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A GEOMETRIC INTERPRETATION OF SEGAL'S INEQUALITY $\|e^{X+Y}\| \le \|e^{X/2}e^Y e^{X/2}\|$

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ABSTRACT. It is shown that the exponential mapping of the manifold of positive elements of a C^*-algebra (provided with its natural connection) increases distances (when measured in the natural Finsler structure). The proof relies on Segal's inequality $\|e^{X+Y}\| \le \|e^{X/2}e^Y e^{X/2}\|$, valid for all symmetric X, Y in any C^*-algebra. In turn, this geometric inequality implies Segal's' inequality.

Let A be a C^*-algebra with 1 and denote by A^+ the set of positive invertible elements of A. Then A^+ is an open subset of A^s, the real Banach space of symmetric elements in A, and therefore, the tangent space T_aA^+ to the manifold A^+ at $a \in A^+$ can be identified to A^s. The manifold A^+ carries a natural Finsler metric (see [1]) defined by $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$ for $X \in TA_a^+$. This norm corresponds to the following interpretation: assume A is faithfully represented in a Hilbert space $(L, \langle\ ,\ \rangle)$, and for each $a \in A^+$, define an inner product in L by $\langle \xi, \eta\rangle_a = \langle a\xi, \eta\rangle$. On the other hand, each $X \in TA_a^+$ determines the bilinear form $B(\xi, \eta) = \langle X\xi, \eta\rangle$ on L. Then the Finsler norm $\|X\|_a$ coincides with the norm of the bilinear form B in the Hilbert space $(L, \langle\ ,\ \rangle_a)$.

The group G of invertible elements of A acts on A^+ by $\mathcal{M}_g a = (g^*)^{-1}ag^{-1}$, $(g \in G,\ a \in A^+)$ making A^+ into a reductive homogeneous space (see [2]) with the natural connection given by

$$D_X Y = X(Y) - \tfrac{1}{2}(Xa^{-1}Y + Ya^{-1}X),$$

where $X(Y)$ denotes the derivative of the field Y in the direction X in the Banach space A^s. In this connection, the geodesic γ with $\gamma(0) = a$ and $\dot\gamma(0) = X$ has the form $\gamma(t) = e^{tXa^{-1}/2}ae^{ta^{-1}X/2}$.

Further, for each $g \in G$ and $a \in A^+$, the map g is an isometry from the Hilbert space $(L, \langle\ ,\ \rangle_a)$ onto $(L, \langle\ ,\ \rangle_{\mathcal{M}_g a})$ and consequently the tangent map $(T\mathcal{M}_g)_a \colon TA_a^+ \to TA^+_{\mathcal{M}_g a}$ is an isometry for the Finsler metric.

The geometry of A^+ in this setting was studied in [1] where, in particular, the following result is proved [1, Theorem 6.3]: *the distance $d(a, b)$ in the Finsler metric defined by*

$$d(a, b) = \inf\{\text{length}(\gamma);\ \gamma \text{ joins } a \text{ to } b\},$$

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is given by $d(a, b) =$ *length of the unique geodesic in* A^+ *joining* a *to* b, *i.e.*, $d(a, b) = \|X\|_a$ *where* $b = e^{Xa^{-1}/2} a e^{a^{-1}X/2}$.

Notice that the Finsler structure in A^+ does not come from a Riemannian metric. However, A^+ shares with Riemannian manifolds of nonpositive curvature the following metric property, which is the main result of this note.

Theorem 1. *For each* $a \in A^+$, *the exponential map* $\exp_a \colon TA_a^+ \to A^+$ *increases distances in the sense that*

$$(*) \qquad d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a$$

for all $X, Y \in TA_a^+$.

Proof. Since G acts isometrically, it suffices to prove the inequality for $a = 1$. Set $x = \exp_1 X = e^X$, $y = \exp_1 Y = e^Y$. The geodesic that joins x to y in time 1 has the formula

$$\gamma(t) = e^{Zx^{-1}t/2} x e^{x^{-1}Zt/2},$$

where $b = \gamma(1) = e^{Zx^{-1}/2} x e^{x^{-1}Z/2}$. The inequality we are after is

$$\|X - Y\| \leq \|Z\|_x = \|x^{-1/2} Z x^{-1/2}\|$$

or

$$\|\log x - \log y\| \leq \|x^{-1/2} Z x^{-1/2}\|.$$

But

$$\begin{aligned} x^{-1/2} y x^{-1/2} &= x^{-1/2}(e^{Zx^{-1}/2} x e^{x^{-1}Z/2}) x^{-1/2} \\ &= e^{(x^{-1/2}Zx^{-1/2})/2} e^{(x^{-1/2}Zx^{-1/2})/2} = e^{x^{-1/2}Zx^{-1/2}}. \end{aligned}$$

Then $x^{-1/2} Z x^{-1/2} = \log(x^{-1/2} y x^{-1/2})$ so we must prove $\|\log x - \log y\| \leq \|\log(x^{-1/2} y x^{-1/2})\|$ or, changing x into x^{-1},

$$\|\log x + \log y\| \leq \|\log(x^{1/2} y x^{1/2})\|.$$

Replacing x, y by kx, ky with k a large positive number allows us to assume without loss of generality that $\log x \geq 0$ and $\log y \geq 0$. Then, the last inequality is equivalent to

$$\|e^{\log x + \log y}\| \leq \|x^{1/2} y x^{1/2}\|.$$

But this is equivalent to Segal's inequality (see [3, Theorem X.57, p. 260, vol. II], or [4])

$$(**) \qquad \|e^{X+Y}\| \leq \|e^{X/2} e^Y e^{X/2}\|$$

and this concludes the proof of Theorem 1. Obviously all steps in the proof can be reversed, so that $(**)$ implies $(*)$.

As an application of Theorem 1, consider a C^*-algebra A with a distinguished family $p_1, p_2, \ldots, p_n$ of selfadjoint orthogonal projections satisfying $p_i p_j = 0$ if $i \neq j$ and $p_1 + p_2 + \cdots + p_n = 1$. Let $B \subset A$ be the C^*-subalgebra of elements of A that commute with all p_i and $H \subset A$ be the Banach subspace of elements $h \in A$ satisfying $p_i h p_i = 0$ for each i. Let also $E = \{e^h \colon h = h^* \in H\}$.

Theorem 2. *For each* $b > 0$ *in* B, *the distance* (*in the Finsler metric*) *from* b *to the submanifold* $E \subset A^+$ *is attained at* $1 \in E$.

Proof. Set $X = \log b$. By definition $X = X_1 + \cdots + X_n$, where $X_i = p_i X p_i$. Since $\|X\| = \max \|X_i\|$, we can assume that $\|X\| = \|X_1\|$, and accordingly, we choose a faithful representation of A in a Hilbert space L with the additional property that, setting $L = L_1 \oplus \cdots \oplus L_n$ with $L_i = p_i(L)$, the subspace L_1 contains a "norming eigenvector" for X_1, i.e., a unit vector $\xi \in L_1$ with $X_1 \xi = \pm \|X_1\| \xi$. Let Y be an arbitrary selfadjoint element of H. Then, by the definition of H, $Y\xi \in L_2 \oplus \cdots \oplus L_n$ and therefore $X\xi = X_1 \xi$ is perpendicular to $Y\xi$. As a consequence we have

$$d(b, 1) = \|X\| = \|X\xi\| \leq \|X\xi - Y\xi\| \leq \|X - Y\|.$$

Then using Theorem 1, we conclude that $d(b, 1) \leq d(b, e^Y)$ and we are done.

Remark. Notice that the tangent map to exp also increases norms. In fact it suffices to show this property for $a = 1$. For that we estimate

$$\left\| \frac{e^{Y+tZ} - e^Y}{t} \right\|_{e^Y} = \frac{1}{|t|} \|e^{-Y/2} e^{Y+tZ} e^{-Y/2} - I\|$$

using Segal's inequality $\|e^{-Y/2} e^{Y+tZ} e^{-Y/2}\| \geq \|e^{tZ}\|$. Assume that $t > 0$ and that $\max \operatorname{Spec}(Z) = \|Z\|$. Then $\|e^{tZ}\| = e^{t\|Z\|} \geq 1$. Hence in this case

$$\frac{1}{|t|} \|e^{-Y/2} e^{Y+tZ} e^{-Y/2} - I\| = \frac{1}{t}(\|e^{tZ}\| - 1) \geq \frac{1}{t}(e^{t\|Z\|} - 1) \geq \|Z\|.$$

Then

$$\lim_{t \to 0+} \left\| \frac{e^{Y+tZ} - e^Y}{t} \right\|_{e^Y} \geq \|Z\|.$$

For other Z's, change Z into $-Z$.

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