UPPER BOUNDS ON HOMOLOGICAL DIMENSIONS

B.L. OSOFSKY

The homological dimension of a module M_R is often related to the cardinality of a set of generators for M or for right ideals of R. In this note, upper bounds for this homological dimension are obtained in two situations.

In [8] Jensen has shown that, for any ring R whose finitely generated right ideals are countably related, if any right ideal of R is generated by $\aleph_n$ elements, then the right global dimension of R exceeds the weak global dimension by at most $n+1$. In section 1 we show that the condition that finitely generated right ideals are countably related may be deleted, and Jensen's theorem will still hold.

In [3] Berstein showed that a direct limit of modules over a countable directed system has dimension at most one more than the supremum of the dimensions of the modules. This is also an immediate consequence of Roos [14], Theorem 1. In section 2 we show that a direct limit of modules over a directed system of cardinality $\aleph_n$ has dimension at most $n+1$ more than the supremum of the dimensions of the modules. Balcerzyk showed this for a directed union in [2].

All rings R will have identity 1; all modules will be unital right R-modules. For a module M, $hd_R(M)$ (or $hd(M)$ if no confusion arises) will denote the homological dimension of M. gl. d(R) will denote the right global dimension of R and w. gl. d(R) its weak global dimension. A basic tool for calculating upper bounds on homological dimensions is the following proposition of Auslander.

PROPOSITION 0. 1. *Let $\mathscr{I}$ be a non-empty well-ordered set, M a right R-module, $\{N_i \mid i \in \mathscr{I}\}$ a family of submodules of M such that $N_i \subseteq N_j$ for $i \leq j$. If $M = \cup_{i \in \mathscr{I}} N_i$ and $hd_R(N_i / \cup_{j<i} N_j) \leq n$ for all $i \in \mathscr{I}$, then $hd_R(M) \leq n$.*

Proof. This is proposition 3 of [1].

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PROPOSITION 0. 2. *Let* $B_R \subseteq A_R$. *If* $hd(A) > hd(B)$, *then* $hd(A) = hd(A/B)$. *If* $hd(A) < hd(B)$, *then* $hd(A/B) = hd(B) + 1$. *If* $hd(A) = hd(B)$, *then* $hd(A/B) \leq 1 + hd(A)$.

Proof. This is Theorem 1. 2 of Kaplansky [10].

We observe that $hd(B)$ cannot exceed both $hd(A)$, and $hd(A/B)$, and $hd(B) = hd(A/B)$ implies $hd(B) = hd(A)$.

COROLLARY 0. 3. *Under the hypotheses of Proposition* 0. 1, *if* $hd_R(\cup_{j<i}N_j) \leq m$ *for all* $i \in \mathscr{I}$, *then* $hd_R(M) \leq m+1$.

Proof. Since $hd_R(\cup_{j<i}N_j) \leq m$ and $hd_R(N_i) = hd_R(\cup_{j<i+1}N_j) \leq m$, $hd_R(N_i/\cup_{j<i}N_j) \leq m+1$ by Proposition 0. 2. Hence $hd_R(M) \leq m+1$ by Proposition 0. 1.

§ 1. **Dimension of a flat module.** In this section we get an upper bound on the homological dimension of a flat R-module in terms of the number of generators of ideals of R.

DEFINITION : R is called $\aleph_n$-noetherian if every right ideal of R is generated by a set of elements of cardinality $\leq \aleph_n$.

LEMMA 1. 1. *Let* F *be an* R*-module generated by a set of cardinality* $\leq \aleph_n$. *If* R *is* $\aleph_n$*-noetherian, then every submodule of* F *is* $\aleph_n$*-generated.*

Proof. Jensen's proof in [7] for the case $n=0$ goes through exactly in this case.

Let F be a free R-module. We say $B \subseteq F$ is a $*$-submodule of $F \Longleftrightarrow$ for all $\{b_i | 1 \leq i \leq n\} \subseteq B$, there is a $u \in \mathrm{Hom}_R(F, B)$ such that $u(b_i) = b_i$ for $1 \leq i \leq n$. By [4], p. 65, B is a $*$-submodule of $F \Longleftrightarrow F/B$ is flat.

LEMMA 1. 2. *Let* M *be a countably related flat* R*-module. Then* $hd_R(M) \leq 1$.

Proof. See Jensen [7], Lemma 2.

THEOREM 1. 3. *Let* F *be a free* R*-module with basis* $\{x_i | i \in \mathscr{I}\}$, B *a* $*$*-submodule of* F. *Then any submodule* T *of* B *generated by* $\aleph_k$ *elements can be embedded in a* $*$*-submodule* T^* *of* B *such that* T^* *is also* $\aleph_k$*-generated. If* $k \in \omega$, *any* $\aleph_k$*-generated* $*$*-closed submodule has homological dimension* $\leq k$.

Proof. We use transfinite induction on k.

Basis : $k = 0$.

Let $\{t_i \mid i \in \omega\}$ generate $T \subseteq B$. We inductively define T_i, u_i so that $T_0 = t_0R$ and

i) $t_i \in T_i$

ii) $T_j \subseteq T_i$ for all $j \leq i$

iii) T_i is a finitely generated submodule of B

iv) $u_i : F \longrightarrow T_{i+1}$ is the identity on T_i.

Assume we have T_i. Let $u' : F \longrightarrow B$ be any map which is the identity on $T_i + t_{i+1}R$. Define $u_i : F \longrightarrow B$ by $u_i(x_j) = 0$ if $T_i + t_{i+1}R$ has zero projection on x_jR ; $u_i(x_j) = u'(x_j)$ otherwise. Since $T_i + t_{i+1}R$ is finitely generated, $u_i(x_j) = 0$ for all but a finite number of $j \in \mathscr{I}$. Hence $T_{i+1} = u_i(F)$ is finitely generated and $\{T_j \mid 0 \leq j \leq i+1\}$ satisfies i) $\longrightarrow$ iv).

Set

$$T^* = \cup_{i=0}^{\infty} T_i.$$

Since any finite subset of T^* is contained in some T_i, $u_i : F \longrightarrow T_{i+1} \subseteq T^*$ is a map leaving it fixed. Hence T^* is a $*$-submodule of B. Clearly T^* is countably generated and contains T. Since F/T^* is flat, by Lemma 1. 2, T^* must be projective.

Induction step. Let $\{t_\gamma \mid \gamma < \aleph_k\}$ generate T. Set

$$T_\gamma = [t_\gamma R + \cup_{\beta<\gamma} T_\beta]^*$$

for all $\gamma < \aleph_k$. If for all $\beta < \gamma$, T_β is generated by at most $\aleph_0$ times cardinality of β elements, then $t_\gamma R + \cup_{\beta<\gamma} T_\beta$ is generated by $\aleph_0 \cdot$cardinality γ elements. Hence by the induction hypothesis, T_γ exists and is generated by at most $\aleph_0 \cdot$cardinality γ elements. Set

$$T^* = \cup_{\gamma<\aleph_k} T_\gamma.$$

Then $T^* \supseteq T$ and T^* is $\aleph_k$-generated. Since a linearly ordered chain of $*$-submodules is a $*$-submodule, T^* is the required module.

If $k \in \omega$, since $\cup_{\beta<\gamma} T_\beta$ is a $*$-submodule of F generated by a set of $\aleph_{k-1}$ elements, by the induction hypothesis $hd_R(\cup_{\beta<\gamma} T_\beta) \leq k-1$. By Corollary 0. 2, $hd_R(T^*) \leq k$.

COROLLARY 1. 4. *Let R be $\aleph_n$-noetherian. Then*

$$gl.\ d(R) \leq w.\ gl.\ dim\,(R) + n + 1.$$

Proof. Let I be any right ideal of R. Since R is $\aleph_n$-noetherian, we can find a projective resolution

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow R/I \longrightarrow 0$$

of R/I where each F_n is free on $\aleph_n$ generators. Let w. gl. dim $(R) = k$. Then the image of F_k is flat, so the image of F_{k+1} is a $*$-submodule of F_k generated by at most $\aleph_n$ elements. Then $hd_R(\text{Image } F_{k+1}) \leq n$ by the theorem, so $hd_R(R/I) \leq k+1+n$. The global dimension theorem (Auslander [1]) completes the proof.

A module M is flat if and only if M is a direct limit of finitely generated free modules. (See [11].) Of course, direct limit cannot be replaced by directed union here. However:

COROLLARY 1. 5. *Let R be $\aleph_n$-noetherian. Then any flat R-module is the directed union of $\aleph_n$-generated flat submodules.*

Proof. Let N be any $\aleph_n$-generated submodule of the flat module M, and let $F = \sum_{i\in\mathscr{I}} \oplus\, x_i R$ be a free module mapping onto M with kernel K. We wish to find a flat submodule $N' \subseteq M$ such that $N \subseteq N'$ and N' is generated by $\aleph_n$ elements. We apply a snaking argument of Kaplansky [9]. N is contained in the image of $\aleph_n$ of the x_i, say $\sum_{i\in\mathscr{I}_0} x_i R$. Let $T_0 = K \cap \sum_{i\in\mathscr{I}_0} x_i R$. T_0 is $\aleph_n$-generated by Lemma 1. 1. Hence T_0 is contained in an $\aleph_n$-generated $*$-closed submodule T_1.

Assume T_l has been defined so that T_l is an $\aleph_n$-generated, $*$-closed submodule of K containing T_j for all $j < l$. Since T_l is $\aleph_n$-generated, it is contained in $\sum_{i\in\mathscr{I}_l} x_i R$ for some set $\mathscr{I}_l \subseteq \mathscr{I}$ of cardinality at most $\aleph_n$. By Lemma 1. 1, $K \cap \sum_{i\in\mathscr{I}_l} x_i R$ is $\aleph_n$-generated. Set $T_{l+1} = (K \cap \sum_{i\in\mathscr{I}_l} x_i R)^*$. Set

$$T = \cup_{l=0}^{\infty} T_l$$

$$\mathscr{I}' = \cup_{l=0}^{\infty} \mathscr{I}_l.$$

Then T is an ascending union of $*$-closed submodules of F and so $*$-closed.

Moreover, $x \in K \cap \sum_{i\in\mathcal{J}'} x_i R$ implies $x \in \sum_{i\in\mathcal{J}_l} x_i R \cap K$ for some l, so $x \in T_{l+1}$. Hence

$$T = K \cap \sum_{i\in\mathcal{J}'} x_i R$$

and $\sum_{i\in\mathcal{J}'} x_i R/T$ is isomorphic to a flat, $\aleph_n$-generated submodule of M containing N. M is the directed union of such submodules.

We note that Corollary 1. 4 is the best possible result, for by Osofsky [12], an $\aleph_n$-noetherian valuation ring which is not $\aleph_{n-1}$-noetherian has global dimension $n+2$. Its weak dimension is 1. By Pierce [13] a free boolean ring on $\aleph_n$ generators has global dimension $n+1$, and its weak dimension is 0. It is unknown if the number of generators of the flat submodules can be reduced in 1. 5. We can show, however, that in the case $n=0$, Corollary 1. 5 is best possible. The following modification of an example of Kaplansky is due to C.U. Jensen.

Let R be the subring of the ring of all continuous functions from $\boldsymbol{Q}$ to $\boldsymbol{Q}$ generated by functions composed of a finite number of linear pieces. Then R is countable, and for all $r \in R$, $r^{-1}(0)$ consists of a finite number of components. Let I be the ideal of R consisting of functions vanishing in a neighborhood of 0. Let $f_0 = 0$, and for $n \geq 1$, let f_n be 1 on $(-\infty, -2/n] \cup [2/n, \infty)$, linear on $[-2/n, -1/n] \cup [1/n, 2/n]$, and 0 on $[-1/n, 1/n]$. Then $\{f_{n+1} - f_n \mid n \in \omega\}$ forms a dual basis for I, so I is projective. We show that no finitely generated submodule of I is flat.

Let $g_1, \cdots, g_n \in I$, and let $[a, b]$ be the largest neighborhood of 0 on which all of the g_i vanish. By multiplying by -1 if necessary, we may assume all of the g_i are non-negative on a neighborhood to the right of b, and at least one of the g_i is strictly positive there. Let h be non-zero on (a, b) and zero elsewhere. Then $hg_i = 0$ for all i. If $\sum g_i R$ is flat, the kernel of an epimorphism from a free module to it is a $*$-submodule, so there exist $\{p_{ij} \mid 1 \leq i,\ j \leq n\} \subseteq R$ such that $0 = \sum_{i=1}^n g_i p_{ij}$, and $hp_{ij} = h\delta_{ij}$ for all j, $1 \leq j \leq n$. Then

$$0 = \sum_{i=1}^n g_i \left(\sum_{j=1}^n p_{ij}\right)$$

and $p_{ij} = \delta_{ij}$ on $[a, b]$. Hence $\sum_{j=1}^n p_{ij} = 1$ on $[a, b]$, so in a neighborhood to the right of b, each $g_i(\sum p_{ij}) \geq 0$, and some $g_i(\sum p_{ij}) > 0$, a contradiction.

COROLLARY 1. 6. Let R have cardinality $\aleph_n$. Then the left and right global dimensions of R can differ by at most $n+1$.

Proof. Each dimension is equal to or greater than the weak dimension, and exceeds it by at most $n+1$.

In Corollary 1. 6, no smaller bound is possible, for Jategaonkar [6] has an example of a left hereditary ring of cardinality $\aleph_n$ and right global dimension $= n+2$.

§ 2. **Dimension of a direct limit.** Let D be a directed set, $\{R_i, \pi_i^j \mid i, j \in D\}$ a directed system of rings, $\{M_i, \xi_i^j \mid i, j \in D\}$ a directed system of groups such that each M_i is an R_i-module and $\xi_i^j(mr) = \xi_i^j(m)\pi_i^j(r)$ for all $m \in M_i$, $r \in R_i$. Let $M = \varinjlim M_i$, $R = \varinjlim R_i$.

PROPOSITION 2. 1. *If D is countable, then $hd_R(M) \leq 1 + sup\, hd_{R_i}(M_i)$ and $gl.\ d(R) \leq 1 + sup\, gl.\ d(R_i)$.*

Proof. See Berstein [3].

We generalize this result to any cardinal $\aleph_n$, where $n \in \omega$.

Let $D' = \{d_\gamma \mid \gamma < \aleph_n\}$ be cofinal in D. Then the direct limit over D is the same as over D', so we may assume $D' = D$.

LEMMA 2. 2. *Assume $n > 0$. Then there exist directed subsets $\{E_\gamma \mid \gamma < \aleph_n\}$ of D such that*

$$D = \cup_{\gamma < \aleph_n} E_\gamma$$

$$E_\beta \subseteq E_\gamma \ \textit{for}\ \beta < \gamma$$

$$\textit{Cardinality}\ E_\gamma = \aleph_{n-1}\ \textit{for all}\ \gamma < \aleph_n.$$

Proof. Assume E_β has been defined for all $\beta < \gamma$. Let $E_{\gamma,0} = \{d_\alpha \mid \alpha < \aleph_{n-1} + \gamma\} \cup \cup_{\beta<\gamma} E_\beta$. Then $E_{\gamma,0}$ has cardinality $\aleph_{n-1}$. Once $E_{\gamma,i}$ has been defined, define $E_{\gamma,i+1} = E_{\gamma,i} \cup F_i$, where F_i is the image of a function from $E_{\gamma,i} \times E_{\gamma,i} \longrightarrow D$ taking (x, y) to an upper bound of x and y. Then each $E_{\gamma,i}$ has cardinality $\aleph_{n-1}$, and so does $E_\gamma = \cup_{i=0}^{\infty} E_{\gamma,i}$. Clearly $D = \cup_{\gamma<\aleph_n} E_\gamma$ and each E_γ is a directed subset of D.

THEOREM 2. 3. *If a set of cardinality $\aleph_n$ is cofinal in D, then*

$$hd_R(M) \leq n + 1 + sup\, hd_{R_i}(M_i)$$

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$$gl.\ d(R) \leq n + 1 + \sup gl.\ d(R_i).$$

Proof. Berstein in [3] reduces the situation to the case where each $R_i = R$ and $\pi_i^j =$ the identity by first using induction to get each M_i R_i-projective and then using the fact that M_R is the direct limit of $\{M_i \otimes_{R_i} R\}$. So we may assume all M_i are R-modules. We now use induction on n. The basis is Proposition 2. 1. Now assume $n > 0$.

By Lemma 2. 2, $D = \cup_{\gamma < \aleph_n} E_\gamma$, where each E_γ is a directed set of cardinality $\aleph_{n-1}$. Now

$$M_\gamma = \varinjlim_{\cup_{\beta<\gamma} E_\beta} M_i = \sum_{i \in \cup_{\beta<\gamma} E_\beta} \oplus M_i \Big/ \sum_{\substack{i, j \in \cup_{\beta<\gamma} E_\beta \\ m_i \in M_i}} (m_i - \xi_i^j m_i) R$$

and by induction $hd_R(M_\gamma) \leq \sup hd_R(M_i) + n$. By Proposition 0. 2,

$$hd_R\Big(\sum_{i \in \cup_{\beta<\gamma} E_\beta} (m_i - \xi_i^j m_i) R\Big) \leq \sup_{i \in \cup_{\beta<\gamma} E_\beta} hd_R(M_i) + n - 1.$$

By Corollary 0. 3,

$$n + \sup hd_R(M_i) \geq hd_R\Big(\cup_{\gamma < \aleph_n} \sum_{\substack{i \in \cup_{\beta<\gamma} E_\beta \\ m_i \in M_i}} (m_i - \xi_i^j m_i) R\Big).$$

Also, $hd_R(\sum \oplus M_i) = \sup hd_R(M_i)$.

But

$$M = \varinjlim_D M_i = \sum_{i \in D} \oplus M_i \ / \ \cup_{\gamma < \aleph_n} \sum (m_i - \xi_i^j m_i) R,$$

so by Proposition 0. 2, $hd_R(M) \leq \sup hd_R(M_i) + n + 1$.

The statement about global dimensions follows since any left ideal of R is the direct limit of left ideals of the R_i (take inverse images of the maps from the $R_i \longrightarrow R$).

Theorem 2. 3 obtains the best possible bounds, for any ideal in a valuation ring is a direct limit of projective ideals, and if it is generated by a set of $\aleph_n$ elements but no set of smaller cardinality, its dimention is $n + 1$. And a free boolean ring on $\aleph_n$ generators (global dimension $n + 1$ by [13]) is a direct limit of semi-simple Artinian rings.

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Institute for Advanced Study and
Rutgers, The State University