An Exploration on Non-regular Rado Sequences

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Abstract

In this paper, we discuss the existence of monochromatic solutions to linear equations in different colorings of the natural numbers. We improve the upper bounds for the least number of colors needed to construct colorings with no monochromatic solution to equations represented by some non-regular sequences.

1 Introduction

We first introduce the notion of a regular sequence defined by Richard Rado in [1].

Definition 1. *Given a sequence of non-zero integers $a = (a_i)_{i=1}^n$, a is regular if for every $c \in \mathbb{N}$, every c-coloring of $\mathbb{N}$ has a monochromatic solution to*

$$a_1x_1 + \cdots + a_nx_n = 0$$

For a non-regular sequence $a = (a_i)_{i=1}^n$, we define an integer c_a such that there exists a c_a-coloring of $\mathbb{N}$ with no monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$, while every $(c_a - 1)$-coloring of $\mathbb{N}$ has a monochromatic solution. In other words, c_a is the minimum number of colors needed to construct a coloring of $\mathbb{N}$ that has no monochromatic solutions to the equation represented by a.

For a proof of the existence of an upper bound on c_a, consult the book by Gasarch, Kruskal, and Parrish.[2] The proof shows that c_a can be upper bounded by the smallest prime number p where $p > \sum_{i=0}^n |a_i|$. This bound has a complexity of $\Omega(\sum_{i=0}^n |a_i|)$. In this paper, we improve the bound for non-regular sequences of length less than 4.

This paper is organized as follows. In Section 2, we present some assumptions we can make on non-regular sequences based on theorems that are proved in the past. In Section 3, we prove that $c_a = 2$ for every non-regular sequence a of length 2. In Section 4, we prove a constant upper bound for a special family of sequences of length 3: $(1, 1, -k)$, where $k \in \mathbb{N}$. In Section 5, we extend the technique we use in Section 4 to prove a logarithmic bound for generic non-regular sequences of length 3 under some conditions.

2 Preliminaries

In this section, we introduce some existing theorems and notations that we use throughout this paper.

Theorem 1. *(Rado's theorem). Given a sequence of nonzero integers $a = (a_i)_{i=1}^n$, the following are equivalent:*

- *a is regular*
- *there exists a subset of a that sums to 0*

Corollary 1. *A sequence of nonzero integers $a = (a_i)_{i=1}^n$ is non-regular if and only if none of the subsets of a sums to 0.*

Definition 2. *A sequence of nonzero integers $a = (a_i)_{i=1}^n$ is trivial if there is no positive integer solution to*

$$a_1x_1 + \cdots + a_nx_n = 0$$

Since every coloring of natural numbers for a trivial sequence has no monochromatic solution, we only focus on colorings of nontrivial sequences in this paper.

We also assume that the solutions to any linear equation are unordered solutions, that is, if $(x_1, ..., x_n)$ is a solution to $a_1x_1 + \cdots + a_nx_n = 0$, then every permutation of $(x_1, ..., x_n)$ is a solution to the same equation.

3 Sequences of Length 2

We want to show that for every non-regular sequence $a = (a_1, a_2)$, $c_a = 2$. Before we prove this, we first need to prove a lemma that allows us to use some assumptions on the non-regular sequences of length 2.

Lemma 1. *For every non-regular sequence of $a = (a_1, a_2)$, there exist co-prime integers b_1, b_2, where $b_1 > b_2 > 0$, such that the solutions to $a_1x_1 + a_2x_2 = 0$ are the same as the solutions to $b_1x_1 = b_2x_2$.*

Proof. Since we assume that a is nontrivial, a_1 and a_2 must have different signs so that $a_1x_1 + a_2x_2 = 0$ has solutions. Therefore, $a_1x_1 + a_2x_2 = 0$ has the same solutions as

$$|a_1|x = |a_2|y$$

By symmetry of this equation, we can assume that $|a_1| > |a_2|$ without loss of loss of generality.

Let d be the great common divisor of $|a_1|$ and $|a_2|$. we make $b_1 = \frac{|a_1|}{d}, b_2 = \frac{a_2}{d}$. b_1 and b_2 are co-prime and $b_1 > b_2 > 0$.

□

Theorem 2. *For all $a, b \in \mathbb{N}$, if a and b are co-prime and $a > b$, there exists a 2-coloring of $\mathbb{N}]$ where $ax = by$ has no monochromatic solution.*

Proof. We propose the following 2-coloring:

$$COL(a^e r) = e \bmod 2$$

where e is the largest integer exponent possible.

Suppose for the sake of contradiction that there exists a monochromatic solution (x_1, x_2) that satisfies $ax = by$.

Let $x = ae_1{}^{r_1}$, $y = ae_2{}^{r_2}$.

Now we can rewrite $ax = by$ as the following:

$$a_1 \cdot a^{e_1}r_1 = b \cdot a^{e_2}r_2$$

We do a case analysis on the relationship between e_1 and e_2.

Case 1: $e_1 + 1 > e_2$:

Since $a, e_2 > 0$ we can divide both sides by a^{e_2}. Now we get

$$a^{e_1+1-e_2}r_1 = br_2$$

r_2 is not divisible by a because e_2 is the largest integer exponent. a_2 is co-prime with a, so a_2r_2 is not divisible by a, which contradicts with the left hand side of this equality.

Case 2: $e_1 + 1 < e_2$:

Since $a, e_1 + 1 > 0$ divide both sides by a^{e_1+1}, now we get

$$r_1 = ba^{e_2-e_1-1}r_2$$

Since r_1 is not divisible by a, the left hand side is not divisible by a, which contradicts with right hand side of this equality.

Case 3: $e_1 + 1 = e_2$:

Since $e_1 \equiv e_2 \mod 2$, this is a contradiction.

Therefore, our proposed coloring gives no monochromatic solution to $ax = by$. □

Corollary 2. *For every non-regular sequence $a = (a_1, a_2)$, $c_a = 2$.*

4 Special Sequences of Length 3

In this section, we investigate a special family of non-regular sequences of Length 3: $(1, 1 - k)$, where $k \geq 4$.

4.1 2-coloring $\mathbb{N}$ for $(1, 1, -k)$

4.1.1 An Upper Bound

We prove that for every integer $k \geq 4$, $c_{(1,1,-k)} \geq 2$ by deriving an upper bound on how many numbers we need to color to guarantee that every 2-coloring has monochromatic solution.

Theorem 3. *For every $k \geq 4, k \in \mathbb{Z}$, every 2-coloring of $[k^3 - 4k^2 + 5k]$ has monochromatic solution to $x + y - kz = 0$.*

Proof. Suppose for the sake of contradiction that there is a 2-coloring of $[k^3 - 4k^2 + 5k]$, which we denote as COL, where there is no monochromatic solution to $[k^3 - 4k^2 + 5k]$.

First we want to show that if there is no monochromatic solution, then $COL(k) = COL(k^3 - 4k^2 + 5k)$.

Since $k + k = k \cdot 2$, $(k, k, 2)$ is a solution. Then $COL(2) \neq COL(k)$ because there is no monochromatic solution.

Since $(2k - 2) + 2 = k \cdot 2$, $(2k - 2, 2, 2)$ is a solution. Then $COL(2) \neq COL(2k - 2)$. We can also conclude that $COL(k) = COL(2k - 2)$ because there are only 2 colors.

Since $(2k^2 - 3k) + k = k \cdot (2k - 2)$, $(2k^2 - 3k, k, 2k - 2)$ is a solution. Then $COL(2k^2 - 3k) \neq COL(k)$
Since $(k^2 - 2k + 2) + (2k - 2) = k \cdot k$, $(k^2 - 2k + 2, 2k - 2, k)$ is a solution. Then $COL(k^2 - 2k + 2) \neq COL(k)$. Similarly, we can conclude $COL(k^2 - 2k + 2) = COL(2k^2 - 3k)$.

Finally, $(k^3 - 4k^2 + 5k) + (2k^2 - 3k) = k \cdot (k^2 - 2k + 2)$, so $(k^3 - 4k^2 + 5k, 2k^2 - 3k, k^2 - 2k + 2)$ is a solution. Therefore, $COL(k^3 - 4k^2 + 5k) \neq COL(2k^2 - 3k)$ and $COL(k^3 - 4k^2 + 5k) = COL(k)$.

Next, we want to show that $COL(ak^2 - (2a - 1)k) \neq COL(k)$ for every $a \in \mathbb{N}$. We proceed by induction on a.

Base case: When $a = 1$, $COL(k^2 - k) \neq COL(k)$ because $(k^2 - k, k, k)$ is a solution.

Inductive case: Suppose $COL(ak^2 - (2a - 1)k) \neq COL(k)$, we want to show that $COL((a + 1)k^2 - (2a + 1)k) \neq COL(k)$.

First we know that $COL(ak^2 - (2a - 1)k) = COL(k^2 - k)$ because there are only 2 colors.

Since $ak^2 - (2a - 1)k + k^2 - k = k \cdot ((a + 1)k - 2a)$,

$(ak^2 - (2a - 1)k, k^2 - k, (a + 1)k - 2a)$ is a solution. Then $COL((a + 1)k - 2a) \neq COL(k^2 - k)$ and $COL((a + 1)k - 2a) = COL(k)$.

Since $(a + 1)k^2 - (2a + 1)k + k = k \cdot ((a + 1)k - 2a)$, $(a + 1)k^2 - (2a + 1)k, k, (a + 1)k - 2a)$ is a solution. Then $COL((a + 1)k^2 - (2a + 1)k) \neq COL(k)$. This proves our inductive hypothesis.

Since $k \geq 4$, $k - 2 \in \mathbb{N}$.

$$COL((k - 2)k^2 - (2k - 5)k) = COL(k^3 - 4k^2 + 5k) \neq COL(k)$$

This contradicts a statement we proved earlier in this proof. Therefore, the Theorem holds.

□

4.1.2 A Lower Bound

As an extension, we also have a lower bound on the least number of numbers we need to color to guarantee that every 2-coloring has monochromatic solution.

Theorem 4. *For every $k \geq 4, k \in \mathbb{Z}$, there exists a 2-coloring of $[\left\lceil \frac{(\lceil \frac{k}{2} \rceil)k}{2} \right\rceil - 1]$ that has no monochromatic solution to $x + y - kz = 0$.*

Proof. We show that such a coloring exists by presenting the following coloring and proving that there is no monochromatic solution to $x + y - kz = 0$

$$COL(i) = \begin{cases} RED & \text{if } i \leq \left\lceil \frac{k}{2} \right\rceil - 1 \\ BLUE & \text{otherwise} \end{cases}$$

First we want to show that there cannot be any red monochromatic solution, which means there is no solution where x, y, z are all in range $[1, \lceil \frac{k}{2} \rceil - 1]$.

Since $z \geq 1$, $x + y \geq k$. Then either x or y must be at least $\frac{k}{2}$, which contradicts the fact that both of them must be at most $\lceil \frac{k}{2} \rceil - 1$

Next, we want to show that there cannot be any blue monochromatic solution, which means there is no solution where x, y, z are all in range $[\lceil \frac{k}{2} \rceil, \left\lceil \frac{(\lceil \frac{k}{2} \rceil)k}{2} \right\rceil - 1]$.

Since $z \geq \lceil \frac{k}{2} \rceil$, $x + y \geq \lceil \frac{k}{2} \rceil k$. Then either x or y must be at least $\frac{(\lceil \frac{k}{2} \rceil)k}{2}$, which contradicts the fact that both of them must be at most $\left\lceil \frac{(\lceil \frac{k}{2} \rceil)k}{2} \right\rceil - 1$ numbers. □

Corollary 3. *For every integer $k \geq 4$, let equation A be $x + y - kz = 0$,*

- *For every 2-coloring of $[k^3 - 4k^2 + 5k]$, there is a monochromatic solution to A.*
- *There exists a 2-coloring of $[\left\lceil \frac{(\lceil \frac{k}{2} \rceil)k}{2} \right\rceil - 1]$ that has no monochromatic solution to A.*

4.2 3-coloring $\mathbb{N}$ for $(1, 1, -k)$

We show that for every $k \geq 4$, $c_{(1,1,-k)} \leq 3$ by constructing a 3-coloring that has no monochromatic solutions to $x + y = kz$.

Theorem 5. *For every integer $k \geq 4$, there is a 3-coloring of $\mathbb{N}$ that has no monochromatic solution to $x + y - kz = 0$.*

Proof. We propose the following 3-coloring of $\mathbb{N}$:

$$COL(x) = COL\left(\left(\frac{k}{2}\right)^e r\right) = \begin{cases} red & \text{if } e \bmod 3 = 0 \\ green & \text{if } e \bmod 3 = 1 \\ blue & \text{if } e \bmod 3 = 2 \end{cases}$$

where r is a real number in $[1, \frac{k}{2})$, and e is largest integer possible.

In any solution (x, y, z), $\frac{kz}{2} \leq max(x, y) < kz$

Let $z = \left(\frac{k}{2}\right)^a b$. Then $\left(\frac{k}{2}\right) z = \left(\frac{k}{2}\right)^{a+1} b$ and $kz \leq \left(\frac{k}{2}\right)^{a+2} b$ because $k \leq \frac{k^2}{4}$ for every integer $k \geq 4$.

Based on the proposed coloring, any number between $\frac{kz}{2}$ and kz cannot be in the same color as z. Therefore, there cannot be a monochromatic solution.

□

Corollary 4. *For every integer $k \geq 4$, $c_{(1,1,-k)} = 3$.*

5 General Sequences of Length 3

Now we look at more generic cases of non-regular sequences of length 3. In this section, we find better upper bounds on c_a given certain conditions.

Before we show the bounds, we first go over some assumptions we can make without loss of generality.

In any nontrivial sequence $a = (a_1, a_2, a_3)$, a_1, a_2, and a_3 cannot all have the same sign, which means two of them have the same sign and the other one has the opposite sign. Therefore, $a_1x_1+a_2x_2+a_3x_3 = 0$ has the same solution as

$$|a_1|x + |a_2|y = |a_3|z$$

Now we can impose some conditions on the relationship between a_1, a_2, and a_3.

5.1 $|a_3| = \gamma(|a_1| + |a_2|)$ for some constant $\gamma > 1$

Theorem 6. *For all $a, b, c \in \mathbb{N}$ where $c = \gamma(a + b)$ for some known constant $\gamma > 1$, there exists a $O(\log(c))$-coloring such that $ax + by = cz$ has no monochromatic solution.*

Proof. We propose the following ($\lceil\log(c)\rceil$+1)-coloring of $\mathbb{N}$:

$$COL(x) = COL\left(\gamma^e r\right) = e \bmod (\lceil\log(c)\rceil + 1)$$

where e is the largest integer possible to make r be a real number in range $[1, \gamma)$.

In any solution (x, y, z), $\frac{c}{a+b}z \leq max(x, y) < cz$.

Based on the proposed coloring, any number between γz and $\gamma^{(\lceil\log(c)\rceil+1-1)}z$ cannot be in the same color as z.

$$\gamma^{(\lceil\log(c)\rceil+1-1)}z = \gamma^{\lceil\log(c)\rceil}z = \gamma cz > cz$$

Since $\gamma = \frac{c}{a+b}$, any number between $\frac{c}{a+b}z$ and cz cannot be in the same color as z. Therefore, there is no monochromatic solution.

□

5.2 $min(|a_1|, |a_2|) = \gamma|a_3|$ for some constant $\gamma > 1$

Similarly, we use the same approach and get a similar bound for c_a when $min(|a_1|, |a_2|) > |a_3|$.

Theorem 7. *For all $a, b, c \in \mathbb{N}$ where $min(a, b) = \gamma c$ for some known constant $\gamma > 1$, there exists a $O(\log(a + b))$-coloring such that $ax + by = cz$ has no monochromatic solution.*

Proof. We propose the following ($\lceil\log(a + b)\rceil$+1)-coloring of $\mathbb{N}$:

$$COL(x) = COL\left(\gamma^e r\right) = e \bmod (\lceil\log(c)\rceil + 1)$$

where e is the largest integer possible to make r be a real number in range $[1, \gamma)$.

In any solution (x, y, z), $\frac{min(a,b)}{c}max(x, y) \leq z < (a + b)max(x, y)$.

Based on the proposed coloring, any number between $\gamma max(x, y)$ and $\gamma^{(\lceil\log(a+b)\rceil+1-1)}max(x, y)$ cannot be in the same color as $max(x, y)$.

$$\gamma^{(\lceil\log(a+b)\rceil+1-1)}max(x, y) = \gamma^{\lceil\log(a+b)\rceil}z = \gamma(a + b)max(x, y) > (a + b)max(x, y)$$

Since $\gamma = \frac{min(a,b)}{c}$, any number between $\frac{min(a,b)}{c}max(x, y)$ and $(a + b)max(x, y)$ cannot be in the same color as z. Therefore, there is no monochromatic solution. □

With these extra assumptions, we improve the linear upper bounds for c_a with logarithmic upper bounds for non-regular sequences of length 3 where $|a_3|$ in not in range $[min(|a_1|, |a_2|), |a_1| + |a_2|]$.

Corollary 5. *For all non-regular sequence $a = a_1, a_2, a_3$, c_a has an upper bound of $\lceil\log(\sum_{i=1}^{3}|a_i|)\rceil + 1$ if the one of the following is true:*

1. *$|a_3| = \gamma(|a_1| + |a_2|)$ for some constant $\gamma > 1$*
2. *$min(|a_1|, |a_2|) = \gamma|a_3|$ for some constant $\gamma > 1$*

6 Future Work

In Section 3, there is a gap between the lower bound and the upper bound for the least number of natural numbers needed to guarantee a monochromatic solution in every 2-coloring. A tight asymptotic bound may be obtained by further examining the coloring patterns.‘ In Section 5, there is a remaining case where c is in the range of $(b, a+b)$ that is not solved.

In addition, it would be interesting to study the same typ of bounds for linear equations where the monochromatic solutions of interest must have distinct numbers.

References

[1] Richard Rado. Notes on combinatorial analysis. *Proceedings of the London Math Society*, 48:122–160, 1943. `http://www.cs.umd.edu/~gasarch/vdw/vdw.html`. Includes Gallai's theorem and credits him.

[2] William Gasarch, Clyde Kruskal, and Andy Parrish. Van der waerden's theorem: Variants and applications, 2012. `https://www.cs.umd.edu/users/gasarch/TOPICS/vdw/GKPbook.pdf`.