A quantum probability explanation for the inverse fallacy

Riccardo Franco

Jerome Busemeyer

Indiana University

Send Correspondence to Riccardo Franco,

riccardo.franco@polito.it

Abstract

This article presents a quantum analysis of the inverse fallacy, which is a well known cognitive heuristic empirically found in intuitive judgments of probabilities. We show that a simple quantum probability formalism, previously used to account for the conjunctive fallacy, also can be used to describe the inverse fallacy. Quantum probabilities are based on the squared magnitudes of inner products between vectors in a vector space. The inverse fallacy follows simply from the fact that the squared magnitude is independent of the direction that one computes the inner product.

1. Introduction

It is now a well established fact that human probability judgments do not always follow the classic rules of probability (Tversky & Kahneman, 1982). Under some special circumstances, human judges are insensitive to base rate information (Kahneman & Tversky, 1972), they judge the conjunction of two events as more likely than one of the constituent events (Tversky & Kahneman, 1983), and they are insensitive to the direction of conditional probabilities (Koehler, 1996; Villejoubert & Mandel, 2002). These violations suggest that human judgments obey a different type of probabilistic logic. Recently, Franco (Franco, 2007) introduced quantum probability as a new formalism for explaining the conjunctive fallacy. The purpose of this article is to apply this quantum explanation to the base rate and inverse fallacy problems.

Why consider a quantum approach? Of course quantum mechanics was originally developed to solve some paradoxical findings in physics. However, later on the theory was organized into a more abstract foundation by Von Neumann (Von Neumann, 1932), who then realized that it provided a new foundation for probability theory [1]. Von Neumann discovered that classic probability theory imposes strong constraints that are not required by quantum probability theory. The explanation for the paradoxical results found in physics required this generalized theory of probability. The same may be true for the paradoxical findings with human probability judgments – classic probability may be too constrained, and the more general quantum probability formalism may be needed to represent the complexity of human judgments under uncertainty.

In this article, we first review Franco's (Franco, 2007) quantum explanation for the conjunctive

[1] This refers to Kolmogorov's (1933; 1956) probability theory, upon which Bayes rule is based.

fallacy. Then, we show how this same explanation applies to the inverse and base rate fallacies. Although this article is focused on probability judgments, we note that another quantum approach has been introduced by Busemeyer, Wang, & Townsend (2006) to describe choice behaviour in the field of decision making.

2. The quantum model of the conjunctive fallacy

The most often cited example of the conjunctive fallacy found by Kahneman and Tversky (1982) is the Linda problem. Judges are initially given a story (S: 'Linda was a philosophy major, she is bright and concerned with issues of discrimination and social justice'). Then they are asked to judge the likelihood of three subsequent events: Event A is 'Linda is active in the Feminist movement,' event B is 'Linda is a bank teller,' and event AB is 'Linda is active in the feminist movement and she is a bank teller.' The major finding is that humans tend to judge event AB to be greater than event B even though the latter contains the former.

It is helpful to first review the classic probability analysis of this judgment problem. Classic probability represents events as sets in a field of sets, assigns probabilities to these sets, and probabilities are real numbers between zero and one. The event B is a set of possible outcomes for Linda and the event ~B is represented by the set complement of B. Similarly, the event A is another set of possible outcomes for Linda, and ~A is the complement of this set. The event AB is the intersection of the two sets A and B: AB = $A \cap B$. Furthermore, the event B can be decomposed into two mutually exclusive and exhaustive subsets: $B = B \cap A \cup B \cap \sim A$. The probability, $Pr[B|S]$, is the probability that Linda is a bank teller given the information provided by the story S. The probabilities, $Pr[B \cap A \mid S]$ and $Pr[B \cap \sim A \mid S]$, are the joint probabilities of the two intersections given the story S. According to classic probability theory, $Pr[B|S] = Pr[B \cap A|S] + Pr[B \cap \sim A|S] \geq Pr[B \cap A|S]$.

For comparison with the quantum model, it is helpful to express the joint probabilities in the product form: Pr[B∩A|S] = Pr[A|S]·Pr[B|A] and Pr[B∩~A|S] = Pr[~A|S]·Pr[B|~A]. Here we added the Markov assumption that Pr[B|A,S] = Pr[B|A] and Pr[B|~A,S] = Pr[B|~A]. Under this Markov assumption, then

$$\Pr[B|S] = \Pr[B|A]\cdot\Pr[A|S] + \Pr[B|\sim A]\cdot\Pr[\sim A|S], \quad (1)$$

$$\Pr[\sim B|S] = \Pr[\sim B|A]\cdot\Pr[A|S] + \Pr[\sim B|\sim A]\cdot\Pr[\sim A|S]. \quad (2)$$

The above two equalities are referred to as the law of total probability in classic probability theory. The conjunction fallacy is also a violation of the law of total probability.

Next we review Franco's (Franco, 2007) quantum model for this judgment problem. Quantum probability represents events by (unit length) vectors in a vector space, forms inner products (called amplitudes) between vectors, and event probabilities equal the squared magnitude of the amplitudes.[2] Franco's (Franco, 2007) quantum model works within a 2 dimensional vector space.[3] The judgment process starts with an initial state vector in this 2-d space, denoted |S⟩, representing the judge's mental state concerning the problem immediately after hearing the story S.[4]

Let's first consider the process when the judge is asked about the joint event AB. In this case, it is assumed that the individual evaluates these two events sequentially, first judging the event A and then judging the event B. The answer about this first event is determined by two vectors: a vector |A⟩ representing an answer of 'true' or 'yes' to the question about A, and another vector |~A⟩ orthogonal to |A⟩, representing an answer of 'false' or 'no' to this question about A. The inner product between |S⟩ and |A⟩ is the amplitude ⟨A|S⟩, and its squared magnitude, $|\langle A|S\rangle|^2$, corresponds to the probability that

[2] More generally, quantum probability defines events as subspaces of a Hilbert space, forms projections onto the subspaces, and determines probabilities by squared length of projections. However for this application, we only need to use subspaces that are rays defined by vectors.

[3] Quantum probability is a very general theory from which one can construct a specific model. Franco (2008) has constructed a simple 2-d model using the principles of this general theory.

[4] We use Dirac notation. The 'ket' symbol |X⟩ corresponds to a column vector; the 'bra' symbol ⟨Y| corresponds to a row vector; the 'bra-ket' symbol ⟨A| S⟩ corresponds to an inner product, which represents the similarity between two vectors.

the answer is 'yes' to A; the inner product between $|S\rangle$ and $|{\sim}A\rangle$ is the amplitude $\langle{\sim}A|S\rangle$, and its squared magnitude, $|\langle{\sim}A|S\rangle|^2$, corresponds to the probability that the answer is 'no' to A. If the answer about the first event is 'yes' to A, then the state of mind changes to $|A\rangle$, but if the answer about the first event is 'no' to A, then the state of mind changes to $|{\sim}A\rangle$. After answering about the first event, the next event is evaluated. The event B corresponds to a vector denoted $|B\rangle$, and the event ~B corresponds to vector $|{\sim}B\rangle$ orthogonal to $|B\rangle$. If the answer about the first event was 'yes' to A, then the probability of answering 'yes' again to B is $|\langle B|A\rangle|^2$, and the probability of answering 'no' to B is $|\langle{\sim}B|A\rangle|^2$. Thus the joint probability of answering yes both times to AB equals the product of the two probabilities, $|\langle B|A\rangle|^2 \cdot |\langle A|S\rangle|^2$. This corresponds exactly with the joint probability $\Pr[A \cap B|S] = \Pr[B|A] \cdot \Pr[A|S]$ from classical probability theory by setting $|\langle A|S\rangle|^2 = \Pr[A|S]$ and $|\langle B|A\rangle|^2 = \Pr[B|A]$.

Finally, consider the process when the judge is asked only about the event B. In this case, the person is not asked to report any judgment about A, and so the question is answered simply by transferring directly from the initial state $|S\rangle$ to either $|B\rangle$ or $|{\sim}B\rangle$. Therefore, the answer is determined by the inner products $\langle B|S\rangle$ and $\langle{\sim}B|S\rangle$: the probability of answering 'yes' to B equals $|\langle B|S\rangle|^2$, and the probability of answering 'no' to B equals $|\langle{\sim}B|\ S\rangle|^2$.

The relation between $|\langle B|S\rangle|^2$ and $|\langle B|A\rangle|^2 \cdot |\langle A|S\rangle|^2$ is the key question, and the answer is obtained by using some simple geometry. Note that the initial state $|S\rangle$ is a vector in a two dimensional space spanned by the two orthonormal vectors $|A\rangle$ and $|{\sim}A\rangle$ that also exist within this same two dimensional space. This allows us to express $|S\rangle$ in terms of its projections, $\langle A|S\rangle$ and $\langle{\sim}A|S\rangle$, on the axes $|A\rangle$ and $|{\sim}A\rangle$ respectively (see Figure 1 which illustrates this)

$$|S\rangle = |A\rangle \cdot \langle A|S\rangle + |{\sim}A\rangle \cdot \langle{\sim}A|S\rangle. \qquad (3)$$

Insert Figure 1 about here

Now we can express the inner products $\langle B|S\rangle$ and $\langle \sim B|S\rangle$ in terms of this new definition for $|S\rangle$:

$$\langle B|S\rangle = \langle B|A\rangle\cdot\langle A|S\rangle + \langle B|\sim A\rangle\cdot\langle \sim A|S\rangle, \tag{4}$$

$$\langle \sim B|S\rangle = \langle \sim B|A\rangle\cdot\langle A|S\rangle + \langle \sim B|\sim A\rangle\cdot\langle \sim A|S\rangle. \tag{5}$$

The above two equalities are called *the law of total amplitude*. They look very similar to the law of total probability (Equations 1 and 2), but there is an important difference. These equations are defined in terms of amplitudes (which in general can be complex numbers with a magnitude less than or equal to one), and the final probability equals the squared amplitude. When we square the amplitude for Equation 4, we obtain[5]

$$\begin{aligned} \Pr[B|S] &= |\langle B|S\rangle|^2 = |\langle B|A\rangle\cdot\langle A|S\rangle + \langle B|\sim A\rangle\cdot\langle \sim A|S\rangle|^2 \\ &= [\langle B|A\rangle\cdot\langle A|S\rangle + \langle B|\sim A\rangle\cdot\langle \sim A|S\rangle]\cdot[\langle B|A\rangle\cdot\langle A|S\rangle + \langle B|\sim A\rangle\cdot\langle \sim A|S\rangle]^* \\ &= |\langle B|A\rangle|^2\cdot\langle A|S\rangle|^2 + |\langle B|\sim A\rangle|^2\cdot|\langle \sim A|S\rangle|^2 \\ &\quad + \langle B|A\rangle\cdot\langle A|S\rangle\cdot\langle B|\sim A\rangle^*\cdot\langle \sim A|S\rangle^* + \langle B|A\rangle^*\cdot\langle A|S\rangle^*\cdot\langle B|\sim A\rangle\cdot\langle \sim A|S\rangle. \\ &= \Pr[B|A]\cdot\Pr[A|S] + \Pr[B|\sim A]\cdot\Pr[\sim A|S] + \textit{Interference}, \end{aligned} \tag{6}$$

where *Interference* = $\langle B|A\rangle\cdot\langle A|S\rangle\cdot\langle B|\sim A\rangle^*\cdot\langle \sim A|S\rangle^* + \langle B|A\rangle^*\cdot\langle A|S\rangle^*\cdot\langle B|\sim A\rangle\cdot\langle \sim A|S\rangle$ can be a positive or negative number. To account for the conjunctive fallacy, Franco (Franco, 2007) assumed that the interference term was negative (see Franco, 2007, for further justification for the interference effect).

One interesting new prediction implied by this quantum theory is that the conjunctive fallacy depends on the order that the two events are processed in the conjunction: the AB order is predicted to produce the fallacy, but the BA order is not. This follows from the fact that with the BA order, a

[5] Recall that for complex numbers $|x|^2 = x\cdot x^*$, where $x = a + b\cdot i$ and $x = a - b\cdot i$, and a and b are real.

judgment of event B occurs first followed by a judgment of event A, and the probability for the joint event equals $|\langle A|B\rangle|^2 \cdot |\langle B|S\rangle|^2 \leq |\langle B|S\rangle|^2$. See Franco (2008) for more discussion about this prediction.

3. The Bayes' rule and the inverse fallacy

A typical task in probability judgements is, given the evidence of a datum A, to estimate the probability of a certain hypothesis B. This conditional probability Pr[B|A] is also called the *posterior probability*. The normative method for calculating the posterior probability is given by the Bayes' rule, which requires the knowledge of: 1) the *likelihoods*, or featural evidences, which are the conditional probabilities Pr[A|B] and Pr[A|~B] assigned to the datum A given the hypothesis B and its negation ~B respectively; 2) the *base rate* or prior probability Pr[B] relevant to the hypothesis B, which expresses the uncertainty about B before the datum is taken into account. It refers to the (base) class probabilities unconditioned on featural evidence. According to Bayesian reasoning, people can only make optimal probability assessments if they take into consideration both the likelihood and the base-rate, through the Bayes' rule:

$$\Pr[B|A]=\Pr[A|B]\ \Pr[B]/\ \Pr[A] \tag{7}$$

where Pr[A] can be computed with the following formula of total probability, corresponding to equation (1). In other words, Bayes' rule tells how to update or revise beliefs in light of new evidence a posteriori. Suppose for example that a clinician knows that: 1) only 5% of the overall population suffers from disease X: this are the base rates Pr[B]=0.05 and Pr[~B] =0.95, 2) 85% of patients who have the disease show the symptom: this is the likelihood Pr[A|B] =0.85, 3) 25% of healthy patients show the symptom: this is the likelihood Pr[A|~B]=0.25. According to Bayes's rule, the posterior probability that a patient who shows the symptom is contaminated by the disease can be calculated as follows: first we use equation (1) to compute Pr[A] =0.85 ×0.05 + 0.25 ×0.95 =0.28, and then the Bayes' rule (7) to obtain Pr[B|A]= 0.85×0.05 / 0.28 =0.15, which means that there is only a 15%

chance that the patient examined has the disease even if he presents a highly diagnostic symptom.

The inverse fallacy (Koehler, 1996; Villejoubert & Mandel, 2002) is the erroneous assumption that Pr[A|B]= Pr[B|A], which is only true in particular situations. It is also called conversion error (Wolfe, 1995), the confusion hypothesis (Macchi, 1995), the Fisherian algorithm (Gigerenzer & Hoffrage, 1995), the conditional probability fallacy and the prosecutor's fallacy (Thompson & Schumann, 1987). In particular, this last name evidences the fact that both expert and non-expert judges often confuse a given conditional probability with its inverse probability. In a different filed (Meehl & Rosen, 1955), it is evidence that clinicians consider that the probability of the presence of a symptom given the diagnosis of a disease is on its own a valid criterion for diagnosing the disease in the presence of the symptom. This result has been experimentally demonstrated in Hammerton (1973) and Liu (1975), where it has been found that the median judgment of Pr[B|A] is almost equal to the presented value of the inverse probability, Pr[A|B]. Similarly,(Eddy, 1982) investigated how physicians estimated the probability that a woman has breast cancer, given a positive result of a mammogram. Approximately 95% of clinicians surveyed gave a numerical answer close to the inverse probability.

4. Quantum explanation of the inverse fallacy

The same two dimensional quantum model described above for the conjunctive fallacy can be used for the inverse fallacy. We no longer focus our attention on the total probabilities Pr[B], Pr[~B], but rather on the conditional probabilities Pr[A|B], Pr[~A|B], Pr[A|B] and Pr[A|~B] (likelihoods). For example (see section 3), if B is the disease and A the symptom, Pr[A|B] is the probability that patients with the disease show the symptom, and in the example of section 3 is equal to 0.85; of course this means that Pr[~A|B] =0.15. The conditional probability Pr[A|B] assumes that B is true. According to classic probability theory, the link between the likelihoods Pr[A|B], Pr[A|~B] and the posterior probabilities Pr[B|A], Pr[~B|A] is given by Bayes' rule, Equation (7).

In the quantum framework the conditional probability of A given B can be obtained by writing the square modulus of the inner product $\langle A|B\rangle$: $Pr[A|B] = |\langle A|B\rangle|^2$. But according to vector space algebra, the inner product $\langle A|B\rangle$ equals the complex conjugate of the inner product $\langle B|A\rangle$, i.e., $\langle B|A\rangle^* = \langle A|B\rangle$. The crucial fact is that the square modulus of a complex number $|X|^2$ is equal to the square modulus of its complex conjugate $|X^*|^2$. Thus we have the identity[6]

$$Pr[B|A] = |\langle B|A\rangle|^2 = |\langle B|A\rangle^*|^2 = |\langle A|B\rangle|^2 = Pr[A|B].$$

Thus the link between posterior probabilities and the likelihoods can be obtained by using the properties of the inner product, leading to

$$\langle B|A\rangle = \langle A|B\rangle^* \rightarrow Pr[B|A]= Pr[A|B], \qquad (8)$$

$$\langle \sim B|A\rangle = \langle A|\sim B\rangle^* \rightarrow Pr[\sim B|A]= Pr[A|\sim B], \qquad (9)$$

and similarly

$$\langle B|\sim A\rangle = \langle \sim A|B\rangle^* \rightarrow Pr[B|\sim A]= Pr[\sim A|B], \qquad (10)$$

$$\langle \sim B|\sim A\rangle = \langle \sim A|\sim B\rangle^* \rightarrow Pr[\sim B|\sim A]= Pr[\sim A|\sim B]. \qquad (11)$$

The last formulas evidence that the quantum formalism *naturally leads* to the inverse fallacy. These have been confirmed in experiments of inversion fallacy, such as (Koehler, 1996; Villejoubert & Mandel, 2002), where the vast majority (80%) of participants provided no more than four Bayesian estimates out of a total of 24 estimates.

5. Failure of Additivity

Classic probability and quantum probability make different predictions concerning the addition of conditional probabilities. According to classic probability theory

---

[6] This identity only holds for the two dimensional representation used by Franco (2008). It does not necessarily hold for higher dimensional quantum models of these judgments.

$$P[B|A] + P[\sim B|A] = 1, \quad (12)$$

$$P[A|B] + P[A|\sim B] \neq 1 \quad (13)$$

Equation 13 can be higher or lower than 1 (except under unusual circumstances when the two conditionals happen to sum exactly to one). The quantum formalism prescribes, that

$$Pr[B|A]+ Pr[\sim B|A] =1, \quad (14)$$

$$Pr[A|B]+ Pr[A|\sim B] =1. \quad (15)$$

Equation (14) follows from the quantum principle that if one is in state $|A\rangle$, and asked about a question about B, then the state must transfer from $|A\rangle$ to either $|B\rangle$ or $|\sim B\rangle$; Equation 15 follows from the properties of inner products described above for the inverse fallacy. Experimental tests of these differential predictions for human judgment are complicated, however, because they depend on the probabilities that the experimenter actually presents to the judge.

It is important to note that in experimental situations, the judge is provided information by the experimenter about some of the probabilities, like for example Pr[A|B] and Pr[A|~B] in (Koehler, 1996; Villejoubert & Mandel, 2002), whose sum can be higher or lower than 1. Now suppose subjects commit inverse fallacy and judge Pr[B|A]= Pr[A|B]. If the probabilities presented by the experimenter satisfy $Pr[A|B] + Pr[A|\sim B]<1$, then the subjective estimates Pr[B|A] and Pr[~B|A] will sum to less than one; similarly if the presented probabilities satisfy $Pr[A|B]+ Pr[A|\sim B] >1$, then the subjective estimates should sum to more than one. In other words, the initial probabilities Pr[A|B] and Pr[A|~B] have been provided by the experimenter as initial data, and have not been estimated by the agents. In this case we say that the given information is not fully consistent with the quantum regime: however, the subjects use such information to judge the conditional probabilities Pr[B|A] and Pr[~B|A], violating the additivity (Equations 12 and 15).

Finally the quantum formalism, through equation (16), leads to the equalities Pr[B|~A] = Pr[~B|A] and Pr[B|A]= Pr[~B|~A]. This can be easily shown, for example in the first case, by noting that

$Pr[B|\sim A]= Pr[\sim A|B]=1-Pr[A|B] =1-Pr[B|A]=Pr[\sim B|A]$. These simple equations, valid only in the quantum regime, provide a new test for evidence that agents in such regime tend to estimate probabilities in a quantum way.

6. Base-rate fallacy

The base rate fallacy (Kahneman & Tversky, 1972), also called base rate neglect, is a logical fallacy that manifests itself as the tendency to disregard base rates (prior probabilities) of events while making conditional probability judgments. This occurs when people feel that the base rate information is not salient to the decision problem (Nisbert, Borgida, Crandall & Reed, 1986). People use base rates only when the base rates have a causal connotation to the event (Kahneman & Tversky, 1972) or when they seem to be more relevant/specific to the event than other given information. In a study done by (Kahneman & Tversky, 1972), subjects were given the following problem: a cab was involved in a hit and run accident at night. Two cab companies, the Green (85%) and the Blue (15%), operate in the city. A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time. What is the probability that the cab involved in the accident was Blue rather than Green? We define the following events: datum A: 'the witness identified the cab as Blue' and hypothesis B: 'it was really a Blue cab'. The correct answer is obtained through the Bayes' rule (7): $Pr[B|A]= 0{,}80\times0{,}15/ (0{,}80\times0{,}15+0{,}20\times0{,}85)=0.41$. Most subjects gave probabilities over 50%, and some gave answers over 80%. According to the base-rate fallacy, this can be explained with the fact that people tend to neglect the base rate, or prior probability, $Pr[B]$.

Some researchers have considered the inverse fallacy as the result of people's tendency to consistently undervalue if not ignore the base-rate information presented as a proxy for prior probabilities: see for example (Dawes, Mirels, Gold & Donahue, 1993). Other researchers, however, have proposed that the base-rate effect was originating from the inverse fallacy and not the reverse (Wolfe, 1995). In support of this notion, in (Wolfe, 1995), Experiment 3, it was found that participants who were trained to distinguish Pr[A|B] from Pr[B|A] were less likely to exhibit base rate neglect compared to a control group. However, experimental deviations from the simple pattern of inversion fallacy evidence that base rates are not completely ignored (Koehler, 1996).

7 . Conclusions

In this article we showed that a quantum model for the conjunctive fallacy also provides a simple explanation of the inverse fallacy which in turn explains the base rate fallacy. This result makes stronger the point of view of quantum cognition, which provides a new explanation for the behaviour of agents in bounded rationality regime and their estimated probabilities, which follow different laws from the classic probability theory.

Further work remains to be done to account for order effects on probability judgments based on several pieces of evidence (e.g., Hogarth & Einhorn, 1992; Wang, 2006). More recent work of (Franco, 2008a) suggests that judges may encode the information and the judgements in more than two dimensions, allowing for more complex quantum algorithms (Franco, 2008b) to explain intuitive probability judgments.

References

Adelman L., Tolcott M.A. & Bresnick T.A., (1993). Examining the Effect of Information Order on Expert Judgment, *Organizational Behavior and Human Decision Processes, 56*, 348-369

Busemeyer, J. R., Wang, Z., & Townsend, J. T. (2006). Quantum dynamics of human decision making. *Journal of Mathematical Psychology, 50*, 220-241.

Dawes, R. M., Mirels, H. L., Gold, E., & Donahue, E. (1993). Equating inverse probabilities in implicit personality judgments. *Psychological Science*, 4, 396-400.

Eddy, D. M. (1982). Probabilistic reasoning in clinical medicine: Problems and opportunities. In D. Kahneman, P. Slovic, & A. Tversky (Eds.), *Judgment under uncertainty: Heuristics and biases* (pp. 249-267). Cambridge: Cambridge University Press.

Kahneman, D., & Tversky, A. (1972). *On prediction and judgment*. ORI Research Monograph, 12.

Koehler, J. J. (1996). The base rate fallacy reconsidered: Descriptive, normative and methodological challenges. *Behavioral & Brain Sciences*, *19*, 1-53.

Kolmogorov, A. (1956) *Foundations of the theory of probability*. NY: Chelsea.

Franco, R. (2007). The conjunction fallacy and interference effects, arXiv:0708.3948v1 [physics.gen-ph], to appear in *Journal of Mathematical Psychology*

Franco R. (2008a), Belief revision in quantum decision theory: gambler’s and hot hand fallacies, 2008, arXiv:0801.4472v1

Franco R. (2008b). Grover’s algorithm and human memory, 2008, arXiv:0804.3294v1 [quant-ph] [physics.gen-ph]

Gigerenzer, G., & Hoffrage, U. (1995). How to improve Bayesian reasoning without instruction: Frequency formats. *Psychological Review, 102*, 684704.

Hammerton, M. (1973). A case of radical probability estimation. Journal *of Experimental Psychology, 101,* 252-254.

Hogarth R.M. & Einhorn H. J., (1992). Order effects in belief updating: The belief adjustment model, Cognitive Psychology, 24, 1-55

Liu, A. Y. (1975). Specific information effect in probability estimation. *Perceptual & Motor Skills, 41,* 475-478.

Macchi, L. (1995). Pragmatic aspects of the base rate fallacy. The *Quarterly Journal of Experimental Psychology, 48A*, 188-207.

Markoczy L. & Goldberg J., (1998). Women and taxis and dangerous judgments: Content sensitive use of base-rate information, *Managerial and Decision Economics, 19* (7/8), 481493

Meehl, P., & Rosen, A. (1955). Antecedent probability and the efficiency of psychometric signs of patterns, or cutting scores. *Psychological Bulletin, 52,* 194-215.

Nisbert R., Borgida E., Crandall R. & Reed H., (1986). Popular induction: Information is not necessarily informative, *Judgement under Uncertainty: Heuristics and Biases*, Cambridge University Press, 101-128.

Thompson W. C. & Schumann E. L. (1987). Interpretation of Statistical Evidence in Criminal Trials: The Prosecutor's Fallacy and the Defense Attorney's Fallacy, *Law and Human Behavior, Vol. 11*, No. 3 (Sep., 1987), pp. 167-187

Tubbs R., Gaeth G.J., Levin I.P. & Van Osdol, L. A., (1993). Belief updating with consistent and inconsistent evidence, *Journal of Behavioral Decision Making, 6*, 257-269

Tversky, A. & Kahneman, D. (1982). Judgments of and by representativeness. In D. Kahneman, P. Slovic and A. Tversky (Eds.), *Judgment under uncertainty: Heuristics and biases*. Cambridge, UK: Cambridge UniversityPress.

Tversky A. & Kahneman D., (1983). Extension versus intuititve reasoning: The conjunction fallacy in probability judgment. *Psychological Review, 90*, 293315.

Villejoubert, G., & Mandel, D. R. (2002). The inverse fallacy: An account of deviations from Bayes's Theorem and the additivity principle. *Memory & Cognition, 30*, 171-178.

Von Neumann, J. (1932). *Mathematical Foundations of Quantum Mechanics*, Beyer, R. T., trans., Princeton Univ. Press.

Wang P., (2006). The order effect in human abductive reasoning: an empirical and computational study. *Journal of experimental and theoretical artificial intelligence, 18* (2), 215- 247.

Wolfe, C. R. (1995). Information seeking on Bayesian conditional probability problems: A fuzzy-trace theory account. *Journal of Behavioral Decision Making, 8*, 85-108.

Figure 1: Representation of vector |S⟩ in terms of vectors |A⟩ and |~A⟩. Note that this figure uses only real valued vectors, but quantum theory also allows complex valued vectors.