Homework 6

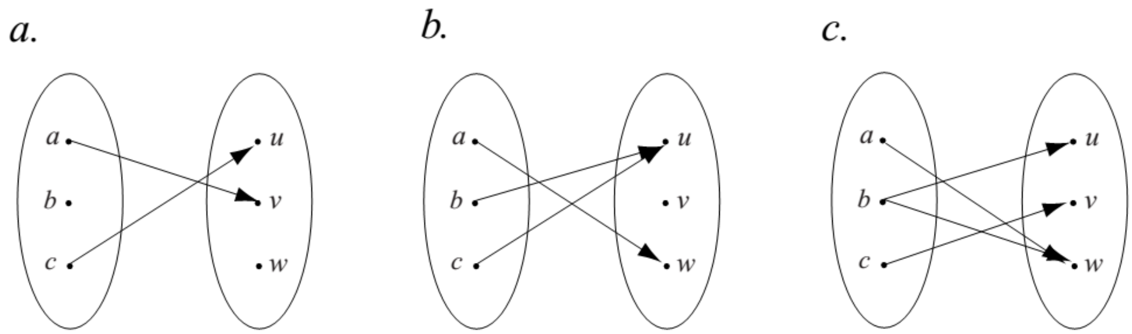
Due on Tuesday, 6/13, 1:15 PM in class

Name________________________________ PID________________________________

Honor Code Pledge: I certify that I am aware of the Honor Code in effect in this course and observed the Honor Code in the completion of this homework.

Signature_____________________________

(15') 1. Let $X = \{a, b, c\}$ and $Y = \{u, v, w\}$. Determine whether each of the following arrow diagrams defines a function from X to Y, and explain your answers in a few words.



Solution:

(a) No, because b is an element in X and is not related to any element in Y.

(b) Yes, because every element in X is related to exact one element in Y.

(c) No, because b is an element in X and is related to two element in Y, namely u and w.

(20') 2. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{u, v, w, x, y\}$, and define $h: X \to Y$ as follows:

$$h(1) = v,\ h(2) = x,\ h(3) = v,\ h(4) = v,\ h(5) = y.$$

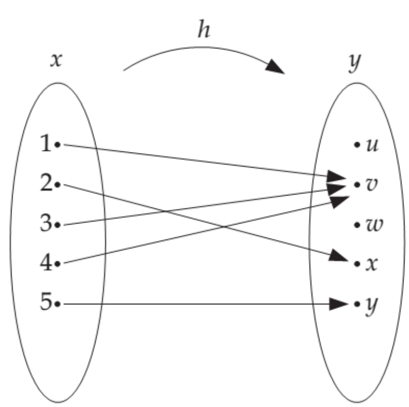
(a) Draw an arrow diagram for h.

(b) Let $A = \{1, 2\}$, $C = \{x, v\}$, $D = \{w\}$, and $E = \{w, y\}$. Find

$$h(A),\ h(X),\ h^{-1}(C),\ h^{-1}(D),\ h^{-1}(E), \text{ and } h^{-1}(Y).$$

Solution:

(a)



(b) $h(A) = \{v,x\}$, $h(X) = \{v,x,y\}$, $h^{-1}(C) = \{1,2,3,4\}$,
$h^{-1}(D) = \emptyset$, $h^{-1}(E) = \{5\}$, $h^{-1}(Y) = \{1,2,3,4,5\}$.

(15') 3. Let S be the set of all strings in 0's and 1's, and define a function $f : S \rightarrow \mathbf{Z}$ as follows:

for each string s in S, $f(s)$ = the number of 0's in s.

(a) What is $f(101011)$? $f(00100)$?

(b) Is f injective? Prove or give a counterexample.

(c) Is f surjective? Prove or give a counterexample.

Solution:

(a) $f(101011) = 2$; $f(00100) = 4$.

(b) No. Counterexample: $f(100) = f(101011)=2$ but string 100 ≠ string 101011.

(c) No. Counterexample: -1 is an element in the co-domain **Z**, but no string can have the number of 0's as -1.

(20') 4. Let S be the set of all strings in 0's and 1's, and define a function $g : S \rightarrow \mathbf{Z}^+ \cup \{0\}$ as follows: (Note that $\mathbf{Z}^+$ denotes the set of all positive integers, so $\mathbf{Z}^+ \cup \{0\}$ denotes the set of all non-negative integers.)

for all strings s in S, $g(s)$ = the number of 1's in s.

(a) What is $g(001000)$? $g(111001)$? $g(10101)$? $g(0100)$?

(b) Is g injective? Prove or give a counterexample.

(c) Is g surjective? Prove or give a counterexample.

(d) Is g a bijection? If so, find g^{-1}.

Solution:

(a) $g(001000)=1$; $g(111001)=4$; $g(10101)=3$; $g(0100)=1$.

(b) No. Counterexample: $g(001000) = g(0100) = 1$ but string 001000 ≠ string 0100.

(c) Yes. *Proof*: Let n denote a particular but arbitrarily chosen element in the co-domain $\mathbf{Z}^+ \cup \{0\}$. Then, by the definition of $\mathbf{Z}^+ \cup \{0\}$, n is a nonnegative integer. We consider the string x of length $n+1$ that starts with a 0, which is followed by n consecutive 1's. By the definition of g, $g(x) = n$. That is, n has at least one preimage in the domain S. Therefore, g is surjective.

(d) No, because g is not injective.

(20') 5. Define $F : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$ as follows: $F(x, y) = (3y - 1, 1 - x)$ for all (x, y) in $\mathbf{R} \times \mathbf{R}$.

(a) $F(0, 0) = ?$ $F(1, 4) = ?$

(b) Is F injective? Prove or give a counterexample.

(c) Is F surjective? Prove or give a counterexample.

(d) Is F a bijection? If not, explain why not. If yes, find F^{-1}.

Solution:

(a) $F(0, 0) = (-1, 1)$; $F(1, 4) = (11, 0)$.

(b) Yes. *Proof*: Suppose (x_1, y_1) and (x_2, y_2) are two elements in the domain such that $F(x_1, y_1) = F(x_2, y_2)$. By the definition of F, we know $3y_1 - 1 = 3y_2 - 1$ and $1 - x_1 = 1 - x_2$, from which, by basic algebra, we can conclude that $x_1 = x_2$ and $y_1 = y_2$. That is, $(x_1, y_1) = (x_2, y_2)$.

(c) Yes. Suppose (u, v) is a particular but arbitrarily chose element in the co-domain. Then, u and v are both real numbers. Let x = 1-v and y = (u+1)/3. Then, x and y are also both real numbers. So, (x, y) is in the domain. By the definition of F, we have

$$\mathrm{F}(x, y) = \mathrm{F}\left(1 - v, \frac{u+1}{3}\right) = \left(3 \cdot \frac{u+1}{3} - 1, 1 - (1 - v)\right) = (u, v).$$

(d) Yes, because F is both injective and surjective. By the steps in (c), we can find the inverse function

$$\mathrm{F}^{-1}(u, v) = \left(1 - v, \frac{u+1}{3}\right).$$

(10') Let f : **R** → **R** and g : **R** → **R** are two functions defined as follows:

$f(x) = x - 1$; $g(x) = x^2 - 1$.

Then, define $F(x) = f(g(x))$ and $G(x) = g(f(x))$.

(a) What is $F(2)$? $G(2)$?

(b) Write explicit expressions for $F(x)$ and $G(x)$. Simplify the results as much as you can.

Solution:

(a) $F(2) = f(g(2)) = f(3) = 2$; $G(2) = g(f(2)) = g(1) = 0$.

(b) $F(x) = f(g(x)) = f(x^2 - 1) = x^2 - 1 - 1 = x^2 - 2$;

$G(x) = g(f(x)) = g(x - 1) = (x - 1)^2 - 1 = x^2 - 2x$.