Teichmüller's theorem and its applications

by

Daniel Berlyne

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Contents

1 Overview

Teichmüller theory originally arose from Oswald Teichmüller's development of the theory of quasiconformal mappings in the late 1930s,[20] which was later improved upon and more rigorously formulated by Lars Ahlfors and Lipman Bers in the 1950s.[3][5] Central to the theory is the notion of Teichmüller space, which is closely related to moduli space and gives a solid foundation for study of this area. As well as Teichmüller himself, Klein, Fricke, Fenchel and Nielsen were the main contributors to the concept of Teichmüller space, and their work in fact preceded that of Teichmüller. An overview of the history of Teichmüller theory can be found in [1].

The aim of this project is to prove Teichmüller's theorem, which states that given a quasiconformal homeomorphism f between two closed Riemann surfaces of genus at least 2, there exists a canonical unique mapping homotopic to f with minimal dilatation. This is commonly split into two parts: Teichmüller's existence theorem and Teichmüller's uniqueness theorem. The uniqueness part was originally considered by Teichmüller as a generalisation of the already-known result for rectangles proved by Grötzsch in 1928, while proof of existence requires significantly more preparation, and involves delving into the analysis of the Beltrami equation.

We shall therefore work our way towards a proof of the uniqueness theorem first, and to this end we will develop some theory of quasiconformal mappings, quadratic differentials and foliations in the first three sections. In particular, we construct natural coordinates for quadratic differentials, a very useful tool which we will frequently use. We use this to establish a relation between quadratic differentials and foliations, and to define a singular Euclidean metric on a Riemann surface. We also establish a lower bound for the dimension of the vector space of quadratic differentials. With this under our belt, in section 5 we state both parts of Teichmüller's theorem. We then prove Grötzsch's problem, and by adapting this proof we also prove the uniqueness theorem. In section 6 we explore the Beltrami equation, solving it by means of Cauchy and Hilbert transforms. We also prove a key theorem required for the proof of Teichmüller's existence theorem: the measurable Riemann mapping theorem. Section 7 shall then assimilate all of our acquired knowledge to focus on proving the existence theorem, as well as defining Teichmüller space. Having proved Teichmüller's theorem, in section 8 we will then apply it to define a metric on Teichmüller space, which is essential in taking this theory further. Using this metric, we will introduce Teichmüller lines and Teichmüller discs, as well as suggesting some potential follow-on topics.

The primary sources for this project are Farb and Margalit's book [9] and Ahlfors' lecture series [4], and hence the approach we shall adopt is similar to that described in these texts.

2 Quasiconformal mappings

We begin by recalling the definition of a conformal mapping.

Definition 2.1. *A mapping $f\colon U \to \mathbb{C}$ on an open set $U \subseteq \mathbb{C}$ is* conformal *if f is holomorphic with non-vanishing derivative on U.*

Informally, a conformal mapping locally preserves signed angles. The idea of quasiconformality is to generalise the notion of a conformal mapping to allow some distortion of angles in a controlled manner. In many instances this does not affect the validity of theorems concerning conformal mappings, and in fact because quasiconformal mappings are less rigid in their definition, they are much easier to use as a tool.

Before defining a quasiconformal mapping, we shall quickly recap an idea from complex analysis.[4, p. 6] Let $f\colon U \to f(U)$ be a continuously differentiable homeomorphism between open subsets of $\mathbb{C}$, let $z = x + iy$ be a coordinate on U at a point z_0, and let $f(z) = w = u + iv$ be a coordinate on $f(U)$ at $f(z_0)$. Then f induces a linear mapping between the corresponding tangent spaces, and in particular a linear mapping of the differentials

$$du = u_x dx + u_y dy \quad \text{and} \quad dv = v_x dx + v_y dy.$$

From this we can derive the complex form

$$dw = df = f_z dz + f_{\bar{z}} d\bar{z},$$

where $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$. Note that here $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$. Notice too that

$$(\bar{f})_z = (u - iv)_z = \frac{1}{2}((u_x - iv_x) - i(u_y - iv_y)) = \frac{1}{2}((u_x - iv_x) + i(-u_y + iv_y)) = \overline{(f_{\bar{z}})},$$
$$(\bar{f})_{\bar{z}} = (u - iv)_{\bar{z}} = \frac{1}{2}((u_x - iv_x) + i(u_y - iv_y)) = \frac{1}{2}((u_x - iv_x) - i(-u_y + iv_y)) = \overline{(f_z)}.$$

It will also be useful to establish a chain rule for differentiation with respect to z and $\bar{z}$. Indeed,

$$\begin{aligned} d(g \circ f) &= (g_z \circ f) df + (g_{\bar{z}} \circ f) d\bar{f} \\ &= (g_z \circ f) \cdot (f_z dz + f_{\bar{z}} d\bar{z}) + (g_{\bar{z}} \circ f) \cdot ((\bar{f})_z dz + (\bar{f})_{\bar{z}} d\bar{z}) \\ &= (f_z \cdot (g_z \circ f) + (\bar{f})_z \cdot (g_{\bar{z}} \circ f)) dz + (f_{\bar{z}} \cdot (g_z \circ f) + (\bar{f})_{\bar{z}} \cdot (g_{\bar{z}} \circ f)) d\bar{z}, \end{aligned}$$

hence

$$\begin{aligned} (g \circ f)_z &= (g_z \circ f) \cdot f_z + (g_{\bar{z}} \circ f) \cdot (\bar{f})_z \\ (g \circ f)_{\bar{z}} &= (g_z \circ f) \cdot f_{\bar{z}} + (g_{\bar{z}} \circ f) \cdot (\bar{f})_{\bar{z}} \end{aligned} \tag{2.1}$$

describes our desired chain rule.

The linear map df maps the unit circle in the z–plane to an ellipse in the w-plane. We call the amount by which the circle is distorted the *complex dilatation* of f at z_0, which is defined as

$$\mu_f = \frac{f_{\bar{z}}}{f_z},$$

as seen in [11, p. 3]. Notice that f is conformal if and only if it is holomorphic with non-vanishing derivative, or in other words $f_{\bar{z}} \equiv 0$ (which is equivalent to satisfying the Cauchy-Riemann equations) and f_z is nowhere zero. Hence f is conformal if and only if $\mu_f \equiv 0$, that is, there is no distortion.[11, p. 3]

Notice also that $|f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x$ is the Jacobian of f (see page 15 for computation), so f is orientation-preserving if and only if $|f_z| > |f_{\bar{z}}|$, or in other words $|\mu_f| < 1$. [4, p. 6]

Definition 2.2. [11, p. 3] *Let $U \subseteq \mathbb{C}$ be open and let $f\colon U \to f(U)$ be an orientation-preserving continuously differentiable homeomorphism. Then the* dilatation *of f at $z_0 \in U$ is defined to be*

$$\mathrm{Dil}_f(z_0) = \frac{|f_z(z_0)| + |f_{\bar{z}}(z_0)|}{|f_z(z_0)| - |f_{\bar{z}}(z_0)|} = \frac{1 + |\mu_f(z_0)|}{1 - |\mu_f(z_0)|} \geq 1$$

and the maximal dilatation *of f is*

$$K_f = \sup_{z \in U} \mathrm{Dil}_f(z).$$

If $K_f < \infty$ then we call f quasiconformal*, and f is called K-quasiconformal if $K_f \leq K$.*

Note that, in particular, a mapping is conformal if and only if it is 1-quasiconformal.

Locally, the mapping f can be thought of as a stretch in a particular direction. The dilatation of f gives its stretch factor — however, we still do not know the direction of maximal stretch. Let $Df(z_0)$ be the $\mathbb{R}$-linear map given by the Jacobian matrix. Then

$$Df(z_0)(u) = f_z(z_0)u + f_{\bar{z}}(z_0)\bar{u}$$

so by the triangle inequality

$$|Df(z_0)(e^{i\theta})| \leq |f_z(z_0)| + |f_{\bar{z}}(z_0)|.$$

This is an equality if and only if

$$\frac{f_{\bar{z}} e^{-i\theta}}{f_z e^{i\theta}} = \mu_f e^{-2i\theta} = |\mu_f| e^{\mathrm{i}\arg\mu_f} e^{-2i\theta}$$

is real and positive at z_0, or in other words $\theta = \frac{1}{2}\arg\mu_f$. Hence f stretches maximally in direction $\frac{1}{2}\arg\mu_f$. (This argument is based on [14, p. 60].)

It is possible to extend the definition of quasiconformality to mappings between Riemann surfaces:

Definition 2.3. *A mapping $f\colon X \to Y$ between Riemann surfaces is said to be K-quasiconformal at $p \in X$ if there exist charts $z\colon U_p \to \mathbb{C}$ and $w\colon V_q \to \mathbb{C}$ around p and $q = f(p)$, respectively, such that $w \circ f \circ z^{-1}$ is K-quasiconformal in the usual sense. Then f is called K-quasiconformal if it is K-quasiconformal at all points $p \in X$.*

Suppose f is a quasiconformal map and g is a conformal map. Then we claim that $K_{g\circ f} = K_f = K_{f\circ g}$ where the compositions make sense. Indeed, by the chain rule (2.1) and since $\mu_g = 0$ we have

$$\begin{aligned}\mu_{g\circ f} = \frac{(g\circ f)_{\bar z}}{(g\circ f)_z} &= \frac{f_{\bar z}(g_z\circ f) + \bar f_{\bar z}(g_{\bar z}\circ f)}{f_z(g_z\circ f) + \bar f_z(g_{\bar z}\circ f)} \\ &= \frac{f_{\bar z} + \bar f_{\bar z}(\mu_g\circ f)}{f_z + \bar f_z(\mu_g\circ f)} \\ &= \frac{f_{\bar z}}{f_z} \\ &= \mu_f,\end{aligned}$$

and so $K_{g\circ f} = K_f$. On the other hand,

$$\begin{aligned}\mu_{f\circ g} &= \frac{g_{\bar z}(f_z\circ g) + \bar g_{\bar z}(f_{\bar z}\circ g)}{g_z(f_z\circ g) + \bar g_z(f_{\bar z}\circ g)} \\ &= \frac{\mu_g(f_z\circ g) + (\overline{g_z}/g_z)(f_{\bar z}\circ g)}{f_z\circ g + (\overline{g_{\bar z}}/g_z)(f_{\bar z}\circ g)} \\ &= \frac{f_{\bar z}\circ g}{(g_z/\overline{g_z})(f_z\circ g) + \overline{\mu_g}(f_{\bar z}\circ g)} \\ &= \frac{\overline{g_z}}{g_z}\frac{f_{\bar z}\circ g}{f_z\circ g} \\ &= \frac{\overline{g_z}}{g_z}\mu_f\circ g,\end{aligned}$$

hence $|\mu_{f\circ g}| = |\mu_f\circ g|$, and so $K_{f\circ g} = K_f$. Therefore Definition 2.3 is independent of the choice of charts, since transition maps are conformal.

Proposition 2.4. [9, p. 298] *Let X be a Riemann surface. Then the set of quasiconformal homeomorphisms $X \to X$ forms a group under composition, denoted $QC(X)$.*

Proof. It is sufficient to show that the inverse of a quasiconformal homeomorphism is quasiconformal and the composition of two quasiconformal homeomorphisms is quasiconformal. Let $f, g\colon X \to X$ be K_f-quasiconformal and K_g-quasiconformal homeomorphisms, respectively. We claim that $K_{f^{-1}} = K_f$, meaning f^{-1} is quasiconformal. To show this, write f in local coordinates as $f(x+iy) = u(x+iy) + iv(x+iy)$. Then we can write f^{-1} in the form $f^{-1}(u+iv) = x(u+iv) + iy(u+iv)$. We obtain the following relation between the partial derivatives of f and f^{-1}:

$$\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} = \frac{1}{u_x v_y - v_x u_y} \begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}$$

which implies

$$\begin{aligned}
(f^{-1})_z = \frac{1}{2}\left((f^{-1})_u - i(f^{-1})_v\right) &= \frac{1}{2}\left(\frac{v_y - iu_y + iv_x + u_x}{u_x v_y - v_x u_y}\right) \\
&= \frac{1}{2}\left(\frac{f_x - if_y}{u_x v_y - v_x u_y}\right) \\
&= \frac{f_z}{u_x v_y - v_x u_y}
\end{aligned}$$

and similarly

$$(f^{-1})_{\bar{z}} = \frac{-f_{\bar{z}}}{u_x v_y - v_x u_y}$$

hence $\mu_{f^{-1}}(z) = -\mu_f(f^{-1}(z))$ and so $\mathrm{Dil}_{f^{-1}}(z) = \mathrm{Dil}_f(f^{-1}(z))$ at each point, therefore in particular $K_{f^{-1}} = K_f$.

We also claim that $K_{g\circ f} \leq K_g K_f$, meaning $g \circ f$ is $K_g K_f$-quasiconformal. This is proved using the chain rule (2.1). First note that

$$\mathrm{Dil}_g \mathrm{Dil}_f = \left(\frac{1+|\mu_g|}{1-|\mu_g|}\right)\left(\frac{1+|\mu_f|}{1-|\mu_f|}\right) = \frac{(|g_z| + |g_{\bar{z}}|)(|f_z| + |f_{\bar{z}}|)}{(|g_z| - |g_{\bar{z}}|)(|f_z| - |f_{\bar{z}}|)}.$$

On the other hand,

$$\mu_{g\circ f} = \frac{(g\circ f)_{\bar{z}}}{(g\circ f)_z} = \frac{f_{\bar{z}}(g_z \circ f) + \bar{f}_{\bar{z}}(g_{\bar{z}} \circ f)}{f_z(g_z \circ f) + \bar{f}_z(g_{\bar{z}} \circ f)} = \frac{f_{\bar{z}}(g_z \circ f) + \overline{f_z}(g_{\bar{z}} \circ f)}{f_z(g_z \circ f) + \overline{f_{\bar{z}}}(g_{\bar{z}} \circ f)},$$

therefore

$$\begin{aligned}
\mathrm{Dil}_{g\circ f} = \frac{1+|\mu_{g\circ f}|}{1-|\mu_{g\circ f}|} &= \frac{|f_z(g_z\circ f) + \overline{f_{\bar{z}}}(g_{\bar{z}}\circ f)| + |f_{\bar{z}}(g_z\circ f) + \overline{f_z}(g_{\bar{z}}\circ f)|}{|f_z(g_z\circ f) + \overline{f_{\bar{z}}}(g_{\bar{z}}\circ f)| - |f_{\bar{z}}(g_z\circ f) + \overline{f_z}(g_{\bar{z}}\circ f)|} \\
&\leq \frac{|f_z||g_z\circ f| + |\overline{f_{\bar{z}}}||g_{\bar{z}}\circ f| + |f_{\bar{z}}||g_z\circ f| + |\overline{f_z}||g_{\bar{z}}\circ f|}{|f_z||g_z\circ f| + |\overline{f_{\bar{z}}}||g_{\bar{z}}\circ f| - |f_{\bar{z}}||g_z\circ f| - |\overline{f_z}||g_{\bar{z}}\circ f|} \\
&= \frac{|f_z||g_z\circ f| + |f_{\bar{z}}||g_{\bar{z}}\circ f| + |f_{\bar{z}}||g_z\circ f| + |f_z||g_{\bar{z}}\circ f|}{|f_z||g_z\circ f| + |f_{\bar{z}}||g_{\bar{z}}\circ f| - |f_{\bar{z}}||g_z\circ f| - |f_z||g_{\bar{z}}\circ f|} \\
&= \frac{(|g_z\circ f| + |g_{\bar{z}}\circ f|)(|f_z| + |f_{\bar{z}}|)}{(|g_z\circ f| - |g_{\bar{z}}\circ f|)(|f_z| - |f_{\bar{z}}|)},
\end{aligned}$$

and so $K_{g\circ f} \leq K_g K_f$, as required. ☐

3 Quadratic differentials

Quadratic differentials are required for the statement of Teichmüller's theorem. They also feature prominently throughout this paper, so we will spend some time getting to know them.

Definition 3.1. (Generalisation of [9, p. 389].) *Let $\{z_\alpha\colon U_\alpha \to \mathbb{C}\}$ be an atlas for a Riemann surface X. A* holomorphic (m,n)–differential *on X is a collection of expressions $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$ such that each $\phi_\alpha\colon z_\alpha(U_\alpha) \to \mathbb{C}$ is holomorphic with finitely many zeros, and*

$$\phi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^m\left(\overline{\frac{dz_\beta}{dz_\alpha}}\right)^n = \phi_\alpha(z_\alpha) \tag{3.1}$$

for any pair of coordinate charts z_α, z_β.

A $(2,0)$–differential is known as a *quadratic differential*, which we shall scrutinise in this section. Later on we will also encounter $(-1,1)$–differentials, which are called *Beltrami differentials.*[11, p. 8]

An alternative way to describe a holomorphic quadratic differential q on X is as a holomorphic map from the holomorphic tangent bundle of X to $\mathbb{C}$.[9, p. 309] Suppose that q is given by $q(z) = \phi(z)dz^2$ in some local coordinates around $x_0 \in X$ with $z(x_0) = z_0$. Let v be a tangent vector to X at x_0, given by $\alpha \in \mathbb{C} \approx T_{z_0}(\mathbb{C})$. Then we say

$$q(v) = \phi(z_0)\alpha^2. \tag{3.2}$$

In particular, it follows that $q(v) = q(-v)$ for all $v \in T_{x_0}(X)$.

3.1 Natural coordinates

It is often useful to define local coordinates for a holomorphic quadratic differential q on a compact Riemann surface X such that it has the form $q(z) = z^n dz^2$ for some $n \geq 0$. These are called *natural coordinates*. The procedure for producing these is as follows.

The first part of this construction closely follows [14, p. 28]. Let $p \in X$ be a point such that $q(p) \neq 0$, and choose a chart $z\colon U \to \mathbb{C}$ around p such that $z(p) = 0$. Write $q(z) = \phi(z)dz^2$ in this chart. Since q has finitely many zeros, we may assume that it has no zeros in this chart. Define a function $\eta\colon z(U) \to \mathbb{C}$ by

$$\eta(z) = \int_0^z \sqrt{\phi(w)}dw.$$

Since ϕ is non-vanishing on U, we are able to choose a branch of the square root function, so this is well-defined.

We then have $\eta'(z) = \sqrt{\phi(z)}$ non-zero on U, hence η is an immersion. Moreover, restricting to a small enough neighbourhood $W \subseteq z(U)$ containing 0, we can assume that η is injective. The *natural coordinate* at p is then defined to be $\zeta := \eta \circ z$. Notice that

$$d\zeta^2 = (\eta'(z)dz)^2 = \phi(z)dz^2,$$

hence $q(\zeta) = d\zeta^2$, as desired.

Suppose ω is another natural coordinate at p, defined on a chart overlapping with that of ζ, and consider the transition map $h := \omega \circ \zeta^{-1}$. Then $\omega = h \circ \zeta$ and on the overlap we have

$$d\zeta^2 = d\omega^2 = h'(\zeta)^2 d\zeta^2.$$

Hence $h'(\zeta)^2 = 1$, which means $\omega = h(\zeta) = \pm\zeta + c$ for some constant $c \in \mathbb{C}$. Natural coordinates are therefore unique up to translation and rotation through an angle of π.

Now let $p \in X$ be a zero of q of order n. In this case it is possible to find a local coordinate ζ such that $q(\zeta) = \zeta^n d\zeta^2$. The following construction is based on [19, §6]. Again, choose a chart $z\colon U \to \mathbb{C}$ around p such that $z(p) = 0$, and write $q(z) = \phi(z)dz^2$ in this chart. Then $\phi(z)$ has a zero of order n at 0, hence it has Laurent expansion

$$\phi(z) = \sum_{k=n}^{\infty} a_k z^k = z^n \sum_{k=0}^{\infty} a_{k+n} z^k$$

with $a_n \neq 0$. Thus, defining η as before, we have

$$\eta(z) = \int_0^z w^{\frac{n}{2}} \left(\sum_{k=0}^{\infty} a_{k+n} w^k \right)^{\frac{1}{2}} dw = z^{\frac{n+2}{2}} \sum_{k=0}^{\infty} b_k z^k$$

for some b_k. Note that the term inside the parentheses, which we shall call S, can be assumed to be non-zero if a small enough neighbourhood of the origin is taken. Thus we can choose a particular branch of the square root of S to give a new Laurent series, so the latter equality is justified. However, $\eta(z)$ is only defined away from the origin because of the $z^{\frac{n+2}{2}}$ term. Now define

$$\zeta := z \left(\frac{n+2}{2} \sum_{k=0}^{\infty} b_k z^k \right)^{\frac{2}{n+2}},$$

choosing a particular branch of the right hand side in a small enough neighbourhood of the origin, so that $\zeta^{\frac{n+2}{2}} = \frac{n+2}{2}(\eta \circ z)$. Then, differentiating both sides and squaring, we have

$$\left(\frac{n+2}{2} \right)^2 \zeta^n d\zeta^2 = \left(\frac{n+2}{2} \right)^2 \eta'(z)^2 dz^2 = \left(\frac{n+2}{2} \right)^2 \phi(z) dz^2.$$

Therefore we have $q(\zeta) = \zeta^n d\zeta^2$, as desired. Again, we call this a natural coordinate at p.

Note than in particular, ζ is zero at p, since we know q has a zero at p.

Taking another natural coordinate ω at p, defined on a chart overlapping with that of ζ, consider the transition map $h = \omega \circ \zeta^{-1}$. Then

$$\zeta^n d\zeta^2 = \omega^n d\omega^2 = h(\zeta)^n h'(\zeta)^2 d\zeta^2,$$

hence

$$\zeta^{\frac{n}{2}} = h(\zeta)^{\frac{n}{2}} h'(\zeta),$$

whereby integration of both sides gives

$$\omega^{\frac{n+2}{2}} = h(\zeta)^{\frac{n+2}{2}} = \zeta^{\frac{n+2}{2}} + c$$

for some constant $c \in \mathbb{C}$. But since our natural coordinates are constructed around a zero of q, any natural coordinate must have a zero at this point, so $c = 0$. Hence

$$\omega = \exp\left(\frac{2\pi i k}{n+2}\right)\zeta$$

for some $k \in \{1, 2, \ldots, n+1\}$. Therefore this time the natural coordinate is unique up to rotation through an angle of $\frac{2\pi k}{n+2}$ for $k \in \{1, 2, \ldots, n+1\}$.

Remark. *Given a Riemann surface X, the set of holomorphic quadratic differentials on X forms a vector space (this follows trivially from the definition of a holomorphic quadratic differential).*[9, p. 317] *We denote this by QD(X). We can obtain a lower bound on the dimension of QD(X), but first we require some knowledge of foliations and their relation to quadratic differentials.*

3.2 Induced metric

Natural coordinates for a holomorphic quadratic differential q on a Riemann surface X induce a singular Euclidean metric on the surface, that is, a metric which is flat everywhere except at a finite number of points (the zeros of q).[9, p. 312] In particular, if $q(z) = z^n dz^2$ in natural coordinates at a point $p \in X$ for some $n \geq 0$, then the area form of this metric is

$$\frac{1}{2i}|z^n|\overline{dz} \wedge dz = (x^2 + y^2)^{\frac{n}{2}} dx \wedge dy$$

and the length form is

$$|z^n|^{\frac{1}{2}}|dz| = (x^2 + y^2)^{\frac{n}{4}} \sqrt{dx^2 + dy^2}.$$

The metric is described geometrically by gluing together $n + 2$ flat rectangles around p; see Figure 3.1.

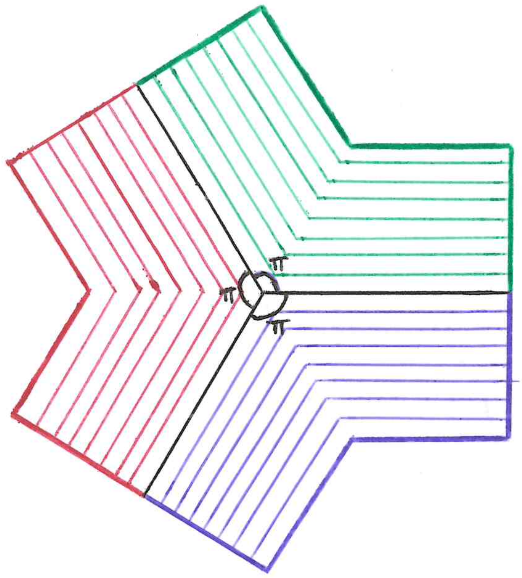

Figure 3.1: Three rectangles glued around a point. All lines depicted are straight.

In this way, we can speak of the Euclidean area of q, denoted $\mathrm{Area}(q)$, and the Euclidean length of a path in X with respect to q, denoted ℓ_q.

4 Foliations

We shall now introduce foliations on Riemann surfaces. In particular, we will explain how holomorphic quadratic differentials induce foliations. This will give a geometric description of how quadratic differentials behave.

Definition 4.1. [9, p. 301] *Given a closed surface X, a* singular foliation *$\mathcal{F}$ is a partition of X into a disjoint union of 1-dimensional subsets called the* leaves *of $\mathcal{F}$ and a finite set of points called the* singular points *of $\mathcal{F}$, such that:*

1. *Each non-singular $p \in X$ has a smooth chart from a neighbourhood of p to $\mathbb{R}^2$ which maps leaves to horizontal line segments, and the transition maps between two such charts take horizontal lines to horizontal lines.*

2. *Each singular $p \in X$ has a smooth chart from a neighbourhood of p to $\mathbb{R}^2$ which maps leaves to the level sets of an n-pronged saddle, for some $n \geq 3$: see Figure 4.1.*

Furthermore, a foliation is called orientable *if each of the leaves can be given an orientation in such a way that the orientations are locally consistent with each other.*

Remark. *A singular foliation is not orientable if it has a singular point with an odd number of prongs.*

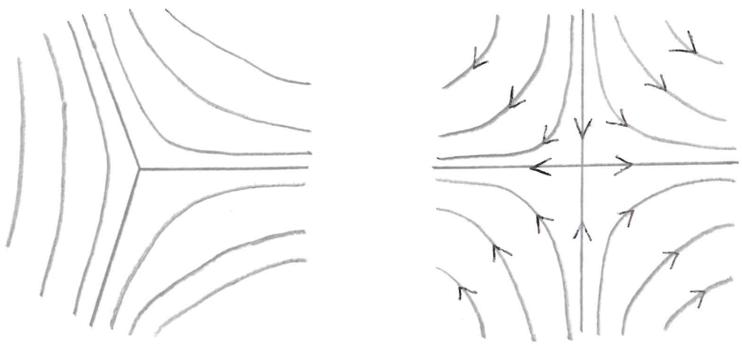

Figure 4.1: A 3-pronged singular point, and a 4-pronged singular point with orientation.

The Euler–Poincaré formula described below determines the total number of prongs in any foliation of a given surface.[9, pp. 301–302]

Proposition 4.2 (Euler–Poincaré Formula). *Let X be a closed surface with singular foliation, let S be the set of singular points of this foliation, and denote the number of prongs of a singularity $s \in S$ by P_s. Then*

$$2\chi(X) = \sum_{s \in S} 2 - P_s.$$

We can equip singular foliations with measures. This will allow us to associate foliations with quadratic differentials in a meaningful way.

Definition 4.3. [9, p. 302] *Let X be a surface with foliation $\mathcal{F}$. A smooth path α in X is said to be* transverse *to $\mathcal{F}$ if it is transverse to each leaf of $\mathcal{F}$ and does not pass through any singular points.*

Definition 4.4. [9, pp. 302–303] *Let $\alpha, \beta \colon I \to X$ be smooth paths transverse to $\mathcal{F}$. A* leaf-preserving isotopy *from α to β is a map $H \colon I \times I \to X$ such that:*

1. *$H(I \times \{0\}) = \alpha$ and $H(I \times \{1\}) = \beta$;*

2. *$H(I \times \{t\})$ is transverse to $\mathcal{F}$ for each $t \in I$;*

3. *$H(\{0\} \times I)$ and $H(\{1\} \times I)$ are each contained in a single leaf.*

(See Figure 4.2.)

Definition 4.5. [9, p. 303] *A* transverse measure *on a foliation $\mathcal{F}$ is a function μ which assigns a positive real number to each smooth path $\alpha \colon I \to X$ transverse to $\mathcal{F}$. It is absolutely continuous with respect to the Lebesgue measure on I and is invariant under*

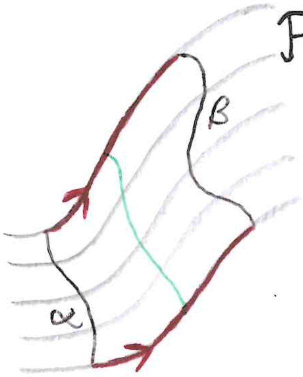

Figure 4.2: A leaf-preserving isotopy.

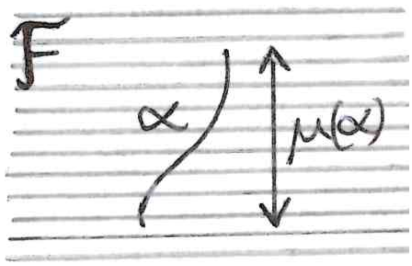

Figure 4.3: A transverse measure.

leaf-preserving isotopy. A measured foliation *is then defined to be a foliation equipped with a transverse measure.*

For example, consider the foliation $\mathcal{F}$ on the complex plane consisting of lines parallel to the real axis. We can then define a transverse measure μ on $\mathcal{F}$ by taking the height of a smooth path transverse to $\mathcal{F}$, as pictured in Figure 4.3.

4.1 The foliations for a quadratic differential

We shall now define two foliations arising from a holomorphic quadratic differential. Using natural coordinates, we will then be able to define transverse measures on these to turn them into measured foliations. This construction is based on [9, p. 310].

Definition 4.6. *The* horizontal foliation *for a holomorphic quadratic differential q on a Riemann surface X is the foliation whose singular points are the zeros of q and whose leaves are the smooth paths in X with tangent vectors that evaluate to positive real numbers under q. The* vertical foliation *for q is defined in the same way, except taking paths whose tangent vectors evaluate to negative real numbers under q.*

By using natural coordinates, we can take charts around each point $p \in X$ such that q has the form $q(z) = dz^2$ if $q(p) \neq 0$, and $q(z) = z^n dz^2$ if p is a zero of order n.

Suppose we have the first case, and let v be a tangent vector at p, given by $\alpha \in \mathbb{C}$ in this chart. Then $q(v) = \alpha^2$, which is a positive real number if and only if α is a non-zero real number, and is a negative real number if and only if α is a non-zero imaginary number. Hence in the given chart, the horizontal foliation for q is given by the horizontal lines in $\mathbb{C}$ and the vertical foliation is given by the vertical lines. The transverse measures for these are those induced by $|dy|$ and $|dx|$, respectively.

Proposition 4.7. *Let p be a zero of order n of a holomorphic quadratic differential q on* *a Riemann surface X. Then p is an $(n+2)$-pronged singular point in the horizontal and vertical foliations for q.*

Proof. (This is a proof of a comment made in [9, p. 310].) Suppose p is a zero of q of order n, and let v be a tangent vector at a point $x \in X$ near enough to p that it is still contained in the chart given by natural coordinates z. Suppose v is given by $\alpha \in \mathbb{C}$ in this chart. Then $q(v) = z(x)^n \alpha^2$. Suppose $z(x) - z(p) = z(x) = \alpha$, that is, x lies on a leaf passing through p. Then $q(v) = \alpha^{n+2}$ is a positive real number if and only if $\alpha = r\zeta^k$ for some $k \in \{0, \ldots, n+1\}$, where ζ is an $(n+2)$th root of unity and r is a positive real number. Hence in the given chart, the horizontal foliation for q is given by an $(n+2)$-pronged singular point with prongs pointing in the directions of the roots of unity. The vertical foliation is obtained by rotating the horizontal one through an angle of $\frac{\pi}{n+2}$. □

Remark. *We choose quadratic differentials in this construction because of the property that $q(-v) = q(v)$ for any vector v in the tangent space, which we showed earlier. Because of this property, quadratic differentials do not induce an orientation on the constructed foliations, so we avoid the problem of non-orientability near odd-pronged singularities.*

4.2 The dimension of QD(X)

We are now ready to find a lower bound on the dimension of QD(X) for a Riemann surface X.[9, p. 317] Let P be a finite set of points in X, and denote by $K_P(X)$ the vector space of meromorphic functions $f\colon X \to \mathbb{C}$ which only have simple poles, each at a point of P. Then, by the Riemann–Roch theorem (handled in more detail in [10, §III.4.8]):

Theorem 4.8. *Let X be a closed Riemann surface of genus g and let P be a finite set of points in X. Then*

$$\dim_{\mathbb{C}}(K_P(X)) \geq |P| + 1 - g.$$

This then allows us to determine a bound on QD(X).[9, p. 317]

Proposition 4.9. *Let X be a closed Riemann surface of genus g. Then*

$$\dim_{\mathbb{C}}(\mathrm{QD}(X)) \geq 3g - 3.$$

Proof. (This proof elaborates on the one given in [9, p. 318].) Let q_0 be an element of QD(X) with only simple zeros. The fact that such a q_0 exists is a useful result in Teichmüller theory, but it is also non-trivial. A proof can be found in [15]. The horizontal foliation for such a q_0 has three prongs at each singularity, so by the Euler–Poincaré formula, q_0 has $4g - 4$ zeros. Let P denote the set of these zeros. Then by Theorem 4.8, we have $\dim_{\mathbb{C}}(K_P(X)) \geq 3g - 3$.

Define a map $\mathrm{QD}(X) \to K_P(X)$ by $q \mapsto \frac{q}{q_0}$. We claim that $\frac{q}{q_0}$ is a well-defined meromorphic function on X with only simple poles, which are the points of P where q does not have a zero. Indeed, given an atlas $\{z_\alpha \colon U_\alpha \to \mathbb{C}\}$ for X, q and q_0 can be described in terms of collections of expressions $\{\phi_\alpha(z_\alpha)dz_\alpha^2\}$ and $\{\psi_\alpha(z_\alpha)dz_\alpha^2\}$, respectively, such that:

1. Each $\phi_\alpha \colon z_\alpha(U_\alpha) \to \mathbb{C}$ is holomorphic with finitely many zeros;

2. For any two coordinate charts z_α, z_β, we have

$$\phi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^2 = \phi_\alpha(z_\alpha);$$

and similarly for the ψ_α. Therefore $\frac{q}{q_0}$ is a collection of expressions $\{\frac{\phi_\alpha(z_\alpha)}{\psi_\alpha(z_\alpha)}\}$ such that:

1. Each $\frac{\phi_\alpha}{\psi_\alpha} \colon z_\alpha(U_\alpha) \to \mathbb{C}$ is meromorphic with only simple poles, each occurring in P;

2. For any two coordinate charts z_α, z_β, we have

$$\frac{\phi_\beta(z_\beta)}{\psi_\beta(z_\beta)} = \frac{\phi_\alpha(z_\alpha)}{\psi_\alpha(z_\alpha)}.$$

In particular, property 2 shows that $\frac{q}{q_0}$ agrees on overlaps of charts. Hence we have proved our claim. This map is also surjective, since any function $f \in K_P(X)$ can be reached by choosing q such that it is zero in the appropriate points of P and so that $\frac{q}{q_0}$ agrees with f everywhere else. Hence it is a vector space isomorphism, so it follows that $\dim_{\mathbb{C}}(K_P(X)) = \dim_{\mathbb{C}}(\mathrm{QD}(X))$, which completes the proof. □

Remark. *In fact, it turns out that* $\dim_{\mathbb{C}}\mathrm{QD}(X) = 3g - 3$. *Again, this is a consequence of the Riemann–Roch theorem, and is proved in* [10, §III.5].

5 Teichmüller's uniqueness theorem

We are now ready to give exact statements of Teichmüller's uniqueness and existence theorems, and with a little preparation we will be able to prove uniqueness.

5.1 Statement of Teichmüller's uniqueness and existence theorems

Definition 5.1. [9, p. 320] *Let X and Y be closed Riemann surfaces of genus g. A* Teichmüller mapping *of* horizontal stretch factor $K > 0$ *is a homeomorphism $f \colon X \to Y$ such that there exist quadratic differentials q_X and q_Y on X and Y, respectively, with the following properties:*

1. *If $p \in X$ is a zero of q_X, then $f(p) \in Y$ is a zero of q_Y;*

2. *If $p \in X$ is not a zero of q_X, then there exist natural coordinates ζ_X for q_X at p and ζ_Y for q_Y at $f(p)$ such that*

$$(\zeta_Y \circ f \circ \zeta_X^{-1})(x + iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y,$$

or, in terms of z and $\bar{z}$,

$$(\zeta_Y \circ f \circ \zeta_X^{-1})(z) = \frac{1}{2}\left(\left(\frac{K+1}{\sqrt{K}}\right) z + \left(\frac{K+1}{\sqrt{K}}\right) \bar{z}\right).$$

We call such q_X and q_Y the initial differential *and* terminal differential*, respectively.*

Note that the dilatation of f is $\mathrm{Dil}_f = K$ if $K \geq 1$ and $\mathrm{Dil}_f = \frac{1}{K}$ if $K < 1$, so the dilatation is in some sense the same as the horizontal stretch factor, since we can swap horizontal and vertical directions if necessary to ensure $K \geq 1$.

Recall that when we stated Teichmüller's theorem at the beginning we spoke of a unique mapping with minimal dilatation. Our definition of a Teichmüller mapping makes this more precise:

Theorem 5.2 (Teichmüller's existence theorem)**.** [9, p. 321] *Let $f\colon X \to Y$ be a homeomorphism between closed Riemann surfaces of genus $g \geq 2$. Then there exists a Teichmüller mapping $h\colon X \to Y$ homotopic to f.*

Theorem 5.3 (Teichmüller's uniqueness theorem)**.** [9, p. 322] *Let $h\colon X \to Y$ be a Teichmüller mapping between closed Riemann surfaces of genus $g \geq 2$. Then for any quasiconformal homeomorphism $f\colon X \to Y$ homotopic to h, we have $K_h \leq K_f$, with equality if and only if $f = h$.*

Remark. *These theorems are also true in the case $g = 1$, except with the condition for equality of dilatations relaxed to $f \circ h^{-1}$ conformal. This is a special case which must be handled separately; see* [9, §10-11].

5.2 Grötzsch's problem

We begin by proving a special case of the uniqueness theorem known as Grötzsch's problem,[9, p. 325] which shall pave the way for the main proof.

Theorem 5.4 (Grötzsch's Problem)**.** *Consider the two rectangles $X = [0, a] \times [0, 1]$ and $Y = [0, Ka] \times [0, 1]$ in $\mathbb{R}^2$ for some $K \geq 1$. Let $f\colon X \to Y$ be an orientation-preserving homeomorphism which is smooth everywhere except at most a finite number of points, and which takes horizontal sides to horizontal sides and vertical sides to vertical sides. Then $K_f \geq K$, with equality if and only if f is affine.*

Proof. (Based on [9, pp. 325–327].) We first show that

$$J_f \mathrm{Dil}_f \geq |f_x|^2 \tag{5.1}$$

where J_f is the Jacobian determinant of f (we give an alternative proof to that in [9]). Considering f as a function $\mathbb{C} \to \mathbb{C}$ in the form $f(x+iy) = u(x+iy) + iv(x+iy)$, the Jacobian is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

with determinant $J_f = u_x v_y - v_x u_y > 0$ (since f is orientation-preserving). Notice that

$$\begin{aligned} u_x v_y - v_x u_y &= \tfrac{1}{4}(4u_x v_y - 4v_x u_y) \\ &= \tfrac{1}{4}(u_x^2 + 2u_x v_y + v_y^2 - u_x^2 + 2u_x v_y - v_y^2 \\ &\qquad + v_x^2 - 2v_x u_y + u_y^2 - v_x^2 - 2v_x u_y - u_y^2) \\ &= \tfrac{1}{4}(u_x + v_y)^2 - \tfrac{1}{4}(u_x - v_y)^2 + \tfrac{1}{4}(v_x - u_y)^2 - \tfrac{1}{4}(v_x + u_y)^2 \\ &= \tfrac{1}{4}|u_x + v_y + iv_x - iu_y|^2 - \tfrac{1}{4}|u_x - v_y + iv_x + iu_y|^2 \\ &= |\tfrac{1}{2}(f_x - if_y)|^2 - |\tfrac{1}{2}(f_x + if_y)|^2 \\ &= |f_z|^2 - |f_{\bar{z}}|^2 \end{aligned}$$

so that

$$\begin{aligned} J_f \mathrm{Dil}_f &= (|f_z|^2 - |f_{\bar{z}}|^2)\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \\ &= (|f_z| + |f_{\bar{z}}|)^2 \\ &= (|\tfrac{1}{2}(f_x - if_y)| + |\tfrac{1}{2}(f_x + if_y)|)^2 \\ &\geq |f_x|^2 \quad \text{by the triangle inequality.} \end{aligned}$$

Next we show that

$$\int_X |f_x| dA \geq K\mathrm{Area}(X). \tag{5.2}$$

First integrate with respect to x only to give

$$\int_0^a |f_x| dx \geq \left| \int_0^a f_x dx \right| = |f(a,y) - f(0,y)| \geq Ka,$$

with the last inequality following from the fact that f sends vertical sides to vertical sides. Then integrate both sides with respect to y from 0 to 1 to give the desired inequality.

Now we have the following chain of inequalities:

$$(K\mathrm{Area}(X))^2 \leq \left(\int_X |f_x| dA \right)^2 \tag{5.3}$$

$$\leq \left(\int_X \sqrt{J_f} \sqrt{\mathrm{Dil}_f} dA \right)^2 \tag{5.4}$$

$$\leq \left(\int_X J_f dA \right) \left(\int_X \mathrm{Dil}_f dA \right) \tag{5.5}$$

$$\leq (K \mathrm{Area}(X))(K_f \mathrm{Area}(X)) \tag{5.6}$$

where (5.3) follows from (5.2), (5.4) follows from (5.1), (5.5) follows from the Cauchy–Schwarz inequality, and (5.6) is due to the definition of K_f as the supremum of Dil_f over all values of $z \in \mathbb{C}$ and the equality $\int_X J_f dA = K\mathrm{Area}(X)$. Hence, by taking out a factor of $K\mathrm{Area}(X)^2$, we are left with $K_f \geq K$, as required.

Finally, we shall prove that $K_f = K$ if and only if f is affine. We prove this in more detail than [9]. Suppose first that f is affine. Then it can be expressed as a function $\mathbb{R}^2 \to \mathbb{R}^2$ in the form $f(x, y) = (bx + cy + d, \beta x + \gamma y + \delta)$ for $b, c, d, \beta, \gamma, \delta \in \mathbb{R}$. Since f maps $(0, 0)$ to $(0, 0)$, we have $d = \delta = 0$. Since f maps $(0, 1)$ to $(0, 1)$ we have $c = 0$ and $\gamma = 1$. Since f maps $(a, 0)$ to $(Ka, 0)$ we have $b = K$ and $\beta = 0$. Hence $f(x, y) = (Kx, y)$. Therefore

$$\mathrm{Dil}_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{|\frac{1}{2}(f_x - if_y)| + |\frac{1}{2}(f_x + if_y)|}{|\frac{1}{2}(f_x - if_y)| - |\frac{1}{2}(f_x + if_y)|} = \frac{|K+1| + |K-1|}{|K+1| - |K-1|} = K,$$

recalling that we chose $K \geq 1$. In particular, $K_f = K$.

Conversely, suppose $K_f = K$ and define $A(x, y) = (Kx, y)$. Then we can assume $K = 1$ by replacing f with $A^{-1} \circ f$. Since $K_f = K$, the inequalities (5.3), (5.4), (5.5), and (5.6) become $\mathrm{Area}(X)^2 \leq \cdots \leq \mathrm{Area}(X)^2$, hence they are all equalities. In particular, $\int_X |f_x| dA = \mathrm{Area}(X) = a$. We claim that this means f takes horizontal lines to horizontal lines. Suppose not; that is, for some fixed y there is an interval $(s, t) \subseteq [0, a]$ on which $|v_x(x, y)| > 0$. Then

$$\begin{aligned}
\int_X |f_x| dA &= \int_0^a \sqrt{u_x^2 + v_x^2} dx \\
&= \int_0^s \sqrt{u_x^2 + v_x^2} dx + \int_s^t \sqrt{u_x^2 + v_x^2} dx + \int_t^a \sqrt{u_x^2 + v_x^2} dx \\
&> \int_0^a |u_x| dx \geq \left| \int_0^a u_x dx \right| = a,
\end{aligned}$$

yielding a contradiction. Furthermore, f is conformal since $K_f = K = 1$, so it preserves the perpendicularity of horizontal and vertical lines. Hence f also takes vertical lines to vertical lines. In particular, fixing $x = x_0$ (or $y = y_0$) gives $u(x_0, y)$ (or $v(x, y_0)$) constant, so u depends only on x, and v only on y. Thus we can write $f(x, y) = (u(x), v(y))$.

Note that since $1 \leq \mathrm{Dil}_f(x, y) \leq K_f = 1$ for all (x, y), we must have $\mathrm{Dil}_f \equiv 1$. So by (5.1), which is now an equality,

$$u_x^2 = |f_x|^2 = J_f = u_x v_y - v_x u_y = u_x v_y.$$

Furthermore, since f is orientation-preserving, it follows that $J_f > 0$, hence $u_x = v_y$ everywhere and $J_f = u_x^2 = v_y^2$. By equality of (5.5), we have the equality case of the Cauchy–Schwarz inequality, so

$$\frac{\sqrt{J_f}}{\sqrt{\mathrm{Dil}_f}} = \sqrt{J_f} = u_x = v_y$$

is constant. Equality of (5.6) then implies $J_f \equiv K = 1$, and since $f(0,0) = (0,0)$, it then follows that $u(x) = x$ and $v(y) = y$. So f is the identity, and in particular it is affine. □

5.3 Proof of the uniqueness theorem

Most of the adjustments that are required to be able to adapt the proof of Grötzsch's problem for the uniqueness theorem are trivial. However, justifying the analogue of inequality (5.2) is not so simple, and we require a lemma to do so (we prove a more general version of [9, p. 328]).

Lemma 5.5. *Let $h\colon X \to Y$ be a Teichmüller mapping between closed Riemann surfaces with initial differential q_X and terminal differential q_Y. Let $f\colon X \to Y$ be a homeomorphism which is homotopic to h. Then there exists a constant $M \geq 0$ such that any path $\alpha\colon I \to X$ embedded in a leaf of the horizontal foliation for q_X satisfies*

$$\ell_{q_Y}(f(\alpha)) \geq \ell_{q_Y}(h(\alpha)) - M.$$

Proof. (Based on [9, p. 328].) Choose a homotopy $H\colon X \times I \to Y$ from f to h. Then $\ell_{q_Y} \circ H(\{x\} \times I)\colon X \to \mathbb{R}$ is continuous, and as X is compact it attains a maximum $N \geq 0$. Let $\alpha\colon I \to X$ be a path embedded in a leaf of the horizontal foliation for q_X, and let $\alpha_0(t) = H(\alpha(0), 1-t)$ and $\alpha_1(t) = H(\alpha(1), t)$ for $t \in I$. Then the concatenation $\alpha_0 \star f(\alpha) \star \alpha_1$ is endpoint-preserving homotopic to $h(\alpha)$. Thus its ℓ_{q_Y}–length must be at least that of $h(\alpha)$, as α is embedded in a horizontal leaf of q_X and so $h(\alpha)$ is embedded in a horizontal leaf of q_Y. Therefore

$$\begin{aligned}\ell_{q_Y}(h(\alpha)) &\leq \ell_{q_Y}(\alpha_0 \star f(\alpha) \star \alpha_1)\\ &= \ell_{q_Y}(\alpha_0) + \ell_{q_Y}(f(\alpha)) + \ell_{q_Y}(\alpha_1)\\ &\leq \ell_{q_Y}(f(\alpha)) + 2N\end{aligned}$$

Hence $\ell_{q_Y}(f(\alpha)) \geq \ell_{q_Y}(h(\alpha)) - M$, where $M = 2N$. □

Now we can prove the analogous inequality.[9, p. 328]

Proposition 5.6. *Let $h\colon X \to Y$ be a Teichmüller mapping between closed Riemann* *surfaces with initial differential q_X, terminal differential q_Y and horizontal stretch factor K. Let $f\colon X \to Y$ be a homeomorphism which is homotopic to h and which is smooth everywhere except a finite number of points. Then*

$$\int_X |f_x| dA \geq \sqrt{K}\,\mathrm{Area}(q_X). \tag{5.7}$$

Proof. (Closely follows [9, p. 329].) Define $\alpha_{p,L}$ to be the path in X which is horizontal with respect to q_X, has length $2L$ and is centred at $p \in X$. Then define $\delta\colon X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by setting $\delta(p, L)$ to be the integral of $|f_x|$ along $\alpha_{p,L}$. This is not defined within a horizontal distance of L of a zero of q_X, however since there are only finitely many zeros, the set on which δ is undefined has measure zero. As we are going to integrate δ over X, we can therefore ignore this set.

Notice that $\delta(p, L) = \ell_{q_Y}(f(\alpha_{p,L}))$ where it is defined, and the Teichmüller map h has horizontal stretch factor K, hence $\ell_{q_Y}(h(\alpha_{p,L})) = 2L\sqrt{K}$. Then applying Lemma 5.3 gives some $M \geq 0$ independent of p and L such that $\ell_{q_Y}(f(\alpha_{p,L})) \geq 2L\sqrt{K} - M$. Hence

$$\begin{aligned} \int_X \delta(p, L) dA &= \int_X \ell_{q_Y}(f(\alpha_{p,L})) dA \\ &\geq \int_X 2L\sqrt{K} - M dA \\ &= (2L\sqrt{K} - M)\mathrm{Area}(q_X). \end{aligned}$$

Notice that by Fubini's theorem,

$$\int_X \delta(p, L) dA = \int_X \left(\int_{-L}^{L} |f_x| dx \right) dA = 2L \int_X |f_x| dA,$$

hence

$$\int_X |f_x| dA \geq \left(\sqrt{K} - \frac{M}{2L} \right) \mathrm{Area}(q_X),$$

and so letting L tend to infinity completes the proof. □

We can now adapt the proof of Grötzsch's problem for Teichmüller's uniqueness theorem as follows (we elaborate on the comment given in [9, p. 330]):

1. Define the horizontal and vertical directions to be those induced by the horizontal and vertical foliations for q_X and q_Y; in other words, replace the standard Euclidean coordinates x and y with the natural coordinates for the initial and terminal differentials.
2. Replace inequality (5.2) with the one in Proposition 5.6.
3. Assume without loss of generality that horizontal stretch factor $K \geq 1$, so that K can be replaced with K_h.

4. The chain of inequalities (5.3) to (5.6) becomes

$$
\begin{aligned}
(\sqrt{K_h}\mathrm{Area}(q_X))^2 &\leq \left(\int_X |f_x| dA\right)^2 \\
&\leq \left(\int_X \sqrt{\mathrm{Dil}_f}\sqrt{J_f} dA\right)^2 \\
&\leq \left(\int_X J_f dA\right)\left(\int_X \mathrm{Dil}_f dA\right) \\
&\leq \mathrm{Area}(q_Y) K_f \mathrm{Area}(q_X) \\
&= \begin{vmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{vmatrix} \mathrm{Area}(q_X) K_f \mathrm{Area}(q_X) = K_f \mathrm{Area}(q_X)^2.
\end{aligned}
$$

5. Replace the affine mapping A with the Teichmüller mapping h. (Notice that since f is in the same homotopy class as h, then $h^{-1} \circ f$ is in the same homotopy class as the identity map, which is a Teichmüller mapping with maximal dilatation 1.)

6 The Beltrami equation

The aim of this section is to prove the measurable Riemann mapping theorem, an important result which is crucial to the proof of Teichmüller's existence theorem. It roughly states that given a suitable function μ, we can find a quasiconformal homeomorphism with complex dilatation μ. In other words, we aim to solve the Beltrami equation, namely

$$f_{\bar{z}} = \mu f_z, \tag{6.1}$$

where μ is taken to be a measurable function with $||\mu||_\infty \leq k < 1$ almost everywhere. We will define what this means by setting some standard notation.

6.1 Notation

We will denote the space of k-times continuously differentiable functions on Ω by $C^k(\Omega)$, often abbreviated to C^k as we will always be using $\Omega = \mathbb{C}$. The space of k-times differentiable functions with compact support is then denoted by C_0^k.

We also define two other kinds of spaces:

Definition 6.1. *For $p \in [1, \infty]$, define L^p-space as*

$$L^p(\mathbb{C}) = \{h\colon \mathbb{C} \to \mathbb{C} \textit{ measurable} : ||h||_p < \infty\}/\sim,$$

where we define

$$||h||_p = \left(\int_{\mathbb{C}} |h|^p dxdy \right)^{\frac{1}{p}} \quad \text{if } p < \infty,$$
$$||h||_\infty = \operatorname*{ess\,sup}_{\mathbb{C}} |h| = \inf \{a \in \mathbb{R} : \{|h| > a\} \text{ has measure zero}\},$$

and $\sim$ *is the equivalence relation identifying any two functions that are equal almost everywhere.*

Definition 6.2. *A function* $h\colon \mathbb{C} \to \mathbb{C}$ *is* Hölder continuous *with* Hölder exponent α *if there exists some constant* $c > 0$ *such that*

$$|h(z_1) - h(z_2)| \leq c|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in \mathbb{C}$. *We denote the space of Hölder continuous functions by* $H_\alpha(\mathbb{C})$.

Again, we often abbreviate $L^p(\mathbb{C})$ and $H_\alpha(\mathbb{C})$ to L^p and H_α.

6.2 The Cauchy and Hilbert transforms

As part of our analysis of the Beltrami equation, we introduce two important transforms: the Cauchy transform and the Hilbert transform.

Definition 6.3. [11, p. 48] *Fix* $h \in L^p(\mathbb{C})$ *for* $p > 2$. *The* Cauchy transform Ph *of* h *is defined by*

$$Ph(w) = -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) dxdy.$$

Lemma 6.4. [11, p. 48] Ph *is Hölder continuous with Hölder exponent* $1 - \frac{2}{p}$. *That is, the Cauchy transform is a function* $P\colon L^p(\mathbb{C}) \to H_{1-\frac{2}{p}}(\mathbb{C})$.

Proof. (Based on [11, p. 49].) We have

$$\begin{aligned} |Ph(w)| &= \frac{1}{\pi} \left| \int_{\mathbb{C}} h(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) dxdy \right| \\ &\leq \frac{1}{\pi} \int_{\mathbb{C}} |h(z)| \left| \frac{w}{z(z-w)} \right| dxdy \\ &= \frac{|w|}{\pi} \int_{\mathbb{C}} |h(z)| \frac{1}{|z(z-w)|} dxdy \end{aligned}$$

and then by the Hölder inequality (defined in [17, p. 63]),

$$|Ph(w)| \leq \frac{|w|}{\pi} \left(\int_{\mathbb{C}} |h(z)|^p dxdy \right)^{\frac{1}{p}} \left(\int_{\mathbb{C}} \frac{1}{|z(z-w)|^q} dxdy \right)^{\frac{1}{q}} \tag{6.2}$$

where the p is the one in $L^p(\mathbb{C})$ and $\frac{1}{p}+\frac{1}{q}=1$. Since $p>2$, it follows that $1<q<2$. Now substitute $z=wu$ (this step is not in [11]). Note that $dzd\bar{z}=w\bar{w}dud\bar{u}=|w|^2dud\bar{u}$ and

$$dzd\bar{z}=(dx+idy)(dx-idy)=dxdx-idxdy+idydx+dydy=-2idxdy,$$

so, writing u as z again, we have

$$\int_{\mathbb{C}}\frac{1}{|z(z-w)|^q}dxdy=\int_{\mathbb{C}}\frac{|w|^2}{|wz(wz-w)|^q}dxdy=|w|^{2-2q}\int_{\mathbb{C}}\frac{1}{|z(z-1)|^q}dxdy. \tag{6.3}$$

We claim that the last integral is convergent (this is stated without proof in [11]). To show this, we must consider three separate cases: behaviour near ∞, 0 and 1. By considering the asymptotic behaviour near these three points, proof of the claim reduces to showing the following are convergent:

1. $\displaystyle\int_{|z|>1}\frac{1}{|z|^{2q}}dxdy$ (we can treat $z-1$ as z near ∞)
2. $\displaystyle\int_{|z|<\frac{1}{2}}\frac{1}{|z|^q}dxdy$ (we can ignore $\frac{1}{|z-1|^q}$ near 0 as it behaves nicely)
3. $\displaystyle\int_{|z-1|<\frac{1}{2}}\frac{1}{|z-1|^q}dxdy$ (we can ignore $\frac{1}{|z|^q}$ near 1 as it behaves nicely).

Write z in polar form as $z=re^{i\theta}$. Then integral 1 can be written as

$$\begin{aligned}\int_{r>1}\frac{r}{|re^{i\theta}|^{2q}}d\theta dr &= \int_1^\infty\int_0^{2\pi}\frac{r}{r^{2q}}d\theta dr\\ &= \int_1^\infty\frac{2\pi}{r^{2q-1}}dr,\end{aligned}$$

which is convergent as $q>1$. Integral 2 can be written as

$$\begin{aligned}\int_{r<\frac{1}{2}}\frac{r}{|re^{i\theta}|^{q}}d\theta dr &= \int_0^{\frac{1}{2}}\int_0^{2\pi}\frac{r}{r^{q}}d\theta dr\\ &= \int_0^{\frac{1}{2}}\frac{2\pi}{r^{q-1}}dr,\end{aligned}$$

which is convergent as $q<2$. Similar reasoning shows integral 3 is convergent. Hence the whole integral is convergent, so it can be regarded as a constant depending only on p. Therefore we can define a constant K_p as

$$K_p:=\frac{1}{\pi}\left(\int_{\mathbb{C}}\frac{1}{|z(z-1)|^q}dxdy\right)^{\frac{1}{q}}.$$

Combining this with (6.2) and (6.3) then gives

$$|Ph(w)| \leq |w| ||h||_p |w|^{\frac{2-2q}{q}} K_p = ||h||_p |w|^{1-\frac{2}{p}} K_p. \tag{6.4}$$

Defining $\widetilde{h}(z) = h(z + w_1)$, we have

$$P\widetilde{h}(w_2 - w_1) = -\frac{1}{\pi} \int_{\mathbb{C}} h(z + w_1) \left(\frac{1}{z + w_1 - w_2} - \frac{1}{z} \right) dxdy,$$

then substituting z for $z + w_1$ gives

$$\begin{aligned} P\widetilde{h}(w_2 - w_1) &= -\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z - w_2} - \frac{1}{z - w_1} \right) dxdy \\ &= Ph(w_2) - Ph(w_1). \end{aligned}$$

Then we can apply (6.4) to $\widetilde{h}$ to give

$$|Ph(w_1) - Ph(w_2)| \leq ||h||_p K_p |w_1 - w_2|^{1-\frac{2}{p}} \tag{6.5}$$

as desired. $\square$

Definition 6.5. [4, p. 52] *Fix $h \in C_0^2$. The* Hilbert transform *Th of h is defined by*

$$Th(w) = \lim_{\varepsilon \to 0} \left(-\frac{1}{\pi} \int_{|z-w|>\varepsilon} \frac{h(z)}{(z-w)^2} dxdy \right).$$

Lemma 6.6. [4, p. 52] *Given $h \in C_0^2$, we have $Th \in C^1$, so that the Hilbert transform is a function $T\colon C_0^2 \to C^1$. Moreover,*

1. $(Ph)_{\bar{z}} = h$;
2. $(Ph)_z = Th$;
3. $\int_{\mathbb{C}} |Th|^2 dxdy = \int_{\mathbb{C}} |h|^2 dxdy.$

Proof. (Elaboration of [4, pp. 52–53].) First note that, writing $w = u + iv$, we have $(Ph)_{\bar{w}} = \frac{1}{2}((Ph)_u + i(Ph)_v)$ and $(Ph)_w = \frac{1}{2}((Ph)_u - i(Ph)_v)$. Then we have

$$\begin{aligned} (Ph)_u &= \lim_{\substack{\delta \to 0 \\ \delta \in \mathbb{R}}} \frac{Ph(w+\delta) - Ph(w)}{\delta} \\ &= \lim_{\delta \to 0} \frac{1}{\delta} \left(-\frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z - (w+\delta)} - \frac{1}{z} \right) dxdy + \frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) dxdy \right) \\ &= \lim_{\delta \to 0} \frac{1}{\delta} \left(-\frac{1}{\pi} \int_{\mathbb{C}} h(z+\delta) \left(\frac{1}{z-w} - \frac{1}{z+\delta} \right) dxdy + \frac{1}{\pi} \int_{\mathbb{C}} h(z) \left(\frac{1}{z-w} - \frac{1}{z} \right) dxdy \right) \\ &= \lim_{\delta \to 0} \left(-\frac{1}{\pi} \int_{\mathbb{C}} \frac{h(z+\delta) - h(z)}{\delta} \frac{1}{z-w} + \frac{1}{\delta} \left(\frac{h(z)}{z} - \frac{h(z+\delta)}{z+\delta} \right) dxdy \right) \end{aligned}$$

$$= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_x}{z-w} dxdy - \frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{h(z)}{z}\right)_x dxdy.$$

Defining $B_R = [-R, R] \times [-R, R]$ and applying Stokes' theorem, the last integral can be expressed as

$$\int_{\mathbb{C}} \left(\frac{h(z)}{z}\right)_x dxdy = \lim_{R\to\infty}\int_{B_R} \left(\frac{h(z)}{z}\right)_x dxdy$$
$$= \lim_{R\to\infty}\int_{\partial B_R} \frac{h(z)}{z} ds,$$

where s is a unit-speed parameter for ∂B_R. But h has compact support, so it vanishes outside a compact set. Hence for R large enough, the integral over ∂B_R vanishes, and in particular the limit as R tends to infinity is zero. So in fact

$$(Ph)_u = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_x}{z-w} dxdy.$$

Similarly,

$$(Ph)_v = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_y}{z-w} dxdy,$$

and so

$$(Ph)_{\bar{w}} = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_{\bar{z}}}{z-w} dxdy \quad \text{and} \quad (Ph)_w = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_z}{z-w} dxdy$$

(these two formulae are simply stated in [4] without any proof). Now let γ_ε be the circle with centre w and radius ε, and let Ω_ε be $\mathbb{C} \setminus \{z : |z-w| < \varepsilon\}$, that is everything except the region γ_ε encloses. Note that

$$dhdz = (h_z dz + h_{\bar{z}} d\bar{z})dz = h_z dzdz + h_{\bar{z}} d\bar{z}dz = -h_{\bar{z}} dzd\bar{z}$$

(not mentioned in [4]). Then using this and Stokes' theorem,

$$-\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_{\bar{z}}}{z-w} dxdy = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{h_{\bar{z}}}{z-w} dzd\bar{z}$$
$$= -\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{dhdz}{z-w}$$
$$= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{dhdz}{z-w} \tag{6.6}$$
$$= \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} \frac{hdz}{z-w}. \tag{6.7}$$

The change of sign in step (6.6) is because Ω_ε and the ball $B(0, \varepsilon)$ of radius ε and centre 0 induce opposite orientations on γ_ε (not mentioned in [4]), noting that integrating over $\mathbb{C}$ amounts to integrating over $B(0, \varepsilon)$ and taking the limit as $\varepsilon \to \infty$. Step (6.7) uses

Stokes' theorem and also that

$$\begin{aligned}
d\left(\frac{hdz}{z-w}\right) &= \frac{dhdz}{z-w} + hd\left(\frac{1}{z-w}\right)dz \\
&= \frac{dhdz}{z-w} + h\left(\left(\frac{1}{z-w}\right)_z dz + \left(\frac{1}{z-w}\right)_{\bar{z}} d\bar{z}\right)dz \\
&= \frac{dhdz}{z-w} - \frac{h}{(z-w)^2}dzdz \\
&= \frac{dhdz}{z-w}.
\end{aligned}$$

Finally, by Cauchy's integral formula (6.7) is just $h(w)$, so we have proved part 1. Similarly

$$\begin{aligned}
-\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_z}{z-w}dxdy &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{dhd\bar{z}}{z-w} \\
&= -\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{dhd\bar{z}}{z-w},
\end{aligned}$$

but this time we have

$$d\left(\frac{hd\bar{z}}{z-w}\right) = \frac{dhd\bar{z}}{z-w} - \frac{hdzd\bar{z}}{(z-w)^2},$$

so

$$\begin{aligned}
-\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \frac{dhd\bar{z}}{z-w} &= \lim_{\varepsilon\to 0}\left(-\frac{1}{2\pi i}\int_{\Omega_\varepsilon} d\left(\frac{hd\bar{z}}{z-w}\right) + \frac{1}{2\pi i}\int_{\Omega_\varepsilon}\frac{hdzd\bar{z}}{(z-w)^2}\right) \\
&= \lim_{\varepsilon\to 0}\left(-\frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{hd\bar{z}}{z-w} + \frac{1}{2\pi i}\int_{\Omega_\varepsilon}\frac{hdzd\bar{z}}{(z-w)^2}\right) \\
&= \lim_{\varepsilon\to 0}\left(-\frac{1}{\pi}\int_{\Omega_\varepsilon} \frac{h(z)}{(z-w)^2}dxdy\right) \\
&= Th(w).
\end{aligned}$$

This proves part 2.

Now observe that

$$P(h_z)(w) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h_z}{z-w}dxdy + \frac{1}{\pi}\int_{\mathbb{C}}\frac{h_z}{z}dxdy = Th(w) - Th(0)$$

and apply parts 1 and 2 to h_z to give

$$\begin{aligned}
(Th)_{\bar{z}} &= P(h_z)_{\bar{z}} = h_z \\
(Th)_z &= P(h_z)_z = T(h_z) = P(h_{zz}) + T(h_z)(0),
\end{aligned}$$

which shows that $Th \in C^1$. Since h has compact support, then as z approaches infinity, $Ph = O(1)$. So we can use integration by parts where appropriate in the following calculation without producing an extra term (not mentioned in [4]):

$$\begin{aligned}
\int_{\mathbb{C}} |Th|^2 dxdy &= -\frac{1}{2i}\int_{\mathbb{C}} (Th)(\overline{Th}) dz d\bar{z} \\
&= -\frac{1}{2i}\int_{\mathbb{C}} (Ph)_z \overline{(Ph)_z} dz d\bar{z} \\
&= -\frac{1}{2i}\int_{\mathbb{C}} (Ph)_z (\overline{Ph})_{\bar{z}} dz d\bar{z} \\
&= \frac{1}{2i}\int_{\mathbb{C}} Ph(\overline{Ph})_{z\bar{z}} dz d\bar{z} \\
&= \frac{1}{2i}\int_{\mathbb{C}} Ph(\overline{(Ph)_{\bar{z}}})_{\bar{z}} dz d\bar{z} \\
&= \frac{1}{2i}\int_{\mathbb{C}} (Ph)\bar{h}_{\bar{z}} dz d\bar{z} \\
&= -\frac{1}{2i}\int_{\mathbb{C}} (Ph)_{\bar{z}}\bar{h} dz d\bar{z} \\
&= \int_{\mathbb{C}} h\bar{h} dxdy \\
&= \int_{\mathbb{C}} |h|^2 dxdy.
\end{aligned}$$

Thus we have shown part 3. □

Remark. *In proving the first two parts, we in fact only needed to use $h \in C_0^1$.*

Part 3 of this lemma shows that T is an isometry on C_0^2, and in fact, since C_0^2 is dense in L^2, it follows by continuity that T can be extended to an isometry on the whole of L^2. [4, p. 53] But we want to extend T to L^p for $p > 2$, since we would like to compare it with P, which is not defined for $p = 2$.

We solve this problem by introducing the *Calderon–Zygmund inequality*, which says that part 3 of the lemma can be replaced with

$$||Th||_p \leq C_p ||h||_p$$

for any $p > 2$, with $C_p \to 1$ as $p \to 2$. A proof of this inequality can be found in [4, §V.D]. We can use this to obtain some additional properties of the Cauchy and Hilbert transforms.[4, p. 53] But first of all, we need to define what it means for an equation to hold in the sense of distributions.

Definition 6.7. *The* distribution *of $f\colon \mathbb{C} \to \mathbb{C}$ is the linear functional $f[\cdot]\colon C_0^1(\mathbb{C}) \to \mathbb{C}$ defined by integrating against test functions $\phi \in C_0^1(\mathbb{C})$:*

$$f[\phi] := \int_{\mathbb{C}} f\phi dxdy.$$

Differentiation in the sense of distributions is defined by $Df[\phi] := -f[D\phi]$.

Lemma 6.8. *The following hold in the sense of distributions for $h \in L^p$ with $p > 2$:*

1. $(Ph)_{\bar{z}} = h$;

2. $(Ph)_z = Th$.

Proof. (Based on [4, p. 53].) We want to show that for all test functions $\phi \in C_0^1$,

$$\int_{\mathbb{C}} (Ph)\phi_{\bar{z}} dxdy = -\int_{\mathbb{C}} \phi h dxdy \quad \text{and} \quad \int_{\mathbb{C}} (Ph)\phi_z dxdy = -\int_{\mathbb{C}} \phi Th dxdy.$$

These equations hold when $h \in C_0^2$ by using integration by parts and Lemma 6.6, and we can approximate h in the L^p-norm by a sequence $h_n \in C_0^2$. Therefore it only remains to show that Th and Ph are also approximated by Th_n and Ph_n. But by the Calderon–Zygmund inequality $||Th - Th_n||_p \leq C_p||h - h_n||_p$, and $|P(h - h_n)| \leq K_p||h - h_n||_p|z|^{1-\frac{2}{p}}$ by (6.4), so this is true. □

6.3 Solving the Beltrami equation

We have now laid the groundwork required to be able to solve the Beltrami equation. Initially we will deal with the case where μ has compact support, and $||\mu||_\infty \leq k < 1$. This means the solution f will be analytic at infinity (that is, $f(\frac{1}{z})$ will be analytic at zero).

Recall that the C_p from the Calderon–Zygmund inequality approaches 1 as $p \to 2$ from above, so for $p > 2$ small enough we have $kC_p < 1$. Fix such a p. Then we have the following theorem.[4, p. 54]

Theorem 6.9. *Under the above assumptions, there exists a unique solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$ such that $f(0) = 0$ and $f_z - 1 \in L^p(\mathbb{C})$.*

In order to prove this, we need Weyl's lemma (proved in [8, §9.4]), which states:

Lemma 6.10 (Weyl's Lemma)**.** *If f is a solution of Laplace's equation $\Delta f = f_{xx} + f_{yy} = 0$ in the sense of distributions, then it is also a smooth solution of Laplace's equation in the non-distributional sense.*

Proof of theorem. (Based on [4, p. 54].) We will first prove uniqueness. Suppose f is such a solution. Then $f_{\bar{z}} = \mu f_z \in L^p$ and $P(f_{\bar{z}})$ exists, and by Lemma 6.8 the function $F = f - P(f_{\bar{z}})$ satisfies $F_{\bar{z}} = 0$ in the sense of distributions. But $F_{\bar{z}} = \frac{1}{2}(F_x + iF_y)$, so we also have $F_{xx} + iF_{yx} = 0$ and $F_{xy} + iF_{yy} = 0$ in the sense of distributions. Multiplying the last equation by i and subtracting from the other then gives $F_{xx} + F_{yy} = 0$, hence

F solves the Laplace equation in the sense of distributions. Then by Weyl's lemma, F is harmonic. Since $f_z - 1 \in \mathrm{L}^p$, then $F_z - 1 \in \mathrm{L}^p$, which can only be satisfied if $F_z = 1$ by Liouville's theorem. Moreover, the normalisation $f(0) = 0$ together with $F_{\bar{z}} = 0$ implies $F = z$, so $f = P(f_{\bar{z}}) + z$ and $f_z = T(f_{\bar{z}}) + 1 = T(\mu f_z) + 1$.

If g is another solution, then $f_z - g_z = T(\mu(f_z - g_z))$, and so by the Calderon–Zygmund inequality, $||f_z - g_z||_p \leq kC_p||f_z - g_z||_p$. But we are assuming that $kC_p < 1$, so $f_z - g_z = 0$ almost everywhere. Therefore $f - g$ is constant, and by the normalisation we have $f = g$.

Now we prove existence (using a different method to [4]). Consider the operator given by $S(h) = T\mu + T(\mu h)$, where $h \in L^p$. Then

$$S(f) - S(g) = T(\mu f) - T(\mu g) = T(\mu(f - g)),$$

so by the Calderon–Zygmund inequality

$$||S(f) - S(g)||_p \leq C_p||\mu(f - g)||_p \leq kC_p||f - g||_p.$$

But we assumed that $kC_p < 1$, hence S is a contraction, meaning we can apply the contraction mapping theorem on L^p-space[1] to conclude that there is exactly one solution h to $S(h) = h$. That is, there is exactly one h such that $h = T(\mu h) + T\mu$. Since μ has compact support and $||\mu||_\infty < 1$, then $\mu(h + 1) \in \mathrm{L}^p$ and so $P(\mu(h + 1))$ is well-defined and continuous. For such a solution h, it follows that $f = P(\mu(h + 1)) + z$ is the solution of the Beltrami equation, since

$$f_z = (P(\mu(h + 1))_z + 1 = T(\mu(h + 1)) + 1 = h + 1 \tag{6.8}$$

$$\text{and} \quad f_{\bar{z}} = (P(\mu(h + 1)))_{\bar{z}} = \mu(h + 1). \tag{6.9}$$

We also have $f_z - 1 = h \in \mathrm{L}^p$. □

Definition 6.11. [4, p. 55] *The solution $f = P(\mu(h + 1)) + z = P(f_{\bar{z}}) + z$ given in the above proof is called the* normal solution *of the Beltrami equation.*

6.4 The measurable Riemann mapping theorem

We claim that the normal solution to the Beltrami equation is a quasiconformal homeomorphism with complex dilatation μ. To show this, we will need a few more lemmas.

Lemma 6.12. [11, p. 52] *Let μ_n, μ be as in the hypothesis of* Theorem 6.9 *with uniformly bounded supports, and let f_n, f be the respective normal solutions. Suppose that $\mu_n \to \mu$ almost everywhere. Then $||(f_n)_z - f_z||_p \to 0$ and $f_n \to f$ uniformly on compact sets.*

[1] This is a complete metric space under the L^p-norm; see [18, p. 5]

Proof. (Elaboration of [11, pp. 52–53].) Applying the Calderon–Zygmund inequality to the solution h of $h = T(\mu h) + T\mu$ determined in the proof of Theorem 6.9, we have $||h||_p \leq kC_p||h||_p + C_p||\mu||_p$, and then solving for $||h||_p$ gives

$$||h||_p \leq \frac{C_p}{1 - kC_p}||\mu||_p.$$

Furthermore, letting f be the normal solution, we have

$$\begin{aligned} ||f_{\bar{z}}||_p = ||\mu(h+1)||_p &\leq ||\mu||_p||h||_p + ||\mu||_p \\ &\leq ||\mu||_p \frac{C_p}{1 - kC_p}||\mu||_p + ||\mu||_p \\ &\leq \frac{kC_p}{1 - kC_p}||\mu||_p + \frac{1 - kC_p}{1 - kC_p}||\mu||_p \\ &= \frac{1}{1 - kC_p}||\mu||_p. \end{aligned} \tag{6.10}$$

Hence applying the Hölder condition (6.4) on P gives

$$\begin{aligned} |f_n - f| = |P((f_n)_{\bar{z}}) - P(f_{\bar{z}})| &\leq K_p||(f_n)_{\bar{z}} - f_{\bar{z}}||_p|z|^{1-\frac{2}{p}} \\ &\leq \frac{K_p}{1 - kC_p}||\mu_n - \mu||_p|z|^{1-\frac{2}{p}}. \end{aligned}$$

But $\mu_n \to \mu$ almost everywhere, so $f_n \to f$ uniformly. Then noticing that (6.8) gives $f_z = T(\mu f_z) + 1$, we also have

$$\begin{aligned} ||(f_n)_z - f_z||_p = ||T(\mu_n(f_n)_z - \mu f_z)||_p &= ||T(\mu_n((f_n)_z - f_z)) + T(\mu_n - \mu)f_z)||_p \\ &\leq ||T(\mu_n((f_n)_z - f_z))||_p + ||T((\mu_n - \mu)f_z)||_p \\ &\leq kC_p||(f_n)_z - f_z||_p + C_p||(\mu_n - \mu)f_z||_p. \end{aligned}$$

But $\mu_n \to \mu$ almost everywhere and $kC_p < 1$, so $||(f_n)_z - f_z||_p \to 0$, as required. □

Additionally, we obtain the following useful estimate by integrating (6.8):

$$\begin{aligned} |f(w_1) - f(w_2)| &\leq |P(\mu(h+1))(w_1) - P(\mu(h+1))(w_2)| + |w_1 - w_2| \\ &\leq K_p||\mu(h+1)||_p|w_1 - w_2|^{1-\frac{2}{p}} + |w_1 - w_2| \\ &\leq \frac{K_p}{1 - kC_p}||\mu||_p|w_1 - w_2|^{1-\frac{2}{p}} + |w_1 - w_2|. \end{aligned} \tag{6.11}$$

Lemma 6.13. [4, p. 56] *If μ has a derivative in the sense of distributions, and $\mu_z \in L^p$ for some $p > 2$, then $f \in C^1$.*

To prove this lemma, we require the following result, which we quote without proof. A full proof of it is given in [4, §V.B].

Lemma 6.14. *If g and h are continuous and have locally integrable derivatives in the sense of distributions that satisfy $g_{\bar{z}} = h_z$, then there exists $f \in C^1$ such that $f_z = g$ and $f_{\bar{z}} = h$.*

Proof of Lemma 6.13. (Closely follows [4, p. 56].) Solving the Beltrami equation amounts to solving the system of equations $f_z = \lambda, f_{\bar{z}} = \mu\lambda$ for λ. By the previous lemma, this is possible for $f \in C^1$ if

$$\lambda_{\bar{z}} = (\mu\lambda)_z = \lambda_z\mu + \lambda\mu_z$$

or, dividing through by λ,

$$(\log\lambda)_{\bar{z}} = \mu(\log\lambda)_z + \mu_z. \tag{6.12}$$

By the same contraction mapping theorem argument that we used in the proof of Theorem 6.9, the equation $h = T(\mu h) + T(\mu_z)$ has a unique solution $h \in \mathrm{L}^p$. Then we define $\sigma = P(\mu h + \mu_z) + c$, where c is a constant chosen so that $\sigma \to 0$ as $z \to \infty$. Then σ is continuous and

$$\begin{aligned}\sigma_z &= T(\mu h + \mu_z) = h \\ \sigma_{\bar{z}} &= \mu h + \mu_z = \mu\sigma_z + \mu_z,\end{aligned}$$

so $\lambda = e^\sigma$ solves (6.12) as required. Moreover, if we normalise the solution f by setting $f(0) = 0$, then we obtain the normal solution, since as $z \to \infty$, $\sigma \to 0$ and so $\lambda \to 1$, meaning $f_z \to 1$ and thus $f_z - 1 \in \mathrm{L}^p$. □

The following construction is based on [4, pp. 56–57]. Recall from section 2 that the inverse f^{-1} of a K-quasiconformal homeomorphism is again K-quasiconformal and has complex dilatation $\mu_{f^{-1}}(z) = -\mu_f(f^{-1}(z))$, hence we have $|\mu_{f^{-1}} \circ f| = |\mu_f|$. For convenience of notation we shall denote μ_f by μ and $\mu_{f^{-1}}$ by $\widetilde{\mu}$. Recall from the proof of Grötzsch's problem that the Jacobian determinant of f is $|f_z|^2 - |f_{\bar{z}}|^2$. Therefore, changing coordinates from x, y to ζ, η via f, we have

$$\begin{aligned}\int_{\mathbb{C}} |\widetilde{\mu}|^p d\zeta d\eta &= \int_{\mathbb{C}} |\mu|^p(|f_z|^2 - |f_{\bar{z}}|^2)dxdy \\ &\le \int_{\mathbb{C}} |\mu|^p|f_z|^2 dxdy \\ &= \int_{\mathbb{C}} |\mu|^{p-2}|f_{\bar{z}}|^2 dxdy \\ &\le \left(\int_{\mathbb{C}} |\mu|^p dxdy\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{C}} |f_{\bar{z}}|^p dxdy\right)^{\frac{2}{p}} \\ &= ||\mu||_p^{p-2}||f_{\bar{z}}||_p^2,\end{aligned}$$

and so by (6.10),

$$||\widetilde{\mu}||_p \leq ||\mu||_p^{\frac{p-2}{p}} (1 - kC_p)^{-\frac{2}{p}} ||\mu||_p^{\frac{2}{p}} = (1 - kC_p)^{-\frac{2}{p}} ||\mu||_p.$$

Then applying (6.11) to f^{-1} with $w_1 = f(z_1)$ and $w_2 = f(z_2)$ gives

$$|z_1 - z_2| \leq K_p (1 - kC_p)^{-1-\frac{2}{p}} ||\mu||_p |f(z_1) - f(z_2)|^{1-\frac{2}{p}} + |f(z_1) - f(z_2)|. \tag{6.13}$$

Now we can prove the theorem we have been building up to. [4, p. 57]

Theorem 6.15. *For any μ with compact support and $||\mu||_\infty \leq k < 1$, the normal solution f to the Beltrami equation is a quasiconformal homeomorphism with complex dilatation μ.*

Proof. (Based on [4, p. 57] and [2, pp. 391–392].) We can find a sequence $\mu_n \in C^1$ such that $\mu_n \to \mu$ almost everywhere, $|\mu_n| \leq k$ and $\mu_n = 0$ outside some disc, since μ has compact support and $||\mu||_\infty \leq k$. Then the corresponding normal solutions f_n tend to f uniformly by Lemma 6.12. The f_n satisfy (6.13), so f does too, hence f is injective, since putting $f(z_1) = f(z_2)$ in (6.13) gives $|z_1 - z_2| \leq 0$. Furthermore, the f_n are C^1 by Lemma 6.13, so f is too. Since $f_z - 1 \in \mathrm{L}^p$, it follows that $f_z \to 1$ as $z \to \infty$, which is enough to show f is surjective.The inverses f_n^{-1} are then C^1 and $f_n^{-1} \to f^{-1}$ uniformly, so f^{-1} is C^1 too, and thus f is a homeomorphism. Moreover, as a uniform limit of quasiconformal mappings, f is itself quasiconformal. We also claim that $f_z \neq 0$ almost everywhere, so that μ_f is defined almost everywhere and is equal to μ. This is true because if $f_z = 0$ then $f_{\bar{z}} = 0$ too, and so the Jacobian $|f_z|^2 - |f_{\bar{z}}|^2 = 0$, which cannot happen on a set of non-zero measure. □

We now drop the assumption that μ has compact support to reach our goal: the measurable Riemann mapping theorem.[11, p. 54]

Theorem 6.16 (Measurable Riemann Mapping Theorem)**.** *For any $\mu \in L^\infty(\mathbb{C})$ with $||\mu||_\infty < 1$, there exists a unique quasiconformal homeomorphism f^μ with complex dilatation μ which fixes $0, 1, \infty$.*

Proof. (Elaboration of [11, pp. 54–55] and [2, pp. 395–396].) First suppose μ has compact support, that is $\mu = 0$ in a neighbourhood of infinity. Let g^μ denote the normal solution to the Beltrami equation. Then g^μ fixes zero and infinity, so setting

$$f^\mu(z) = \frac{g^\mu(z)}{g^\mu(1)}$$

gives a solution which also fixes 1. By Theorem 6.15, g^μ is a quasiconformal homeomorphism with complex dilatation μ, hence so is f^μ.

Now suppose $\mu = 0$ in a neighbourhood of zero. We want to reduce this to the first case, so apply a change of coordinates from z to $\frac{1}{z}$, using the change of coordinates formula for a Beltrami differential to give

$$\lambda(z) = \mu\left(\frac{1}{z}\right)\frac{z^2}{\bar{z}^2}.$$

Then $\lambda = 0$ in a neighbourhood of infinity, so there exists a quasiconformal homeomorphism f^λ with complex dilatation λ which fixes $0, 1, \infty$. We claim that

$$f^\mu(z) = \frac{1}{f^\lambda(\frac{1}{z})}$$

is a quasiconformal homeomorphism with complex dilatation μ which fixes $0, 1, \infty$ (this is stated without proof in [11] and [2]). Indeed, it fixes $0, 1, \infty$ since f^λ does, and is a quasiconformal homeomorphism since f^λ is, so we only need to verify its complex dilatation. Let $i(z) = \frac{1}{z}$. Then, using the chain rule stated in (2.1),

$$\begin{aligned}
f^\mu_{\bar{z}} = (i \circ f^\lambda \circ i)_{\bar{z}} &= (f^\lambda \circ i)_{\bar{z}}(i_z \circ f^\lambda \circ i) + \overline{(f^\lambda \circ i)}_{\bar{z}}(i_{\bar{z}} \circ f^\lambda \circ i) \\
&= (f^\lambda \circ i)_{\bar{z}}(i_z \circ f^\lambda \circ i) \\
&= i_{\bar{z}}(f^\lambda_z \circ i) + \bar{i}_{\bar{z}}(f^\lambda_{\bar{z}} \circ i) \\
&= \bar{i}_{\bar{z}}(g_{\bar{z}} \circ i) \\
&= -\frac{1}{\bar{z}^2} f^\lambda_{\bar{z}}\left(\frac{1}{z}\right) \\
&= -\frac{1}{\bar{z}^2}\lambda\left(\frac{1}{z}\right) f^\lambda_z\left(\frac{1}{z}\right) \\
&= -\frac{1}{z^2}\mu(z) f^\lambda_z\left(\frac{1}{z}\right),
\end{aligned}$$

and similarly

$$f^\mu_z = -\frac{1}{z^2} f^\lambda_z\left(\frac{1}{z}\right),$$

hence $f^\mu_{\bar{z}} = \mu f^\mu_z$ as required.

Finally, consider the general case. Note that any μ can be written in the form $\mu = \eta + \nu$, where η and ν vanish near infinity and zero, respectively. We claim that the required f^μ is given by $f^\mu = f^\lambda \circ f^\nu$, where

$$\lambda = \left(\frac{\eta}{1 - \bar{\nu}\mu} \cdot \frac{f^\nu_z}{\overline{f^\nu}_{\bar{z}}}\right) \circ (f^\nu)^{-1}$$

(this is stated without proof in [11] and [2]). Indeed, since $\eta = 0$ in a neighbourhood of infinity, it follows that $\mu = \nu$ in a neighbourhood of infinity, and so $\bar{\nu}\mu = \bar{\nu}\nu = |\nu|^2 < 1$. Hence $\lambda = 0$ in a neighbourhood of infinity, so we can use the first case to conclude that f^λ exists, and by the second case f^ν exists too. Then $f^\lambda \circ f^\nu$ is a quasiconformal homeomorphism which fixes $0, 1, \infty$ since f^λ and f^ν are. The complex dilatation is μ:

$$\begin{aligned}
(f^\lambda \circ f^\nu)_{\bar{z}} &= f^\nu_{\bar{z}}(f^\lambda_z \circ f^\nu) + \overline{f^\nu}_{\bar{z}}(f^\lambda_{\bar{z}} \circ f^\nu) \\
&= \nu f^\nu_z(f^\lambda_z \circ f^\nu) + \overline{f^\nu}_{\bar{z}}((\lambda f^\lambda_z) \circ f^\nu) \\
&= \nu f^\nu_z(f^\lambda_z \circ f^\nu) + \frac{\eta}{1 - \bar{\nu}\mu} f^\nu_z(f^\lambda_z \circ f^\nu) \\
&= \frac{\nu - \nu\bar{\nu}\mu + \eta}{1 - \bar{\nu}\mu} f^\nu_z(f^\lambda_z \circ f^\nu) \\
&= \frac{\mu - \nu\bar{\nu}\mu}{1 - \bar{\nu}\mu} f^\nu_z(f^\lambda_z \circ f^\nu) \\
&= \frac{\mu(1 - \bar{\nu}\mu) + \eta\bar{\nu}\mu}{1 - \bar{\nu}\mu} f^\nu_z(f^\lambda_z \circ f^\nu) \\
&= \mu \left(f^\nu_z(f^\lambda_z \circ f^\nu) + \frac{\eta\bar{\nu}}{1 - \bar{\nu}\mu} f^\nu_z(f^\lambda_z \circ f^\nu) \right) \\
&= \mu \left(f^\nu_z(f^\lambda_z \circ f^\nu) + \frac{\eta\bar{\nu}}{1 - \bar{\nu}\mu} f^\nu_z \left(\left(\frac{1}{\lambda} f^\lambda_{\bar{z}} \right) \circ f^\nu \right) \right) \\
&= \mu \left(f^\nu_z(f^\lambda_z \circ f^\nu) + \bar{\nu}\overline{f^\nu}_{\bar{z}}(f^\lambda_{\bar{z}} \circ f^\nu) \right) \\
&= \mu \left(f^\nu_z(f^\lambda_z \circ f^\nu) + \bar{\nu}\overline{f^\nu_z}(f^\lambda_{\bar{z}} \circ f^\nu) \right) \\
&= \mu \left(f^\nu_z(f^\lambda_z \circ f^\nu) + \overline{f^\nu_{\bar{z}}}(f^\lambda_{\bar{z}} \circ f^\nu) \right) \\
&= \mu \left(f^\nu_z(f^\lambda_z \circ f^\nu) + \overline{f^\nu}_z(f^\lambda_{\bar{z}} \circ f^\nu) \right) \\
&= \mu(f^\lambda \circ f^\nu)_z.
\end{aligned}$$

Quasiconformal homeomorphisms with a given complex dilatation are unique up to composition with a conformal mapping $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, that is, a Möbius transformation. But since our f^μ fixes three points, the only possible such Möbius transformation is the identity. Hence f^μ is unique. □

Remark. *Additionally, this f is smooth wherever μ is, and varies complex analytically with respect to μ (see* [2]*).*

7 Teichmüller's existence theorem

7.1 Teichmüller space: two definitions

We shall finally introduce a fundamental part of Teichmüller theory which will be required to prove the existence theorem: Teichmüller space.

Definition 7.1. [7, p. 8] *Let γ be a geodesic in hyperbolic space $\mathbb{H}^2$. Then the complement of γ in $\mathbb{H}^2$ consists of two components. The closure of one of these components is called a* hyperbolic half-space. *A surface X with complete, finite-area hyperbolic metric is then* *said to have* totally geodesic boundary *if X admits an atlas $\{\phi_\alpha \colon U_\alpha \to H_\alpha\}$ of charts to hyperbolic half-spaces H_α bounded by geodesics γ_α such that $\phi_\alpha(U_\alpha \cap \partial X) = \phi_\alpha(U_\alpha) \cap \gamma_\alpha$ for each α.*

Definition 7.2. [9, pp. 263–264] *Let S be a compact surface of genus at least 2. A* hyperbolic structure *on S is a diffeomorphism $\phi \colon S \to X$ for some surface X with a complete, finite-area hyperbolic metric and totally geodesic boundary. The map ϕ is called a* marking*, and the pair (X, ϕ) is called a* marked hyperbolic surface*. Given two hyperbolic structures $\phi_i \colon S \to X_i$ for $i = 1, 2$, the homeomorphism $\phi_2 \circ \phi_1^{-1} \colon X_1 \to X_2$ is called the* change of marking map*.*

Two hyperbolic structures $\phi_i \colon S \to X_i$ for $i = 1, 2$ are said to be homotopic if there is an isometry $I \colon X_1 \to X_2$ such that $I \circ \phi_1$ and ϕ_2 are homotopic.

Definition 7.3. [9, p. 264] *The* Teichmüller space *of a compact surface S of genus at least 2 is the set of homotopy classes of hyperbolic structures on S. In other words,*

$$\mathrm{Teich}(S) = \{(X, \phi)\}/\sim,$$

where two marked hyperbolic spaces are equivalent if and only if the hyperbolic structures they define are homotopic.

Remark. *By the classification of surfaces, S is determined up to homeomorphism by its genus, so we can simply write S_g for the surface of genus g. Moreover, it turns out that* $\mathrm{Teich}(S_g) \approx \mathbb{R}^{6g-6}$ *for $g \geq 2$, a fact that will be useful in proving Teichmüller's existence theorem. This arises from Fenchel–Nielsen coordinates, which are studied in depth in* [9, §10.6].

As well as this one, there is also a second equivalent definition of Teichmüller space involving representations of the fundamental group of the surface. It will be useful to be able to switch between the two while proving Teichmüller's existence theorem. First we need to state exactly what we mean by a representation.

Definition 7.4. *A* projective representation *of $\pi_1(S_g)$ in $\mathbb{H}^2$ is a group homomorphism $\rho \colon \pi_1(S_g) \to \mathrm{PSL}(2, \mathbb{R})$. Such a representation is called* faithful *if it is injective, and* discrete *if its image is discrete in* $\mathrm{PSL}(2, \mathbb{R})$*.*

We denote the space of discrete faithful representations by $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2, \mathbb{R}))$. The group $\mathrm{PGL}(2, \mathbb{R})$ acts on this space by conjugation as follows: given $\gamma \in \pi_1(S_g)$ and $h \in \mathrm{PGL}(2, \mathbb{R})$, we define $(h \cdot \rho)(\gamma) = h\rho(\gamma)h^{-1}$. Then $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PGL}(2, \mathbb{R})$ is the quotient by this action.[9, p. 269]

Proposition 7.5. [9, p. 269] *For $g \geq 2$, there is a natural bijective correspondence between the quotient* $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ *and* $\mathrm{Teich}(S_g)$.

Proof. (Sketch; see [9, pp. 269–270] for more detail.) Let $[(X, \phi)]$ be a point in $\mathrm{Teich}(S_g)$. If $\widetilde{X}$ is the metric universal cover of X, then by the uniformisation theorem there is an isometric identification $\widetilde{X} \to \mathbb{H}^2$ since $g \geq 2$, and the group of deck transformations is identified with a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Moreover, the group $\pi_1(X)$ acts isometrically and properly discontinuously on $\widetilde{X}$, and the marking ϕ identifies $\pi_1(S_g)$ with $\pi_1(X)$ and hence with the group of deck transformations of $\widetilde{X}$. In this way we obtain a discrete faithful representation $\rho\colon \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$. Choosing the equivalence class of this ρ in $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ completes the description of one direction of the desired bijection.

To define an inverse, first take $[\rho] \in \mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$. We claim that ρ is a covering space action on $\mathbb{H}^2$, and so $X := \mathbb{H}^2/\rho(\pi_1(S_g))$ has fundamental group $\pi_1(S_g)$. Then by classification of surfaces X is diffeomorphic to S_g. We obtain a particular marking by considering the homomorphism $\rho_*\colon \pi_1(S_g) \to \pi_1(X)$ induced by ρ, and then taking the unique homotopy class of homotopy equivalences from S_g to X that realises ρ_*. $\square$

One useful corollary of this proposition is that it allows us to define a topology on $\mathrm{Teich}(S_g)$.[9, pp. 270–271] Giving $\pi_1(S_g)$ the discrete topology and $\mathrm{PSL}(2,\mathbb{R})$ its standard topology as a Lie group, we can give the set of homomorphisms $\mathrm{Hom}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))$ the compact-open topology. We then give $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))$ the subspace topology, and in turn $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ is endowed with the quotient topology. The above bijective correspondence then induces a topology on $\mathrm{Teich}(S_g)$, which is known as the *algebraic toplogy*.

7.2 Proof of Teichmüller's existence theorem

We now have all the tools we need to prove Teichmüller's existence theorem. Recall the statement:

Theorem (Teichmüller's Existence Theorem)**.** *Let $f\colon X \to Y$ be a homeomorphism between closed Riemann surfaces of genus $g \geq 2$. Then there exists a Teichmüller mapping $h\colon X \to Y$ homotopic to f.*

The rest of the section shall be dedicated to proving this step by step, and follows the method used in [9, pp. 330–336].

Define a norm on the vector space $\mathrm{QD}(X)$ of quadratic differentials on X by

$$||q|| = \int_X |q|.$$

This norm induces a metric and hence also a topology on $\mathrm{QD}(X)$. Denote the open unit ball with respect to this norm by $\mathrm{QD}_1(X)$. The first step of our proof shall be to define an exponential map $\Omega\colon \mathrm{QD}_1(X) \to \mathrm{Teich}(S_g)$, whereby proof of the theorem shall amount to proving surjectivity of this map.

Given $q \in \mathrm{QD}_1(X)$, we define

$$K = \frac{1 + ||q||}{1 - ||q||} > 1.$$

Then we can construct a Teichmüller map $h\colon X \to Y$ with initial differential q and horizontal stretch factor K for some Riemann surface Y as follows:

Let Z denote the set of zeros of q, and consider $X' = X \setminus Z$. Note that X' is still a Riemann surface; we see this by observing that its complex structure arises via the natural coordinates of q restricted to X'. Denote the underlying topological surfaces of X and X' by S and S', respectively. Composing each chart of X' with the affine map

$$h(x + iy) = \sqrt{K}x + \frac{i}{\sqrt{K}}y$$

gives a new atlas on S', yielding a new Riemann surface Y'. Now, by the removable singularity theorem (defined in [12, p. 5]), the complex structure Y' on S' extends uniquely to a complex structure Y on S. Moreover, there is an induced homeomorphism $h\colon X \to Y$ (and an induced quadratic differential on Y), which by construction is a Teichmüller mapping with initial differential q and horizontal stretch factor K. We can regard X as a point $\mathcal{X} \in \mathrm{Teich}(S_g)$ by identifying X with S_g, and then by regarding h as a marking $h\colon S_g \to Y$ we obtain a point $\mathcal{Y} = [(Y, h)]$ in $\mathrm{Teich}(S_g)$. This construction defines a function $\Omega\colon \mathrm{QD}_1(X) \to \mathrm{Teich}(S_g)$.

Suppose Ω is surjective; this means that for every $\mathcal{Z} = [(Z, f)] \in \mathrm{Teich}(S_g)$ there exists some $q \in \mathrm{QD}_1(X)$ such that $\Omega(q) = \mathcal{Z}$, that is there is a Teichmüller mapping $h\colon X \to Z$ in the homotopy class of the homeomorphism $f\colon X \to Z$ (identifying X with S_g). Hence surjectivity of Ω is sufficient to prove Teichmüller's existence theorem.

In order to prove surjectivity, we must first show that Ω is continuous and proper.

Proposition 7.6. [9, p. 331] *The map* $\Omega\colon \mathrm{QD}_1(X) \to \mathrm{Teich}(S_g)$ *is continuous.*

Proof. (Based on [9, p. 335].) Let X be a Riemann surface of genus $g \geq 2$, homeomorphically identified with S_g so that X represents a point $\mathcal{X} \in \mathrm{Teich}(S_g)$. We shall show continuity of Ω by splitting it into $\Omega = \Omega_2 \circ \Omega_1$, where $\Omega_1\colon \mathrm{QD}_1(X) \to \mathrm{L}^\infty(U)$ and $\Omega_2\colon \mathrm{L}^\infty(U) \to \mathrm{Teich}(S_g)$, with U being the upper half-plane; we then show Ω_1 and Ω_2 are continuous.

First we define Ω_1. Let q denote the coefficient of a quadratic differential in $\mathrm{QD}_1(X)$. We can think of X as a quotient $U/\pi_1(X)$ of the upper half-plane by conformal automorphisms as follows:

Let $\widetilde{X} \xrightarrow{\pi} X$ be the universal cover of X. Then by the uniformisation theorem, $\widetilde{X}$ is isomorphic to U since $g \geq 2$, and X is isomorphic to $U/\mathrm{Deck}(\widetilde{X} \xrightarrow{\pi} X)$. But the group of deck transformations is isomorphic to $\pi_1(X)$, so X is isomorphic to $U/\pi_1(X)$.

Given some chart of X on which q is defined, we can take analytic continuations[2] of q to the whole of X and lift it to obtain a holomorphic function $\widetilde{q}$ on U. Every deck transformation g satisfies $\pi \circ g = \pi$, so q is left unchanged by deck transformations, but g changes coordinates z in U to $g(z)$, hence $\widetilde{q}$ must satisfy the change of coordinates formula

$$\widetilde{q}(z) = \widetilde{q}(g(z)) \left(\frac{dg}{dz}\right)^2, \tag{7.1}$$

so that $\widetilde{q} = (\widetilde{q} \circ g)(g')^2$ for each deck transformation g. Since every such g corresponds to some $[\gamma] \in \pi_1(X)$, we can think of each deck transformation as an action of $\pi_1(X)$ on U. In this sense, $\widetilde{q}\colon U \to \mathbb{C}$ is a $\pi_1(X)$-equivariant function. In fact, $\widetilde{q} \in \mathrm{L}^\infty(U)$, recalling that q was chosen to be the coefficient of a bounded holomorphic quadratic differential. (The above construction of $\widetilde{q}$ is not explained in detail in [9].)

If we fix the covering map $U \to X$ and the fundamental domain in U, then we obtain a well-defined map $\Omega_1\colon \mathrm{QD}_1(X) \to \mathrm{L}^\infty(U)$ by defining

$$\begin{aligned} &\Omega_1(0) = 0 \\ &\Omega_1(q)(z) = ||q|| \frac{\overline{\widetilde{q}(z)}}{|\widetilde{q}(z)|} \qquad \text{if } q \neq 0. \end{aligned}$$

We define Ω_1 like this because we want $\Omega_1(q)$ to behave like a Beltrami coefficient. Indeed, $\widetilde{q}$ transforms by dz^2, so $\overline{\widetilde{q}}$ and $|\widetilde{q}|$ transform by $\overline{dz}^2$ and $dz\overline{dz}$, respectively, and thus $\Omega_1(q)$ transforms by $\frac{\overline{dz}}{dz}$.

Continuity of Ω_1 follows by the $\pi_1(X)$-equivariance of each $\Omega_1(q)$: if we change $q \in \mathrm{QD}_1(X)$ by a small amount in one chart, then it must obey the same change of coordinates formula as given in (7.1), so the function $\Omega_1(q)$ also changes by a small amount.

We then define $\Omega_2\colon \mathrm{L}^\infty(U) \to \mathrm{Teich}(S_g)$ by appealing to the measurable Riemann mapping theorem. Let $\mu \in \mathrm{L}^\infty(U)$ and reflect over the real axis to give a new $\mu \in \mathrm{L}^\infty(\mathbb{C})$. We can assume $||\mu||_\infty < 1$ since for any $q \in \mathrm{QD}_1(X)$ we will have $||\Omega_1(q)||_\infty = ||q|| < 1$. Then the measurable Riemann mapping theorem gives a unique quasiconformal homeomorphism $f^\mu\colon \mathbb{C} \to \mathbb{C}$ which fixes $0, 1, \infty$ and has complex dilatation μ. It is also smooth wherever μ is and varies complex analytically with respect to μ. Consider X as a representation $\rho\colon \pi_1(X) \to \mathrm{Isom}^+(\mathbb{H}^2)$ by identifying $\mathrm{PSL}(2, \mathbb{R})$ with the group $\mathrm{Isom}^+(\mathbb{H}^2)$ of orientation-

[2]If f_1 and f_2 are analytic functions on domains R_1 and R_2, respectively, such that $R_1 \cap R_2 \neq \emptyset$ and $f_1 = f_2$ on $R_1 \cap R_2$, then f_2 is called an analytic continuation of f_1 to R_2 and vice versa. Moreover, if it exists, the analytic continuation of f_1 to R_2 is unique.

preserving isomorphisms of $\mathbb{H}^2$. Restricting f^μ to U, if we conjugate every element in the image of ρ by f^μ, we obtain a new representation, which gives a new Riemann surface X'. In particular, f^μ induces a homeomorphism $X \to X'$ which is smooth almost everywhere. We can then regard X' as a point of $\mathrm{Teich}(S_g)$ by identifying X with S_g. This procedure defines our Ω_2, which is continuous because f^μ varies complex analytically with respect to μ.

We need now only check that $\Omega_2 \circ \Omega_1 = \Omega$. Let $q \in \mathrm{QD}_1(X)$ and write $\widetilde{q}$ in polar form as $\widetilde{q}(u) = re^{i\theta}$ for $u \in U$. Then $\Omega(q) \in \mathrm{Teich}(S_g)$ is obtained from X by stretching by a factor of $\frac{1+||q||}{1-||q||}$ in direction $-\frac{1}{2}\theta$ at that point, by definition of the Teichmüller map which Ω constructs. On the other hand,

$$\Omega_1(q)(u) = ||q||\frac{re^{-i\theta}}{|re^{i\theta}|} = ||q||e^{-i\theta},$$

so the $f^{\Omega_1(q)}$ determined in the definition of Ω_2 has complex dilatation $\mu = ||q||e^{-i\theta}$ at u, and therefore dilatation $K = \frac{1+||q||}{1-||q||}$. In section 2 we showed that $f^{\Omega_1(q)}$ can then be described as a stretch by a factor of K in direction $\frac{1}{2}\arg\mu = -\frac{1}{2}\theta$. Therefore $\Omega = \Omega_2 \circ \Omega_1$, as required. □

Proposition 7.7. [9, p. 331] *The map* $\Omega\colon \mathrm{QD}_1(X) \to \mathrm{Teich}(S_g)$ *is proper.*

Proof. (Closely follows [9, pp. 331–332].) Define $\kappa\colon \mathrm{Teich}(S_g) \to \mathbb{R}$ by

$$\kappa(\mathcal{Y}) = \inf\left\{ K_h : \begin{array}{l} h\colon X \to Y \text{ is a quasiconformal} \\ \text{homeomorphism isotopic to} \\ \text{the change of marking} \end{array} \right\}$$

where $\mathcal{Y} \in \mathrm{Teich}(S_g)$ is represented by a marked Riemann surface Y. We claim that κ is continuous.

Indeed, given two nearby elements $\mathcal{Y}, \mathcal{Y}'$ of $\mathrm{Teich}(S_g)$, we can represent them by nearby elements of $\mathrm{DF}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PGL}(2,\mathbb{R})$ due to the bijective correspondence described earlier in this section. Then by taking $\mathcal{Y}'$ sufficiently close to $\mathcal{Y}$, we can find a K-quasiconformal mapping between fundamental domains of these representations for any $K > 1$, with K approaching 1 as $\mathcal{Y}'$ approaches $\mathcal{Y}$. By definition of $\kappa(\mathcal{Y})$, there exists a quasiconformal homeomorphism $h\colon X \to Y$ isotopic to the change of marking with $K_h = \kappa(\mathcal{Y}) + \varepsilon$ for $\varepsilon > 0$ arbitrarily small. We can therefore find a $K(\kappa(\mathcal{Y}) + \varepsilon)$-quasiconformal homeomorphism $X \to Y'$ isotopic to the change of marking by composing these two quasiconformal mappings. Since ε is arbitrarily small and K can be made arbitrarily close to 1 by taking $\mathcal{Y}'$ close enough to $\mathcal{Y}$, it follows that $\kappa(\mathcal{Y}')$ can be made arbitrarily close to $\kappa(\mathcal{Y})$ by taking $\mathcal{Y}'$ close enough to $\mathcal{Y}$.

Now let $A \subset \mathrm{Teich}(S_g)$ be compact and let $q \in \Omega^{-1}(A)$. By definition of Ω, there exists a Teichmüller map $h\colon X \to \Omega(q)$ isotopic to the change of marking with dilatation $K_h = \frac{1+||q||}{1-||q||}$. By Teichmüller's uniqueness theorem, any quasiconformal homeomorphism $X \to \Omega(q)$ isotopic to the change of marking must have dilatation at least K_h, as any such map is then homotopic to h. Since κ is continuous, $\kappa|_A$ attains a maximum $M \geq 0$, so $M \geq K_h = \frac{1+||q||}{1-||q||}$. Solving for $||q||$, we get $||q|| \leq \frac{M-1}{M+1} < 1$, hence $\Omega^{-1}(A)$ is contained in the closed ball of radius $\frac{M-1}{M+1}$ centred on the origin in $\mathrm{QD}_1(X)$. Then, since $\Omega^{-1}(A)$ is closed (as Ω is continuous), $\Omega^{-1}(A)$ is therefore compact, and hence Ω is proper. $\square$

In summary, we have shown that $\Omega\colon \mathrm{QD}_1(X) \to \mathrm{Teich}(S_g)$ is injective (by Teichmüller's uniqueness theorem), proper and continuous. We also showed that $\mathrm{QD}(X)$ has real dimension at least $6g-6$, hence $\mathrm{QD}_1(X)$ does too, and so it contains a subspace homeomorphic to $\mathbb{R}^{6g-6}$. Additionally, we know that $\mathrm{Teich}(S_g) \approx \mathbb{R}^{6g-6}$. We can therefore regard Ω as a map $\mathbb{R}^{6g-6} \to \mathbb{R}^{6g-6}$ by restricting as appropriate.

Now we can apply Brouwer's invariance of domain theorem (expounded in [6]) to Ω, which states that any injective continuous map $\mathbb{R}^n \to \mathbb{R}^n$ is an open map. Since Ω is also proper, it is therefore a closed map too; indeed, suppose $U \subseteq \mathbb{R}^{6g-6}$ is closed and $\Omega(U)$ is not closed. Then we can pick a point $p \in \overline{\Omega(U)} \setminus \Omega(U)$. Take a compact ball around p and pull back to a compact subset V of $\mathbb{R}^{6g-6}$. By compactness we can choose a sequence $\Omega(a_k)$ convergent to p, and by compactness of V there is a subsequence of a_k convergent in V. In fact, by continuity of Ω this subsequence is convergent to $\Omega^{-1}(p)$. But then $\Omega^{-1}(p) \in U$ by closedness, so $p \in \Omega(U)$, contradicting our assumption.

Hence $\Omega(\mathbb{R}^{6g-6}) \subseteq \mathbb{R}^{6g-6}$ is both open and closed, hence is the whole of $\mathbb{R}^{6g-6}$. Thus Ω is surjective, as required.

8 Applications

8.1 The Teichmüller metric

The most important and direct application of Teichmüller's theorem is in defining a metric on Teichmüller space. This then allows us to extend this theory to study moduli space, tackle problems in dynamics and even foray into string theory and computer graphics.[16][13]

Definition 8.1. [9, p. 337] *Let $\mathcal{X}, \mathcal{Y} \in \mathrm{Teich}(S_g)$ be points in Teichmüller space represented by marked Riemann surfaces X and Y, respectively, and let $f\colon X \to Y$ be the change of marking map. Let $h\colon \mathcal{X} \to \mathcal{Y}$ be the unique Teichmüller mapping in the homotopy class of f given by Teichmüller's theorem. Then the* Teichmüller distance *between $\mathcal{X}$ and $\mathcal{Y}$ is*

$$d_{\mathrm{Teich}}(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}\log(K_h).$$

Proposition 8.2. [9, p. 337] *Teichmüller distance defines a complete metric on* $\text{Teich}(S_g)$ *called the* Teichmüller metric*.*

Proof. (Elaboration of [9, pp. 337–338].) By definition $d_{\text{Teich}}(\mathcal{X},\mathcal{Y}) = 0$ if and only if there is a Teichmüller mapping $h\colon X \to Y$ with dilatation 1 homotopic to the change of marking. The change of marking is therefore homotopic to a conformal map, hence $\mathcal{X} = \mathcal{Y}$. Recall that the inverse of a K-quasiconformal homeomorphism is again a K-quasiconformal homeomorphism, so $d_{\text{Teich}}(\mathcal{X},\mathcal{Y}) = d_{\text{Teich}}(\mathcal{Y},\mathcal{X})$. It therefore only remains to prove the triangle inequality. Let K_1 and K_2 be the dilatations of the Teichmüller mappings defining $d_{\text{Teich}}(\mathcal{X},\mathcal{Y})$ and $d_{\text{Teich}}(\mathcal{Y},\mathcal{Z})$. Then composing these Teichmüller mappings gives a K_1K_2-quasiconformal homeomorphism which is homotopic to the Teichmüller mapping defining $d_{\text{Teich}}(\mathcal{X},\mathcal{Z})$. Denote the dilatation of this Teichmüller mapping by K. Then by Teichmüller's uniqueness theorem, $K \leq K_1K_2$, so

$$\frac{1}{2}\log(K) \leq \frac{1}{2}\log(K_1K_2) = \frac{1}{2}\log(K_1) + \frac{1}{2}\log(K_2).$$

Now we show completeness. Let $\mathcal{X} \in \text{Teich}(S_g)$ be a point represented by a marked Riemann surface X and recall the map $\Omega\colon \text{QD}_1(X) \to \text{Teich}(S_g)$ we defined in the proof of Teichmüller's existence theorem. By definition of this map, if $\mathcal{Y} \in \text{Teich}(S_g)$ is a distance $\frac{1}{2}\log(K)$ from $\mathcal{X}$ then $||\Omega^{-1}(\mathcal{Y})|| = \frac{K-1}{K+1}$. Furthermore, $\frac{K-1}{K+1}$ is a continuous increasing function for $K \geq 1$ bounded below by 0 (attained when $K = 1$) and above by 1 and $\frac{1}{2}\log(K)$ is increasing, hence Ω^{-1} takes the closed ball in $\text{Teich}(S_g)$ of radius $\frac{1}{2}\log(K)$ centred on $\mathcal{X}$ to the closed ball in $\text{QD}_1(X)$ of radius $\frac{K-1}{K+1}$ centred on 0. Since $\frac{K-1}{K+1}$ never reaches 1, these balls are bounded and therefore compact. But Ω^{-1} is a homeomorphism, so this means closed balls in $\text{Teich}(S_g)$ are compact. Hence $(\text{Teich}(S_g), d_{\text{Teich}})$ is a proper metric space, thus it is complete. □

8.2 Teichmüller lines

We will now describe how to construct *Teichmüller lines*,[9, pp. 322–323] which turn out to be in direct correspondence with geodesics in $(\text{Teich}(S_g), d_{\text{Teich}})$.

Let X be a closed Riemann surface, q a holomorphic quadratic differential on X and $K > 1$ a real number. Let $X' = X \setminus Z$, where Z is the set of zeros of q, and let S and S' be the topological surfaces underlying X and X', respectively. Note that X' is still a Riemann surface, as we can define its complex structure by restricting q to X' and taking natural coordinates. Now define $f\colon \mathbb{C} \to \mathbb{C}$ by

$$f(x+iy) = \sqrt{K}x + i\frac{1}{\sqrt{K}}y,$$

and compose this with the charts on X' to give a new complex structure for S', which defines a new Riemann surface Y'. Then by the removable singularity theorem we can extend this complex structure uniquely to the whole of S to give a Riemann surface Y. The map f induces a Teichmüller mapping $X \to Y$ with horizontal stretch factor K, and therefore we obtain a point in Teich(S). By varying K from 0 to ∞ we obtain a set of points in Teich(S) which we call a *Teichmüller line*. In particular, $K = 1$ gives the point $\mathcal{X} \in \text{Teich}(S)$ represented by X.

If we then take the bijective correspondence between $(0, \infty)$ and $\mathbb{R}$ given by associating K with $\frac{1}{2}\log(K)$, we obtain an embedding $\mathbb{R} \hookrightarrow \text{Teich}(S)$, which is isometric by definition of d_{Teich}. Teichmüller lines are therefore bi-infinite geodesics with respect to the Teichmüller metric. In fact:

Theorem 8.3. [9, p. 339] *Every geodesic in* $(\text{Teich}(S_g), d_{\text{Teich}})$ *for* $g \geq 2$ *is part of a Teichmüller line.*

Proof. (Based on [9, p. 339].) Let γ be a geodesic connecting two points $\mathcal{X}, \mathcal{Z} \in \text{Teich}(S_g)$ represented by marked Riemann surfaces X, Z. That is, any point $\mathcal{Y}$ on γ represented by Y satisfies $d(\mathcal{X}, \mathcal{Y}) + d(\mathcal{Y}, \mathcal{Z}) = d(\mathcal{X}, \mathcal{Z})$, or

$$\log(K_{\mathcal{X}\mathcal{Y}} K_{\mathcal{Y}\mathcal{Z}}) = \log(K_{\mathcal{X}\mathcal{Y}}) + \log(K_{\mathcal{Y}\mathcal{Z}}) = \log(K_{\mathcal{X}\mathcal{Z}}),$$

where $K_{\mathcal{X}\mathcal{Y}}, K_{\mathcal{Y}\mathcal{Z}}, K_{\mathcal{X}\mathcal{Z}}$ are the horizontal stretch factors of the Teichmüller maps $h_{\mathcal{X}\mathcal{Y}}$, $h_{\mathcal{Y}\mathcal{Z}}$, $h_{\mathcal{X}\mathcal{Z}}$ homotopic to the changes of marking. Then $K_{\mathcal{X}\mathcal{Y}} K_{\mathcal{Y}\mathcal{Z}} = K_{\mathcal{X}\mathcal{Z}}$. Note that $h_{\mathcal{Y}\mathcal{Z}} \circ h_{\mathcal{X}\mathcal{Y}}$ has dilatation $K_{\mathcal{Y}\mathcal{Z}} K_{\mathcal{X}\mathcal{Y}}$, so it has the same dilatation as $h_{\mathcal{X}\mathcal{Z}}$. These maps must therefore be equal by Teichmüller's uniqueness theorem, and in particular the initial differential for $h_{\mathcal{X}\mathcal{Y}}$ is the same as the one for $h_{\mathcal{X}\mathcal{Z}}$. Therefore the Teichmüller line passing through $\mathcal{X}$ and $\mathcal{Y}$ is the same as the one passing through $\mathcal{X}$ and $\mathcal{Z}$. Hence we have shown that any point on a geodesic connecting two points is on the Teichmüller line through those two points, as required. $\square$

Corollary 8.4. [9, p. 339] *There is a unique Teichmüller geodesic between any two points of Teichmüller space.*

Proof. [9, p. 339] By Teichmüller's uniqueness theorem, there is a unique Teichmüller line passing through any two points, hence the result follows directly from the previous theorem. $\square$

8.3 Teichmüller discs and beyond

A *Teichmüller disc* is then defined to be a closed 1-dimensional complex submanifold of $\text{Teich}(S_g)$ isometric to the unit disc with the Poincaré metric.[4, p. 97] The factor of $\frac{1}{2}$

in the definition of the Teichmüller metric was introduced specifically with this isometry in mind. Given any pair of points in a Teichmüller disc, the geodesic between them is contained in the disc. The study of Teichmüller discs and geodesic flows on them is an important topic in dynamical systems, in particular the dynamics of rational billiards. The interested reader might look to Zorich's survey [21] for more on this.

Another major application of Teichmüller's theorem is in providing a way of defining and working with moduli space of Riemann surfaces, an object which is a fundamental part of many areas of geometry and topology. The Nielsen–Thurston classification of homeomorphisms of a compact orientable surface is a key theorem in this area. These topics are discussed in detail in [9, §12-13].

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