

Helmholtz's Theorem

Is this solution unique?

YES, as long as the vector field $\mathbf{F}(\mathbf{r})$ itself goes to zero at infinity

If the divergence $D(\mathbf{r})$ and the curl $\mathbf{C}(\mathbf{r})$ of a vector function $\mathbf{F}(\mathbf{r})$ are specified, and if they both go to zero faster than $1/r^2$ as r goes to infinity, and if $\mathbf{F}(\mathbf{r})$ itself goes to zero as r goes to infinity, then $\mathbf{F}(\mathbf{r})$ is uniquely given by

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}$$

MAXWELL'S EQUATIONS

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's equations specify the divergence and curl of the electric and magnetic fields. Using Helmholtz's theorem, we can then determine the electric and magnetic fields from Maxwell's equations.

MAXWELL'S EQUATIONS - ELECTROSTATICS

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{E} = 0$$

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\nabla \left(\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathfrak{R}} dV' \right) \\ &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathfrak{R}^2} \hat{\mathfrak{R}} dV' \end{aligned}$$

$$\mathfrak{R} = \mathbf{r} - \mathbf{r}'$$

Coulomb's Law!

For a continuous surface charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{\mathfrak{R}^2} \hat{\mathfrak{R}} da'$$

For a continuous line charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\mathbf{r}')}{\mathfrak{R}^2} \hat{\mathfrak{R}} dl'$$

For a collection of discrete charges,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\mathfrak{R}_i^2} \hat{\mathfrak{R}}_i$$

ELECTROSTATICS

ELECTROSTATICS

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 \quad \text{Helmholtz Theorem}$$

$$\nabla \times \mathbf{E} = 0$$

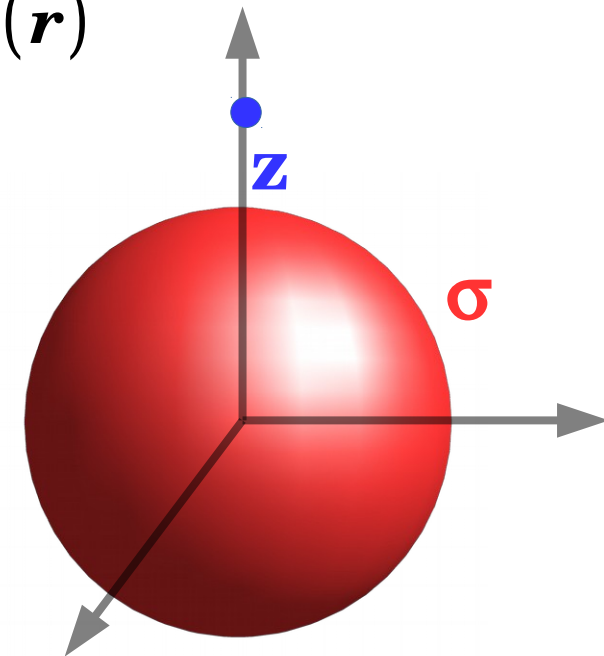
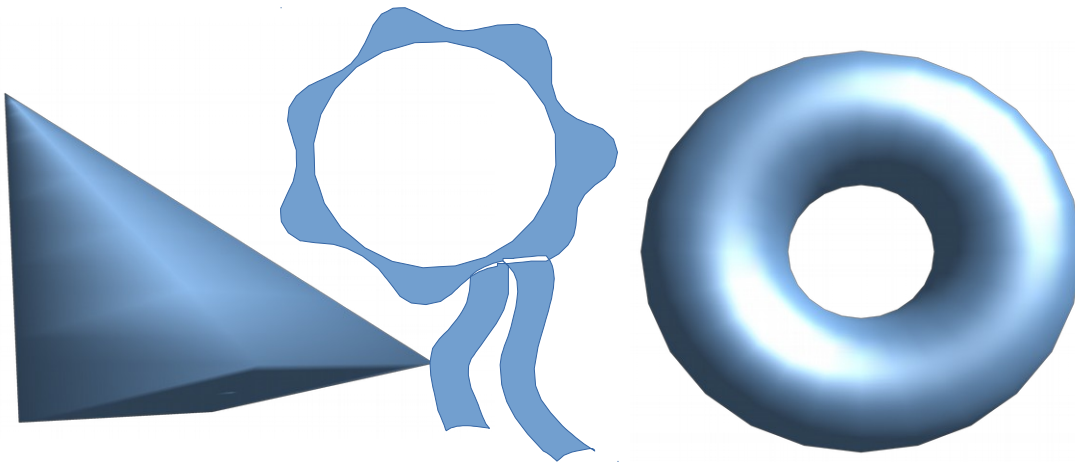
Maxwell's Equations
for Electrostatics

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathcal{R}^2} \hat{\mathcal{R}} dV'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\mathcal{R}_i^2} \hat{\mathcal{R}}_i$$

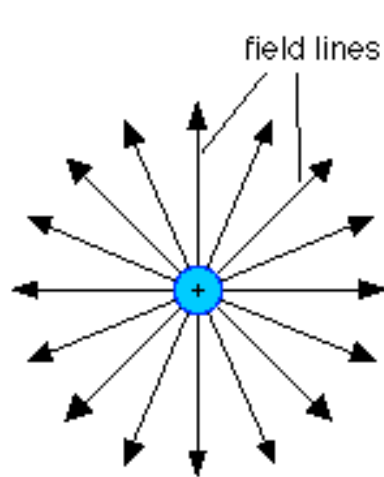
Coulomb's Law

Force on a test charge Q , $\mathbf{F}(\mathbf{r}) = Q \mathbf{E}(\mathbf{r})$

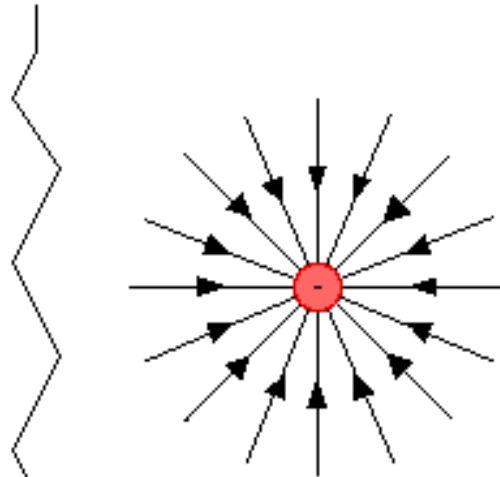


In principle, we're done with Electrostatics!

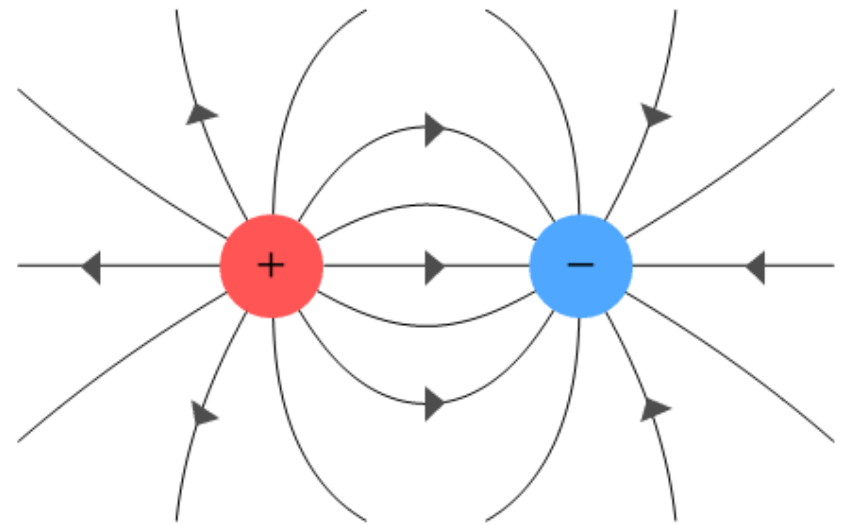
ELECTRIC FIELD LINES



The electric field from an isolated positive charge

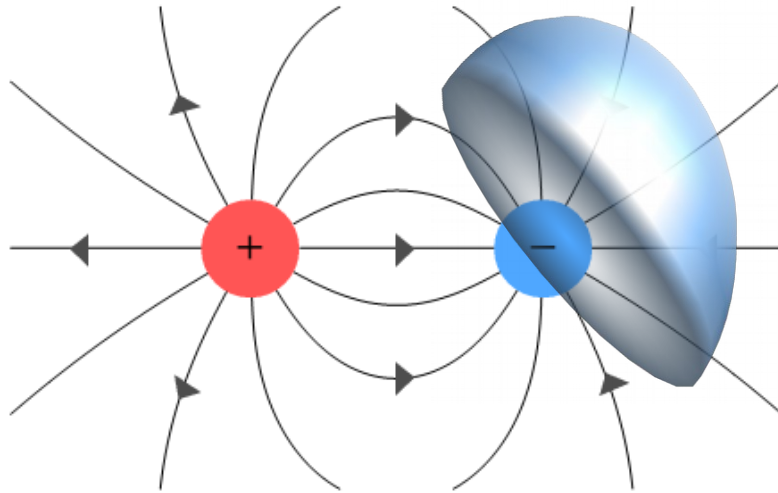


The electric field from an isolated negative charge



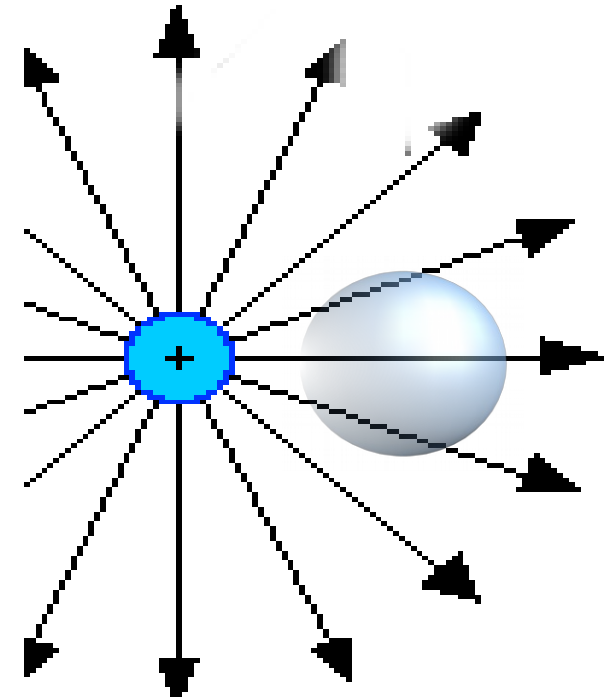
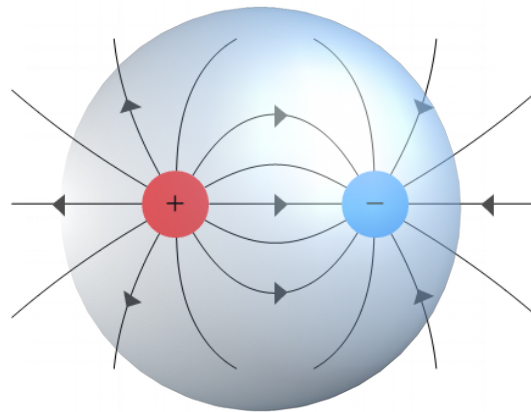
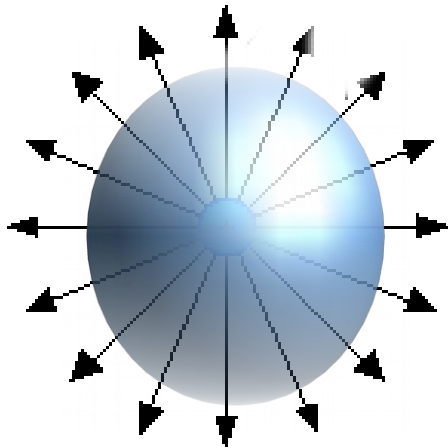
- Field lines begin on positive charges
- Field lines end on negative charges, or they extend upto infinity
- The strength of the field is indicated by the density of the field lines
- Field lines can never cross

ELECTRIC FIELD LINES - CLOSED SURFACE



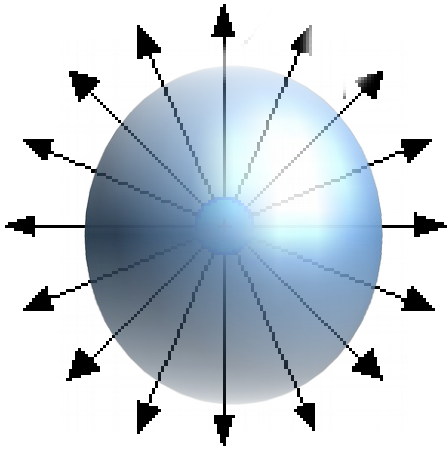
$$\text{Flux } \Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{S}$$

Flux \propto Number of field lines



Flux through a closed surface is a measure of the total charge inside the surface – Gauss's Law

GAUSS'S LAW

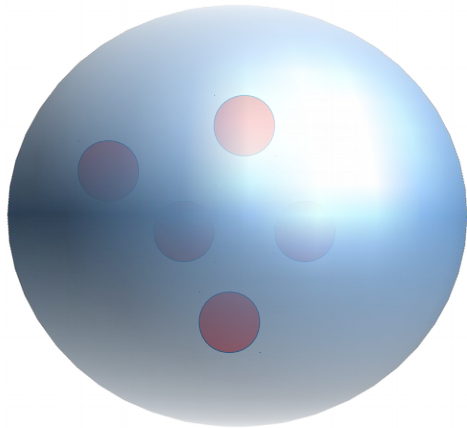


Point charge q at origin

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r^2} \hat{\mathbf{r}} \right) (r^2 \sin\theta d\theta d\phi \hat{\mathbf{r}}) = \frac{q}{\epsilon_0}$$

Principle of superposition

$$\mathbf{E} = \sum_{i=1}^N \mathbf{E}_i$$

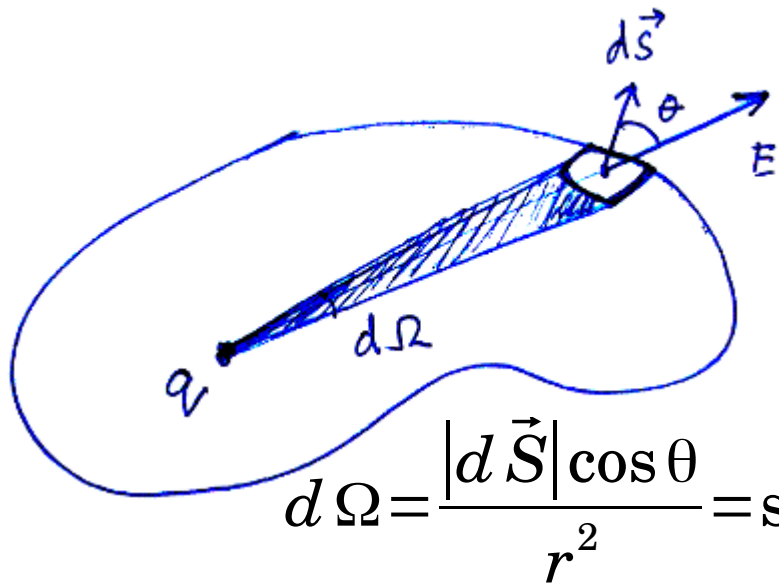


Collection of point charges q_i

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^N \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^N \left(\frac{q_i}{\epsilon_0} \right)$$

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{enc}$$

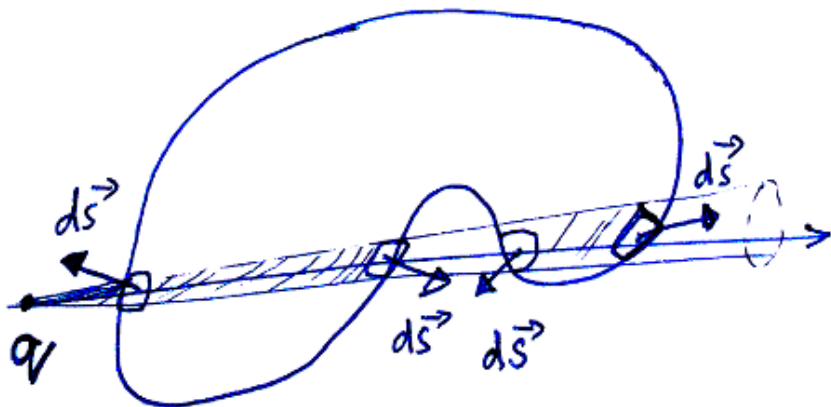
GAUSS'S LAW



$$\begin{aligned}
 \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{S} &= \frac{q}{4\pi\epsilon_0} \int_{\text{surface}} \frac{\hat{\mathbf{r}} \cdot d\mathbf{S}}{r^2} \\
 &= \frac{q}{4\pi\epsilon_0} \int_{\text{surface}} \frac{|d\mathbf{S}| \cos \theta}{r^2} \\
 &= \frac{q}{4\pi\epsilon_0} \int_{\text{surface}} d\Omega \\
 &= \frac{q}{4\pi\epsilon_0} 4\pi = \frac{q}{\epsilon_0}
 \end{aligned}$$

The solid angle subtended by a surface S is defined as the surface area of a unit sphere covered by the surface's projection onto the sphere.

A measure of how large an object appears to an observer looking from that point



If the point is located outside then the contributions exactly cancel

E Use superposition principle ---> Add contribution from each charge

GAUSS'S LAW - DIFFERENTIAL FORM

$$\int_{\text{surface}} \mathbf{E} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{E} dV$$
$$\frac{1}{\epsilon_0} Q_{\text{enc}} = \int_{\text{vol}} \frac{\rho(\mathbf{r})}{\epsilon_0} dV$$

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} \Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

Helmholtz Theorem

$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

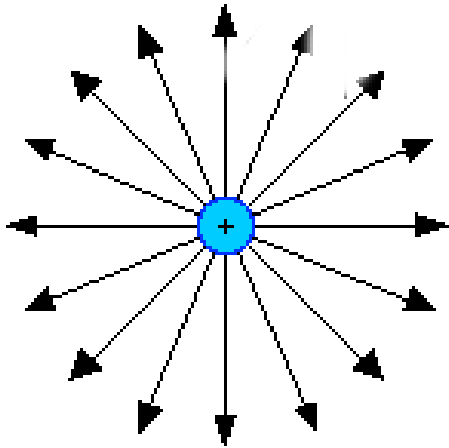
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

Valid for moving charges!

Only for static charges

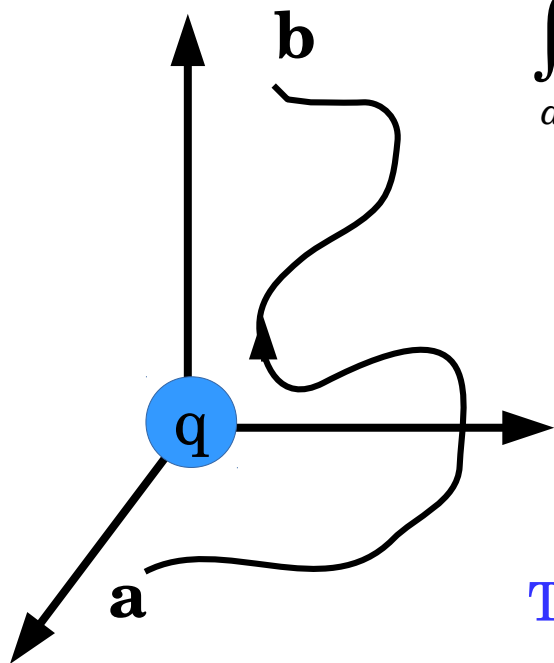
Gauss's Law

CURL OF THE ELECTRIC FIELD



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$\nabla \times \mathbf{E} = 0$$



$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \right) \cdot (dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin\theta d\phi \hat{\boldsymbol{\phi}})$$

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{q}{r^2} dr = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r_a} - \frac{1}{r_b} \right)$$

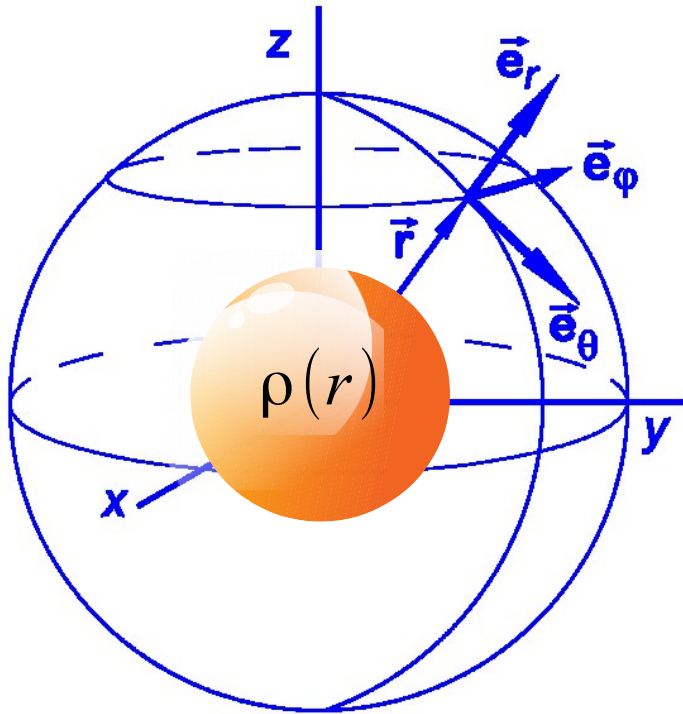
$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \begin{array}{l} \text{Stokes Theorem} \\ \Rightarrow \end{array} \quad \nabla \times \mathbf{E} = 0$$

True for any charge configuration due to superposition!

Valid only for static charges

GAUSS'S LAW + SYMMETRY - SPHERE

Consider a spherically symmetric charge distribution $\rho(\mathbf{r})$



$$E_\phi = 0 \quad \text{Why?}$$

Rotate about the z-axis

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow E_\phi = 0$$

$$E_\theta = 0 \quad \text{Why?}$$

Rotate about the x-axis

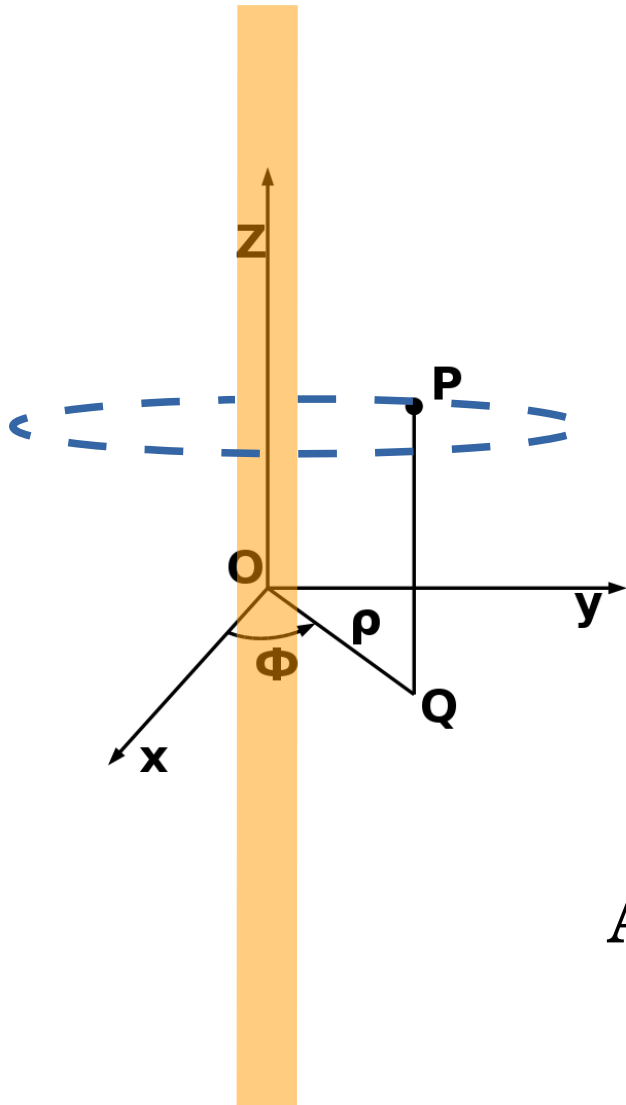
$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow E_\theta = 0$$

Apply Gauss's Law:

$$E_r \cdot 4\pi R^2 = \frac{1}{\epsilon_0} \int_0^R \rho(r) 4\pi r^2 \cdot dr$$

GAUSS'S LAW + SYMMETRY - CYLINDER

Consider a long, narrow wire with a charge per unit length λ



$$E_{\phi} = 0 \quad \text{Why?}$$

Rotate about the z-axis

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow E_{\phi} = 0$$

$$E_z = 0 \quad \text{Why?}$$

Flip about z-axis

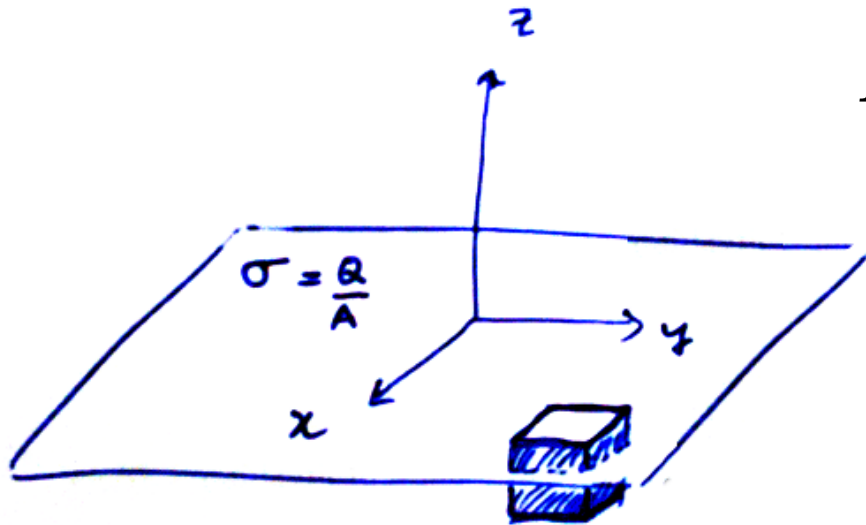
Nothing distinguishes z from -z

$$\Rightarrow E_z = 0$$

Apply Gauss's Law:
$$E_{\rho} \cdot 2\pi\rho = \frac{1}{\epsilon_0} \lambda$$

GAUSS'S LAW + SYMMETRY - SURFACE

Consider an infinite sheet of charge with a surface charge density σ



$$E_{\parallel} (E_x, E_y) = 0 \quad \text{Why?}$$

Rotate the sheet about any point
Translate by any in-plane vector
Field cannot change $\Rightarrow E_{\parallel} = 0$

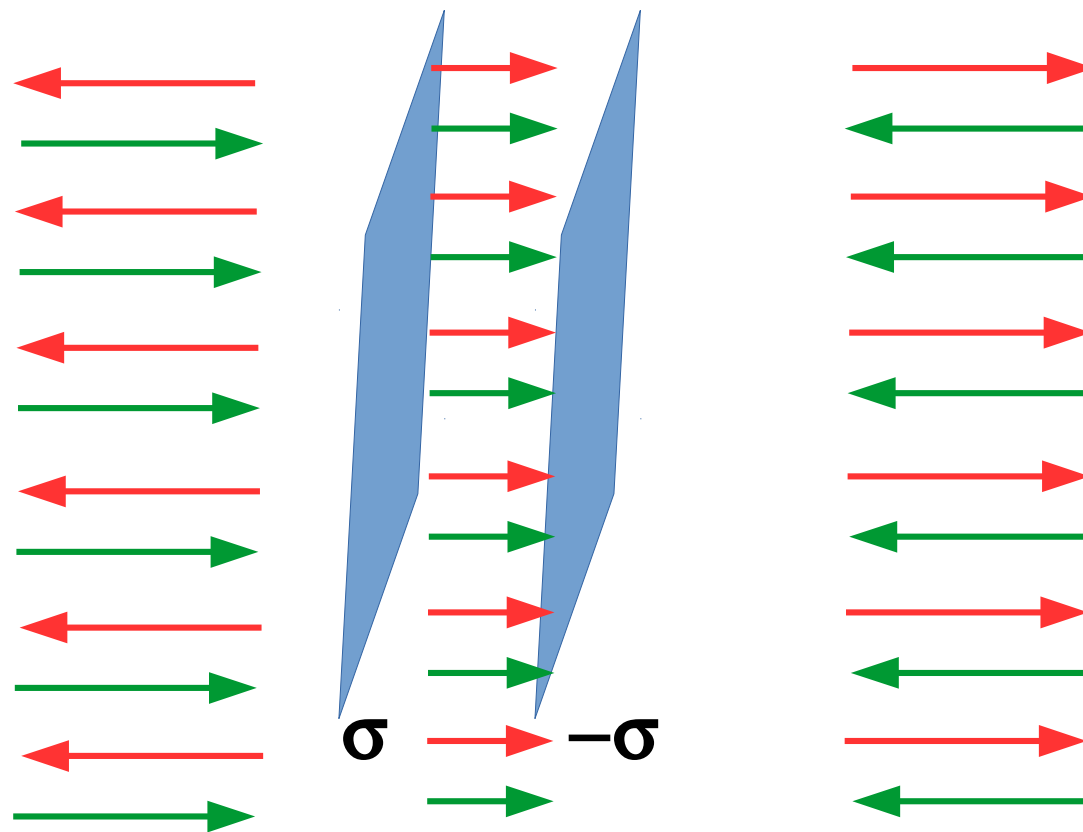
Apply Gauss's Law:

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A$$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}$$

GAUSS'S LAW + SYMMETRY - SURFACE

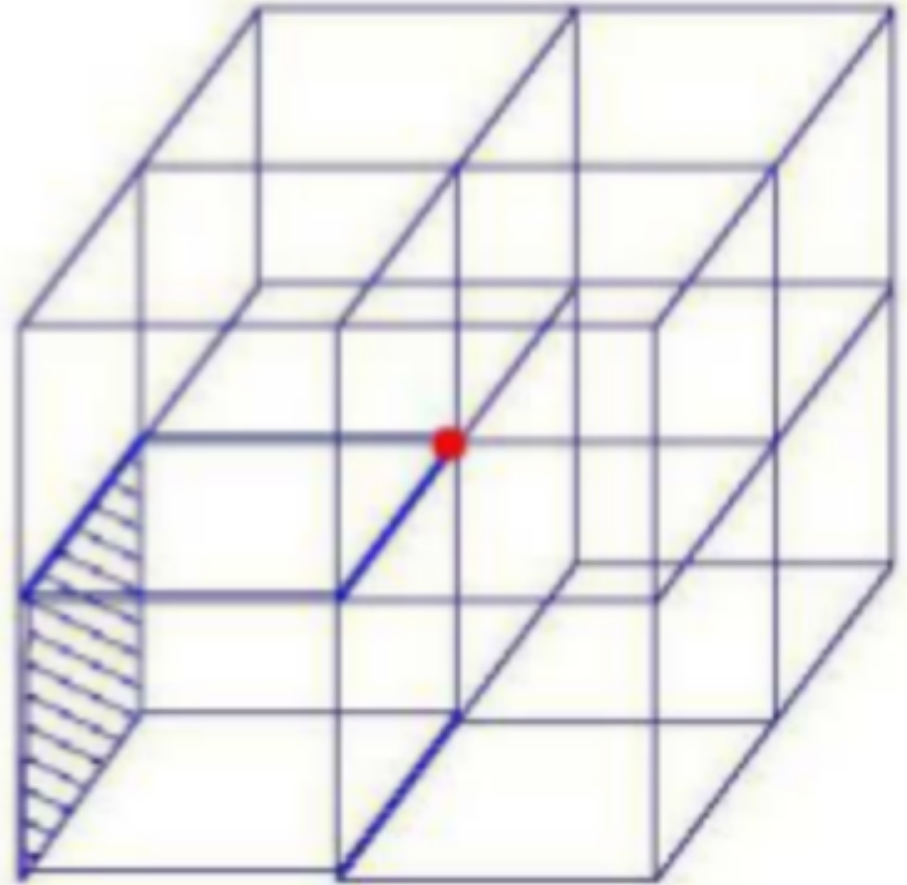
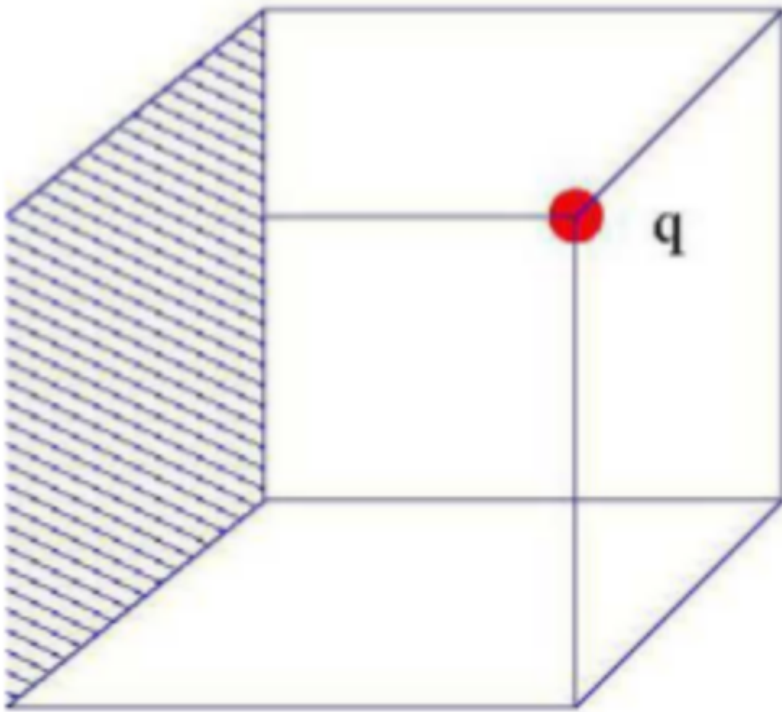
Consider two parallel plates with equal and opposite charge densities $\pm\sigma$. What is the electric field?



$$C = \frac{\epsilon_0 A}{d}$$

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}} + \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}} & \text{between the plates} \\ 0 & \text{everywhere else} \end{cases}$$

GAUSS'S LAW + SYMMETRY



What is the flux of the electric field through the shaded face?

$$\frac{q}{24\epsilon_0}$$

THE ELECTRIC POTENTIAL

Gauss's Law is always true.

It may not always be useful!

If we can take advantage of the symmetries of a problem, Gauss's Law can be a very powerful tool.

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V$$

$V(\mathbf{r}) \equiv$ Electric Potential

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 \quad \Rightarrow \quad \nabla^2 V = -\rho / \epsilon_0 \quad \text{Poisson's Equation}$$

In regions of no charge $\nabla^2 V = 0$ Laplace Equation