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ON THE COHOMOLOGY OF GENERALIZED HOMOGENEOUS SPACES

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Abstract. We observe that work of Gugenheim and May on the cohomology of classical homogeneous spaces G/H of Lie groups applies verbatim to the calculation of the cohomology of generalized homogeneous spaces G/H, where G is a finite loop space or a p-compact group and H is a "subgroup" in the homotopical sense.

We are interested in the cohomology $H^*(G/H; R)$ of a generalized homogeneous space G/H with coefficients in a commutative Noetherian ring R. Here G is a "finite loop space" and H is a "subgroup". More precisely, G and H are homotopy equivalent to ΩBG and ΩBH for path connected spaces BG and BH, and G/H is the homotopy fiber of a based map $f : BH \longrightarrow BG$. We always assume this much, and we add further hypotheses as needed. Such a framework of generalized homogeneous spaces was first introduced by Rector [10], and a more recent framework of p-compact groups has been introduced and studied extensively by Dwyer and Wilkerson [4] and others.

We ask the following question: How similar is the calculation of $H^*(G/H; R)$ to the calculation of the cohomology of classical homogeneous spaces of compact Lie groups? When $R = \mathbf{F}_p$ and H is of maximal rank in G, in the sense that $H^*(H; \mathbf{Q})$ and $H^*(G; \mathbf{Q})$ are exterior algebras on the same number of generators, the second author has studied the question in [8, 9]. There, the fact that $H^*(BG; R)$ need not be a polynomial algebra is confronted and results similar to the classical theorems of Borel and Bott [2, 3] are nevertheless proven. The purpose of this note is to begin to answer the general question without the maximal rank hypothesis, but under the hypothesis that $H^*(BG; R)$ and $H^*(BH; R)$ are polynomial algebras.

In fact, we shall not do any new mathematics. Rather, we shall merely point out that work of the first author [7] that was done before the general context was introduced goes far towards answering the question. Essentially the following theorem was announced in [7] and proven in [5]. We give a brief sketch of its proof and then return to a discussion of its applicability to the question on hand. Let BT^n be a classifying space of an n-torus T^n.

Theorem 1. *Assume the following hypotheses.*

(i) $\pi_1(BG)$ *acts trivially on* $H^*(G/H; R)$.

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(ii) *R is a PID and $H_*(BG;R)$ is of finite type over R.*
(iii) *$H^*(BG;R)$ is a polynomial algebra.*
(iv) *There is a map $e : BT^n \longrightarrow BH$ such that $H^*(BT^n;R)$ is a free $H^*(BH;R)$-module via e^*.*

Then $H^*(G/H;R)$ is isomorphic as an R-module to $\mathrm{Tor}_{H^*(BG;R)}(R, H^*(BH;R))$, regraded by total degree. Moreover, there is a filtration on $H^*(G/H;R)$ such that its associated bigraded R-algebra is isomorphic to $\mathrm{Tor}_{H^*(BG;R)}(R, H^*(BH;R))$.

Proof. The first two hypotheses ensure that $H^*(G/H;R)$ is isomorphic to the differential torsion product $\mathrm{Tor}_{C^*(BG;R)}(R, C^*(BH;R))$. (See, for example, [5, pp. 21–25]. The second hypothesis allows Lemma 3.2 there to be applied with $\mathbf{Z}$ replaced by R, thus allowing the finite type over $\mathbf{Z}$ hypothesis assumed there to be replaced by the finite type over R hypothesis assumed here.) Therefore there is an Eilenberg-Moore spectral sequence that converges from $\mathrm{Tor}_{H^*(BG;R)}(R, H^*(BH;R))$ to $H^*(G/H;R)$. The conclusion of the theorem is that this spectral sequence collapses at E_2 with trivial additive extensions, but not necessarily trivial multiplicative extensions. The last hypothesis and a comparison of spectral sequences argument essentially due to Baum [1] shows that the conclusion holds in general if it holds when $BH = BT^n$. (See [5, pp. 37–38].) Here the strange result [5, 4.1] gives that there is a morphism

$$g : C^*(BT^n;R) \longrightarrow H^*(BT^n;R)$$

of differential algebras such that g induces the identity map on cohomology and annihilates all $\cup_1$-products.

Now the general theory of differential torsion products of [5] kicks in. In modern language, implicit in the discussion of [6, p. 70], there is a model category structure on the category of A-modules for any DGA A over R such that every right A-module M admits a cofibrant approximation of a very precise sort. Namely, for any HA-free resolution $X \otimes_R HA \longrightarrow HM$ of HM, there is a cofibrant approximation $P = X \otimes_R A \longrightarrow M$. Grading is made precise in the cited sources. The essential point is that P is not a bicomplex but rather has differential with many components. When HA is a polynomial algebra and $M = R$, we can take X to be an exterior algebra with one generator for each polynomial generator of HA. Here, asssuming that A has a $\cup_1$-product that satisfies the Hirsch formula ($\cup_1$ is a graded derivation), [5, 2.2] specifies the required differential explicitly in terms of $\cup_1$-products. Using g to replace $C^*(BT^n;R)$ by $H^*(BT^n;R)$ in our differential torsion product, we see that the differential torsion product $\mathrm{Tor}_{C^*(BG;R)}(R, H^*(BT^n;R))$ is computed by exactly the same chain complex as the ordinary torsion product $\mathrm{Tor}_{H^*(BG;R)}(R, H^*(BT^n;R))$. (See [5, 2.3].) The conclusion follows. □

Hypotheses (i) and (ii) in Theorem 1 are reasonable and not very restrictive. Hypothesis (iii) is intrinsic to the method at hand. Note that $H^*(BG;R)$ can have infinitely many polynomial generators, so that G need not be finite. The key hypothesis is (iv). Here the following homotopical version of a theorem of Borel is relevant. It was first noticed by Rector [10, 2.2] that Baum's proof [1] of Borel's theorem is purely homotopical. A generalized variant of Baum's proof is given in [5, pp. 40–42]. That proof applies directly to give the following theorem. We state it for H and G as in the first paragraph. However, we are interested in its applicability to T^n and H in Theorem 1, and we restate it as a corollary in that special case.

Theorem 2. *Let R be a field and assume the following hypotheses.*

(i) $\pi_1(BG)$ *acts trivially on* $H^*(G/H;R)$.
(ii) $H^*(BH;R)$ *and* $H^*(BG;R)$ *are polynomial algebras on the same finite number of generators.*
(iii) $H^*(G/H;R)$ *is a finite dimensional R-module.*

Then $H^*(G/H;R) \cong R \otimes_{H^*(BG;R)} H^*(BH;R)$ *as an algebra and*

$$H^*(BH;R) \cong H^*(BG;R) \otimes_R H^*(G/H;R)$$

as a left $H^*(BG;R)$-module. In particular, $H^*(BH;R)$ is $H^*(BG;R)$-free.

Corollary 3. *Let R be a field and assume given a map $e: BT^n \longrightarrow BH$ that satisfies the following properties, where H/T^n is the fiber of e.*

(i) $\pi_1(BH)$ *acts trivially on* $H^*(H/T^n;R)$.
(ii) $H^*(BH;R)$ *is a polynomial algebra on n generators.*
(iii) $H^*(H/T^n;R)$ *is a finite dimensional R-module.*

Then $H^*(H/T^n;R) \cong R \otimes_{H^*(BH;R)} H^*(BT^n;R)$ *as an algebra and*

$$H^*(BT^n;R) \cong H^*(BH;R) \otimes_R H^*(H/T^n;R)$$

as a left $H^*(BH;R)$-module. In particular, $H^*(BT^n;R)$ is $H^*(BH;R)$-free.

When Corollary 3 applies, its conclusion gives hypothesis (iv) of Theorem 1. We comment briefly on applications to the integral and p-compact settings for the study of generalized homogeneous spaces.

Remark 4. A counterexample of Rector [10] shows that not all finite loop spaces H have (integral) maximal tori. When H does have a maximal torus, hypothesis (iii) of the corollary holds by definition. Assuming that H is simply connected, [9, 3.11] describes for which primes p $H^*(BH;\mathbf{Z})$ is p-torsion free, so that $H^*(BH;\mathbf{F}_p)$ is a polynomial algebra. If R is the localization of $\mathbf{Z}$ at the primes p for which $H^*(H;\mathbf{Z})$ is p-torsion free, then $H^*(BH;R)$ is also a polynomial algebra, and $H^*(BT;R)$ is a free $H^*(BH;R)$-module. That is, hypothesis (iv) of Theorem 1 holds for the localization of $\mathbf{Z}$ away from the finitely many "bad primes" for which $H^*(BH;\mathbf{F}_p)$ is not a polynomial algebra on n generators.

Remark 5. In the p-compact setting, taking $R = \mathbf{F}_p$, Dwyer and Wilkerson [4, 8.13, 9.7] prove that if H is connected, BH is $\mathbf{F}_p$-complete, $H^*(H;\mathbf{F}_p)$ is finite dimensional, and $H^*(H;\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}$ is an exterior algebra on n generators, then there is a map $e: BT^n \longrightarrow BH$ such that $H^*(H/T^n;\mathbf{F}_p)$ is finite dimensional. Here Corollary 3 applies whenever $H^*(BH;\mathbf{F}_p)$ is a polynomial algebra on n generators.

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