BU Algebra Qualifying Exam, Fall 2004
September 4, 2004, SR 242, 9:00-1:00

Dr. Dugas and Dr. Arnold Name________________________

(A) Write an essay (please use complete sentences) outlining the development from the definition of PID to the Jordan Normal Form of matrices over fields.

Clearly state which problems you are submitting in parts (B) through (E)!

(B) Prove 5 of the following 10 statements: (Any theorems you use should be clearly identified, either by name or statement)

B.1: If R is a PID, then R_R is Noetherian.

B.2: Let R be a ring with identity and X, Y non-isomorphic simple R-modules. Then $Hom_R(X, Y) = 0$.

B.3: If M_R is a Noetherian R-module and K a submodule of M, then M/K and K are Noetherian R-modules.

B.4: If G is a finite group of order p^k, p a prime, then G is nilpotent.

B.5: Let R be an integral domain that is not a PID. Then R has an ideal I that is maximal with respect to the property of being non-principal.

B.6: Let $K \subseteq E \subseteq F$ be fields. If E is algebraic over K and F is algebraic over E, then F is algebraic over K.

B.7: Let $K \subseteq F$ be fields with $[F : K] = n < \infty$ and $f(x) \in K[x]$ be an irreducible polynomial of degree $d \geq 2$. If $\gcd(d, n) = 1$, then $f(x)$ has no root in F.

B.8 Let $K \subseteq F$ be fields and let $E = \{a \in F : a \text{ algebraic over } K\}$. Then E is a subfield of F.

B.9: Let K be a Galois extension of Q such that $Gal(F : Q)$ is isomorphic to the symmetric group S_5. Then K is the splitting field of a polynomial of degree 5 over Q.

B.10: Let A, B be 3×3 complex matrices that have the same characteristic and minimal polynomial. Then A, B have the same Jordan normal form.

(C) State accurately and completely (w/o proof) 5 of the following 10 results.

C.1: The Fundamental Theorem of Galois Theory.

C.2: The Artin-Wedderburn Theorem for semisimple rings.

C.3 The Fundamental Theorem for Finitely Generated Modules over PID's.

C.4: The Stacked Basis Theorem.

C.5: The three Sylow Theorems.

C.6: The Orbit-Stabilizer Theorem.

C.7: The structure theorem for finite nilpotent groups.

C.8: Hilbert's Basis Theorem.

C.9: The Jordan-Hölder Theorem for modules.

C.10: Hilbert's Nullstellensatz.

(D) Complete 5 of the 10 indicated definitions

D.1: A module M_R is Artinian, if...

D.2: $K \subseteq F$ is a Galois extension, if...

D.3: The ring R is semi-simple, if...

D.4: The ring R is Dedekind, if...

D.5: The module M_R has a decomposition series, if...

D.6: $K \subseteq F$ is a normal field extension, if...

D.7: The subgroup P of the finite group G is a p-Sylow subgroup, if...

D.8: The ring R is a division ring, if...

D.9: The group G acts on the set T, if...

D.10: If G is a group, then the center of G is...

(E) Work 5 of the following 10 problems. Show your work and explain your solution.

E.1 Find the structure of $(Z \times Z)/span_Z\{(2,6),(-2,1)\}$, where Z is the ring of integers.

E.2: How many elements of order 6 are there in S_6 ? A_6 ?

E.3: Show that a group of order $3 \cdot 23 \cdot 29$ is not simple.

E.4: Show that a group of order 36 is not simple.

E.5: Show that a group of order $105 = 3 \cdot 5 \cdot 7$ has a normal 7-Sylow subgroup.

E.6: How many units are there in $Z/60Z$?

E.7: Why is $a = 1 + \sqrt{5} \in Q[\sqrt{5}]$ an algebraic integer ? What is the minimal polynomial of a ?

E.8: Let $D = Z[\sqrt{5}]$ and $F = Q[\sqrt{5}]$. Show that $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.

E.9: Find $[F : Q]$, where F is the splitting field of $x^4 - 5$ over Q.

E.10: Find the minimal polynomial of $a = \sqrt[3]{2 + \sqrt{2}}$ over Q.