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Diffusion and convection in viscous flow

A. F. Pillow

A survey is made of the way in which a variety of physical scalar and vector quantities are diffused and convected out of fixed regions in moving fluids. In particular, viscous flow itself is viewed as the diffusion and convection of circulation which is generated at the boundaries of the fluid by the no-slip condition. The non-linearity of the problem arises from the fact that the convection field is in part self-generated by the diffused circulation. Ways of overcoming these difficulties are reviewed in the light of the above point of view. New kinematic interpretations are given for the equations governing axisymmetric viscous flow and these are used to determine the flow vector for ring circulation, angular momentum and other relevant physical quantities.

1. Convection and diffusion of scalar quantities

For any scalar quantity Q, the amount of Q in any arbitrary fixed two- [three-] dimensional region S at time t is given by the set function

(1) $$Q(S, t) = \int_S \sigma dS ,$$

where $\sigma(\mathbf{r}, t)$ is the area [volume] density of Q at the point with position vector $\mathbf{r}$ at time t. If $m(\mathbf{r}, t)$ is the area [volume] density of the sources of production of Q, then

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(2a) $$\frac{dQ}{dt} = \int_S \frac{\partial\sigma}{\partial t}\, dS = \int_S m dS - \oint_{\partial S} ds\mathbf{n}.\mathbf{F}$$

or, locally,

(2b) $$\frac{\partial\sigma}{\partial t} + \mathrm{div}\mathbf{F} = m\ ,$$

where $\mathbf{n}$ is the unit *outward* normal to the boundary ∂S of S and $\mathbf{F}(r, t)$ is the flux vector for Q. The flux vector $\mathbf{F}$ is the time rate of discharge of Q per unit length [area] at time t across an infinitesimal line [area] element at $\mathbf{r}$ perpendicular to $\mathbf{F}$.

2. Conservation of area [volume] in an incompressible fluid

Consider first the case where Q is the total mass of fluid in the region S and $\sigma = \rho$, the fluid density. Conservation of mass then requires $m = 0$ in equation (2) and $\mathbf{F}$ becomes the mass flux vector $\rho\mathbf{q}$ where $\mathbf{q}(\mathbf{r}, t)$ is the fluid velocity vector. In particular, if the fluid is incompressible, conservation of mass entails conservation of fluid area [volume] in two[three] dimensions. If Q denotes fluid area [volume], then $\sigma = 1$ and $\mathbf{F} = \mathbf{q}$ in equations (2), so that

(3a) $$\oint_{\partial S} ds\mathbf{n}.\mathbf{q} = 0$$

or, locally,

(3b) $$\mathrm{div}\mathbf{q} = 0\ .$$

In two dimensions, equation (3) permits the introduction of a stream function $\psi(\mathbf{r}, t)$ such that, if $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$,

(4) $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} = \psi_y\mathbf{i} - \psi_x\mathbf{j} = \mathrm{curl}\psi\mathbf{k}\ .$$

The increase in ψ along any curve is the discharge of fluid area per unit time across that curve from left to right. The curves ψ = constant are consequently streamlines.

Similarly, in axisymmetric motion, a stream function $\psi(\mathbf{r}, t)$ can also be introduced, such that the increase in $2\pi\psi$ along an arc in an axial half-plane gives the volume discharge per unit time across the surface generated by the arc when it is rotated about the axis of symmetry. Let (x, σ, ϕ) be cylindrical polar coordinates with the

x-axis as the axis of symmetry. Then

$$(5) \qquad \mathbf{q} = u\hat{x} + v\hat{\sigma} + w\hat{\phi} = \mathrm{curl}[(\tfrac{\psi}{\sigma})\hat{\phi}] + w\hat{\phi} = \tfrac{1}{\sigma}[\psi_\sigma\hat{x} - \psi_x\hat{\sigma}] + w\hat{\phi} .$$

3. Conservation of heat in an incompressible homogeneous fluid

If the fluid has constant specific heat c and constant thermal conductivity K, the heat flux vector $\rho c\mathbf{F}$ is the sum of a diffusive and a convective component. Thus

$$(6) \qquad \mathbf{F} = -k\nabla T + \mathbf{q}T ,$$

where $T(\mathbf{r}, t)$ and $\mathbf{q}(\mathbf{r}, t)$ are the temperature and velocity fields and k, the thermal diffusivity, is defined by $k = K/\rho c$. Conservation of heat requires that m be zero in equation (2). Since the heat density is ρcT, it follows that

$$(7a) \qquad \frac{d}{dt}\left[\int_S T dS\right] = -\oint_{\partial S} ds\mathbf{n}.\mathbf{F}$$

or, locally, for suitably differentiable fields,

$$(7b) \qquad \frac{\partial T}{\partial t} + \mathrm{div}\mathbf{F} = 0 .$$

Since $\mathrm{div}\mathbf{q} = 0$ [equation (3b)], the last equation may also be written in the form

$$(8) \qquad \frac{\partial T}{\partial t} + \mathbf{q}.\nabla T = k\nabla^2 T .$$

In terms of the stream function ψ introduced in equation (4) this last equation takes the form

$$(9) \qquad \frac{\partial T}{\partial t} = \frac{\partial(\psi,T)}{\partial(x,y)} + k\nabla^2 T$$

in two dimensions.

In the axisymmetric case the corresponding equation reads

$$(10) \qquad \frac{\partial T}{\partial t} = \frac{1}{\sigma}\frac{\partial(\psi,T)}{\partial(x,\sigma)} + k\nabla^2 T ,$$

where $\nabla^2 \equiv \dfrac{\partial^2}{\partial\sigma^2} + \dfrac{1}{\sigma}\dfrac{\partial}{\partial\sigma} + \dfrac{\partial^2}{\partial x^2}$.

4. Conservation of circulation in two-dimensional incompressible viscous flow

In two dimensions the motion of a viscous incompressible homogeneous fluid under the action of conservative body forces can be described kinematically entirely in terms of the conservation of fluid area and of circulation.

In a fixed arbitrary region S in two dimensions let $\Gamma(S, t)$ be the total circulation. Then

$$\Gamma = \oint_{\partial S} \mathbf{q}.d\mathbf{s} = \int_S \omega dS , \tag{11}$$

where ω , the vorticity, is the area density of circulation

$$\boldsymbol{\omega} = \text{curl}\mathbf{q} = \omega\mathbf{k} = -(\nabla^2\psi)\mathbf{k} . \tag{12}$$

In a viscous fluid in two dimensions, the flux vector $\mathbf{F}$ for circulation is, in complete analogy with the heat case, given by

$$\mathbf{F} = -\nu\nabla\omega + \mathbf{q}\omega , \tag{13}$$

where ν , the kinematic viscosity, is constant. Again, it will be noted that $\mathbf{F}$ has both a diffusive and a convective component.

The ratio of the magnitude of these terms is called the local *Reynolds* number R . Thus

$$R = \frac{|\mathbf{q}\omega|}{\nu|\text{grad}\omega|}$$

and $R \ll 1$ implies diffusion dominates convection whilst $R \gg 1$ implies that the opposite is true. The local angle α between the two flux terms is given by

$$\cos\alpha = \frac{\mathbf{q}.\text{grad}\omega}{|\mathbf{q}|\ |\text{grad}\omega|} .$$

If the body forces are conservative, there can be no production of circulation within the fluid. Consequently just as in the case of conservation of heat it follows that

$$\frac{d\Gamma}{dt} = \int_S \frac{\partial\omega}{\partial t}\, dS = -\oint ds\mathbf{n}.\mathbf{F} , \tag{14a}$$

where $\mathbf{F}$ is now given by (13). In its local form (14a) becomes

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(14b) $$\frac{\partial\omega}{\partial t} + \mathrm{div}\mathbf{F} = 0 \ .$$

Since, in view of (3b), $\mathrm{div}\mathbf{q} = 0$, equation (14b) may be rewritten as

(15) $$\frac{\partial\omega}{\partial t} + \mathbf{q}.\nabla\omega = \nu\nabla^2\omega \ .$$

Introduction of ψ leads to the alternative form

(16) $$\frac{\partial\omega}{\partial t} = \frac{\partial(\psi,\omega)}{\partial(x,y)} + \nu\nabla^2\omega \ ,$$

where

(17) $$\nabla^2\omega = -\omega \ .$$

In a viscous fluid, circulation diffuses into the fluid from the boundaries, where it is generated by boundary forces arising from the enforcement of the no-slip boundary condition. At fixed solid boundaries

(18) $$\mathbf{q} = 0 \ , \quad \text{i.e.} \quad \psi = \text{constant} \quad \text{and} \quad \mathbf{n}.\nabla\psi = 0 \ .$$

Circulation may also be introduced by the external forcing flow prescribed at infinity. Whilst for a given convection field ψ , the temperature field T in (9) is linear, the essentially non-linear character of viscous flow is revealed by the convective Jacobian in (16). The convection field ψ is generated not only by the, externally provided, forcing flow, but also by the self-convection induced by the distributed circulation which has either been convected from upstream infinity or has diffused in previously from the boundaries and been convected downstream.

The self-convection field is generated in accordance with the Poisson equation (17). The unknown line density of circulation production at the boundary is such that the distribution of circulation within the fluid at any time induces at each point of the boundary a counter-velocity which just nullifies that of the forcing flow past the body at that time.

From the present viewpoint it is equations (5) and (14) that are basic for the kinematic description of two-dimensional viscous fluid flow. Together with the definition of vorticity (12) and the no-slip boundary conditions (18), these equations provide a complete phenomenological definition of a viscous fluid in two dimensions. In this approach, the equation of motion of the fluid is relegated to an auxiliary role in that it serves only to determine the dynamical pressure field $p(\mathbf{r}, t)$ in terms of kinematical quantities and the known force potential per unit mass Λ. Thus

$$-\nabla[p/\rho+\Lambda] = \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}.\nabla\mathbf{q} + \nu\,\mathrm{curl}\,\boldsymbol{\omega} . \tag{19}$$

If the Bernouilli function $H = p/\rho + \Lambda + \frac{1}{2}q^2$ is introduced, then (19) can be written in the alternative form

$$-\nabla H = \frac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \boldsymbol{\omega} + \nu\,\mathrm{curl}\,\boldsymbol{\omega} . \tag{20}$$

The Helmholtz vorticity equation (15) follows directly if the curl operator is applied to equation (20).

5. Steady viscous flow in two dimensions

For two-dimensional steady flow equation (20) simplifies to the form

$$-\nabla H = \mathbf{k} \times \mathbf{F} . \tag{21}$$

It follows that the decrease in H along any curve is equal to the discharge of circulation per unit time across that curve from left to right. In particular, the curves H = constant are the flux lines of the circulation flux vector $\mathbf{F}$. Thus if . $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ then

$$\begin{cases} -H_y = u\omega - \nu\omega_x = F_1 \\ +H_x = v\omega - \nu\omega_y = F_2 \end{cases} . \tag{22}$$

6. Use of conformal mapping

It should be noted that the steady form of (9) or (16) is invariant under conformal mapping. This can rapidly be seen by allowing x and y to be complex and introducing the auxiliary independent complex variables $z = x + iy$ and $z^* = x - iy$. The transformation $z = f(\zeta)$ then provides

$$k\nabla_\zeta^2 T + \frac{\partial(\psi,T)}{\partial(\xi,\eta)} = 0 , \tag{23}$$

$$\nu\nabla_\zeta^2 \omega + \frac{\partial(\psi,\omega)}{\partial(\xi,\eta)} = 0 , \tag{24}$$

(25) $$\nabla_\zeta^2\psi = -\left|\frac{dz}{d\zeta}\right|^2\omega ,$$

where $\nabla_\zeta^2 \equiv \dfrac{\partial^2}{\partial\xi^2} + \dfrac{\partial^2}{\partial\eta^2}$.

The Poisson equation (17) is not invariant, since circulation is the true invariant under conformal mapping and consequently the sources of the ψ field are altered in strength by a factor equal to the reciprocal of the area magnification.

For steady potential flow past a two-dimensional heated body with complex velocity potential $w = \phi + i\lambda$, Boussinesq (1905) [1] noted that (9) reduces to

(26) $$k\nabla_w^2 T - T_\phi = 0$$

in the (ϕ, λ) plane and corresponds in that plane to diffusion in a uniform flow with unit velocity parallel to the real ϕ axis past a correspondingly heated slit on that axis.

7. Heat sources and doublets in uniform convection

In one dimension in a medium moving with velocity U along the x-axis equation (9) takes the form

(27) $$k\frac{d^2T}{dx^2} - U\frac{dT}{dx} = 0 .$$

For a point source at the origin emitting Q units of heat per unit time the appropriate solution is

(28) $$\begin{cases} T = \dfrac{Q}{U\rho c} , & x \geq 0 , \\ T = \dfrac{Q}{\rho Uc}\exp\left(\dfrac{Ux}{k}\right) , & x < 0 . \end{cases}$$

It will be seen that downstream of the source the entire heat is carried away by convection and that the diffusive flux is zero. Upstream of the source the two fluxes are equal but oppositely directed, and both fall off exponentially in magnitude. There is no nett flow of heat towards upstream infinity.

For uniform flow parallel to the x-axis in two dimensions the

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corresponding equation is

(29) $$U\frac{\partial T}{\partial x} = k\nabla^2 T \ .$$

This may be rendered self-adjoint by the substitution $T = \exp\left(\frac{Ux}{2k}\right)\phi$ which yields

$$\nabla^2\phi - \frac{U^2}{4k^2}\phi = 0 \ .$$

The radially symmetric solution of this equation is

$$\phi = AK_0\left(\frac{U}{2k}r\right) + BI_0\left(\frac{U}{2k}r\right) \ ,$$

where A and B are arbitrary constants.

If this is to decay at infinity, $B = 0$. For a source strength Q at the origin we require

$$\int_0^{2\pi} \mathrm{F}.r d\theta = \frac{Q}{\rho c} \ .$$

In the limit $r \to 0$ only the diffusive term contributes significantly. Hence we find

(30) $$T = \frac{Q}{2\pi K}\exp\left(\frac{Ux}{2k}\right)K_0\left(\frac{U}{2k}r\right) \ .$$

It is instructive to consider the behaviour of this solution for large r. Substitution of the result

$$K_n(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}}\exp(-z) \quad \text{as} \quad z \to +\infty$$

gives

$$T \sim \frac{Q}{2\pi K}\left(\frac{\pi k}{Ur}\right)^{\frac{1}{2}}\exp\frac{U}{2k}(x-r) \quad \text{as} \quad r \to +\infty \ .$$

The exponential which now appears typifies a wake-like behaviour about the positive x-axis. Inside the parabola,

$$y^2 = \frac{4k}{U}\left(x+\frac{k}{U}\right) \ ,$$

where the argument of the exponential lies between 0 and -1, the

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temperature is significant and decays algebraically as $x \to +\infty$ only as a result of the sideways spreading of the wake by diffusion. If $r \to +\infty$ for any other value of θ the temperature falls off exponentially. The strongest decay is in the upstream direction for $\theta = \pi$. It is interesting, however, to calculate the rate at which heat is being convected across any line $x = x_0$. Since

$$\int_0^\infty K_0\left(\frac{U}{2k}r\right)dy = \frac{\pi k}{U}\exp\left[-\frac{U}{2k}|x|\right] ,$$

it is still true that the total convective discharge per unit time across $x = x_0$ is Q if $x_0 > 0$, and that this quantity falls off exponentially for $x_0 < 0$ like $\exp\left[-\frac{U}{k}|x|\right]$. For $x_0 > 0$ the nett diffusive discharge across $x = x_0$ must therefore be zero. In the neighbourhood of the centre of the wake the diffusion aids the convection whilst further out it acts in the opposite direction.

An example more directly relevant to the viscous fluid case is that of a heat doublet of strength M at the origin. The corresponding result is

(31) $$T = \frac{M}{2\pi K}\frac{Uy}{2kr}\exp\left(\frac{Ux}{2k}\right)K_1\left(\frac{U}{2k}r\right) .$$

For large r , T again exhibits wake-like behaviour with the same exponential wake factor.

$$T \sim \frac{M}{2\pi K}\left(\frac{\pi k}{Ur}\right)^{\frac{1}{2}}\left(\frac{Uy}{2kr}\right)\exp\frac{U}{2k}(x-r) .$$

Again, it is still strictly true that the convective discharge of dipole moment about $y = 0$ across the line $x = x_0$ is M for $x_0 > 0$, i.e.

$$\int_{-\infty}^{\infty} \rho c U y T dy = M \quad \text{for} \quad x_0 > 0 .$$

For $x_0 < 0$ this discharge falls off exponentially like

$$\exp\left[-\frac{U}{k}|x_0|\right] .$$

8. Brief survey of methods for steady two-dimensional viscous flow

Whilst a variety of exact special solutions of the steady form of equations (16) and (17) are available, the bulk of the methods used have relied upon an initial appropriate physical approximation to either the diffusive or convective term in the circulation flux vector $\mathbf{F}$. Usually such approximations are only locally valid and elaborate asymptotic matching procedures become necessary if a global solution is to be built up iteratively.

The commonest approach amounts to replacing the actual convection field ψ by a known convection field $\lambda(x, y)$ and then studying the resulting equations

(32) $$\nu\nabla^2\omega + \frac{\partial(\lambda,\omega)}{\partial(x,y)} = -\frac{\partial(\psi-\lambda,\omega)}{\partial(x,y)} ,$$

(33) $$\nabla^2\omega = -\omega ,$$

which become linear if the right-hand side of (32) is iteratively approximated. More radical procedures are also listed below.

A. Potential flow

(34) $$\begin{cases} \omega = 0 , \\ \nabla^2\psi = 0 . \end{cases}$$

This is appropriate in uniform flow past bodies for $R \gg 1$ in regions far from the body and its wake.

B. Diffusion negligible $R \gg 1$. Euler flow

(35) $$\begin{cases} \omega = f(\psi) , \\ \nabla^2\psi = -f(\psi) . \end{cases}$$

The commonest cases are $f(\psi) = \text{const.}\ \psi$ and $f(\psi) = \omega_0$ (const.). The latter is appropriate to cellular flow where if there is no production of circulation within a closed streamline C , we must demand $\oint_C ds\mathbf{n}.\nabla\omega = 0$, i.e. $f'(\psi)\oint_C ds\mathbf{n}.\nabla\psi = 0$. If the circumstances are such that $\Gamma \neq 0$, where $\Gamma = \oint_C \mathbf{q}.d\mathbf{s} = \oint_C ds\mathbf{n}.\nabla\psi$, then $f'(\psi) = 0$, i.e.

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$\omega = \omega_0$.

C. Stokes flow $R \ll 1$ (convection negligible) $(\lambda = 0)$

(36)
$$\begin{cases} \nabla^2\omega = 0 \ , \\ \nabla^2\psi = 0 \ . \end{cases}$$

In uniform flow with velocity $U\mathbf{i}$ past a finite body this linearisation is only of local validity near the body where $|q| \ll U$. The effect of the circulation production at the body is that roughly of a circulation dipole at the origin of strength M , say. Thus $\omega \underset{\sim}{\sim} \dfrac{My}{2\pi\nu r^2}$ and if $q \sim U\mathbf{i}$ as $r \to \infty$ then $R = O\left(\frac{rU}{\nu}\right)$ as $r \to \infty$ and hence there is an infinite region where this approximation is inadequate.

D. Oseen flow $(\lambda = Uy)$

This is appropriate for uniform flow with velocity $U\mathbf{i}$ past a finite body. Put $\lambda = Uy$ in (32) and neglect the right-hand side

(37)
$$\begin{cases} \nu\nabla^2\omega + U\dfrac{\partial\omega}{\partial x} = 0 \ , \\ \nabla^2\psi = -\omega \ . \end{cases}$$

These equations are valid in a complete neighbourhood of infinity and provide the basis for an adequate asymptotic description of such flows.

E. Burgers flow

Here λ in (32) is taken as the stream function of potential flow past the body, i.e. $\nabla^2\lambda = 0$. If $\phi + i\lambda$ is the complex potential and $\omega = \Pi\exp(\phi/2\nu)$ then

(38)
$$\begin{cases} \nabla^2\Pi - \dfrac{1}{4\nu^2}(\nabla\phi)^2\Pi = 0 \ , \\ \nabla^2\psi = -\Pi\exp(\phi/2\nu) \ . \end{cases}$$

Separable solutions of these equations are available in simple coordinate systems and use can also be made of conformal mapping.

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F. Prandtl flow $\cos\alpha << 1$

In the immediate neighbourhood of boundaries on streamline bodies and other regions of rapid change across streamlines, the downstream component of the diffusive flux is often negligible compared with its lateral component. For slowly curving streamlines with s denoting distance along the streamline and n distance perpendicular to it, the governing equations within these layers of rapid change are taken as

$$(39)\qquad \begin{cases} \nu\dfrac{\partial^2\omega}{\partial n^2} + \dfrac{\partial(\psi,\omega)}{\partial(s,n)} = 0\ , \\ \dfrac{\partial^2\psi}{\partial n^2} = -\omega\ . \end{cases}$$

A variety of solutions is then available in terms of a similarity variable $\eta = \dfrac{n}{\delta(s)}$ for different layer thickness functions $\delta(s)$. These solutions permit matching to a range of external conditions.

9. Conservation of whirl in three dimensions

The conservation theorems presented in para. 4 can be extended to the general three-dimensional case if a vector quantity Γ (which might well be called the *"whirl"* in a volume V) is introduced.

$$(40)\qquad \Gamma(V,\ t) = \int_V \omega dV\ ,$$

where $\omega = \text{curl}\,q$. The quantity Γ has been called vector circulation by Milne-Thomson [2], but this can cause confusion when ordinary scalar circulation round a closed curve is considered in three dimensions. The meaning of the term "total vorticity" used by Lighthill [3] is clear but it is dimensionally misleading.

The conservation equation (14) is now replaced by a vector equation,

$$(41)\qquad \frac{d\Gamma}{dt} = \int_V \frac{\partial\omega}{\partial t}dV = -\oint_{\partial V} dS\mathbf{n}.\underset{\sim}{F}\ ,$$

in which the flux of the vector quantity Γ now requires a tensor F for its specification. For a viscous fluid, a possible flux tensor $\underset{\sim}{F}$ for whirl is given by

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(42) $$\underset{\sim}{F} = -\nu\nabla\omega + q\omega - \omega q \ .$$

For suitably differentiable q and $\underset{\sim}{F}$ the local form of (41) is the Helmholtz vorticity equation,

(43) $$\frac{\partial\omega}{\partial t} + q.\nabla\omega - \omega.\nabla q = \nu\nabla^2\omega \ .$$

This equation can, of course, be obtained directly from (20) by applying the curl operator and it is on this basis that (41) is justified.

Whilst (41) may be interpreted as saying that there is no production of whirl within V, there will of course be local changes in its vector density ω arising from the stretching and twisting of vortex filaments. It is for this reason that (43) appears somewhat more complicated than its two-dimensional counterpart.

10. Conservation of ring circulation and angular momentum in axisymmetric flow

Introduce cylindrical polar coordinates as in para. 2 and in accord with (5) put

(44) $$q = u\hat{x} + v\hat{\sigma} + w\hat{\phi} = u + w\hat{\phi} = \text{curl}\left[\frac{\psi}{\sigma}\hat{\phi}\right] + w\hat{\phi} \ .$$

We shall assume that all variables are independent of the azimuthal angle ϕ but that the swirl velocity $w\hat{\phi}$ is not necessarily zero. Let $T = \sigma w$ so that $2\pi T$ is the axial circulation and ρT the angular momentum density. Note that $\text{div}\,u = 0$ and put

(45) $$\text{curl}\,u = \zeta\hat{\phi} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial\sigma}\right)\hat{\phi} \ .$$

The equations of motion of a viscous incompressible fluid can then be put in the following form:

(46) $$\frac{\partial u}{\partial t} - u \times \zeta\hat{\phi} = -\nabla H - \nu\text{curl}[\zeta\hat{\phi}] + G \ ,$$

(47) $$\frac{\partial T}{\partial t} + u.\nabla T = -\nu\nabla.\left[-\nabla T + \frac{2\hat{\sigma}}{\sigma}T\right] \ .$$

Here

(48) $$H = p/\rho + \tfrac{1}{2}(u^2+v^2) + \Lambda \quad \text{and} \quad G = h^2\sigma \ ,$$

where h denotes w/σ and is the azimuthal angular velocity. An important consequence of (46) is that so far as the component $\mathbf{u}$ of the fluid velocity in an axial half-plane is concerned, the sole effect of the swirl velocity $w\boldsymbol{\phi}$ may be described in terms of an artificial force field $\mathbf{G}$ (per unit mass) which essentially is the centrifugal force which would arise in an accelerating frame rotating with the local swirl angular velocity. Such a force field $\mathbf{G}$ is not in general conservative. It is closely analogous to the buoyancy force proportional to the temperature excess which arises in free convection problems. It follows that to each axisymmetric flow with swirl there corresponds an equivalent swirl-free flow in an appropriate "centrifugal force" field. These remarks clearly extend to axisymmetric flows viewed from a uniformly rotating frame of reference.

Consider now a ring vortex filament of radius σ with its centre on the x-axis. As the ring moves, the ring vorticity ζ will change, but if the cross-section of the filament has area $\delta\Lambda$, the ring circulation $\zeta(\delta\Lambda)$ will, in the absence of viscosity, remain constant. Further, the volume, $2\pi\sigma(\delta\Lambda)$, of the filament will also remain constant. It follows that the ring circulation volume density $\zeta/2\pi\sigma$, which we will denote by $l/2\pi$, remains constant throughout the motion of the vortex if viscosity is absent. We may obtain an equation for l in a viscous fluid by applying the operator $\frac{1}{\sigma}\mathrm{curl}$ to equation (46).

(49) $$\boldsymbol{\phi}\left[\frac{\partial l}{\partial t} + \mathbf{u}.\nabla l\right] = -\frac{\nu}{\sigma}\mathrm{curl}\ \mathrm{curl}[\sigma l\boldsymbol{\phi}] + \frac{1}{\sigma}\mathrm{curl}\mathbf{G}\ .$$

Now

$$\mathrm{curl}(\sigma l\boldsymbol{\phi}) = \frac{1}{\sigma}\frac{\partial(\sigma^2 l)}{\partial\sigma}\mathbf{x} - \frac{\partial(\sigma l)}{\partial x}\boldsymbol{\sigma}\ .$$

Hence

(50) $$\frac{\partial l}{\partial t} + \mathbf{u}.\nabla l = \nu\left[\nabla^2 l + \frac{2}{\sigma}\boldsymbol{\sigma}.\nabla l\right] + \frac{\partial h^2}{\partial x}\ .$$

If the artificial convection field $\mathbf{u}_1 = \frac{1}{\sigma}\hat{\boldsymbol{\sigma}}$ arising from a uniform line source of fluid on the axis of strength 2π per unit length is introduced, equation (50) can be reduced to the standard form (2b). Note first that

(51) $$\operatorname{div}\mathbf{u}_1 = 2\pi\delta(\sigma) \ .$$

Here $\delta(\sigma)$ denotes a distribution with unit line density along the axis of symmetry where σ (the position vector component perpendicular to that axis) is zero. Equation (50) can then be rephrased in the form

(52) $$\begin{cases} \dfrac{\partial l}{\partial t} + \operatorname{div}\mathbf{J} = -4\pi\nu\delta(\sigma)l \ , \text{ where} \\ \\ \mathbf{J} = (\mathbf{u}-2\nu\mathbf{u}_1)l - \nu\nabla l - h^2\mathbf{\hat{x}} \ . \end{cases}$$

Here $\mathbf{J}$ is the flux vector for ring circulation.

Similarly (47) can also be cast in the form

(53) $$\begin{cases} \dfrac{\partial T}{\partial t} + \operatorname{div}\mathbf{K} = 0 \ , \text{ where} \\ \\ \mathbf{K} = (\mathbf{u}+2\nu\mathbf{u}_1)T - \nu\nabla T \ . \end{cases}$$

Here $\mathbf{K}$ is the flux vector for angular momentum about the $\mathbf{\hat{x}}$-axis of symmetry. If ψ is zero on the axis, the definition of l in terms of ψ takes the related form

(54) $$\operatorname{div}[2\mathbf{u}_1\psi - \nabla\psi] = \sigma^2 l \ .$$

Confirmation of (52) may be obtained by considering the time rate of change of the circulation Γ in a fixed closed circuit C described in an axial half-plane. Consider first the case where C does not contain any points of the $\mathbf{\hat{x}}$-axis of symmetry. Then

$$2\pi\Gamma = 2\pi \oint_C \mathbf{u}.d\mathbf{s} = \int_V l\,dV \ ,$$

where V is the region generated by rotation of the area A enclosed by C around the axis of symmetry. Now

$$\frac{d}{dt}(2\pi\Gamma) = \int_V \frac{\partial l}{\partial t} dV \ .$$

But, since

$$\frac{d}{dt}(2\pi\Gamma) = 2\pi \oint \frac{\partial \mathbf{u}}{\partial t}.d\mathbf{s} \quad \text{and} \quad \mathbf{G} = \sigma\boldsymbol{\hat{\phi}} \times h^2\mathbf{\hat{x}} \ ,$$

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we can use (46) directly to obtain

$$\frac{d}{dt}(2\pi\Gamma) = -2\pi \oint_C \sigma ds \times \boldsymbol{\phi}.[(\mathbf{u}-2\nu\mathbf{u}_1)l - \nu\nabla l - h^2\hat{\mathbf{x}}]$$

$$= - \int_{\partial V} d\mathbf{S}.\mathbf{J} \ .$$

The curve C , being arbitrary, may now be contracted to the neighbourhood of any point P , not on the axis, in the axial half-plane. An application of the divergence theorem then shows that

$$\frac{\partial l}{\partial t} + \mathrm{div}\mathbf{J} = 0 \ .$$

In the limiting case, when C contracts to the neighbourhood of a point P on the axis, proper account must be taken of the discontinuity in $\mathbf{u}_1$ in the integral over ∂V . If this is done then equation (52) is recaptured completely.

A verification of (53) may be obtained in a like manner by appealing to the angular momentum principle. The torque about the $\hat{\mathbf{x}}$-axis of the viscous forces acting on the boundary ∂V of V is given by

$$\mu \oint_C 2\pi\sigma^2\left[\frac{\partial}{\partial x}\left(\frac{T}{\sigma}\right)d\sigma - \sigma\,\frac{\partial}{\partial\sigma}\left(\frac{T}{\sigma^2}\right)dx\right] = -2\pi\mu \oint_C ds \times \boldsymbol{\phi}.\left[-\nabla T + \frac{2\hat{\boldsymbol{\sigma}}}{\sigma}T\right]$$

$$= - \mu \int_{\partial V} d\mathbf{S}.[-\nabla T + 2\mathbf{u}_1 T] \ .$$

The viscous torque gives rise to a rate of increase of angular momentum within V together with a rate of flow of angular momentum out of V . Hence

$$- \mu \int_{\partial V} d\mathbf{S}.[-\nabla T + 2\mathbf{u}_1 T] = \int_V \frac{\partial(\rho T)}{\partial t}dV + \int_{\partial V} \rho T\mathbf{u}.d\mathbf{S} \ .$$

An application of the divergence theorem gives

$$\int_V \left[\frac{\partial T}{\partial t} + \mathrm{div}\mathbf{K}\right]dV = 0 \ .$$

Equation (53) then results if V is contracted to the neighbourhood of the field point P , since the integrand is continuous.

The flux vectors $\mathbf{J}$ and $\mathbf{K}$ in (52) and (53) lend themselves to simple interpretation in terms of diffusion and convection just as before. The new feature is that, in addition to ordinary convection by the flow field, there is a fictitious radial convection field $2\nu u_1$ which now plays a role both in (52) and (53). In the case (52) of ring circulation, the radial convection is inwards and gives rise to a viscosity dependent uniform line sink for ring circulation on the axis, which completely removes the amount of this quantity discharged there by the radial convection field. Note also that when swirl is present there is an additional flux component $-h^2\hat{\mathbf{x}}$ of ring circulation parallel to the axis.

In the case (53) of angular momentum, the fictitious radial convection is outwards and there is no corresponding viscosity dependent fictitious line source of angular momentum on the axis.

Off the axis, the three equations (52), (53), (54) can also be recast in a form similar to (10).

(55) $$\frac{\partial l}{\partial t} = \frac{1}{\sigma}\frac{\partial(\psi,l)}{\partial(x,\sigma)} + \nu\nabla^2 l + \frac{2\nu}{\sigma}\frac{\partial l}{\partial\sigma} + \frac{\partial h^2}{\partial x} ,$$

(56) $$\frac{\partial T}{\partial t} = \frac{1}{\sigma}\frac{\partial(\psi,T)}{\partial(x,\sigma)} + \nu\nabla^2 T - \frac{2\nu}{\sigma}\frac{\partial T}{\partial\sigma} ,$$

(57) $$-\sigma^2 l = \nabla^2\psi - \frac{2}{\sigma}\frac{\partial\psi}{\partial\sigma} .$$

It is also instructive to consider the steady state form of (46). If $\mathbf{F} = F_1\hat{\mathbf{x}} + F_2\hat{\boldsymbol{\sigma}}$ is the flux vector for ring circulation displayed in (52), then under steady conditions (46) gives

(58) $$\begin{cases} F_1 = ul - \nu\dfrac{\partial l}{\partial x} - h^2 = -\dfrac{1}{\sigma}\dfrac{\partial H}{\partial\sigma} , \\ F_2 = \left(v - \dfrac{2\nu}{\sigma}\right)l - \nu\dfrac{\partial l}{\partial\sigma} = \dfrac{1}{\sigma}\dfrac{\partial H}{\partial x} . \end{cases}$$

These equations (58) are analogous to (21) and again $-H$ plays the role of a flux function for $\mathbf{F}$. The increase in $-2\pi H$ along an arc in the (x, σ) plane gives the discharge from left to right across the surface generated by rotating this arc about the axis.

11. Conservation of moment of whirl

The axial component of the moment of whirl in a region V is defined in axisymmetric flow as

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(59) $$\mathfrak{X}M = \frac{1}{2}\int_V \mathbf{r} \times \zeta\boldsymbol{\Phi}\, dV = \frac{\mathfrak{X}}{2}\int_V \sigma^2 l\, dV \,.$$

It can be shown that $\frac{d(\rho M)}{dt}$ is the amount of momentum which is produced within V per unit time parallel to the $\mathfrak{X}$-axis. If $\frac{dM}{dt}$ is calculated from the expression for $\frac{\partial l}{\partial t}$ provided by (52), the resulting integrand can be expressed as a divergence of a possible flux vector F for moment of whirl. The manipulation yields

(60) $$\frac{\partial k}{\partial t} + \mathrm{div}[(\mathsf{u}+2\nu\mathsf{u}_1)k - \nu\nabla k + \tfrac{1}{2}\{u^2-v^2-\sigma^2h^2\}\mathfrak{X} + uv\boldsymbol{\vartheta}] = 0$$

where $\frac{1}{2}\sigma^2 l$, the density of moment of whirl, is denoted by k .

In two dimensions the moment of circulation about the x-axis is given by

$$M = \int_S y\omega\, dS \,,$$

where again it can be shown that $\frac{d(\rho M)}{dt}$ is the amount of momentum parallel to the x-axis produced within S per unit time. A similar calculation provides the flux vector F for moment of circulation

(61) $$\mathsf{F} = y[-\nu\nabla\omega + \mathsf{q}\omega] + \tfrac{1}{2}(u^2-v^2)\mathsf{i} + (\nu\omega+uv)\mathsf{j} \,.$$

With these flux vectors, integral invariants for the flow field can be constructed which describe the total flux through any surface enclosing the flow-producing singularities, in terms of the total strength of those singularities. Such invariant integrals are needed if the constants in the asymptotic expansion of the flow in the far field are to be determined.

12. Some axisymmetric flows which are solely dependent on distance from the axis

Simple illustrations of the types of singularities which occur are provided by the steady state versions of (55), (56) and (57) when there is no dependence upon the axial coordinate x .

(62) $$\frac{\partial^2 l}{\partial\sigma^2} + \frac{3}{\sigma}\frac{\partial l}{\partial\sigma} = 0 ,$$

(63) $$\frac{\partial^2 T}{\partial\sigma^2} - \frac{1}{\sigma}\frac{\partial T}{\partial\sigma} = 0 ,$$

(64) $$\frac{\partial^2 \psi}{\partial\sigma^2} - \frac{1}{\sigma}\frac{\partial \psi}{\partial\sigma} = -\sigma^2 l .$$

The solution $T = A$ of (63) (where A is a constant) arises when there is a uniform line source of angular momentum on the axis with line density $4\pi\nu A$ per unit length. The diffusive flux is zero, but there is steady (fictitious) radial convection carrying the angular momentum produced on the axis out to $\sigma = \infty$.

The solution $T = B\sigma^2$ of (63) (where B is a constant) which describes solid body rotation with angular velocity B arises from infinite production of angular momentum at $\sigma = \infty$. The diffusive flux of angular momentum is inwards and at each station is just balanced by the (fictitious) radial convection of angular momentum outwards. Both are zero on $\sigma = 0$.

Similarly the solutions,

$$l = C , \quad \psi = -\frac{C}{8}\sigma^4 ,$$

of (62) and (64) (where C is a constant) arise from a (fictitious) uniform line sink of ring circulation on the axis with line density $4\pi\nu C$ per unit length. The diffusive flux is zero but there is steady (fictitious) radial convection of ring circulation inwards from $\sigma = \infty$ to the sink on the axis.

The solutions,

$$l = \frac{D}{\sigma^2} , \quad \psi = -\frac{D}{2}\sigma^2 ln(\sigma) .$$

of (62) and (64) describe production of moment of ring circulation (or axial momentum) on the axis.

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University of Queensland,
St Lucia, Queensland.