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LYUBEZNIK RESOLUTIONS AND THE ARITHMETICAL RANK OF MONOMIAL IDEALS

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Abstract. In this paper, we prove that the length of a Lyubeznik resolution of a monomial ideal gives an upper bound for the arithmetical rank of the ideal.

1. Introduction

Let S be a polynomial ring over a field K. Let I be a monomial ideal of S and $G(I) = \{m_1, m_2, \dots, m_\mu\}$ the minimal set of monomial generators of I. In general, it is unknown how to construct a minimal graded free resolution of S/I. In 1960, Taylor [16] discovered a graded free resolution of S/I, which is called the *Taylor resolution* of I:

$$T_\bullet : 0 \longrightarrow T_\mu \xrightarrow{d_\mu} T_{\mu-1} \xrightarrow{d_{\mu-1}} \cdots \xrightarrow{d_1} T_0 \longrightarrow S/I \longrightarrow 0,$$

where

$$T_0 = Se_\emptyset, \ T_s = \bigoplus_{1 \le i_1 < i_2 < \cdots < i_s \le \mu} Se_{i_1 i_2 \cdots i_s},$$

$$d_s(e_{i_1 i_2 \cdots i_s}) = \sum_{j=1}^{s} (-1)^{j-1} \frac{\mathrm{lcm}(m_{i_1}, \dots, m_{i_s})}{\mathrm{lcm}(m_{i_1}, \dots, \widehat{m_{i_j}}, \dots, m_{i_s})} e_{i_1 \cdots \widehat{i_j} \cdots i_s}.$$

Here $e_{i_1 i_2 \cdots i_s}$ $(1 \le i_1 < i_2 < \cdots < i_s \le \mu)$ are free basis elements of T_s, and the degree of $e_{i_1 i_2 \cdots i_s}$ is defined by

$$\deg e_{i_1 i_2 \cdots i_s} = \deg \mathrm{lcm}(m_{i_1}, m_{i_2}, \dots, m_{i_s}).$$

In 1988, Lyubeznik [13] constructed a graded free resolution of S/I as a subcomplex of the Taylor resolution of I. This complex is called a *Lyubeznik resolution*.

We recall the definition of a Lyubeznik resolution. Let $1 \le i_1 < i_2 < \cdots < i_s \le \mu$. If m_q does not divide $\mathrm{lcm}(m_{i_t}, m_{i_{t+1}}, \dots, m_{i_s})$ for all $t < s$ and for all $q < i_t$, then the symbol $e_{i_1 i_2 \cdots i_s}$ is said to be *L-admissible*. The *Lyubeznik resolution* of I is a subcomplex of the Taylor resolution of I generated by all L-admissible symbols. Note that a Lyubeznik resolution of I depends on the order of the generators $m_1, m_2, \dots, m_\mu$. We define the *L-length* of I as the minimum length of Lyubeznik resolutions of I. The Taylor resolution of I is far from being a minimal graded free resolution in general, but a Lyubeznik resolution of I often gives a minimal graded

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free resolution or a graded free resolution whose length is equal to the projective dimension of S/I.

The *arithmetical rank* of I is defined by

$$\operatorname{ara} I := \min \left\{ r \in \mathbb{N} \;:\; \text{there exist } a_1, \ldots, a_r \in I \text{ such that } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I} \right\}.$$

A trivial upper bound for $\operatorname{ara} I$ is the cardinality of the minimal set of monomial generators of I, denoted by $\mu(I) = \mu$, which is equal to the length of the Taylor resolution of I. In this paper, we prove the following theorem:

Theorem 1. *Let I be a monomial ideal of S. If the L-length of I is λ, then*

$$\operatorname{ara} I \leq \lambda.$$

It is known from Lyubeznik [12] that if I is a squarefree monomial ideal, then $\operatorname{pd}_S S/I \leq \operatorname{ara} I$, where $\operatorname{pd}_S S/I$ is the projective dimension of S/I. It is natural to ask when $\operatorname{ara} I = \operatorname{pd}_S S/I$ holds for a squarefree monomial ideal I. The author together with Terai and Yoshida ([9, 10]; see also [11]) has proved that $\operatorname{ara} I = \operatorname{pd}_S S/I$ for squarefree monomial ideals I with $\mu(I) - \operatorname{height} I \leq 2$. Barile [3, 4, 5, 6, 7], Barile and Terai [8], and Schmitt and Vogel [15] also proved the same equality for some classes of squarefree monomial ideals. Since the projective dimension of S/I is equal to the length of the minimal graded free resolution of S/I, we have the following corollary:

Corollary 2. *Let I be a squarefree monomial ideal of S. If the L-length of I is equal to the projective dimension of S/I, then*

$$\operatorname{ara} I = \operatorname{pd}_S S/I.$$

In particular, if the Lyubeznik resolution of I with respect to some order of monomial generators is minimal, then the same assertion is true.

In Section 2, we prove Theorem 1 and several corollaries. In Section 3, we give examples of squarefree monomial ideals I whose L-length is equal to the projective dimension of S/I; see Barile [1, 2]. We also show that for the Stanley–Reisner ideal I of the triangulation of the projective plane with 6 vertices, the L-length of I coincides with $\operatorname{ara} I$. Notice that Yan [17] proved that $\operatorname{ara} I = 4 > 3 = \operatorname{pd}_S S/I$ when $\operatorname{char} K \neq 2$.

2. Proof of Theorem 1

In this section, we prove Theorem 1, which is the main result in this paper.

Proof of Theorem 1. Let $G(I) = \{m_1, m_2, \ldots, m_\mu\}$ be the minimal set of monomial generators of I. We consider the Lyubeznik resolution of I with respect to this order.

To prove the theorem, it is enough to find λ elements $g_1, g_2, \ldots, g_\lambda$ such that

$$\sqrt{(g_1, g_2, \ldots, g_\lambda)} = \sqrt{I}.$$

We set

$$\begin{cases} g_1 = m_1, \\ g_2 = m_2 + \displaystyle\sum_{\substack{[i_1,i_2,\ldots,i_{\lambda-1}]\in L_{\lambda-1} \\ i_1\geq 3}} m_{i_1}m_{i_2}\cdots m_{i_{\lambda-1}}, \\ \qquad\vdots \\ g_\ell = m_\ell + \displaystyle\sum_{\substack{[i_1,i_2,\ldots,i_{\lambda-\ell+1}]\in L_{\lambda-\ell+1} \\ i_1\geq \ell+1}} m_{i_1}m_{i_2}\cdots m_{i_{\lambda-\ell+1}}, \\ \qquad\vdots \\ g_\lambda = m_\lambda + \displaystyle\sum_{\substack{[i_1]\in L_1 \\ i_1\geq \lambda+1}} m_{i_1} = m_\lambda + m_{\lambda+1} + \cdots + m_\mu, \end{cases}$$

where

$$L_s := \left\{ [i_1, i_2, \ldots, i_s] \in \mathbb{N}^s \ : \ \begin{array}{l} 1 \leq i_1 < i_2 < \cdots < i_s \leq \mu, \\ e_{i_1 i_2\cdots i_s} \text{ is } L\text{-admissible} \end{array} \right\}.$$

Put $J = (g_1, g_2, \ldots, g_\lambda)$. We prove that $m_\ell \in \sqrt{J}$ for all $\ell = 1, 2, \ldots, \mu$ by induction on ℓ. We need the following lemma:

Lemma 3. *Suppose* $[i_1, i_2, \ldots, i_s] \in L_s$. *Then:*

1. $[i_{j_1}, \ldots, i_{j_t}] \in L_t$ *for all* $t \leq s$ *and for all* $1 \leq j_1 < \cdots < j_t \leq s$.
2. *If* $i_1 > 1$, *then* $[1, i_1, i_2, \ldots, i_s] \in L_{s+1}$. *In particular, if* $[i_1, i_2, \ldots, i_\lambda] \in L_\lambda$, *then* $i_1 = 1$.
3. *Suppose* $\ell < i_1$. *If* $[\ell, i_1, i_2, \ldots, i_s] \notin L_{s+1}$, *then* $m_\ell m_{i_1} m_{i_2} \cdots m_{i_s}$ *is divisible by at least one of* $m_1, m_2, \ldots, m_{\ell-1}$.

Proof. These follow from the definition of L-admissibleness. □

The case $\ell = 1$ is clear because $m_1 = g_1$. For $\ell = 2$, we consider $m_2 g_2$. Then

$$m_2 g_2 = m_2^2 + \sum_{\substack{[i_1,i_2,\ldots,i_{\lambda-1}]\in L_{\lambda-1} \\ i_1\geq 3}} m_2 m_{i_1} m_{i_2} \cdots m_{i_{\lambda-1}} \in J.$$

Since $[2, i_1, i_2, \ldots, i_{\lambda-1}] \notin L_\lambda$ by Lemma 3 (2), the second term is divisible by m_1 by Lemma 3 (3). Hence $m_2^2 \in J$, and thus $m_2 \in \sqrt{J}$.

We assume $\ell > 2$ and $m_1, m_2, \ldots, m_{\ell-1} \in \sqrt{J}$. Set $\nu = \nu_\ell = \min\{\ell-2, \lambda-2\}$. Then we show that

$$\sum_{[\ell,i_2,\ldots,i_s]\in L_s} m_\ell m_{i_2} \cdots m_{i_s} \in \sqrt{J} \tag{2.1}$$

by descending induction on s ($\lambda - \nu \leq s \leq \lambda - 1$).

First, we consider $m_\ell g_2$. By a similar argument as in the case $\ell = 2$, we have (2.1) for $s = \lambda - 1$.

Next, we assume

$$\sum_{[\ell,i_2,\ldots,i_{s+1}]\in L_{s+1}} m_\ell m_{i_2} \cdots m_{i_{s+1}} \in \sqrt{J} \tag{2.2}$$

and prove (2.1). Then $m_\ell g_{\lambda-s+1} \in J$ implies that

$$m_\ell m_{\lambda-s+1} + \sum_{\substack{[i_1,i_2,\ldots,i_s]\in L_s \\ i_1 \geq \lambda-s+2}} m_\ell m_{i_1} m_{i_2} \cdots m_{i_s} \in J.$$

Since $\lambda - s + 1 \leq \nu + 1 < \ell$ by the definition of ν, we have

$$\sum_{[\ell,i_2,\ldots,i_s]\in L_s} m_\ell^2 m_{i_2} \cdots m_{i_s} + \sum_{\substack{[i_1,i_2,\ldots,i_s]\in L_s \\ i_1 > \ell}} m_\ell m_{i_1} m_{i_2} \cdots m_{i_s} \in \sqrt{J}.$$

The second term can be written in the following form:

$$\sum_{[\ell,i_1,i_2,\ldots,i_s]\in L_{s+1}} m_\ell m_{i_1} m_{i_2} \cdots m_{i_s} + \sum_{\substack{[i_1,i_2,\ldots,i_s]\in L_s \\ [\ell,i_1,i_2,\ldots,i_s]\notin L_{s+1}}} m_\ell m_{i_1} m_{i_2} \cdots m_{i_s}. \tag{2.3}$$

The first term of (2.3) is in $\sqrt{J}$ by assumption (2.2). The second term of (2.3) is in $\sqrt{J}$ by Lemma 3 (3). Therefore (2.1) is also satisfied for s. Hence, (2.1) is satisfied for all $s \geq \lambda - \nu$.

Now, we prove that $m_\ell \in \sqrt{J}$. If $\nu = \ell - 2$, then we consider $m_\ell g_\ell$. By a similar argument as above, we have

$$m_\ell^2 + \sum_{[\ell,i_1,i_2,\ldots,i_{\lambda-\ell+1}]\in L_{\lambda-\ell+2}} m_\ell m_{i_1} m_{i_2} \cdots m_{i_{\lambda-\ell+1}} \in \sqrt{J}.$$

Since (2.1) is satisfied for $s = \lambda - \nu = \lambda - \ell + 2$, we have $m_\ell^2 \in \sqrt{J}$ and so $m_\ell \in \sqrt{J}$ as required. For $\nu = \lambda - 2$, we consider $m_\ell g_\lambda$. By a similar argument as in the case of $\nu = \ell - 2$, we have $m_\ell \in \sqrt{J}$. □

Proof of Corollary 2. By Lyubeznik [12], we have $\operatorname{pd}_S S/I \leq \operatorname{ara} I$. On the other hand, our theorem gives the opposite inequality. □

We also have an upper bound on the arithmetical rank, which was proved by Terai.

Corollary 4 (Terai)**.** *Let I be a squarefree monomial ideal of S, and let $G(I) = \{m_1, m_2, \ldots, m_\mu\}$ be the minimal set of monomial generators of I. We set*

$$l = \max\left\{ l \ : \begin{array}{c} m_{j_1} \neq \operatorname{lcm}(m_{j_1}, m_{j_2}) \neq \cdots \neq \operatorname{lcm}(m_{j_1}, m_{j_2}, \ldots, m_{j_s}) \\ \textit{for some } m_{j_1}, m_{j_2}, \ldots, m_{j_s} \in G(I) \end{array} \right\}.$$

Then we have

$$\operatorname{ara} I \leq l.$$

Proof. Let λ denote the length of a Lyubeznik resolution of I. If $e_{i_1 i_2 \cdots i_\lambda}$ is L-admissible, then

$$m_{i_\lambda} \neq \operatorname{lcm}(m_{i_\lambda}, m_{i_{\lambda-1}}) \neq \cdots \neq \operatorname{lcm}(m_{i_\lambda}, m_{i_{\lambda-1}}, \ldots, m_{i_1})$$

by the definition. Therefore $\lambda \leq l$ holds and Corollary 2 gives the desired inequality. □

The next corollary was proved by Barile [1, 2].

Corollary 5 (Barile [1, Proposition 2.4], [2, Remark 3])**.** *Let I be a squarefree monomial ideal and let $G(I) = \{m_1, m_2, \ldots, m_\mu\}$ be the minimal set of monomial generators of I. If there exists an integer $s > 1$ such that m_1 divides $m_{i_1} \cdots m_{i_s}$ for all $2 \le i_1 < \cdots < i_s \le \mu$, then*

$$\operatorname{ara} I \le s.$$

Proof. The assumption implies that $L_{s+1} = \emptyset$. Then the assertion follows from Corollary 2. □

3. Examples

In this section, we give some examples of Lyubeznik resolutions.

For two L-admissible symbols $e_{i_1 \cdots i_s}$ and $e_{j_1 \cdots j_t}$, we say that

$$e_{i_1 \cdots i_s} \le e_{j_1 \cdots j_t}$$

if $i_1, \ldots, i_s$ is a subsequence of $j_1, \ldots, j_t$. This induces a partial order on the set of all L-admissible symbols. Barile [2, Remark 1] pointed out that a necessary and sufficient condition for a Lyubeznik resolution of I to be minimal is that for all maximal L-admissible symbols $e_{i_1 \cdots i_s}$,

$$\operatorname{lcm}(m_{i_1}, \ldots, m_{i_s}) \ne \operatorname{lcm}(m_{i_1}, \ldots, \widehat{m_{i_j}}, \ldots, m_{i_s}) \quad \text{for all } j = 1, \ldots, s.$$

First, we consider an ideal I whose Lyubeznik resolution is minimal. The first example shows that a Lyubeznik resolution of I is minimal for an ideal I with $\mu(I) - \operatorname{height} I \le 1$.

Example 6 (See [9, Theorem 2.1])**.** Let I be a squarefree monomial ideal with $\mu(I) - \operatorname{pd}_S S/I \le 1$. Then the L-length of I is equal to $\operatorname{pd}_S S/I$. In particular, we have $\operatorname{ara} I = \operatorname{pd}_S S/I$ by Corollary 2.

Moreover we assume that $\mu(I) - \operatorname{height} I \le 1$. The author classified these ideals in [9, Theorem 4.4] with Terai and Yoshida. Then it is easy to see that a Lyubeznik resolution of I is minimal.

Remark 7. For the ideal I in Example 6, there are many proofs of $\operatorname{ara} I = \operatorname{pd}_S S/I$. For example, we can also prove it by the method of Barile [2, Proposition 2].

When $\mu(I) - \operatorname{height} I = 2$, a Lyubeznik resolution of I is not necessarily minimal as the next example shows.

Example 8. Let $I = (m_1, m_2, m_3, m_4)$ be a squarefree monomial ideal with $\mu(I) - \operatorname{height} I = 2$. Assume that S/I is Cohen–Macaulay.

If m_1 divides $m_i m_j$ for all $2 \le i < j \le 4$ upon renumbering the generators, then the Lyubeznik resolution of I with respect to this order is minimal. Otherwise, the L-length of I is larger than the projective dimension of S/I, and thus Lyubeznik resolutions of I are not minimal for any order of generators.

Note that in both cases, $\operatorname{ara} I = \operatorname{pd}_S S/I = 2$ holds by [10, Proposition 4.5].

The next example was considered by Barile [1].

Example 9 (Barile [1, Example 2.6])**.** Let I be the squarefree monomial ideal generated by the following $n+2$ elements:

$$\begin{cases} m_i = x_1x_2x_{2i+1}x_{2i+2}, \quad i = 1, 2, \ldots, n-1, \\ m_n = x_1x_3x_5 \cdots x_{2n-1}x_{2n+1}, \\ m_{n+1} = x_1x_4x_6x_8 \cdots x_{2n-2}x_{2n}x_{2n+1}, \\ m_{n+2} = x_2x_3 \cdots x_{2n}x_{2n+1}. \end{cases}$$

Barile [1, 2] proved that $\operatorname{ara} I = \operatorname{pd}_S S/I = n$. She computed $\operatorname{pd}_S S/I$ by proving that the Lyubeznik resolution of I with this order is minimal.

For another example, Novik [14] proved that a Lyubeznik resolution is minimal for the matroid ideal of a finite projective space.

Secondly, we exhibit several ideals whose Lyubeznik resolutions are not necessarily minimal, but which have L-length equal to the projective dimension. Let λ be the length of the Lyubeznik resolution of I with respect to some order of monomial generators of I. A sufficient condition for $\lambda = \operatorname{pd}_S S/I$ to hold is that one of the L-admissible symbols $e_{i_1 \cdots i_\lambda}$ must satisfy

$$\operatorname{lcm}(m_{i_1}, \ldots, m_{i_\lambda}) \neq \operatorname{lcm}(m_{i_1}, \ldots, \widehat{m_{i_j}}, \ldots, m_{i_\lambda}) \quad \text{for all } j = 1, \ldots, \lambda. \tag{3.1}$$

The next example is a generalization of [10, Lemma 5.1].

Example 10. Let $I = (m_1, m_2, \ldots, m_\mu)$ be a squarefree monomial ideal with $\mu(I) - \operatorname{pd}_S S/I = 2$. We assume that m_im_j is divisible by one of $m_1, m_2, \ldots, m_{\mu-3}$ for all $\mu - 2 \leq i < j \leq \mu$. Then the L-length of I is equal to $\operatorname{pd}_S S/I$. In particular, we have $\operatorname{ara} I = \operatorname{pd}_S S/I$ by Corollary 2.

Remark 11. For an ideal I as in Example 10, we also have $\operatorname{ara} I = \operatorname{pd}_S S/I$ by the result of Schmitt–Vogel [15, Lemma].

The next example was considered by Barile [1, Example 2.7].

Example 12 (Barile [1, Example 2.7])**.** Let I be the squarefree monomial ideal generated by the following 8 elements:

$$x_1x_2x_3, x_1x_4x_5x_6, x_2x_7, x_3x_8, x_1x_9, x_4x_{10}, x_5x_{11}, x_6x_{12}.$$

Barile proved that $\operatorname{ara} I = \operatorname{pd}_S S/I = 6$.

In the left (resp. right) table below, the element in the ith column and jth row is $\beta^S_{i,i+j}(S/I) := \dim_K[\operatorname{Tor}^i_S(K, S/I)]_{i+j}$ (resp. the cardinality of the set $[t_1, \cdots, t_i] \in L_i : \deg e_{t_1 \cdots t_i} = i + j\}$):

	0	1	2	3	4	5	6
0:	1						
1:		6					
2:		1	18	3			
3:		1	7	34	13		
4:			3	17	46	32	6
5:						1	1

	0	1	2	3	4	5	6
0:	1						
1:		6					
2:		1	18	3			
3:		1	7	34	13		
4:			3	17	46	33	8
5:					1	3	1

The difference between these tables arises from the L-admissibility of e_{124678}, e_{123678} and e_{12678}. As these tables show, the Lyubeznik resolution of I is not minimal, but the L-length of I is equal to $\operatorname{pd}_S S/I = 6$.

The next example is a generalization of the ideals in [10, Subsection 4.4].

Example 13. Let j, k, ℓ, n be integers with $1 \leq j \leq k \leq \ell < n-2$. Let I be the squarefree monomial ideal generated by the following n elements:

$$\begin{aligned} m_1 &= x_1 \cdots x_k y_{\ell+1} \cdots y_{n-2}, \\ m_2 &= x_1 \cdots x_k y_j \cdots y_\ell, \\ m_{i+2} &= x_i y_i z_{t_i}, \qquad 1 \leq i \leq n-2. \end{aligned}$$

Set $x_i = 1$ for $k < i \leq n-2$ and $y_i = 1$ for $1 \leq i < j$. Then the product $m_3 \cdots m_n$ is divisible by m_1. We consider the product $m_3 \cdots \widehat{m_i} \cdots m_n$. When $i \leq \ell$, a product $m_2 m_3 \cdots \widehat{m_i} \cdots m_n$ is divisible by m_1. When $i > \ell$, a product $m_3 \cdots \widehat{m_i} \cdots m_n$ is divisible by m_2. This means that the L-length of I is at most $n-2$. Hence $\operatorname{ara} I \leq n-2$.

In particular, we have $\operatorname{ara} I = \operatorname{pd}_S S/I = n-2$ for the following cases:

(1) $z_{t_i} = z_i$ for all $i = 1, 2, \ldots, n-2$.
(2) $z_{t_k} = z_{t_{n-2}} = z_k$, and $z_{t_i} = z_i$ for $i \neq k, n-2$.

In fact, the ideal in case (1) satisfies $\mu(I) - \operatorname{height} I = 2$ (see [10, Subsection 4.4]). In case (2), $e_{1 \cdots \widehat{k+2} \cdots n-1}$ is L-admissible and satisfies (3.1).

Remark 14. Let $m_1, m_2, \ldots, m_n$ be squarefree monomials as in Example 13 and let w be a new variable. Put $T \subset \{3, 4, \ldots, n\}$ with $\sharp T \geq 2$. We set $I' = (m_1', m_2', \ldots, m_n')$, where

$$\begin{cases} m_1' = m_1 w, \\ m_2' = m_2, \\ m_i' = m_i w, & \text{if } i \in T, \\ m_i' = m_i, & \text{if } i \in \{3, 4, \ldots, n\} \setminus T. \end{cases}$$

Then the same assertion as in Example 13 is true. For example, for the ideal I' generated by the following 6 elements, we have $\operatorname{ara} I' = \operatorname{pd}_S S/I' = 4$:

$$x_1 x_2 y_4 w,\ x_1 x_2 y_2 y_3,\ x_1 z_1 w,\ x_2 y_2 z_2 w,\ y_3 z_3,\ y_4 z_2.$$

Moreover, for this ideal I', it seems to be difficult to show $\operatorname{ara} I' = \operatorname{pd}_S S/I'$ by the method of Barile ([1, Proposition 1.1], [2, Propositions 1, 2]).

Finally, we give an ideal I whose L-length is not equal to the projective dimension of S/I but is equal to the arithmetical rank of I. In the following example, we consider the Stanley–Reisner ideal I of the triangulation of the projective plane with 6 vertices. The projective dimension of S/I depends on the characteristic of K, and Yan [17] proved that $\operatorname{ara} I = 4 > 3 = \operatorname{pd}_S S/I$ if $\operatorname{char} K \neq 2$. Our theorem provides the best upper bound for $\operatorname{ara} I$.

Example 15 (Yan [17])**.** Let I be the squarefree monomial ideal generated by the following 10 elements:

$$x_1x_2x_3, x_1x_2x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, x_2x_3x_4, x_2x_4x_6, x_2x_5x_6, x_3x_4x_5, x_3x_5x_6.$$

This ideal is the Stanley–Reisner ideal of the triangulation of the projective plane with 6 vertices. Then a minimal graded free resolution of I is given by the following left (resp. right) diagram if $\operatorname{char} K \neq 2$ (resp. if $\operatorname{char} K = 2$):

	0	1	2	3
0:	1			
1:				
2:		10	15	6

	0	1	2	3	4
0:	1				
1:					
2:		10	15	6	1
3:				1	

Hence, the projective dimension of S/I is given by

$$\operatorname{pd}_S S/I = \begin{cases} 3 & \text{if } \operatorname{char} K \neq 2, \\ 4 & \text{if } \operatorname{char} K = 2. \end{cases}$$

Yan [17] proved that $\operatorname{ara} I = 4$ for any characteristic of K.

On the other hand, the Lyubeznik resolution of I with respect to this order is given by the following diagram:

	0	1	2	3	4
0:	1				
1:					
2:		10	15	18	9
3:			12	9	

In particular, the length is 4. Therefore Theorem 1 implies that the L-length of I coincides with $\operatorname{ara} I$.

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