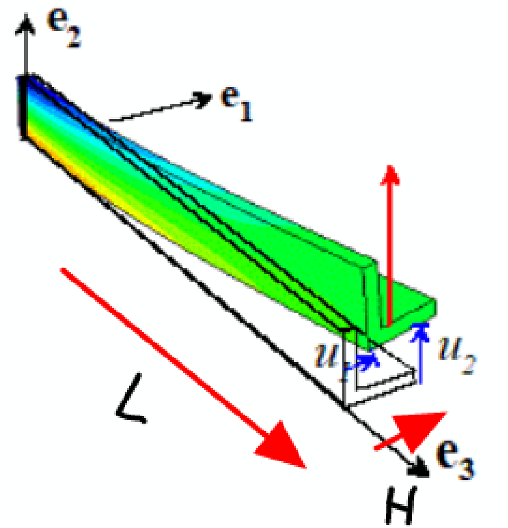
11 Beams

Goal : understand approximations & equations for solids with special shapes

Assumptions : (1) Length >> x-sect dimension
(2) Small deflections
(3) Linear elastic
(4) Neglect twist



[ABAQUS does not assume (2) - (4)]

Two versions of beam theory exist

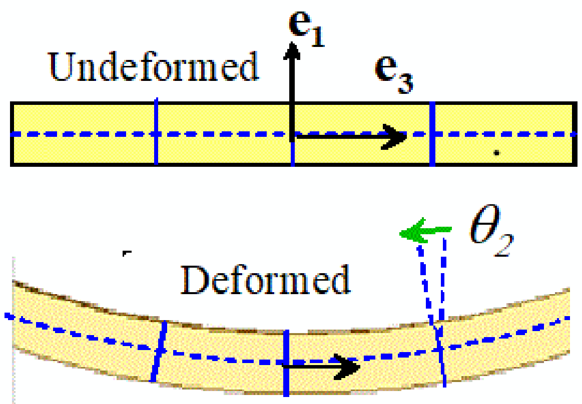
(1) "Euler - Bernoulli" "Cubic Formulation" in ABAQUS } Valid for $L > 15H$

(2) "Timoshenko" "Shear Flexible" in ABAQUS } Valid for $L > 5H$] ABAQUS default

In ABAQUS can change to Euler - Bernoulli in mesh module by useng "element type" module

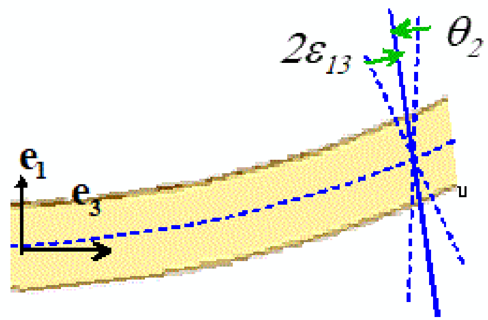
① Bernoulli Theory:

x-sections remain perpendicular to centerline



② Timoshenko

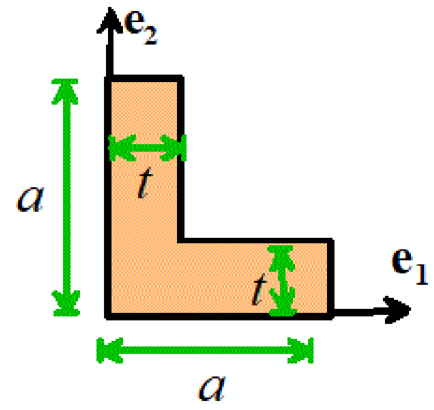
x-secti can rotate relative to centerline



Focus mostly on case ① here

Describing x-sect geometry (elastic beams)

X-sect area $A = \int_A dA$



Centroid :

$$\underline{r} = r_1 \underline{e}_1 + r_2 \underline{e}_2 = \frac{1}{A} \int_A (x_1 \underline{e}_1 + x_2 \underline{e}_2)\, dA$$

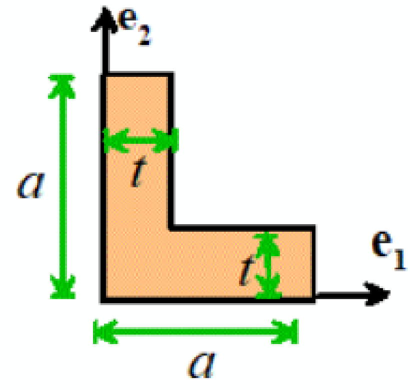
Area moment of inertia tensor

$$[I] = \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix}$$

$$I_{11} = \int_A (x_2 - \bar{r}_2)^2 dA \qquad I_{22} = \int_A (x_1 - \bar{r}_1)^2 dA$$

$$I_{12} = \int_A (x_1 - \bar{r}_1)(x_2 - \bar{r}_2)\, dA$$

Example: Calculate the area moments of inertia for the L section shown



Here $A = \int_0^t \int_0^a dx_1 dx_2 + \int_t^a \int_0^t dx_1 dx_2$

$$\underline{r} = \frac{1}{A}\left\{ \int_0^t \int_0^a (x_1 \underline{e}_1 + x_2 \underline{e}_2) dx_1 dx_2 + \int_t^a \int_0^t (x_1 \underline{e}_1 + x_2 \underline{e}_2) dx_1 dx_2 \right\}$$

I_{11}, I_{22}, I_{12} follow: use MATLAB to do integrals

```
syms x1 x2 a t rbar
integrand = 1
A = simplify(int(int(integrand,x1,[0,a]),x2,[0,t]) + int(int(integrand,x1,[0,t]),x2,[t,a]))
integrand = [x1,x2];
rbar = simplify((int(int(integrand,x1,[0,a]),x2,[0,t]) + int(int(integrand,x1,[0,t]),x2,[t,a]))/A)
integrand = (x2-rbar(2))^2;
I11 = simplify(int(int(integrand,x1,[0,a]),x2,[0,t]) + int(int(integrand,x1,[0,t]),x2,[t,a]))
integrand = (x1-rbar(1))^2;
I22 = simplify(int(int(integrand,x1,[0,a]),x2,[0,t]) + int(int(integrand,x1,[0,t]),x2,[t,a]))
integrand = (x1-rbar(1))*(x2-rbar(2));
I12 = simplify(int(int(integrand,x1,[0,a]),x2,[0,t]) + int(int(integrand,x1,[0,t]),x2,[t,a]))
```

A = $t\,(2\,a - t)$

rbar =

$$\left(\frac{a^2 + a\,t - t^2}{4\,a - 2\,t} \quad \frac{a^2 + a\,t - t^2}{4\,a - 2\,t}\right)$$

I11 =

$$\frac{t\,(5\,a^4 - 10\,a^3\,t + 11\,a^2\,t^2 - 6\,a\,t^3 + t^4)}{12\,(2\,a - t)}$$

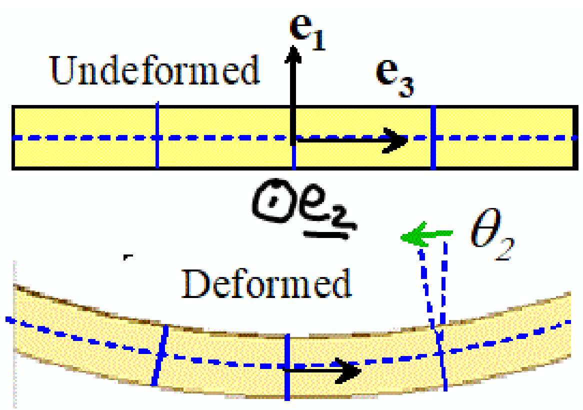
I22 =

$$\frac{t\,(5\,a^4 - 10\,a^3\,t + 11\,a^2\,t^2 - 6\,a\,t^3 + t^4)}{12\,(2\,a - t)}$$

I12 =

$$-\frac{a^2\,t\,(a - t)^2}{4\,(2\,a - t)}$$

Describing deformation



Calculate:

(1) Deflection vector of centroid

$$\underline{u} = u_1(x_3)\,\underline{e}_1 + u_2(x_3)\,\underline{e}_2 + u_3(x_3)\,\underline{e}_3$$

(2) Rotation vector of x-section

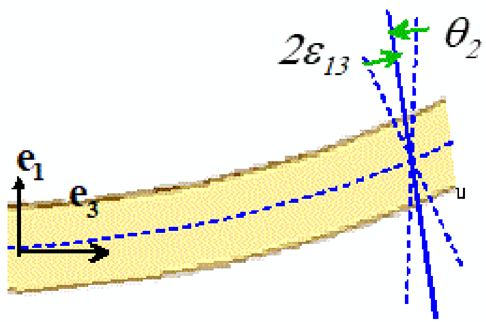
$$\underline{\theta} = \theta_1\,\underline{e}_1 + \theta_2\,\underline{e}_2 + \theta_3\,\underline{e}_3$$

[$\underline{u}$, $\underline{\theta}$ are U, UR in ABAQUS]

These are related:

Euler-Bernoulli $\quad \theta_2 = \dfrac{\partial u_1}{\partial x_3} \qquad \theta_1 = -\dfrac{\partial u_2}{\partial x_3}$

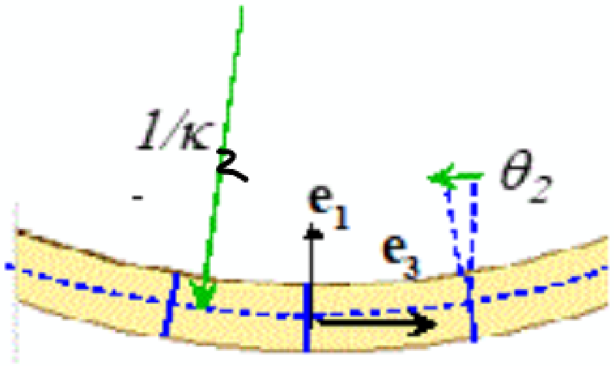
Timoshenko



$$\theta_2 = \frac{\partial u_1}{\partial x_3} - 2\varepsilon_{23} \qquad \theta_1 = -\frac{\partial u_2}{\partial x_3} + 2\varepsilon_{13}$$

ε_{13}, ε_{23} : additional unknowns

Curvature Vector



$$\underline{\kappa} = \frac{d\underline{\theta}}{dx_3} = \kappa_1 \underline{e}_1 + \kappa_2 \underline{e}_2 + \kappa_3 \underline{e}_3$$

$[\kappa_1, \kappa_2]$: Bending κ_3 : Twist (neglect)

Strains: Assume no stretch of centroid

$$\varepsilon_{33} = -\kappa_2 (x_1 - \bar{r}_1) + \kappa_1 (x_2 - \bar{r}_2)$$

$$\varepsilon_{11} = \varepsilon_{22} = -\nu \, \varepsilon_{33}$$

Euler Bernoulli : all other $\varepsilon_{ij} = 0$

Timoshenko has nonzero ε_{13}, ε_{23}

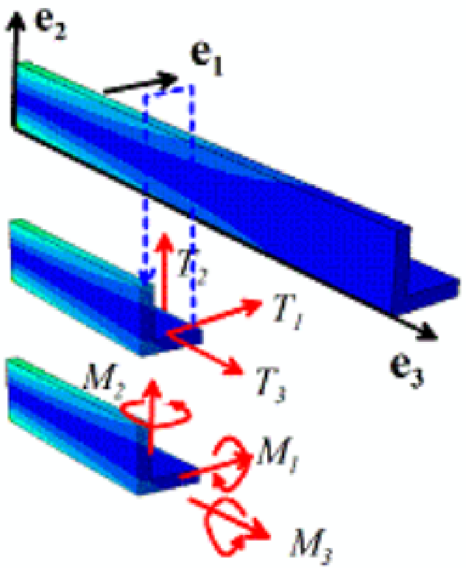
Stresses: $\sigma_{33} = E\,\varepsilon_{33}$

All other $\sigma_{ij} = 0$ in Euler-Bernoulli

In Timoshenko: $\sigma_{13} = 2\mu\,\varepsilon_{13}$ $\sigma_{23} = 2\mu\,\varepsilon_{23}$

μ: Shear modulus (need to specify in ABAQUS)

Internal Forces in beams



Internal force Vector $\underline{T} = T_1 \underline{e_1} + T_2 \underline{e_2} + T_3 \underline{e_3}$

$[T_1, T_2]$: Shear T_3 : Axial

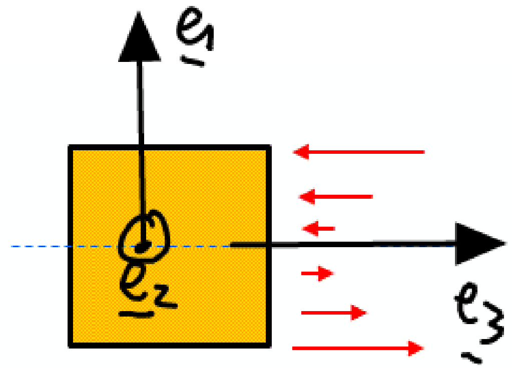
Moment Vector $\underline{M} = M_1 \underline{e_1} + M_2 \underline{e_2} + M_3 \underline{e_3}$

$[M_1, M_2]$: Bending M_3 : Twist (neglect)

Moment - Curvature relations

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = E \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \qquad \underline{M} = E\,[I]\,\underline{k}$$

To see this recall :



$$\left.\begin{aligned} M_2 &= -\int_A \underbrace{(x_1-\bar{r}_1)}_{\text{moment arm}}\ \underbrace{\sigma_{33}\, dA}_{\text{Force}} \\ M_1 &= \int_A (x_2-\bar{r}_2)\,\sigma_{33}\, dA \end{aligned}\right\} \text{moments of stress distribution}$$

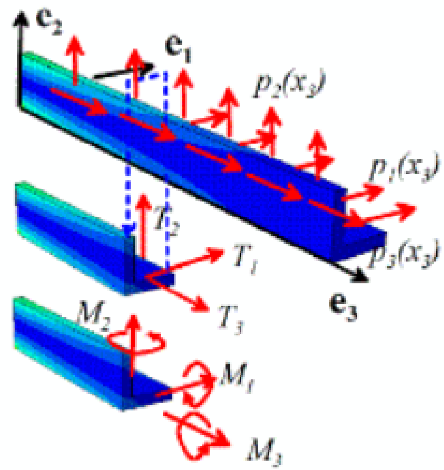
$$\sigma_{33} = E\,\varepsilon_{33} = E\left(-k_2\,(x_1-\bar{r}_1) + k_1\,(x_2-\bar{r}_2)\right)$$

$$\Rightarrow M_2 = -E\int (x_1-\bar{r}_1)\left(-k_2\,(x_1-\bar{r}_1) + k_1(x_2-\bar{r}_2)\right)dA$$

$$= -E\,k_1\,I_{12} + E\,k_2\,I_{22}$$

Equations of motion

Let $\underset{\sim}{p} = p_1 \underset{\sim}{e_1} + p_2 \underset{\sim}{e_2} + p_3 \underset{\sim}{e_3}$ be external force per unit length on beam



Let ρ be mass density

Let $\underset{\sim}{a} = \dfrac{d^2 \underset{\sim}{u}}{dt^2}$ be accel of centroid

Linear momentum : ($\underset{\sim}{F} = m \underset{\sim}{a}$)

$$\frac{d \underset{\sim}{I}}{dx_3} + \underset{\sim}{p} = \rho A \underset{\sim}{a}$$

(3 equations in 1, 2, 3 directions)

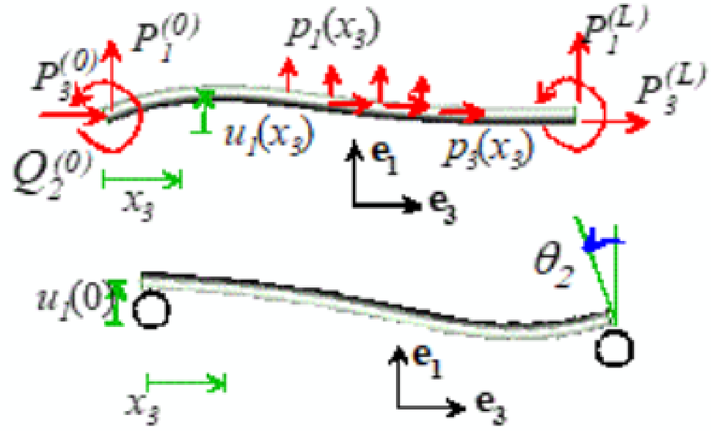
Angular momentum (neglect mass moment of inertia)

$$\frac{dM_1}{dx_3} - T_2 - \theta_1 T_3 = 0 \qquad \frac{dM_2}{dx_3} + T_1 - \theta_2 T_3 = 0$$

Boundary Conditions

Either: Displacement / Rotation

$\underline{u}$, $\underline{\theta}$ given at $x_3=0$, $x_3=L$



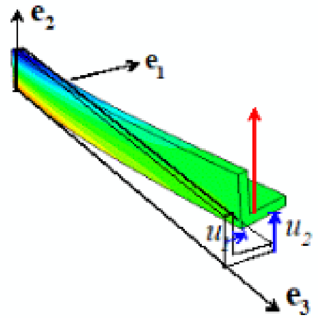
Or: Force / moments given

$$\underline{T} = \underline{P}^{(L)} \text{ at } x_3 = L \qquad \underline{T} = -\underline{P}^{(0)} \text{ @ } x_3 = 0$$

$$\underline{M} = \underline{Q}^{(L)} \text{ at } x_3 = L \qquad \underline{M} = -\underline{Q}^{(0)} \text{ @ } x_3 = 0$$

(Free beam $\underline{T} = \underline{M} = \underline{0}$)

Beams – summary of equations



Goal: Calculate (1) Displacement of centroid $\mathbf{u}(x_3) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$

(2) Rotation of x-section $\boldsymbol{\theta}(x_3) = \theta_1\mathbf{e}_1 + \theta_2\mathbf{e}_2 + \theta_3\mathbf{e}_3$

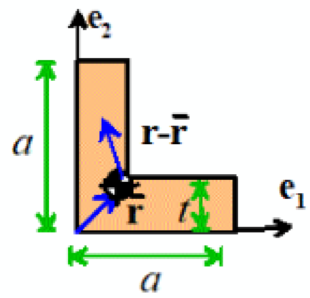
(3) Curvature vector $\boldsymbol{\kappa}(x_3) = d\boldsymbol{\theta} / dx_3$

Neglect twist θ_3 here, but ABAQUS will include twist

Section properties:

$$A = \int_A dA \qquad \bar{\mathbf{r}} = \frac{1}{A}\int_A (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)\, dA$$

$$\mathbf{I} = \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \qquad I_{11} = \int_A (x_2 - \bar{r}_2)^2\, dA \qquad I_{22} = \int_A (x_1 - \bar{r}_1)^2\, dA \qquad I_{12} = \int_A (x_1 - \bar{r}_1)(x_2 - \bar{r}_2)\, dA$$



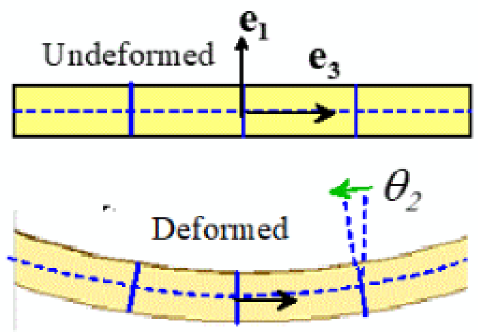
Deformation:

Euler-Bernoulli theory (no shear): $\theta_1 = -du_2 / dx_3 \qquad \theta_2 = du_1 / dx_3$

(Timoshenko theory allows x-sect to rotate relative to neutral axis)

Axial strain: $\varepsilon_{33} = \kappa_1(x_2 - \bar{r}_2) - \kappa_2(x_1 - \bar{r}_1)$

(Timoshenko beam has shear strains)



Stresses: $\sigma_{33} = E\varepsilon_{33}$

Other stresses zero in E-B beams
(Timoshenko beams have shear stresses)

Beams – summary of equations

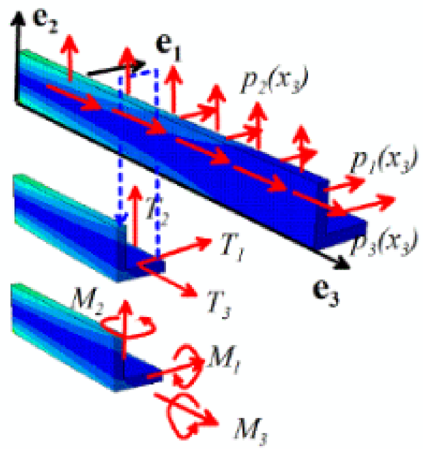
Internal Forces (forces/moments on section normal to $\mathbf{e}_3$:

Force vector $\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3$

Moment vector $\mathbf{M} = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3$

$$M_1 = \int_A \sigma_{33}(x_2 - \bar{x}_2)\,dA \qquad M_2 = -\int_A \sigma_{33}(x_1 - \bar{x}_1)\,dA$$

No twist means $M_3 = 0$



Moment-Curvature relations:

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = E \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}$$

Equations of motion: **F=ma:** $\dfrac{dT_i}{dx_3} + p_i = \rho A a_i$

Angular Momentum: $\dfrac{dM_1}{dx_3} - T_2 - \theta_1 T_3 = 0 \qquad \dfrac{dM_2}{dx_3} + T_1 - \theta_2 T_3 = 0$

Boundary Conditions (at ends):

Fixed end: $\mathbf{u} = \mathbf{0}$

Free to move: $\mathbf{T} = \mathbf{0}$

Clamped end: $\boldsymbol{\theta} = \mathbf{0}$

Free to rotate: $\mathbf{M} = \mathbf{0}$