big(BP_*(B\mathbf{Z}_p), BP_*(X)\big) \to 0.$$

Thus it suffices, for the proof of Theorem 1, to prove the following result about $BP_*BP$-comodules. Let $\mathfrak{BP}_0$ denote the category of $BP_*BP$-comodules which are coherent as $BP_*$-modules; for each finite complex $X$, $BP_*(X) \in \mathfrak{BP}_0$.

THEOREM 1′. *Let* $M \in \mathfrak{BP}_0$, *and put* $L = BP_*(B\mathbf{Z}_p)$. *The* $\mathrm{Tor}_1^{BP_*}(L, M)$ *is a coherent* $BP_*$*-module, and* hom dim $L \otimes_{BP_*} M < \infty$.

The proof will make use of the prime filtration theorem for coherent $BP_*BP$-comodules [7, 8]. Thus we begin with the case $M = BP_*/I_n$, $0 \leq n < \infty$, where $I_n$ denotes the invariant prime ideal $(v_0, v_1, \ldots, v_{n-1})$. Recall $L = BP_*(B\mathbf{Z}_p)$.

LEMMA. $\mathrm{Tor}_1^{BP_*}(L, BP_*/I_n)$ *is a direct sum of* $p^n - 1$ *copies of* $BP_*/I_n$, *and* $L \otimes_{BP_*} BP_*/I_n$ *is* $v_n$*-torsion with homological dimension* $n+1$.

PROOF. We use the standard free resolution, resulting from the Gysin sequence,

$$0 \to F_1 \xrightarrow{\phi} F_0 \to L \to 0,$$

with $F_0$ free on $\{\alpha_n\}_1^\infty$, $F_1$ free on $\{\beta_n\}_1^\infty$ and $\phi(\beta_n) = p\alpha_n + \lambda_1\alpha_{n-1} + \cdots + \lambda_{n-1}\alpha_1$, where the $\lambda_n$'s are the coefficients of the power series $[p]^{BP}(T)$ associated with the formal group law

$$[p]^{BP}(T) = pT + \lambda_1 T + \cdots + \lambda_n T^{n+1} + \ldots.$$

Recall that $I_n = (p, \lambda_1, \ldots, \lambda_{p^n-1})$, and that $\lambda_{p^n-1} = v_n \in BP_{2(p^n-1)}$; the generators $\alpha_n$ and $\beta_n$ are assigned degree $2n-1$.

Now tensor with $BP_*/I_n$, and let $\{\bar{\beta}_n\}$, $\{\bar{\alpha}_n\}$ denote the images of the generators in $F_1 \otimes_{BP_*} BP_*/I_n$ and $F_0 \otimes_{BP_*} BP_*/I_n$. Under the map $\phi \otimes 1$, we have $\phi \otimes 1(\bar{\beta}_i) = 0$ for $i < p^n$, and then relations

$$\begin{cases} \phi \otimes 1\big(\bar{\beta}_{p_n}\big) = \lambda_{p^n-1}\bar{\alpha}_1, \\ \phi \otimes 1\big(\bar{\beta}_{p^n+1}\big) = \lambda_{p^n-1}\bar{\alpha}_2 + \lambda_{p^n}\bar{\alpha}_1, \\ \ldots \end{cases}$$

The conclusions of the lemma now follow easily: since $\lambda_{p^n-1} = v_n \neq 0$ in $BP_*/I_n$, $\mathrm{Tor}_1 \cong \mathrm{Ker}(\phi \otimes 1)$ is free over $BP_*/I_n$ on $\bar{\beta}_1, \ldots, \bar{\beta}_{p^n-1}$, and $L \otimes_{BP_*} BP_*/I_n$ is $v_n$-torsion, hence has homological dimension at least $n+1$ by the Koszul resolution [2], while the free resolution over $BP_*/I_n$ obtained by dividing out $\mathrm{Ker}(\phi \otimes 1)$ forces its homological dimension to be at most $n+1$.

By the prime filtration theorem [7, 8], the general case of Theorem 1′ will follow if we show that for an exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathfrak{BP}_0$, the conclusion of Theorem 1′ holds for $M$ if it holds for $M'$ and $M''$. For this we examine the 6-term exact sequence (with evident abbreviations)

$$0 \to \operatorname{Tor}(L, M') \to \operatorname{Tor}(L, M) \to \operatorname{Tor}(L, M'')$$
$$\overset{\Delta}{\to} L \otimes M' \to L \otimes M \to L \otimes M'' \to 0.$$

Notice that we can give $L \otimes M'$ the diagonal comodule structure: from $\psi_L\colon L \to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} L$ and $\psi_{M'}\colon M' \to \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} M'$ we give $L \otimes M'$ the coaction map

$$L \underset{\mathrm{BP}_*}{\otimes} M' \overset{\psi_L \otimes \psi_{M'}}{\to} \mathrm{BP}_*\mathrm{BP} \underset{\mathrm{BP}_*}{\otimes} L \underset{\mathrm{BP}_*}{\otimes} \mathrm{BP}_*\mathrm{BP} \underset{\mathrm{BP}_*}{\otimes} M'$$
$$\cong \left(\mathrm{BP}_*\mathrm{BP} \underset{\mathrm{BP}_*}{\otimes} \mathrm{BP}_*\mathrm{BP}\right) \underset{\mathrm{BP}_*}{\otimes} \left(L \underset{\mathrm{BP}_*}{\otimes} M'\right) \overset{m \otimes 1}{\to} \mathrm{BP}_*\mathrm{BP} \underset{\mathrm{BP}_*}{\otimes} \left(L \underset{\mathrm{BP}_*}{\otimes} M'\right).$$

By assumption, $\operatorname{Tor}(L, M'')$ is a coherent $\mathrm{BP}_*$-module, and hom dim $M' \otimes L < \infty$; then the Key Lemma applies to $\Delta$, with the conclusion that $\operatorname{Ker}\Delta$ is coherent and $\operatorname{Coker}\Delta$ has finite homological dimension. This gives the desired conclusions for $M$ at once, and so proves Theorem 1′.

Finally, we consider $\mathrm{BP}_*(EG \times_G X)$ with $G = \mathbf{Z}_p$ and $X$ a finite $G$-CW-complex [**1**]. Then the fixed point set $X^G$ is a subcomplex, and $EG \times_G X^G = BG \times X^G$. Thus we obtain an exact triangle

$$\begin{array}{ccc} \mathrm{BP}_*(EG \times_G X, EG \times_G X^G) & \overset{\partial}{\to} & \mathrm{BP}_*(BG \times X^G) \\ & \nwarrow \qquad \swarrow & \\ & \mathrm{BP}_*(EG \times_G X) & \end{array}$$

Now if $G$ acts freely on $Y$, there is a fibre bundle $EG \to EG \times_G Y \to Y/G$ with contractable fibre (in addition to the usual bundle $Y \to EG \times_G Y \to BG$); thus $\mathrm{BP}_*(EG \times_G Y) \cong \mathrm{BP}_*(Y/G)$. Now choose a suitable equivariant regular neighborhood of $X^G$ in $X$, apply excision and the relative version of the observation above for free actions to conclude that the relative group in the exact triangle is coherent. By Theorem 1, hom dim $\mathrm{BP}_*(BG \times X^G) < \infty$ and now the corollary to the key lemma implies that this is also true for $\mathrm{BP}_*(EG \times_G X)$. This completes the proof of Theorem 3.

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