The vorticity equation and its applications

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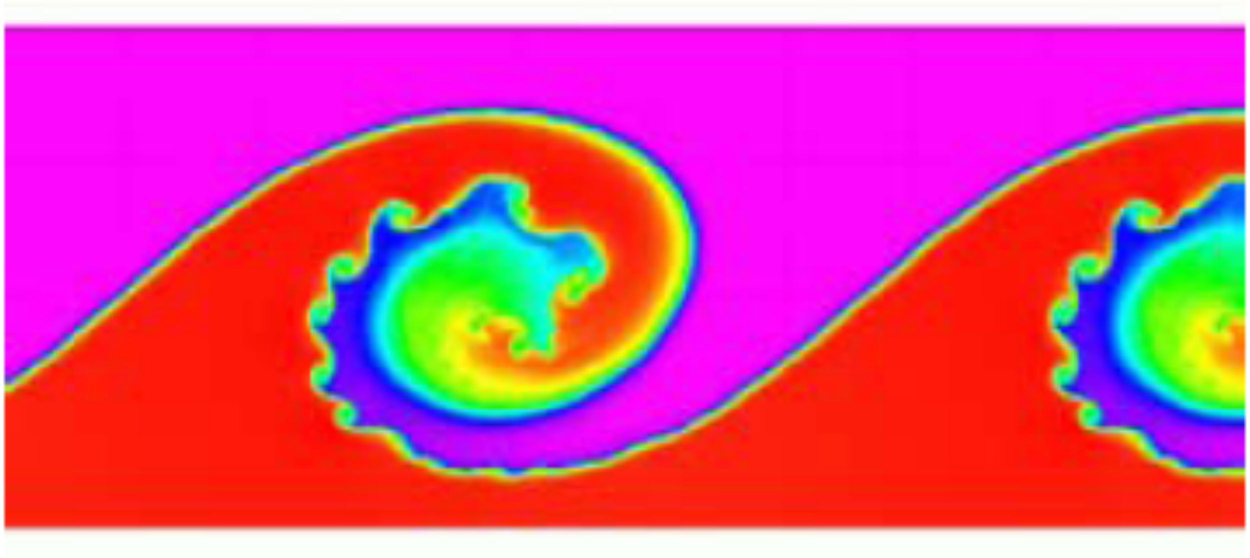
Examples of vortex flows

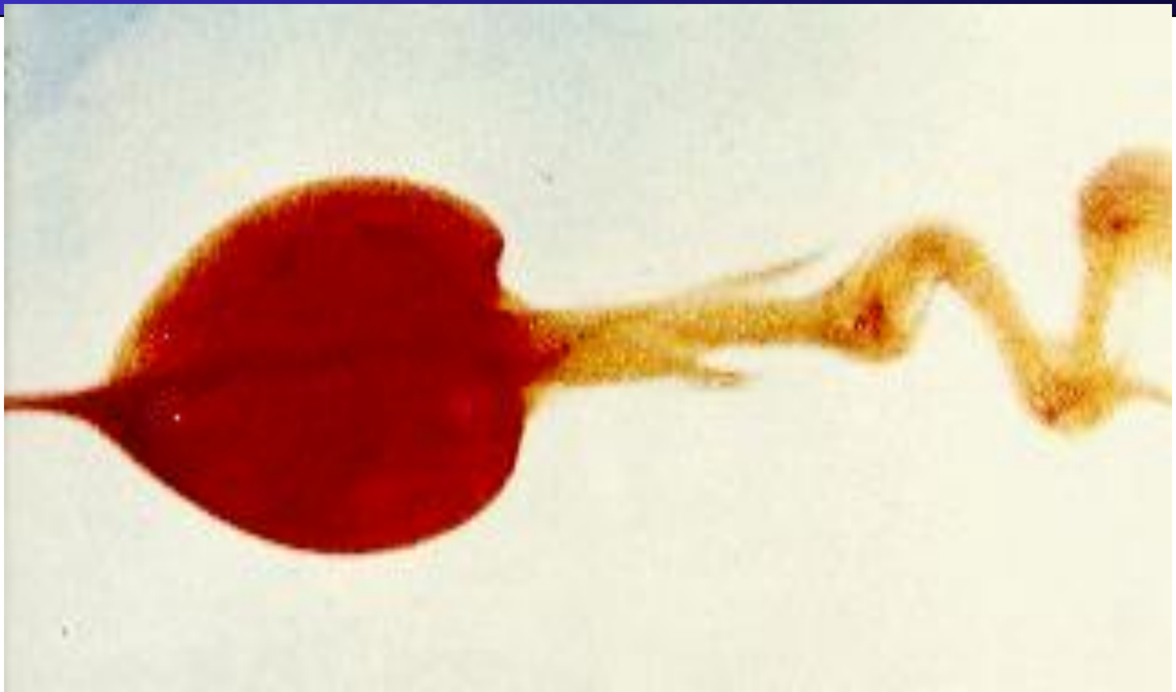


Variable-density mixing layer

Examples of vortex flows

cyclonic vortex in the atmosphere

Examples of vortex flows



VORTEX BREAKDOWN IN THE LABORATORY

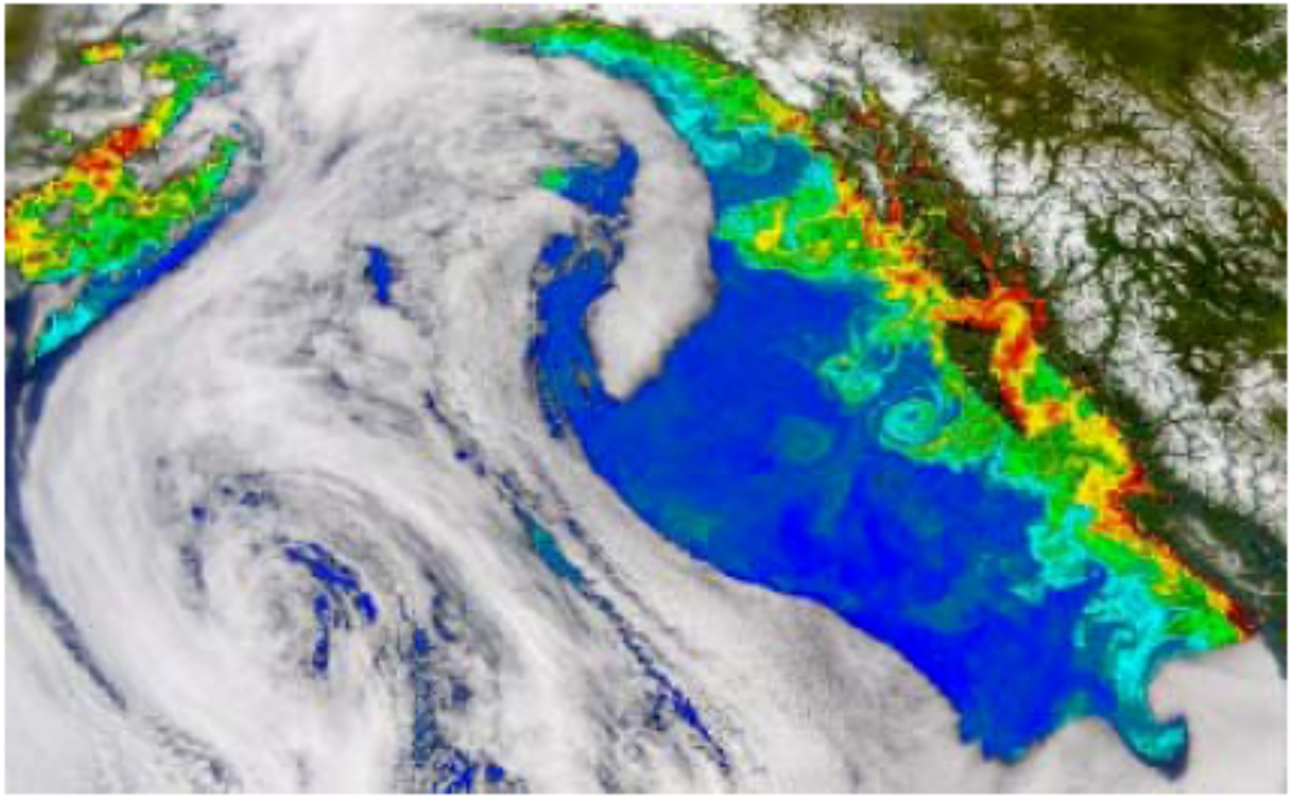
The photo at the right is of a laboratory vortex breakdown provided by Professor Sarpkaya at the Naval Postgraduate School in Monterey, California. Under these highly controlled conditions the bubble-like or B-mode breakdown is nicely illustrated. It is seen in the enlarged version that it is followed by an S-mode breakdown.

Examples of vortex flows

VOLCANIC VORTEX RING

The image at the right depicts a vortex ring generated in the crater of Mt. Etna. Apparently these rings are quite rare. The generation mechanism is bound to be the escape of high pressure gases through a vent in the crater. If the venting is sufficiently rapid and the edges of the vent are relatively sharp, a nice vortex ring ought to form.

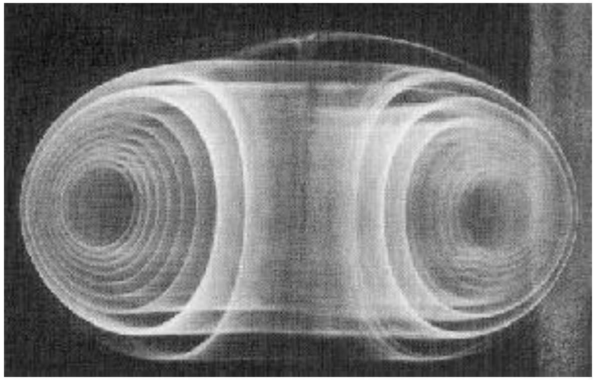
Examples of vortex flows



Vortices in the atmosphere and the oceans – North Pacific

Vortex ring flow

Examples of vortex flows

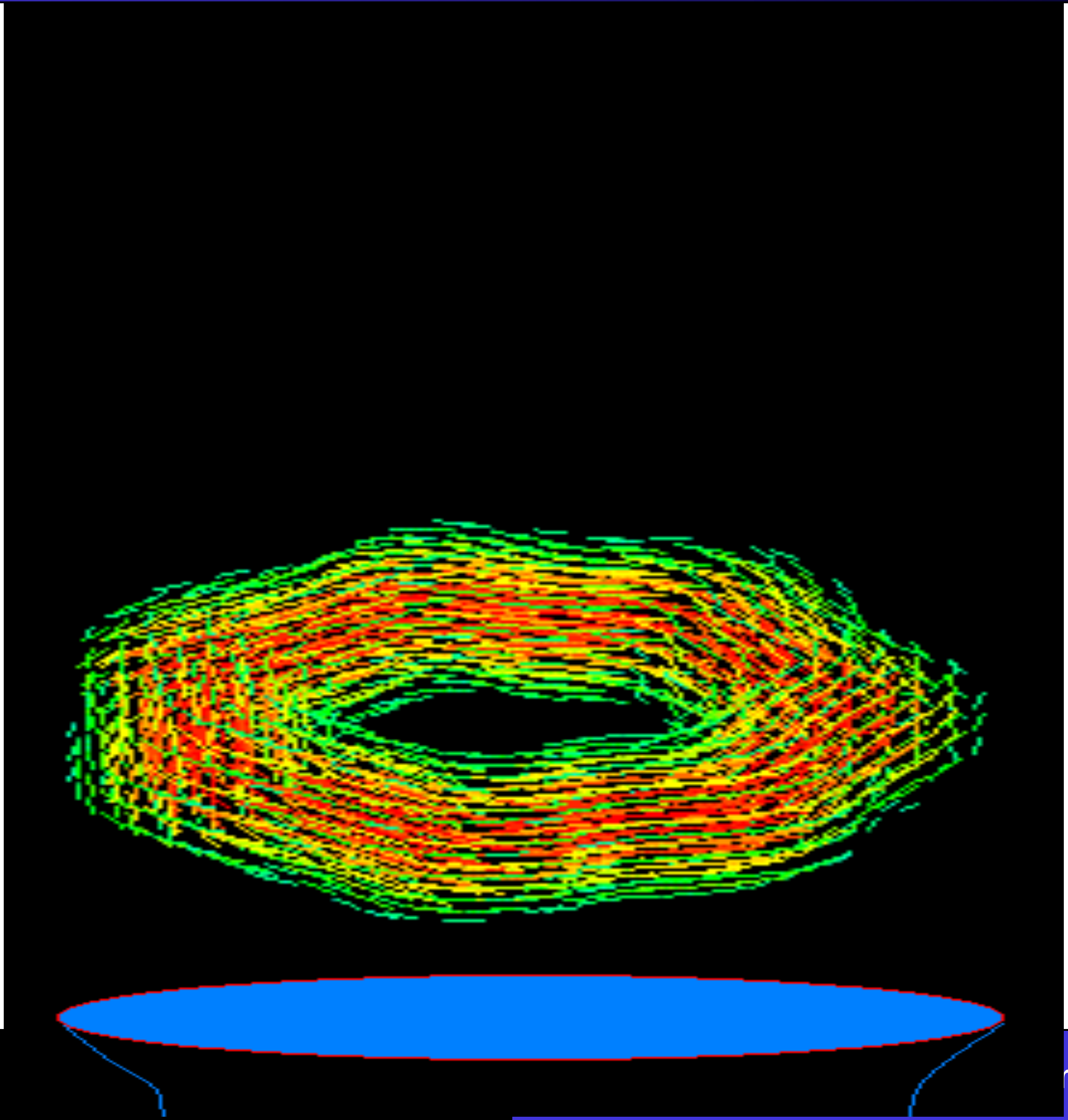


A VORTEX RING
At the right is a vortex ring generated by Professor T.T. Lim and his former colleagues at the University of Melbourne. The visualization technique appears to be by smoke.

Virtual image of a vortex ring flow

www.applied-scientific.com/ MAIN/PROJECTS/NSF00/FAT_RING/Fat_Ring.html -

Force acts impulsively



Overview:

- Derivation of the equation of transport of vorticity
- Describing of the 2D flow motion

  on the basis of vorticity ω and streamfunction ψ instead of the more popular (u,v,p)-system

- Well-known solutions of the system (ω, ψ)

NSE

We start from the **Navier–Stokes equations** for **incompressible Newtonian viscous flow**, given by (in the absence of gravity)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \nu\,(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p \qquad (7.57)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,. \qquad (7.58)$$

These equations of motion were first obtained in 1822 by the French physicist Claude Louis Marie Henri Navier (1785–1836), and later rederived independently in 1845 by the Irish mathematician and physicist George Gabriel Stokes (1819–1903). Note that the *difference* between the **Navier–Stokes equations** and **Euler's equations** for an incompressible ideal fluid is the *second-order* derivative term $\nu\,(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{u}$.

Vorticity

As well as the flow velocity itself, it is useful to define the **vorticity** of a fluid flow which is equal to the curl of the flow velocity.

The **vorticity** is a vector-valued function of position and time defined as

$$\boldsymbol{\omega} := \boldsymbol{\nabla} \times \boldsymbol{u} \,, \tag{1.18}$$

and it is crucially important in the study of **fluid dynamics**.

The vorticity at a point is a measure of the **local rotation**, or **spin**, of a fluid element at that point. Note that the local spin is *not* the same as the **global rotation** of a fluid.

If the flow in a region has zero vorticity, then the flow is described as **irrotational**. **Irrotational flow** is one of the major categories of fluid flows.

In 2-D flow, in the x–y plane of a **Cartesian coordinate system**, the velocity has the form

$$\boldsymbol{u} = [u_x(t,x,y), u_y(t,x,y), 0]^T \,, \tag{1.19}$$

and the vorticity is $\boldsymbol{\omega} = (0, 0, \omega_z)^T$, where

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \,. \tag{1.20}$$

Vorticity transport equation

Helmholtz Equation

Derivation (2-D)

If we neglect viscous forces, the x- and y-components of the 2-D momentum equation can be written as follows.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + g_x \qquad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial y} + g_y \qquad (2)$$

We now take the curl of this momentum equation by performing the following operation.

$$\frac{\partial}{\partial x}\left\{y\text{-momentum } (2)\right\} - \frac{\partial}{\partial y}\left\{x\text{-momentum } (1)\right\}$$

Helmholtz equation

If we assume that ρ is constant (low speed flow), the two pressure derivative terms cancel. Since the gravity components g_x and g_y are generally constant, these also disappear when the curl's derivatives are applied. Using the product rule on the lefthand side, the resulting equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right] = 0$$

We note that the quantity inside the parentheses is merely the z-component of the vorticity $\xi \equiv \partial v/\partial x - \partial u/\partial y$, so the above equation can be more compactly written as

$$\omega_z = \varsigma$$

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + \xi\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right] = 0$$

Vorticity equation on plane

1) $$\frac{\partial}{\partial y}: \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

2) $$\frac{\partial}{\partial x}: \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

2)−1)= $$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \overset{1}{\cancel{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}} + u\overset{3}{\frac{\partial^2 v}{\partial x^2}} + v\overset{4}{\frac{\partial^2 v}{\partial x \partial y}} + \overset{1}{\cancel{\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}}$$

$$- u\overset{3}{\frac{\partial u}{\partial x \partial y}} - \overset{2}{\cancel{\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}}} - v\overset{4}{\frac{\partial^2 u}{\partial y^2}} - \overset{2}{\cancel{\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}}} = -\cancel{\frac{1}{\rho}\frac{\partial p}{\partial x \partial y}} + \cancel{\frac{1}{\rho}\frac{\partial p}{\partial x \partial y}} +$$

$$\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

Taking into account:

$$u = \frac{\partial \Psi}{\partial y}, v = -\frac{\partial \Psi}{\partial x}$$

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x \partial y} = 0$$

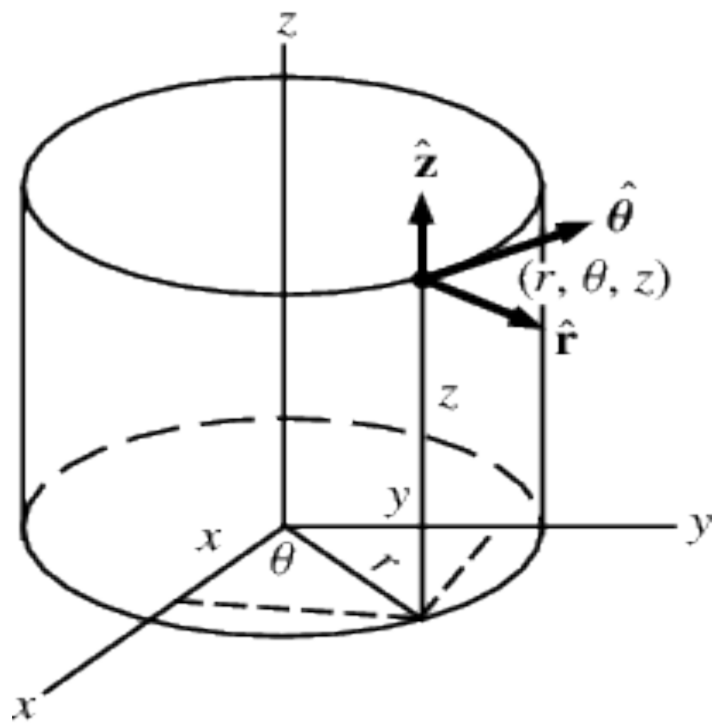
Vorticity equation on plane

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\overset{3}{u}\frac{\partial^2 v}{\partial x^2}-\overset{3}{u}\frac{\partial u}{\partial x \partial y}+\overset{4}{v}\frac{\partial^2 v}{\partial x \partial y}-\overset{4}{v}\frac{\partial^2 u}{\partial y^2}$$

$$=\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$

$$\frac{\partial}{\partial t}\omega+u\frac{\partial \omega}{\partial x}+v\frac{\partial \omega}{\partial y}=\nu\left[\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}\right]$$

Cylindrical coordinate system



In cylindrical coordinates (r , θ ,z) with $\partial / \partial \theta = 0$

-axisymmetric case

Vorticity equation: axisymmetric case

1) $$\frac{\partial}{\partial r}: \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

2) $$\frac{\partial}{\partial z}: \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right]$$

2)−1)=
$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2} + v\frac{\partial^2 v}{\partial z \partial r} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{v}{r}$$
$$- u\frac{\partial u}{\partial z \partial r} - \frac{\partial u}{\partial r}\frac{\partial u}{\partial z} - v\frac{\partial^2 u}{\partial r^2} - \frac{\partial v}{\partial r}\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z \partial r} + \frac{1}{\rho}\frac{\partial p}{\partial z \partial r}$$
$$\nu\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial v}{\partial z} - \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial v}{\partial z}\right]$$

Proof with Mathematica

Taking into account:

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial r}, v = -\frac{1}{r}\frac{\partial \Psi}{\partial z}$$

Continuity equation

$$\frac{\partial (ru)}{\partial z} + \frac{\partial (rv)}{\partial r} = 0$$

$$\frac{\partial (ru)}{\partial z} + \frac{\partial (rv)}{\partial r} = r\frac{\partial u}{\partial z} + r\frac{\partial v}{\partial r} + v =$$

$$r\frac{1}{r}\frac{\partial \Psi}{\partial z \partial r} - r\frac{1}{r}\frac{\partial \Psi}{\partial z \partial r} - \frac{1}{r}\frac{\partial \Psi}{\partial z} + \frac{1}{r}\frac{\partial \Psi}{\partial z} = 0$$

Vorticity equation: axisymmetric case

3 3 4 4

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+u\frac{\partial^2 v}{\partial z^2}-u\frac{\partial u}{\partial z\partial r}+v\frac{\partial^2 v}{\partial z\partial r}-v\frac{\partial^2 u}{\partial r^2}$$

$$=\nu\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{\partial^2}{\partial r^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)-\frac{1}{r^2}\left(\frac{\partial v}{\partial z}-\frac{\partial u}{\partial r}\right)\right]$$

$$\frac{\partial}{\partial t}\omega+u\frac{\partial \omega}{\partial z}+v\frac{\partial \omega}{\partial r}=\nu\left[\frac{\partial^2 \omega}{\partial z^2}+\frac{\partial^2 \omega}{\partial r^2}+\frac{1}{r}\frac{\partial \omega}{\partial r}-\frac{\omega}{r^2}\right]$$

Vorticity transport equation for 2D : q=1- axisymmetric vortices, q=0 – plane vortices

$$\frac{\partial\omega}{\partial t}+v\frac{\partial\omega}{\partial r}+u\frac{\partial\omega}{\partial z}=\nu\left[\frac{\partial^2\omega}{\partial z^2}+\frac{\partial^2\omega}{\partial r^2}+\frac{q}{r}\frac{\partial\omega}{\partial r}-\frac{q\omega}{r^2}\right]$$

The Stokes stream function can be introduced as follows

$$u=\frac{1}{r^q}\frac{\partial\Psi}{\partial r},v=-\frac{1}{r^q}\frac{\partial\Psi}{\partial z}$$

and gives second equation

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{\partial^2\Psi}{\partial z^2}-\frac{q}{r}\frac{\partial\Psi}{\partial r}=-r^q\omega$$

For 3D problem: generalized Helmholtz equation

$$\frac{\partial \omega_x}{\partial t} + v\frac{\partial \omega_x}{\partial x} + u\frac{\partial \omega_x}{\partial y} + w\frac{\partial \omega_x}{\partial z} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} + \nu \Delta \omega_x \quad ,$$

$$\frac{\partial \omega_y}{\partial t} + v\frac{\partial \omega_y}{\partial x} + u\frac{\partial \omega_y}{\partial y} + w\frac{\partial \omega_y}{\partial z} = \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} + \omega_z \frac{\partial v}{\partial z} + \nu \Delta \omega_y \quad ,$$

$$\frac{\partial \omega_z}{\partial t} + v\frac{\partial \omega_z}{\partial x} + u\frac{\partial \omega_z}{\partial y} + w\frac{\partial \omega_z}{\partial z} = \omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y} + \omega_z \frac{\partial w}{\partial z} + \nu \Delta \omega_z \quad ,$$

where $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2$

$$\vec{\omega} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

For 3D problem we can not introduce streamfuction Ψ like for 2D problem.

For 3D problem: generalized Helmholtz equation in cylindrical coordinates

$$\frac{\partial \omega_r}{\partial t} + u_r \frac{\partial \omega_r}{\partial r} + u_\theta \frac{\partial \omega_r}{r\partial \theta} + u_z \frac{\partial \omega_r}{\partial z} = \omega_r \frac{\partial u_r}{\partial r} + \omega_\theta \frac{\partial u_r}{r\partial \theta} + \omega_z \frac{\partial u_r}{\partial z} + \nu\left(\Delta \omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial \omega_\theta}{\partial \theta}\right),$$

$$\frac{\partial \omega_\theta}{\partial t} + u_r \frac{\partial \omega_\theta}{\partial r} + u_\theta \frac{\partial \omega_\theta}{r\partial \theta} + u_z \frac{\partial \omega_\theta}{\partial z} - \frac{u_r \omega_\theta}{r} = \omega_r \frac{\partial u_\theta}{\partial r} + \omega_\theta \frac{\partial u_\theta}{r\partial \theta} + \omega_z \frac{\partial u_\theta}{\partial z} - \frac{u_\theta \omega_\theta}{r} + \nu\left(\Delta \omega_\theta - \frac{\omega_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \omega_r}{\partial \theta}\right),$$

$$\frac{\partial \omega_z}{\partial t} + u_r \frac{\partial \omega_z}{\partial r} + u_\theta \frac{\partial \omega_z}{r\partial \theta} + u_z \frac{\partial \omega_z}{\partial z} = \omega_r \frac{\partial u_z}{\partial r} + \omega_\theta \frac{\partial u_z}{r\partial \theta} + \omega_z \frac{\partial u_z}{\partial z} + \nu \Delta \omega_z,$$

where

$$\omega_r = \frac{\partial u_z}{r\partial \theta} - \frac{\partial u_\theta}{\partial z}, \omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \omega_z = \frac{\partial (r u_\theta)}{r\partial r} - \frac{\partial u_r}{r\partial \theta},$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

For 2D problem:

$(u,v,p) \longrightarrow (\omega, \psi)$

Winning: two variables instead of three

Losses: difficulties with boundary conditions for streamfunction

7.1.3. *Invariants of motion*

When the fluid is inviscid and of constant density and the vorticity is governed by the vorticity equation (7.1), we have four invariants of motion:

$$\text{Energy}: \quad K = \frac{1}{2}\int_{V_\infty} \mathbf{v}^2 \mathrm{d}^3\mathbf{x}$$

$$= \frac{1}{2}\int_D \boldsymbol{\omega}(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x}$$

$$= \int_D \mathbf{v}\cdot(\mathbf{x}\times\boldsymbol{\omega})\,\mathrm{d}^3\mathbf{x}, \quad \text{[Lamb32,}$$

$$\text{Impulse}: \quad \mathbf{P} = \frac{1}{2}\int_D \mathbf{x}\times\boldsymbol{\omega}(\mathbf{x},t)\,\mathrm{d}^3\mathbf{x},$$

$$\text{Angular impulse}: \quad \mathbf{L} = \frac{1}{3}\int_D \mathbf{x}\times(\mathbf{x}\times\boldsymbol{\omega})\,\mathrm{d}^3\mathbf{x},$$

$$\text{Helicity}: \quad H = \int_D \mathbf{v}\cdot\boldsymbol{\omega}\,\mathrm{d}^3\mathbf{x},$$

Vortex flow 2-D plane

7.3.1. *Vorticity equation*

Assuming that the fluid motion is incompressible, the x, y components of the fluid velocity are expressed by using the stream function $\Psi(x, y, t)$ as

$$u = \partial_y \Psi, \qquad v = -\partial_x \Psi, \tag{7.22}$$

(Appendix B.2). The z component of the vorticity is given by

$$\omega = \partial_x v - \partial_y u = -\partial_x^2 \Psi - \partial_y^2 \Psi = -\nabla^2 \Psi, \tag{7.23}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2$. The x, y components of vorticity vanish identically because $w = 0$ and $\partial_z = 0$, where w is the z component of velocity and the velocity field is independent of z.

Streamfunction and (u, v) through vorticity

If the function $\omega(x, y)$ is given, Eq. (7.23) is the Poisson equation $\nabla^2\Psi = -\omega(x, y)$ of the function $\Psi(x, y)$. Its solution is expressed by the following integral form,

$$\Psi(x, y) = -\frac{1}{4\pi}\iint_D \omega(x', y')\log\big((x'-x)^2 + (y'-y)^2\big)\mathrm{d}x'\mathrm{d}y'.$$

and u, v are given by

$$u(x, y) = \partial_y\Psi = \frac{1}{2\pi}\iint_D \frac{(y'-y)\omega(x', y')}{(x'-x)^2 + (y'-y)^2}\,\mathrm{d}x'\mathrm{d}y',$$

$$v(x, y) = -\partial_x\Psi = -\frac{1}{2\pi}\iint_D \frac{(x'-x)\omega(x', y')}{(x'-x)^2 + (y'-y)^2}\,\mathrm{d}x'\mathrm{d}y'.$$

- Solutions, which contain vorticity expressed through delta-functions

Vortex flow.

7.3.4. *Point vortex*

When the vortex domain D of Fig. 7.2(a) shrinks to a point $P = (x_1, y_1)$ by keeping the vortex strength Γ to a finite value k, then we have the following relation,

$$\iint_{\lim D \to P} \omega(x, y)\, dx dy = k.$$

This implies that there is a concentrated vortex at P, and that the vorticity can be expressed by the delta function (see Appendix A.7) as

$$\omega(x, y) = k\delta(x - x_1)\,\delta(y - y_1), \tag{7.38}$$

Substituting this to (7.33) and (7.34), the velocities are

$$u(x, y) = -\frac{k}{2\pi} \frac{y - y_1}{(x - x_1)^2 + (y - y_1)^2}, \tag{7.39}$$

$$v(x, y) = \frac{k}{2\pi} \frac{x - x_1}{(x - x_1)^2 + (y - y_1)^2}, \tag{7.40}$$

This is the same as the right-hand sides of (7.36) and (7.37) with Γ replaced by k, and also the expressions (5.47) and (5.48) (and also (5.73)) if x and y are replaced by $x - x_1$ and $y - y_1$, respectively.

From the definitions of the integral invariants (7.28)–(7.30) and (7.32), we obtain $\Gamma = k$, $X_c = x_1$ and $Y_c = y_1$. Thus, we have the following.

> *The strength k of the point vortex is invariant. In addition, the position (x_1, y_1) of the vortex does not change.*

Namely, the point vortex have no self-induced motion. In other words, a rectilinear vortex does not drive itself.

Vortex flow.

7.3.5. *Vortex sheet*

Vortex sheet is a surface of discontinuity of tangential velocity [Fig. 4.5(a)]. Suppose that there is a surface of discontinuity at $y = 0$ of a fluid flow in the cartesian (x, y, z) plane, and that the velocity field is as follows:

$$\mathbf{v} = \left(\frac{1}{2}U, 0, 0\right) \quad \text{for } y < 0; \quad \mathbf{v} = \left(-\frac{1}{2}U, 0, 0\right) \quad \text{for } y > 0. \quad (7.43)$$

(Directions of flows are reversed from Fig. 4.5(a).) The vorticity of the flow is represented by

$$\boldsymbol{\omega} = (0, 0, \omega(y)), \quad \omega = U\,\delta(y), \qquad (7.44)$$

where $\delta(y)$ is the Dirac's delta function (see A.7). This can be confirmed by using the formula (5.20) (Problem 7.4).

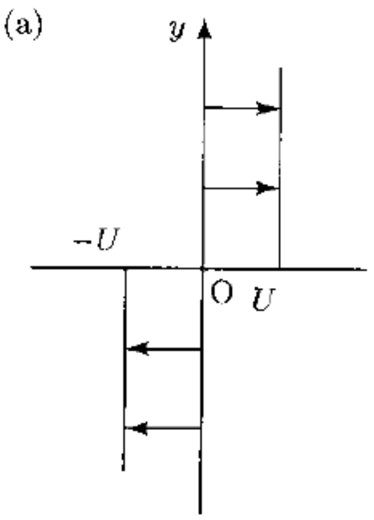


Fig. 4.5. (a) Vortex sheet,

7.6. Axisymmetric vortices with circular vortex-lines

We consider axisymmetric vortices in this section. An axisymmetric flow without swirl is defined by the velocity,

$$\mathbf{v}(\mathbf{x}) = (v_x(x, r), v_r(x, r), 0),$$

in the cylindrical coordinates (x, r, ϕ) (instead of (r, θ, z) of Appendix D.2). A Stokes's stream function $\Psi(x, r)$ can be defined for such axisymmetric flows of an incompressible fluid by

$$v_x = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \quad v_r = -\frac{1}{r}\frac{\partial \Psi}{\partial x}. \tag{7.56}$$

The continuity equation of an incompressible fluid, $\operatorname{div}\mathbf{v} = \partial_x v_x + \frac{1}{r}\partial_r(r v_r) = 0$ (see (D.9)), is satisfied identically by the expressions.

The Biot-Savart Law

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{r}\frac{\partial \Psi}{\partial r} = -r\ \omega$$

$$A_\phi = \frac{1}{r}\Psi(x, r, \phi) = \frac{1}{4\pi}\int \mathrm{d}x' \int \mathrm{d}r' \int_0^{2\pi} r'\mathrm{d}\phi' \ \frac{\omega_\phi(x', r')\cos(\phi' - \phi)}{|\mathbf{x} - \mathbf{y}|},$$

$$u(x,t) = \frac{1}{4\pi}\int \frac{\omega(x',t) \times (x - x')}{|x - x'|^3}\, dx'$$

Using this formula, a vortex particle's velocity can be computed from the vorticity and location of every other particle. Each particle is then advected according to its local velocity, and has its vorticity modified to account for vortex stretching and viscous effects.

- Solutions, which contain vorticity expressed through delta-functions

Vortex flow.

7.6.2. *Circular vortex ring*

When the vortex-line is circular and concentrated on a circular core of a small cross-section, the vortex is often called a *circular vortex ring.* The stream function of a thin circular ring of radius R lying in the plane $x = 0$ is obtained from (7.58) by setting $\omega_\phi(x', r') = \gamma\delta(x')\,\delta(r' - R)$ as

$$\Psi_\gamma(x, r) = \frac{\gamma}{4\pi} rR \int_0^{2\pi} \mathrm{d}\phi \, \frac{\cos\phi}{\left[x^2 + r^2 - 2rR\cos\phi + R^2\right]^{1/2}}, \qquad (7.63)$$

where $\gamma = \iint \omega_\phi \mathrm{d}x'\mathrm{d}r'$ is the strength of the vortex ring.

Vortex flow.

The impulse of the vortex ring of radius R and strength γ is given by (7.9):

$$\mathbf{P} = (P_x, 0, 0), \quad P_x = \int \omega_\phi r^2 \pi \mathrm{d}x\mathrm{d}r = \pi R^2 \gamma.$$

The stream function of a thin-cored vortex ring of radius R and strength γ, defined by (7.58), can be expressed [Lamb32, Sec. 161] as

$$\Psi_\gamma(x,r) = \frac{\gamma}{2\pi}\sqrt{rR}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right], \qquad (7.64)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind of modulus $k^2 = 4rR/(x^2 + (r+R)^2)$, defined by

$$K(k) = \int_0^{\frac{1}{2}\pi} \frac{1}{(1 - k^2\sin^2\xi)^{1/2}}\mathrm{d}\xi, \quad E(k) = \int_0^{\frac{1}{2}\pi}(1 - k^2\sin^2\xi)^{1/2}\mathrm{d}\xi.$$

Considering a thin-cored vortex ring of radius R and a core radius a with circulation γ (Fig. 7.6), Kelvin (1867) gave a famous formula

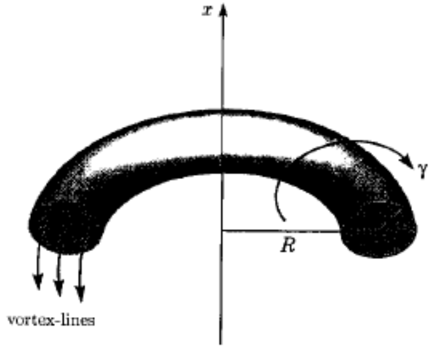


Fig. 7.6. A vortex ring of radius R and a core radius a with circulation γ (a definition sketch).

for its speed as

$$U = \frac{\gamma}{4\pi R}\left[\log\frac{8R}{a} - \frac{1}{4}\right], \qquad (7.65)$$

under the assumption $\epsilon = a/R \ll 1$. Its kinetic energy is given by

$$K = \frac{\rho R\gamma^2}{2}\left[\log\frac{8R}{a} - \frac{7}{4}\right],$$

Hicks (1885) confirmed Kelvin's result for the vortex ring with uniform vorticity in the thin-core. In addition, he calculated the speed of a ring of hollow thin-core, in which there is no vorticity within the core and pressure is constant, as

$$U_h = \frac{\gamma}{4\pi R}\left[\log\frac{8R}{a} - \frac{1}{2}\right]. \qquad (7.66)$$

- Other solutions

Vortex flows. Hill's ring

7.6.1. *Hill's spherical vortex*

Hill's spherical vortex (Hill, 1894) is an axisymmetric vortex of the vorticity field occupying a sphere of radius R with the law $\omega = Ar$ (A: a constant) and zero outside it:

$$\omega_\phi = Ar \quad \left(\sqrt{x^2 + r^2} < R\right); \quad \omega_\phi = 0 \quad \left(\sqrt{x^2 + r^2} > R\right). \qquad (7.60)$$

The total circulation is given by

$$\Gamma = \iint_{R_* < R} Ar \, \mathrm{d}x\mathrm{d}r = \frac{2}{3} AR^3,$$

where $R_* = \sqrt{x^2 + r^2}$. This vortex moves with a constant speed

$$U = \frac{2}{15} R^2 A = \frac{\Gamma}{5R}, \qquad (7.61)$$

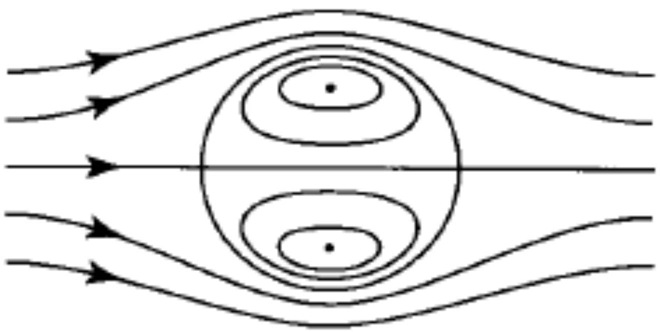


Fig. 7.5. Hill's spherical vortex.

without change of form. In the frame of reference in which the vortex is stationary, the stream function is given by

$$\Psi(x,r) = \begin{cases} \frac{3}{4}Ur^2(1 - R_*^2/R^2) & \text{for } R_* < R, \\ -\frac{1}{2}Ur^2(1 - R^3/R_*^3) & \text{for } R_* > R. \end{cases} \quad (7.62)$$

where $\Psi = -\frac{1}{2}Ur^2$ is the stream function of uniform flow of velocity $(-U, 0, 0)$. Obviously, the spherical surface $R_* = R$ coincides with the stream-line $\Psi = 0$ (Fig. 7.5).

The impulse of Hill's spherical vortex is obtained from (7.9) by using (7.60), and found to have only the axial component P_x:

$$\mathbf{P} = (P_x, 0, 0), \quad P_x = \int \omega_\phi r^2 \pi \mathrm{d}x\mathrm{d}r = 2\pi R^3 U.$$

1 Viscous decay of a line vortex

As an example of the above theory, we study the effects of **viscosity** on a **line vortex**. That is, we take as an **initial condition** the flow pattern of a $1/r$–**vortex** given by

$$\boldsymbol{u} = \frac{\Gamma_0}{2\pi r}\,\hat{\boldsymbol{e}}_\varphi\,, \tag{7.77}$$

where Γ_0 is a constant. This is a solution for a **vortex** in an **inviscid fluid**, with zero **vorticity** except at $r = 0$ where it is infinite. There is, however, a *finite* **circulation** around the origin. The idea is that we can see the *effect of viscosity* on such vortices by using the inviscid solution as an **initial condition** for the **Navier–Stokes equations**

We use polar coordinates *(r, θ)* and assume symmetry

$$\partial / \partial\theta = 0$$

$$\frac{\partial\omega}{\partial t} = \nu\left(\frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r}\right)$$

Solution

$$\omega = \frac{c}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

Further we define constant c

$$\Gamma = \Gamma(r,t)\big|_{t=0} = 2\pi\int \omega r dr\big|_{t=0}$$

$$= \frac{c2\pi}{4\pi\nu t}\int_0^r e^{-\frac{r^2}{4\nu t}} r dr = c$$

and find solution

$$\omega = \frac{\Gamma}{4\pi\nu t} e^{-\frac{r^2}{4\nu t}}$$

Proof with Mathematica

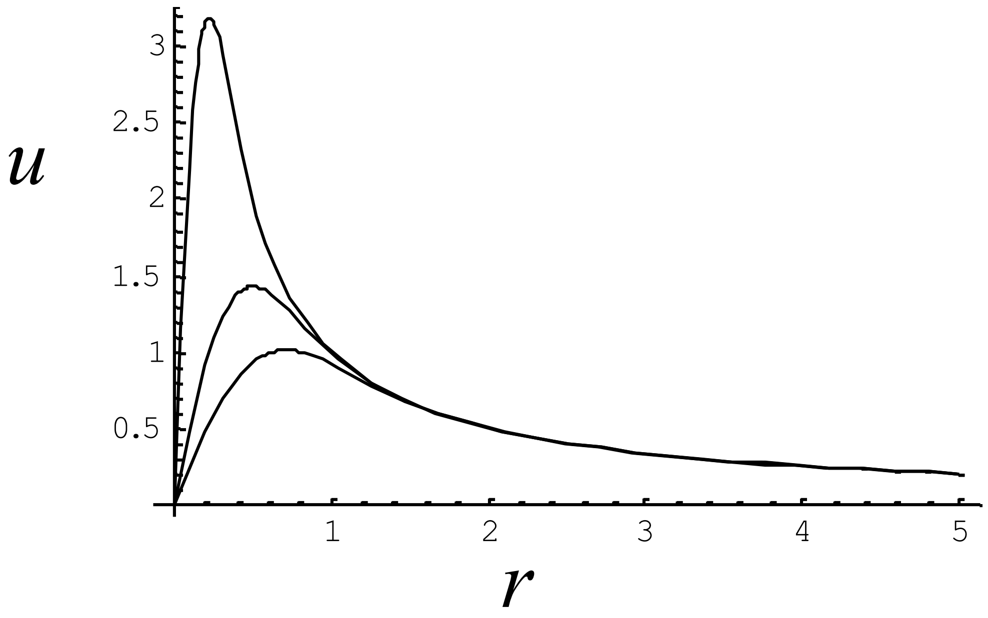
Appropriate tangent velocity

This eventually leads to a solution given by

$$u(r,t) = \frac{1}{r}\int_0^r \omega r dr = \frac{\Gamma}{2\pi r}(1 - e^{-r^2/4\nu t})$$

The first term is just the inviscid solution. We see that at fixed r, as time increases, the **flow velocity** component u_φ decreases and departs from a $1/r$-dependence. In other words, the **vorticity** increases. Close to the axis, but within a **distance** that increases in time [i.e., $r \ll (4\nu t)^{1/2}$], the flow is approximately that of uniform rotation. The intensity of the vortex decreases in time, as the core spreads outwards.

Graph of the function $\frac{1}{x}(1 - e^{-x^2})$



Burgers vortex (a viscous vortex with swirl)

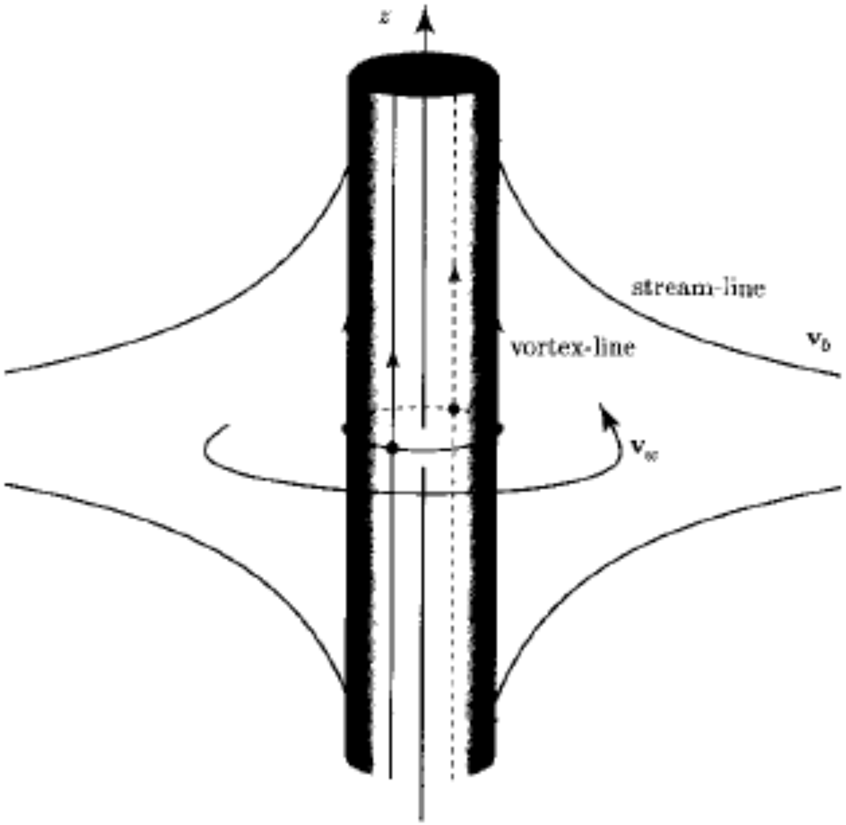


Fig. 7.8. Burgers vortex under an external straining $\mathbf{v}_b$.

Vorticity

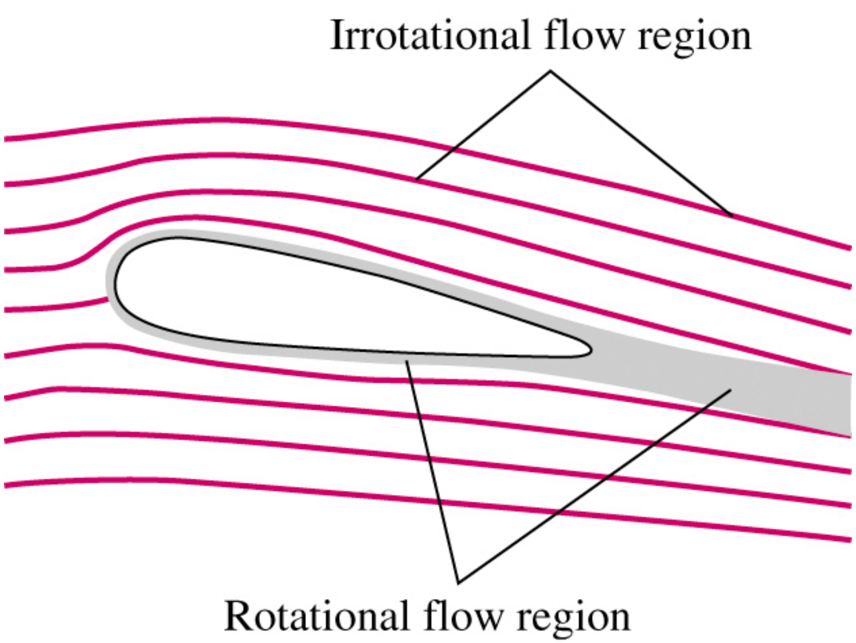
$$\omega(t \to \infty) \to \omega_B(r) = \frac{\Gamma}{\pi l_B^2} \exp[-r^2/l_B^2],$$

where $\Gamma = \int_0^\infty \Omega_0(s)\, 2\pi s\, ds$ denotes the initial total vorticity l_B represents a length scale of the final form. The swirl velocity $v_\theta(r)$ around the Burgers vortex is given as

$$v_\theta(r) = \frac{\Gamma}{2\pi l_B} \frac{1}{\hat{r}} \left(1 - e^{-\hat{r}^2}\right), \quad \hat{r} = r/l_B. \qquad (7.80)$$

As a result of detailed analyses, evidences are increasing to show that strong concentrated vortices observed in computer simulations or experiments of turbulence have this kind of Burgers-like vortex. This implies that in turbulence there exists a mechanism of spontaneous self-formation of Burgers vortices in the statistical sense.

Irrotational Flow Approximation



- Irrotational approximation: vorticity is negligibly small

$$\vec{\zeta} = \nabla \times \vec{V} \cong 0$$

- In general, inviscid regions are also irrotational, but there are situations where inviscid flow are rotational, e.g., solid body rotation (Ex. 10-3)

Irrotational Flow Approximation
2D Flows

- For 2D flows, we can also use the streamfunction
- Recall the definition of streamfunction for planar (x-y) flows

$$U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x}$$

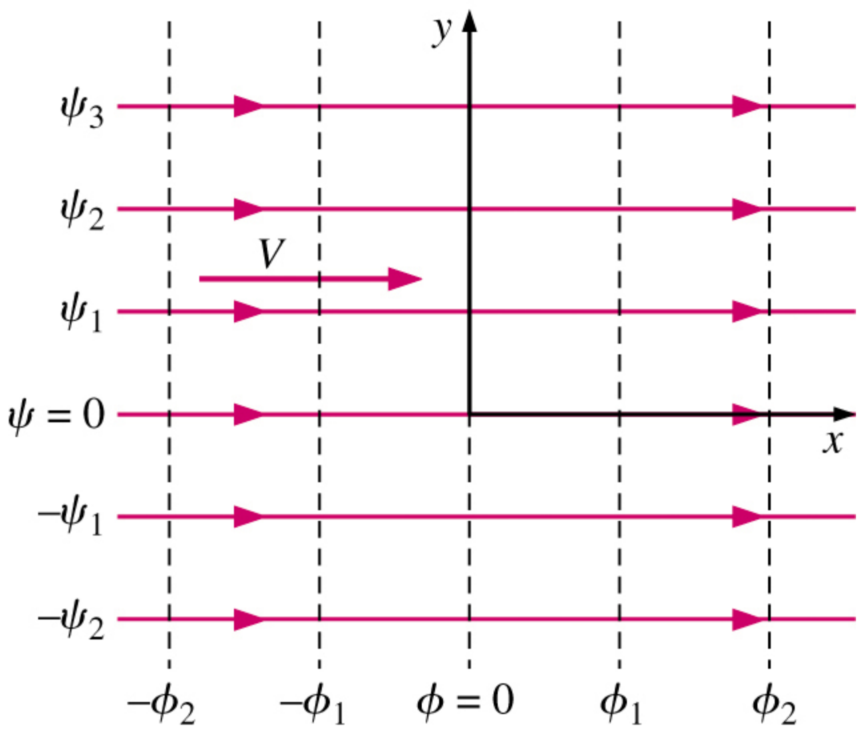
- Since vorticity is zero,

$$\zeta_z = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0$$

$$\boxed{\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0}$$

- This proves that the Laplace equation holds for the streamfunction <u>and</u> the velocity potential

Elementary Planar Irrotational Flows
Uniform Stream



Proof with Mathematica

- In Cartesian coordinates

$$\phi = Vx, \quad \psi = Vy$$

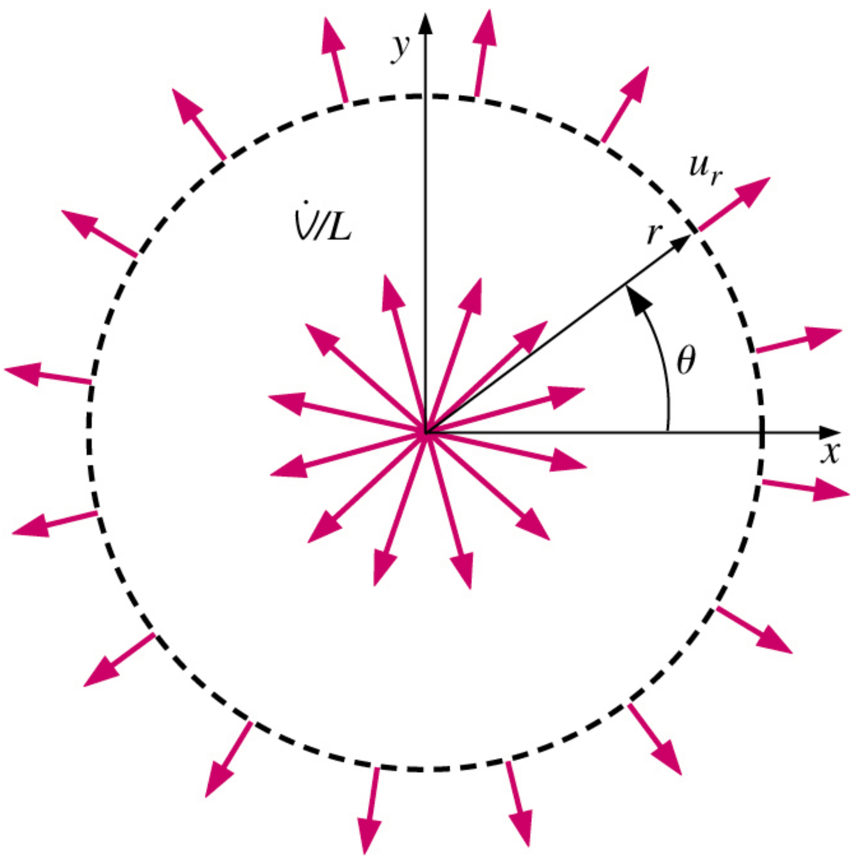
- Conversion to cylindrical coordinates can be achieved using the transformation

$$x = r cos\theta, \quad y = r sin\theta$$

$$\phi = Vr cos\theta, \quad \psi = Vr sin\theta$$

Elementary Planar Irrotational Flows

Line Source/Sink

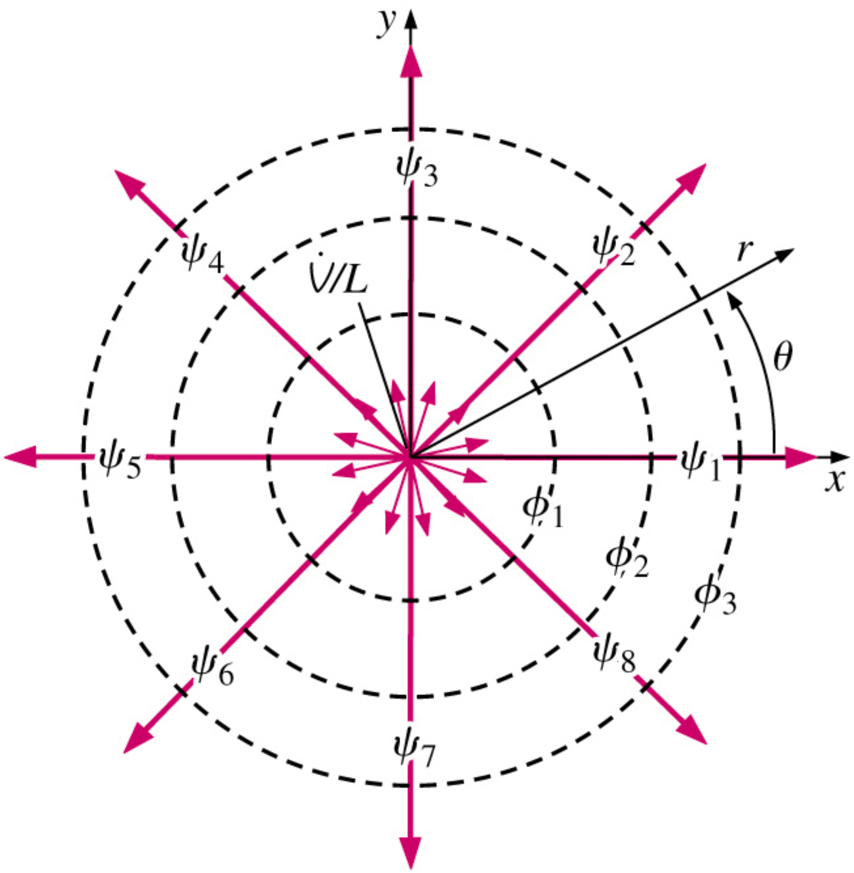


- Potential and streamfunction are derived by observing that volume flow rate across any circle is $\dot{\mathcal{V}}/L$
- This gives velocity components

$$U_r = \frac{\dot{\mathcal{V}}/L}{2\pi r}, \quad U_\theta = 0$$

Elementary Planar Irrotational Flows

Line Source/Sink



Proof with Mathematica

- Using definition of (U_r, U_θ)

$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{\mathcal{V}}/L}{2\pi r}$$

$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

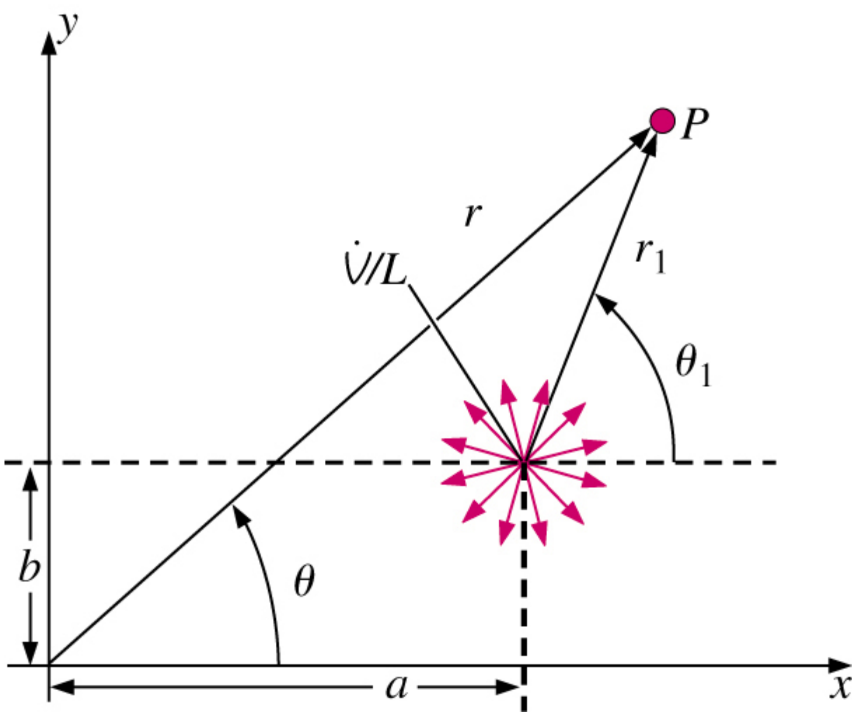
- These can be integrated to give ϕ and ψ

$$\phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r \qquad \psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta$$

Equations are for a source/sink at the origin

Elementary Planar Irrotational Flows

Line Source/Sink

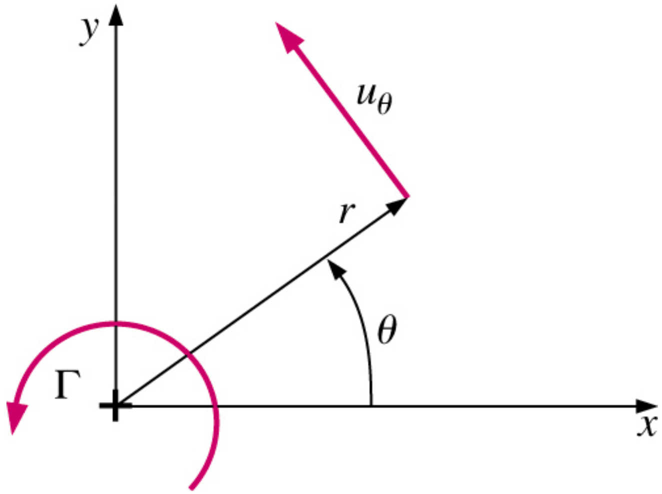


- If source/sink is moved to (x,y) = (a,b)

$$\phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r_1 = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

$$\psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta_1 = \frac{\dot{\mathcal{V}}/L}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

Elementary Planar Irrotational Flows
Line Vortex



Equations are for a source/sink at the origin

- Vortex at the origin. First look at velocity components

$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

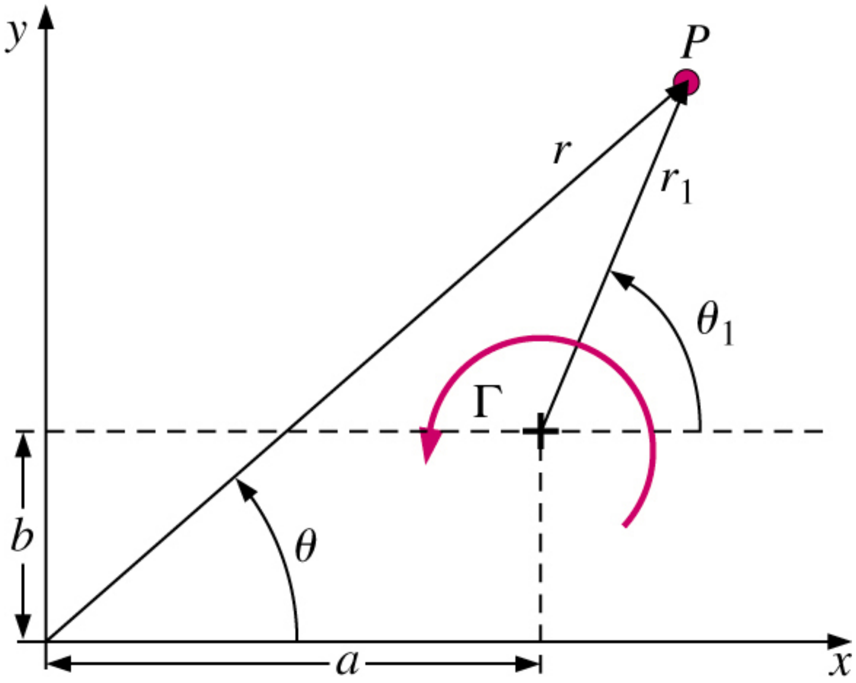
$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

- These can be integrated to give ϕ and ψ

$$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Elementary Planar Irrotational Flows

Line Vortex



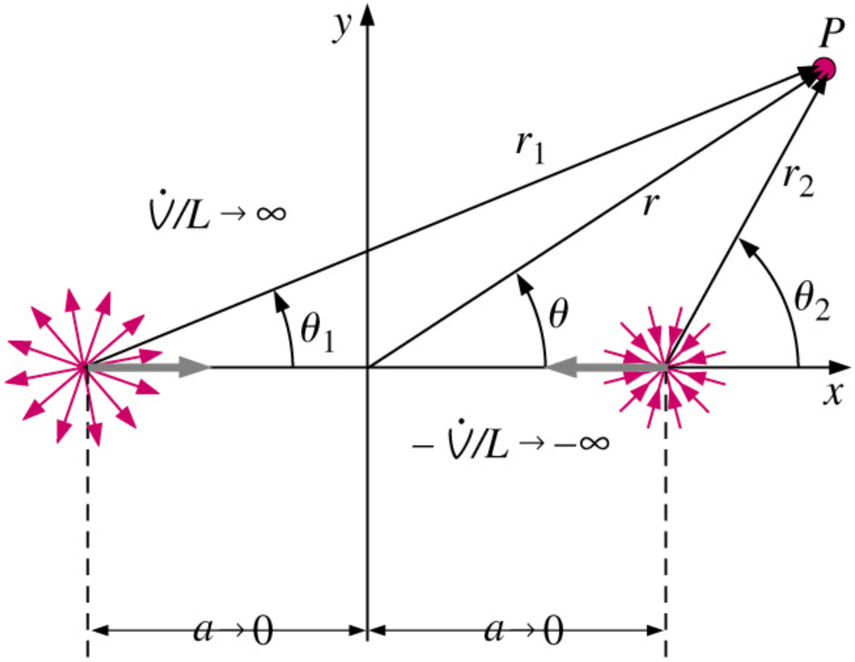
- If vortex is moved to (x,y) = (a,b)

$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

Elementary Planar Irrotational Flows

Doublet



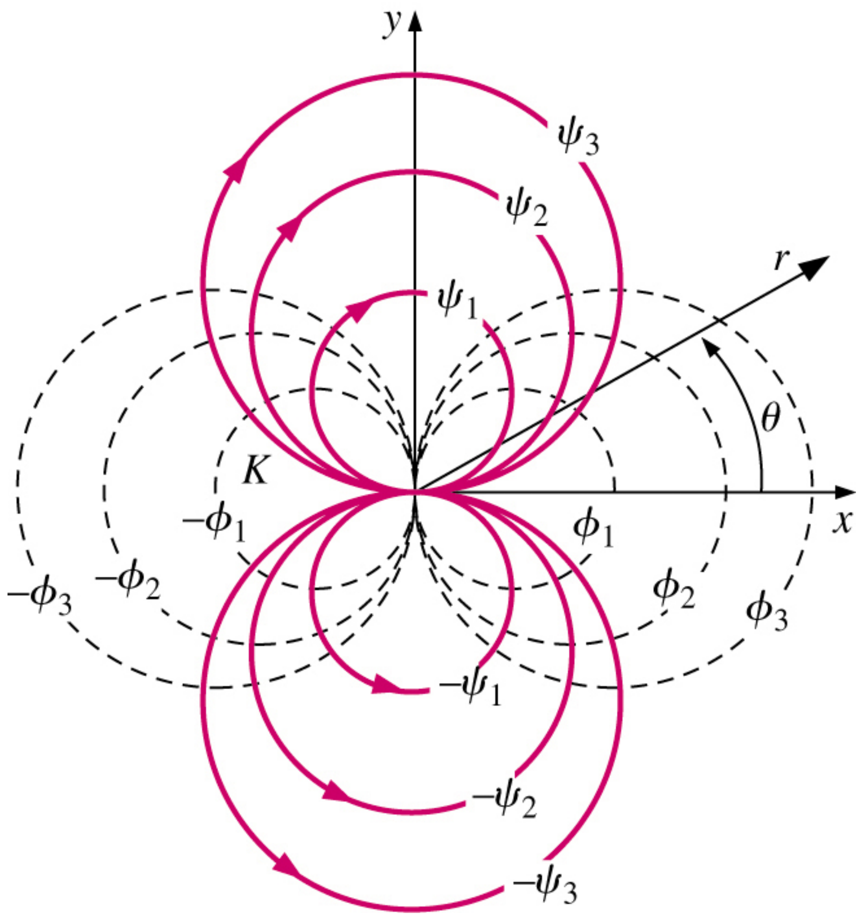
- A doublet is a combination of a line sink and source of equal magnitude
- Source

$$\psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta_1 \qquad \theta_1 = \tan^{-1}\left(\frac{y}{x+a}\right)$$

- Sink

$$\psi = -\frac{\dot{\mathcal{V}}/L}{2\pi}\theta_2 \qquad \theta_2 = \tan^{-1}\left(\frac{y}{x-a}\right)$$

Elementary Planar Irrotational Flows
Doublet



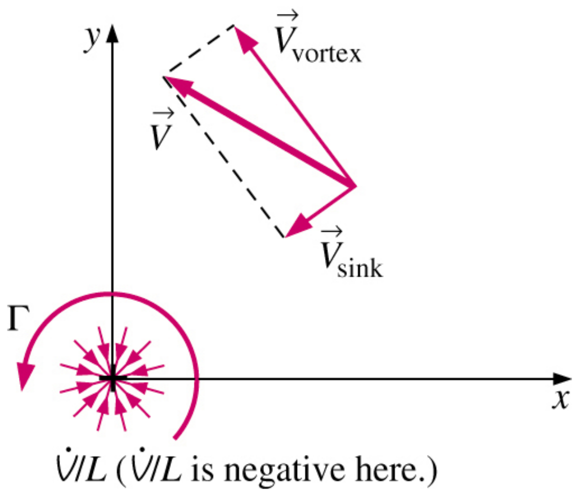
- Adding ψ_1 and ψ_2 together, performing some algebra, and taking $a \rightarrow 0$ gives

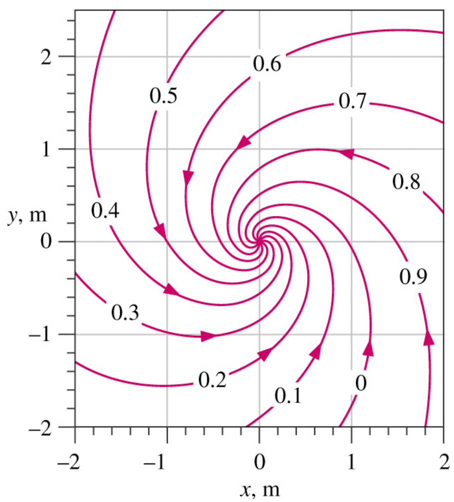
$$\psi = -K\frac{\sin\theta}{r}$$

$$\phi = K\frac{\cos\theta}{r}$$

K is the doublet strength

Examples of Irrotational Flows Formed by Superposition





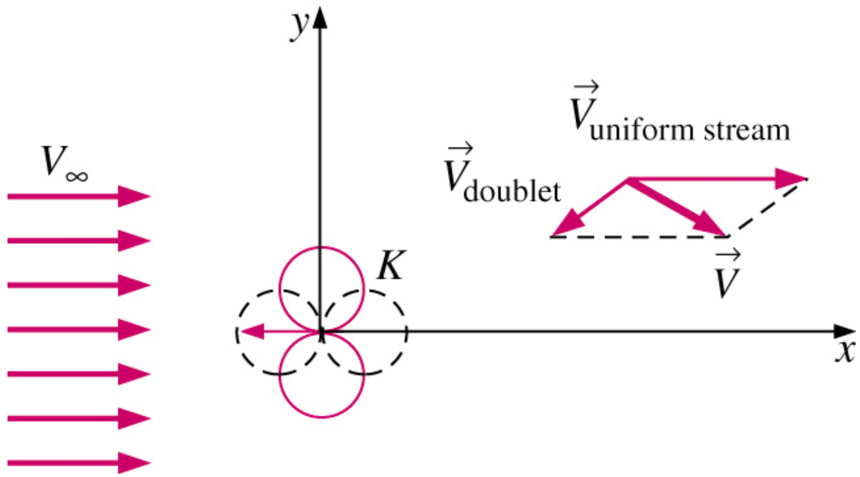
- Superposition of sink and vortex : bathtub vortex

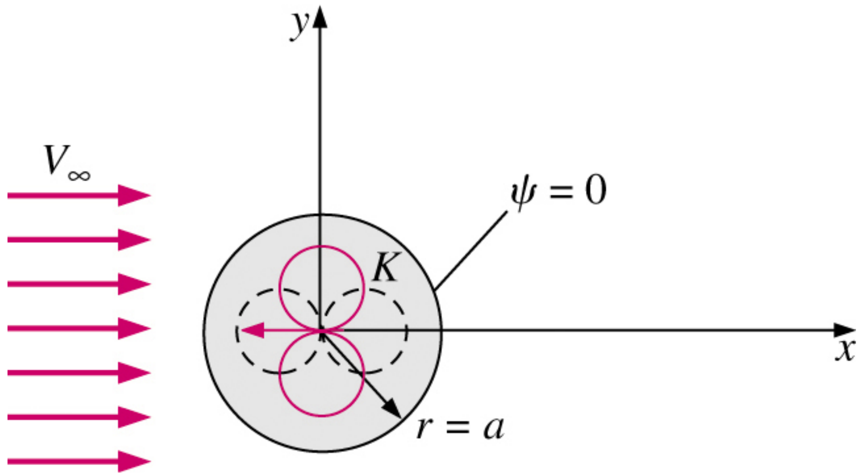
$$\psi = \underbrace{\frac{\dot{\mathcal{V}}/L}{2\pi}\theta}_{\text{Sink}} - \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\text{Vortex}}$$

$$U_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\dot{\mathcal{V}}/L}{2\pi r}$$

$$U_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Examples of Irrotational Flows Formed by Superposition





- Flow over a circular cylinder: Free stream + doublet

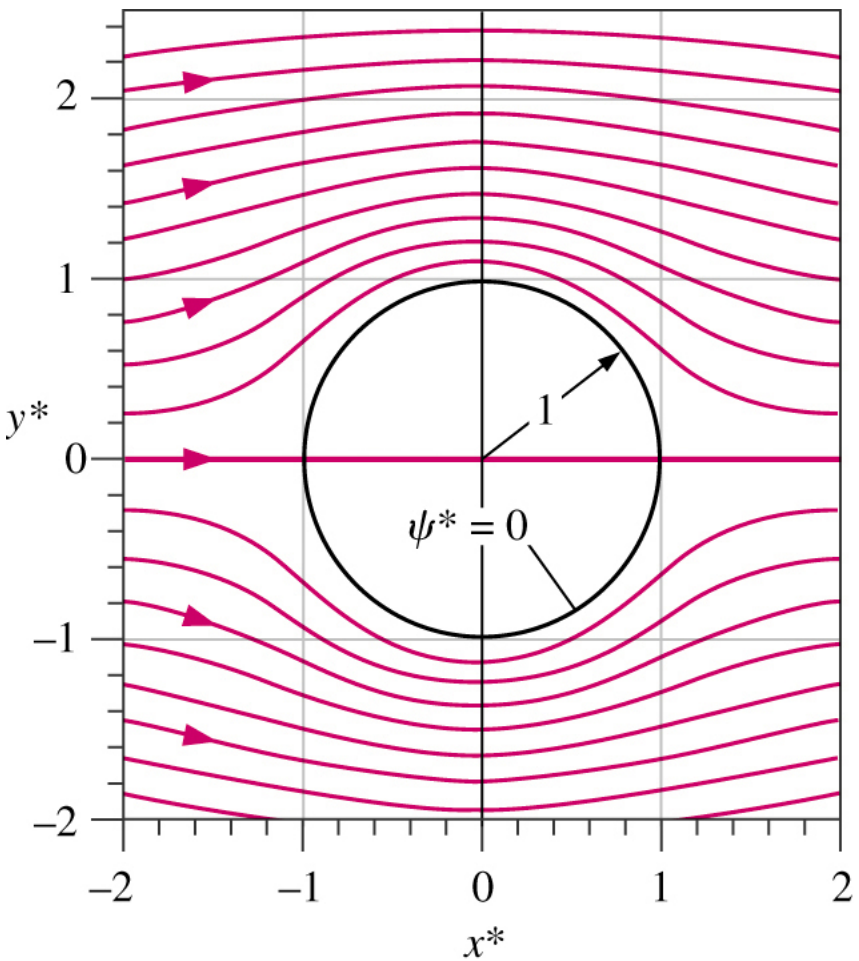
$$\phi = V r \cos\theta + K \frac{\cos\theta}{r}$$

$$\psi = V r \sin\theta - K \frac{\sin\theta}{r}$$

- Assume body is $\psi = 0$ ($r = a$) $\Rightarrow K = Va^2$

$$\psi = V \sin\theta \left(r - a^2/r \right)$$

Examples of Irrotational Flows Formed by Superposition



- Velocity field can be found by differentiating streamfunction

$$U_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V\cos\theta\left(1 - a^2/r^2\right)$$

$$U_\theta = -\frac{\partial \psi}{\partial r} = -V\sin\theta\left(1 + a^2/r^2\right)$$

- On the cylinder surface (r=a)

$$U_r = 0, \quad U_\theta = -2V\sin\theta$$

Normal velocity (U_r) is zero, Tangential velocity (U_θ) is non-zero ⇒slip condition.