DAV University Jalandhar

Department of Physics

Study Material: PHY 330A Electromagnetic Theory

B.Sc (Hons.) Physics 6^{th} sem

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BOOK: Electrodynamics by D J Grifiths, Optics & Lasers By Subramaniam

Modern Optics
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Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 08
Wave propagation in anisotropic media

Now, we will discuss the propagation of electromagnetic waves in anisotropic media. In anisotropic media various interesting phenomena happen, and it is very interesting in the sense that most of the devices that we will be discussing in this course, will be around this anisotropic behavior of the medium through which, the electromagnetic waves will be travelling. We have seen the isotropic medium where the electric field and the displacement vector there in the same direction.

The other aspects of isotropic medium that also we have seen in general the propagation characteristics with that background in mind, we will now look into the various different aspects of the electric field, magnetic field, the displacement vector their relation their orientation in this particular discussion.

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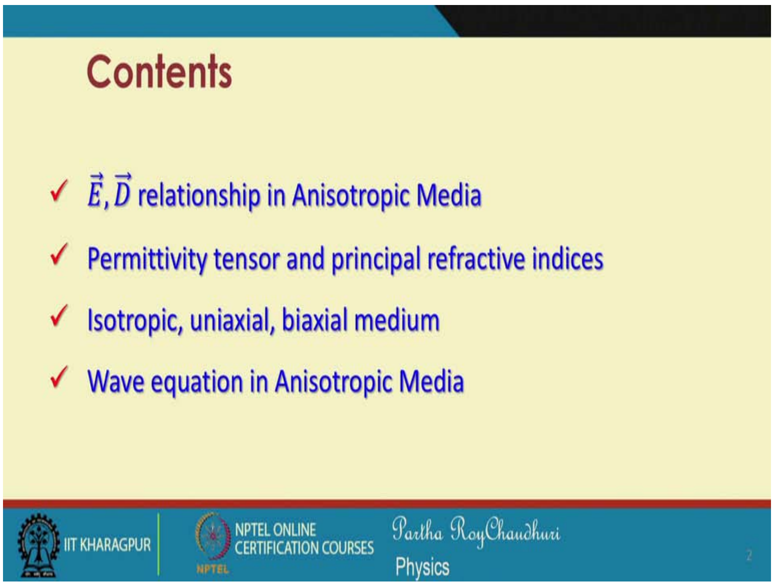

So, the content goes like this, first we look at the E D that is, electrical field and the displacement vector the relationship in a general anisotropic media. And from there we will try to organize the permittivity tensor in a very special case the permittivity tensor will correspond to a tensor, which will which will indicate only the principal refractive indices. Then will make a comparison of the isotropic Uniaxial and Biaxial medium we will defined the various properties of this. Then at the end we will try to bring in the wave equation for anisotropic medium.

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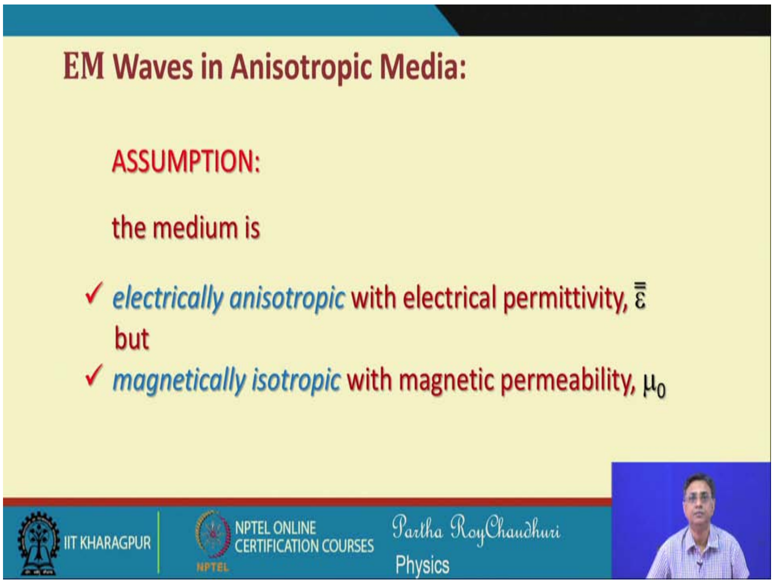

So, in the case of a general anisotropic medium in this discussion we will first make these assumptions that; the medium is electrically isotropic with electrical permittivity given by this epsilon bar that is a tensor, but the medium is magnetically isotropic with magnetic permeability, mu naught which is the free space permeability.

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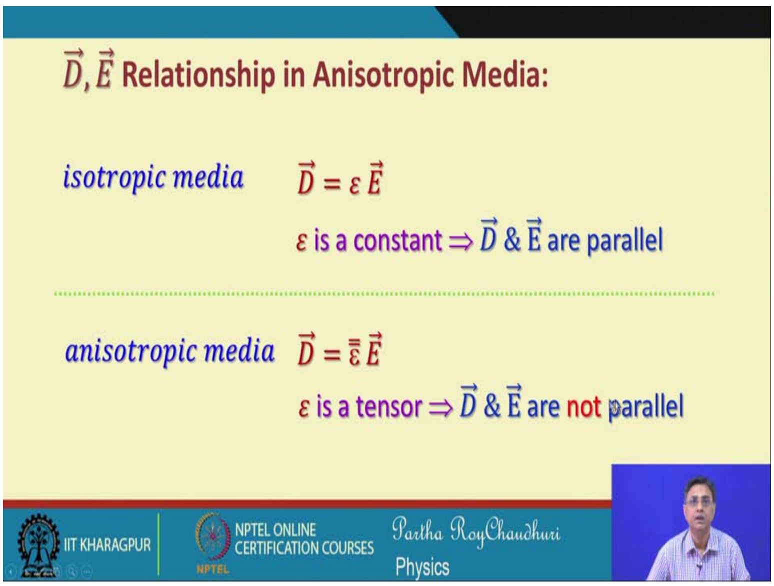

So, looking at the E D relationship in anisotropic medium the constitutive relation in this case is, the displacement vector is equal to the epsilon times the electric field vector, which is for isotropic medium this epsilon is a constant. And as a result it is a scalar being and the D and E are parallel. So, the electric field vector and the displacement vectors will point along the same direction where as in the case of anisotropic media this D and E in general are in different directions.

Epsilon is a tensor in this case which is a 9 by 3 by 3, 9 component tensor and as we have mentioned that D and E are not parallel as a consequence of this tensor property of the permittivity of this of the of the anisotropic material.

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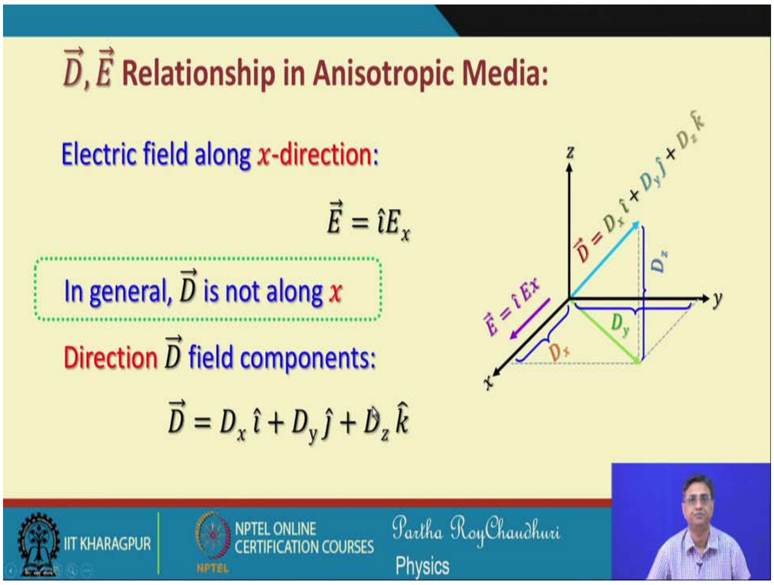

So, first let us look at the E and D behavior in anisotropic medium. Let us suppose that the electric field is incident along x axis. So, that one can represent the electric field vector as E equal to i cap E x and there is no y component or any z component of the electric field. So, electric field is solely along the x axis. In that case, in general the D is not along x axis, it will be somewhere different from the x axis. And this D vector will be represented by D equal to D x i D y j and D z k so; that means, there will be 3 components excited because of this incident electric field, which are in general not along the x axis.

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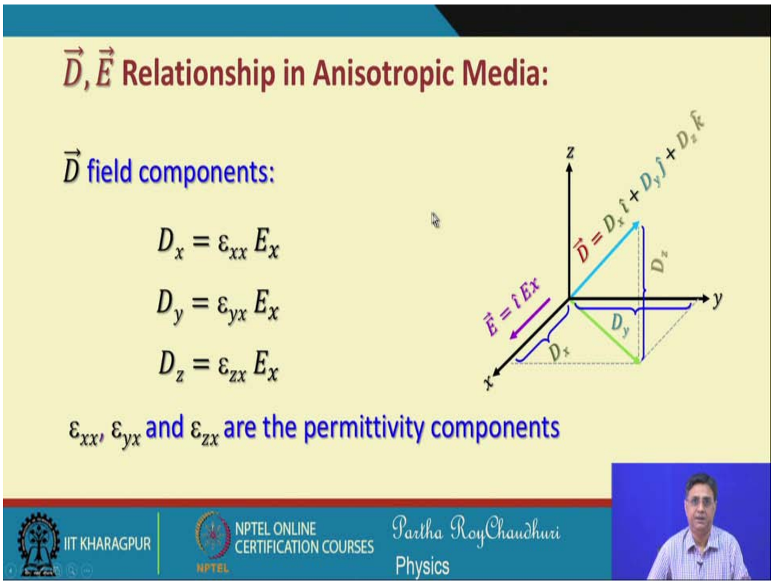

Similarly, we can apply an electric field in the y direction and as well as in the z direction, but let us see what happens when the electric field is applied along the x axis. So, the D component, because the by the very property of the displacement vector that the displacement vector is proportional to the electric field. So, D x that is the component of the electric displacement along the x axis will be proportional to E x x. And this proportionality constant for this particular medium will be represented by epsilon x x. And likewise for D y this will be epsilon y x E x, because this electric field is E x. So, we write in this notation epsilon y x. And similarly for the z component of the displacement we can write the D z equal to epsilon z x into E x.

So, we have been able to decompose the 3 displacement components, arising out of electric field only along the x axis. So, these epsilon x x epsilon y x and epsilon z x are the permittivity components of the medium when the electric field is incident along the x axis.

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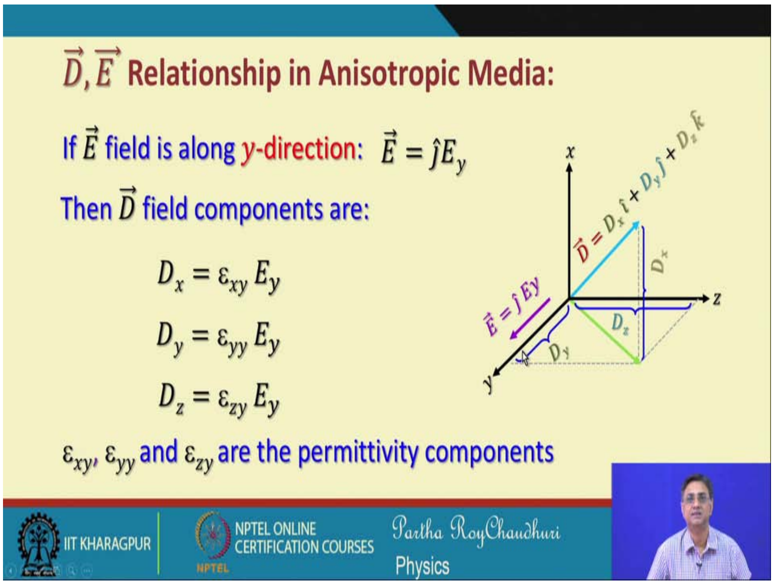

Next, let us consider the behavior when we apply an electric field along the y axis. In the same way the displacement vector will be in general not in the direction of the electric field, but in a in a direction which is different from the direction of the electric field. As a result, in the same way we can write the displacement vector in this case, the x component is equal to epsilon x y E y, for the y component it is epsilon y y E y, and for the z components it is epsilon z y E y.

Here also epsilon x y, epsilon y y and epsilon y z are the respective permittivity components when the electric field is applied along the y axis. We have one more case which will complete the discussion that is if the electric field is incident along the z axis in that case in the same way the displacement vector will be in general along a direction which is not along z axis.

As a result, in the same way we can decompose the displacement vectors along the 3 mutually orthogonal perpend orthogonal axis where D x will be equal to epsilon x z E z, D y will be equal to epsilon y z E z and D z will be equal to epsilon z z E z, where again this epsilon x z, y z and z z are the permittivity components in this case.

So now we will arrange all the 3 electric field vectors to be representing in one arbitrary direction.

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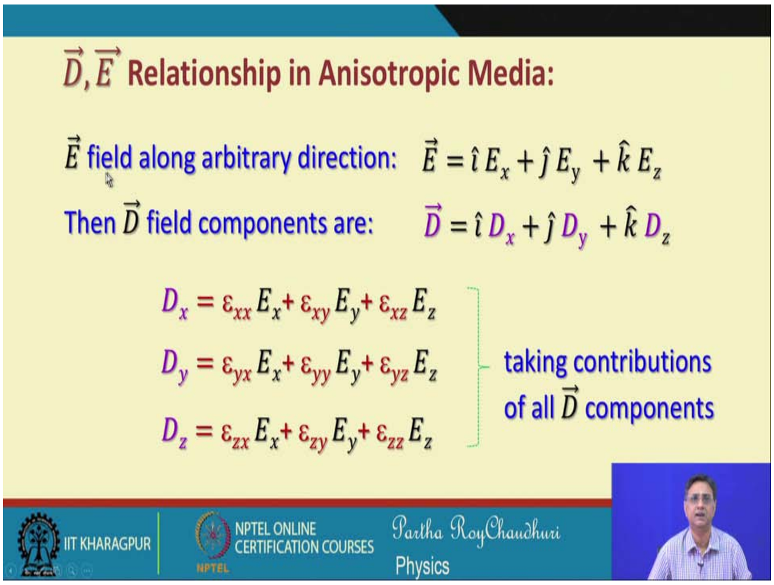

That is, let us consider that the electric field is now along any general direction that is E will have 3 components E x, E y and E z. As a result, the displacement field components will also be represented by D x, D y and D z, but they have a relationship. Because this D x D y and D z all of them will end it is excitation from E x from E y as well as from E z. So, when you apply an electric field E x you get E x x E x E epsilon y z epsilon y x E x and epsilon z x E x. So, these are the 3 components which are due to the electric field E x, these 3 components are due to the electric field E y and these 3 components are the electric are the displacement component which is due to the z component of the electric field. So, these are 3 simultaneous equations.

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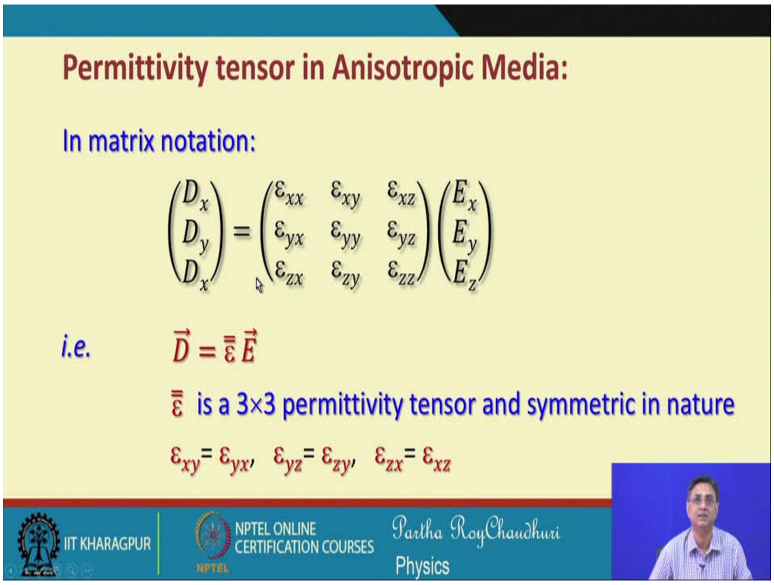

So, which can be which can be written in the in the form of matrix equation that is D x, D y, D z will be equal to this all the all the 9 components 3 by 3 matrix; E x x E x y E x z and so on, multiplied by this E x E y and E z. In a compact notation we can write this matrix equation as D equal to epsilon tensor into the electric field. So, this epsilon tensor as you have seen is a 3 by 3 permittivity tensor. And this tensor you see that it is symmetric in nature; that means, that epsilon x y will be equal to epsilon y x, epsilon y z will be equal to z y and so on, that is epsilon z x will be equal to x z. So, they are symmetric in nature this component will be equal to this component, this one will be equal to this one and this one will be equal to this one. And we will find lot of interesting applications using this particular property.

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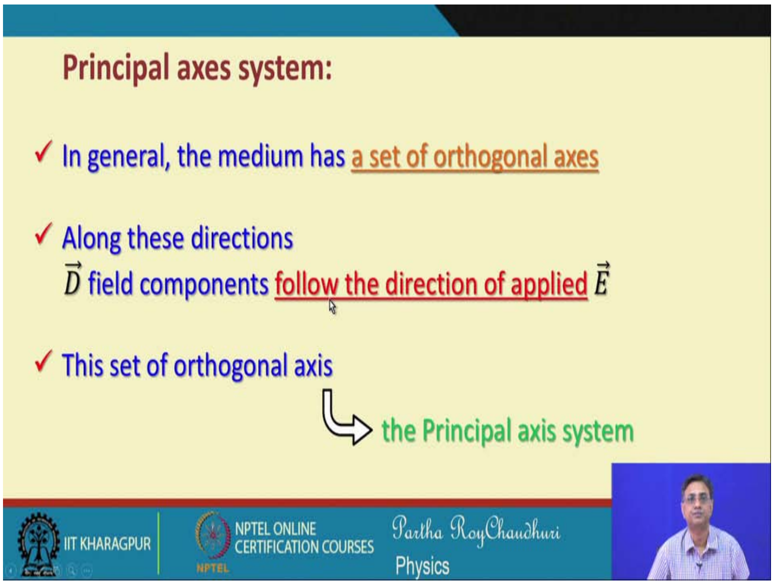

So, what you find so far is that the principal in the principal axes system, in general the medium has a set of set of 3 orthogonal axes. And along these directions D field components follow the direction of the applied E. This is an interesting finding that if we if we search, if we look for a set of orthogonal coordinate axes within the medium, there is one representing there is one which will correspond to the system that in which, the electric field will follow the displacement field will follow the electric field in the same direction. That means, when you apply the electric field along the x direction due will be only along the x direction and so on. Such a set of orthogonal coordinate axis is known as the principle axes system of the medium and that is, the characteristic property of the medium.

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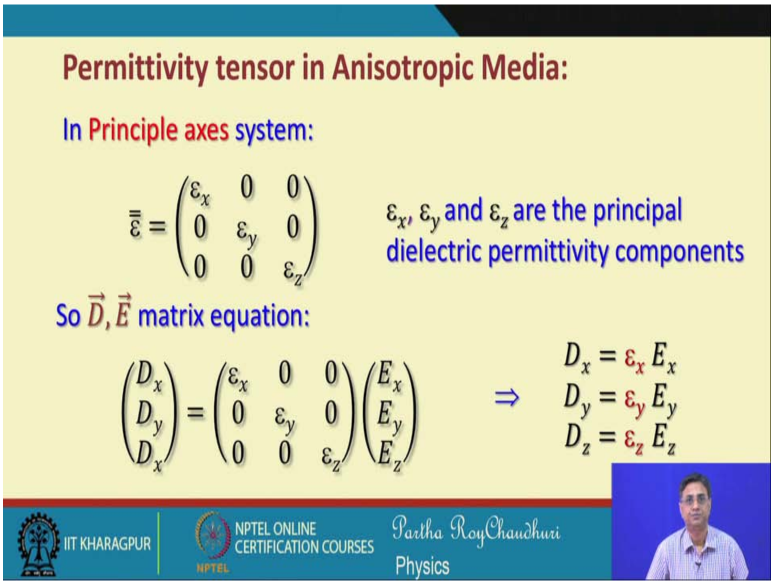

So, in such a principal axes system we can write the epsilon permittivity tensor as epsilon x x, epsilon y y and epsilon z z. If we if we represent with a sort notation then we can write that this is because there is no other components no other cross components x y, y z or z x. So, we can write that epsilon tensor as epsilon x, epsilon y and epsilon z.

These are the 3 principal dielectric permittivity components of the medium so; that means, we could find one orthogonal coordinate system, orthogonal axes within the medium along which the displacement fields will follow the electric field. And if you apply an electric field along the x direction the displacement will be only along the x axes and so on and so forth, for the y and z component z component of the displacement vectors, when you apply electric field along the x and y directions respectively.

So, D and E matrix equation in the case of the principal axes system for the medium can be represented by this equation; that is, D x, D y, D z is now a simple diagonal matrix E x epsilon x, epsilon y and epsilon z multiplied by E x, E y, E z. So, simply there will relate themselves in this way the D x is equal to epsilon x E x D y equal to epsilon y E y and so on.

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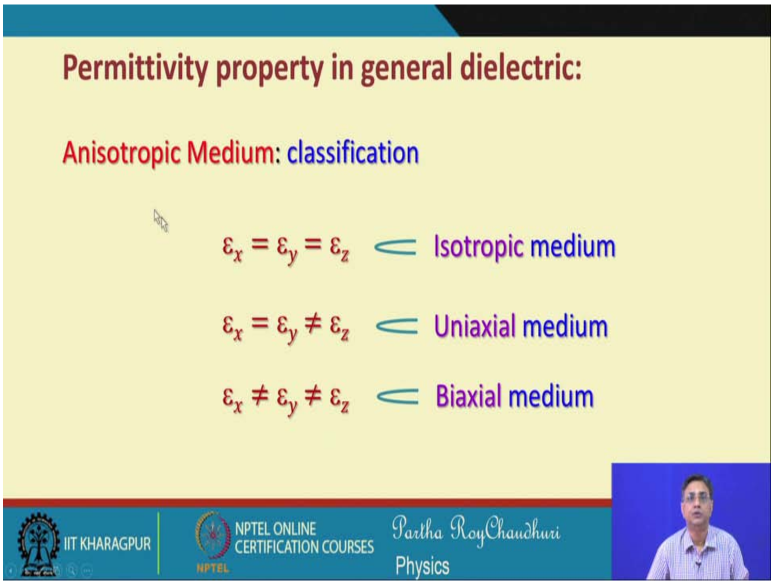

So, for anisotropic medium there are possibilities that E x and E y and E z maybe related in some way or other. For example, if E x E y epsilon x epsilon y and epsilon z all of them are equal; that means, the electromagnetic wave is direction independent. Then you call that the medium is anisotropic medium. So, in anisotropic medium there for all directions, whether it is incident along any electric field is incident along any arbitrary direction or along x direction or along y direction or any other direction.

So, the displacement vector will follow the direction of the electric field, which is anisotropic medium, but there is there is another situation where 2 of them may be equal, but is not equal to the third one. For example, epsilon x is equal to epsilon y, but not equal to epsilon z. In that case this such a medium will be called a Uniaxial medium. Whereas, there is another situation where, where you will find another group of media, where this epsilon x and epsilon y epsilon z all 3 of them are different. In general, such a medium will be called a Biaxial medium; that means, in a Biaxial medium the refractive indices or the permittivity seen by the wave will be all different in the all 3 mutually orthogonal directions.

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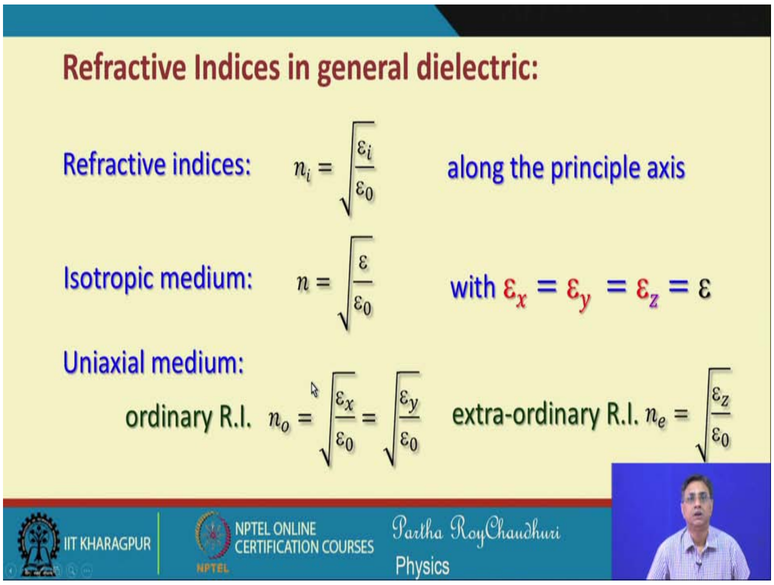

So, the refractive indices we can write in this form that; n i is equal to epsilon i epsilon 0, where this is the refra permittivity. And this is the free space permittivity the ratio and square root of that will represent the refractive index. So, that is along the principal axes. Now, for anisotropic medium this is in general all of them are equal. So, epsilon x equal to epsilon y equal to epsilon z is equal to epsilon; so, which will be represented by only one epsilon for all the directions. For Uniaxial medium; however, there will be one value for which along 2 directions the refractive indices or the permittivity seen will be identical whereas, this quantity will be different for the third direction.

The one along which the refractive index seen by the wave is different is called the extraordinary refractive index. And the wave corresponding to that refractive index will be call the extraordinary wave whereas, the one for which there are 2 directions for which the refractive indices are the same, the permittivity's are the same we call that is the ordinary refractive index. And the wave corresponding to that refractive index direction will be called the ordinary wave.

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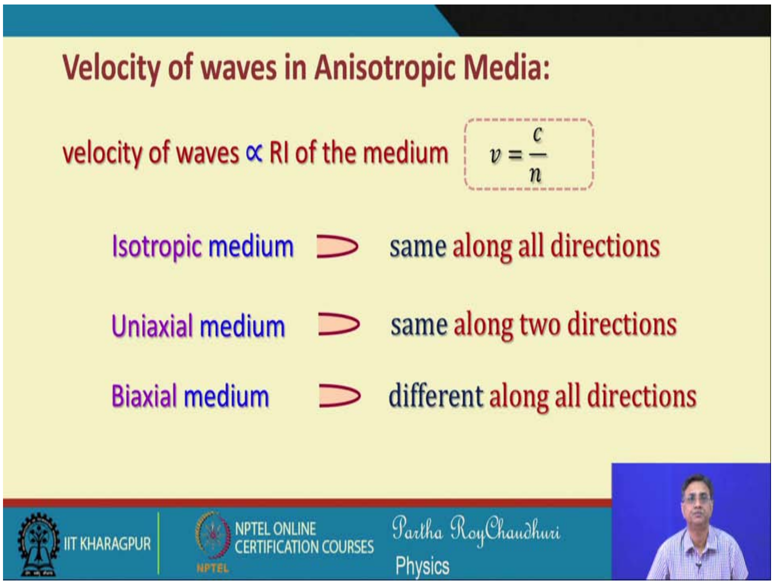

So, to summarize that the velocity of the waves will be proportional to the refractive indices, this is in general true. And for isotropic medium therefore, the velocities of the electromagnetic waves will be same in all directions; whereas, for Uniaxial medium the velocities will be different along 2 different directions, but for Biaxial medium the velocities or different along all the directions.

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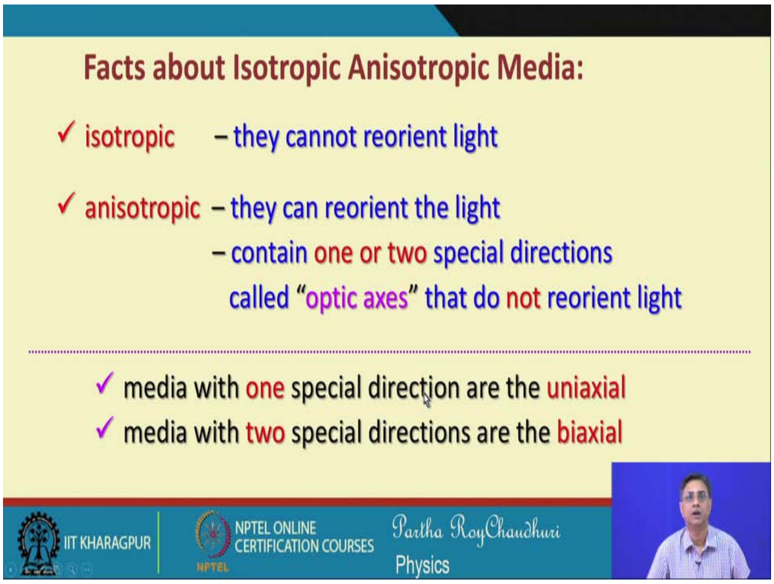

Now, some more facts about the isotropic and anisotropic media, for an isotropic medium there cannot this medium cannot reorient the light, because the light which is incident along a particular direction in such a medium will immerge only along this direction. So, there is no effect, there is no activity of the medium on the travelling electromagnetic waves as regards the direction of propagation. So, the direction of propagation will not be disturbed will not be influenced by the medium; whereas, the situation is different for anisotropic medium. And this property will be extensively used in the case of various modulators and devices, which will be will be discussing later in the later part of this course.

So, for anisotropic medium, the medium can reorient the light that is, the direction of the incident electromagnetic waves can be changed by the anisotropic medium by it is property. And also the velocities can also be also be altered. So in general, such a medium contains one or 2 special directions. And these directions are called the optic axis along which they do not reorient the light. So, in anisotropic medium particularly, in a in a Uniaxial medium there will be directions along reach the electromagnetic waves direction will not be altered. And in the other direction the direction of the electromagnetic waves will be altered.

So, to summarize that a medium with one special direction are called the Uniaxial medium, one special direction means; the direction along which the direction of the propagation of electromagnetic waves will not be altered, such one special direction material or the medium are called the Uniaxial medium, but for a Biaxial medium there will be 2 such special directions along which Biaxial. So there are 2 axes, there are 2 direction along which the electromagnetic waves direction will not be changed, when compared to or with regard to the direction of the incident electromagnetic waves.

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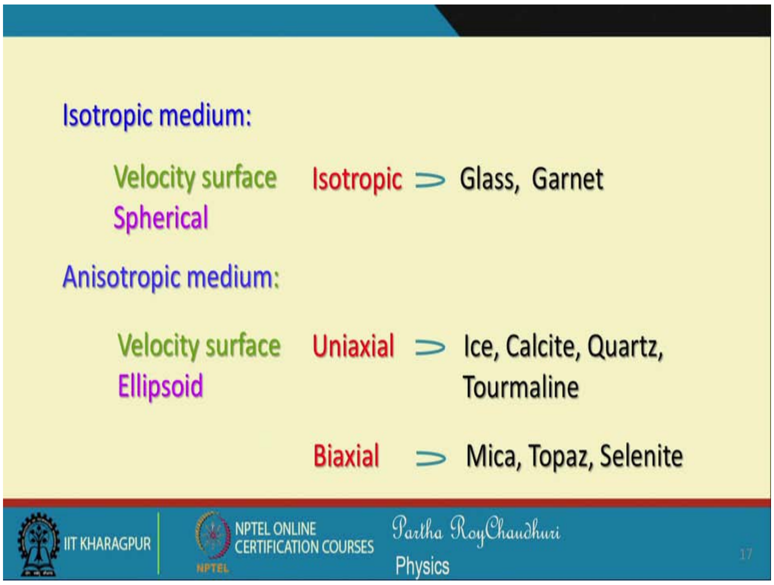

So, for isotropic medium velocity surface will be a spherical that is; if I if I image in the wave front in the all 4 pi solid angle directions, then it will be spherical because the direction the velocity is equal in all directions. The examples of such a medium are ordinary Glass, Garnet and many more. For anisotropic medium the velocity surface will be in general and Ellipsoid. We will discuss more about this Ellipsoid in terms of the index Ellipsoid in the later part of this section. So, for a Uniaxial medium the examples are like Ice, Calcite, Quartz, Tourmaline and these are very widely used in designing an in configuring various optical instruments and devices. Biaxial medium examples are Mica, Topaz, Selenite.

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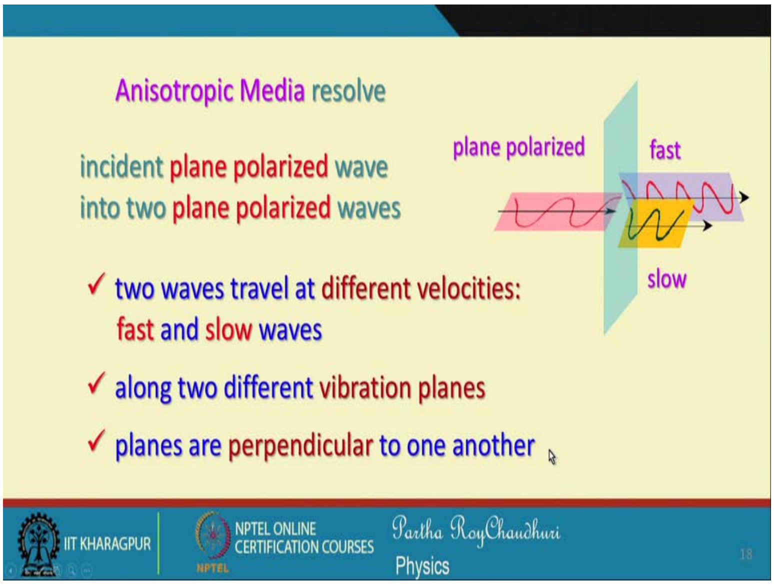

So, in anisotropic medium, incident plane polarized light will decompose into 2 plane polarized light. So, let us suppose that you have an incident plane polarized light this will immerge out from the medium or within the medium, within the medium in 2 mutually orthogonal directions. And in general the 2 waves travel at different velocities these are; the one which is faster will be called the fast waves, and the one which is slower will be the slow waves.

And, in general because they are travelling with different velocities. So, they will develop a different phase difference and this phase difference will be, will be utilized to construct devices. Along the 2 different vibration planes these 2 waves will be travelling; and the planes are perpendicular to each other, the vibration planes are perpendicular to each other.

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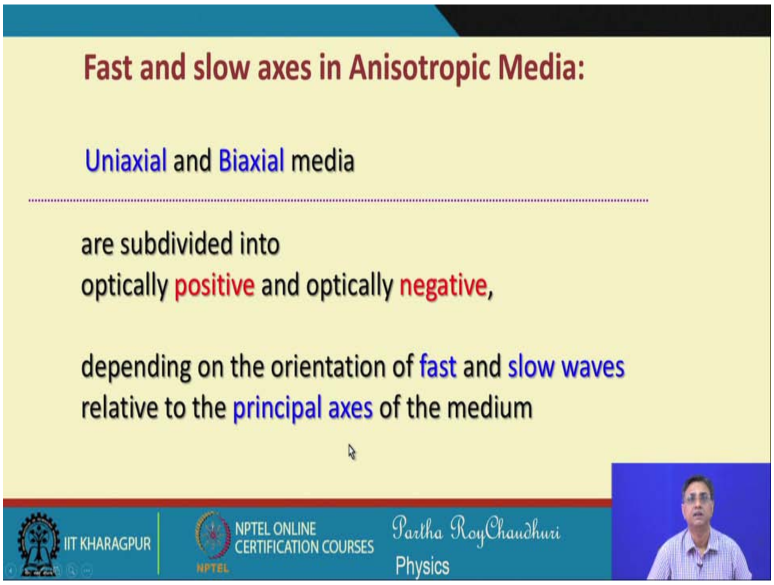

There are some more things so Uniaxial and Biaxial media are against subdivided into positive and negative optically active media depending on the orientation of the fast and slow waves. So, in some medium the one which is fast waves, in some other medium the same will be referred to as a slow waves. So, this is all relative to the principle axes system of the medium.

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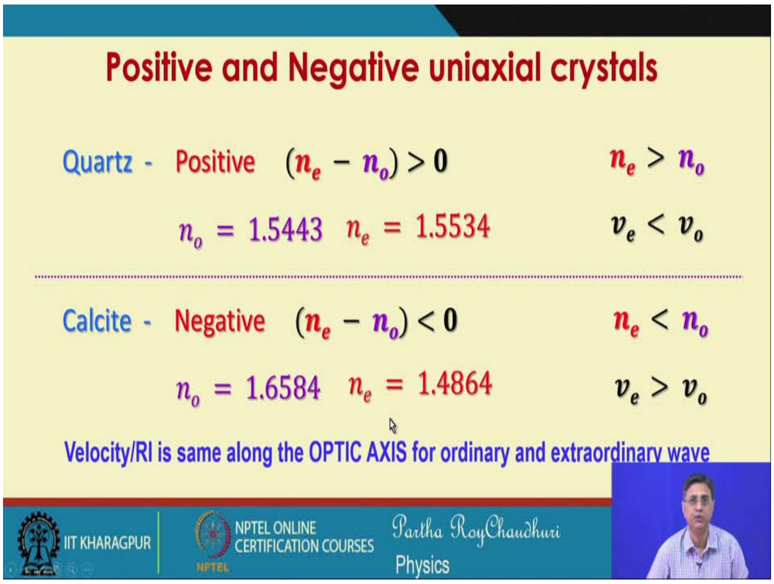

For quartz there is a beautiful example, which is called a positive crystal n e is, is more than n o that is, the extraordinary refractive index is more than ordinary refractive index.

So, the typical values are like this 1.5443, 1.5534. And therefore, the velocity of the of the e, e wave will be less than the velocity of the o wave. For Calcite, which is a negative crystal the situation is just opposite that is n e, n e is less than n o. And the typical values are like this; so you can see that the extraordinary wave travels with a higher velocity then the ordinary wave. So, the velocity or the refractive index is the same along the optic axis for ordinary and extraordinary wave both their same.

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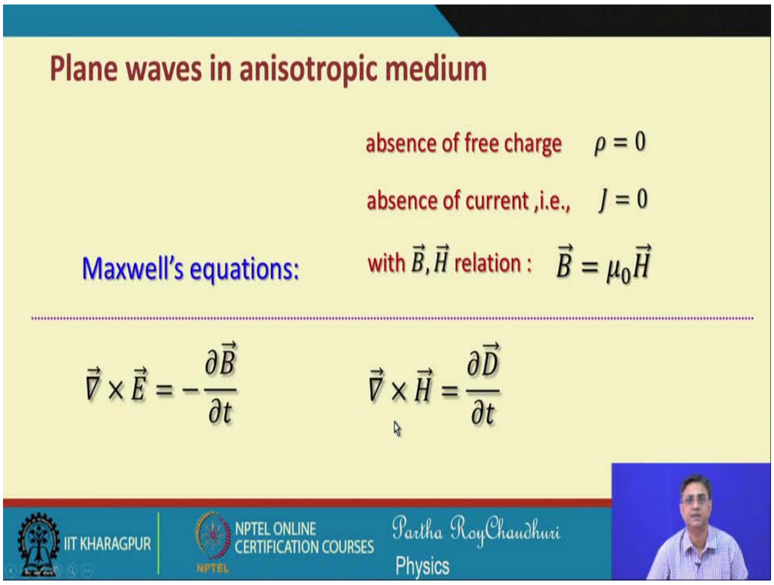

Now, we will we will look at the plane wave equation, solution for this anisotropic medium. And this is the property that we assume that there is no free charge, there is no free current and B and H they are related by this constitutive relation B equal to mu naught H in such a medium. So, these are the 2 Maxwell's curl equation from which; we can we can derive we can organize the wave equation for the anisotropic medium.

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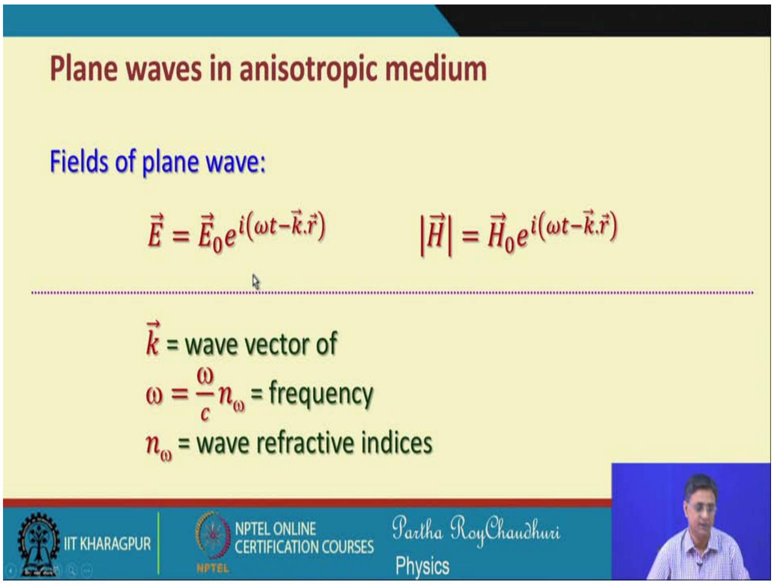

Fields of plane waves are represent we have seen that fields of plane waves are represented by E equal to E naught e to the power of i omega t minus k dot r, and similarly for the h field we can represent in this way.

Where, k is the wave vector and E this is the frequency of the electromagnetic waves and this is the wave refractive indices, we will see that there are 2 refractive indices one is the ray refractive index another is the wave refractive indices.

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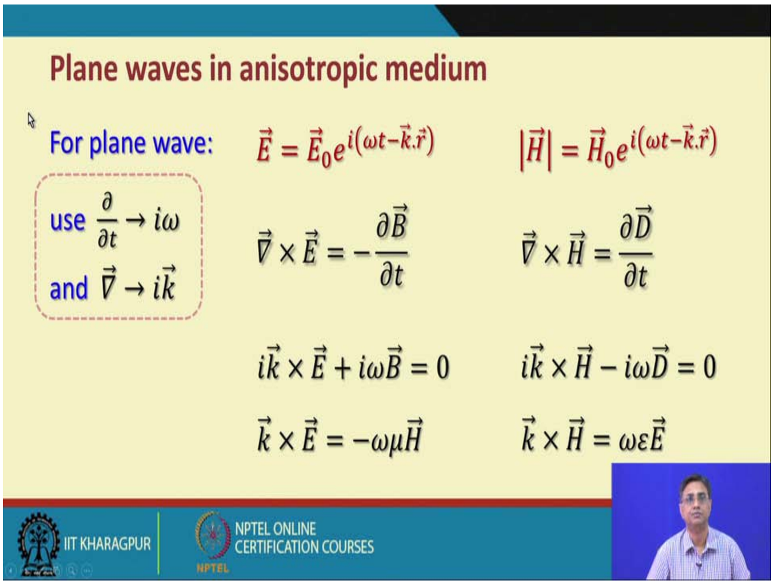

So, if we now use this del del t, as to represent this i omega because this plane wave equation this is true that if you operate this del del t on this equation it will just leave i omega multiplied by this E itself. So, we can write the first equation which is which is which will give you i k cross E. This we have seen earlier again will see in details and i omega B. So, this will give you this equation and from other equation we can write that k cross H equal to omega epsilon E, where epsilon in this case in the case of anisotropic medium is a tensor.

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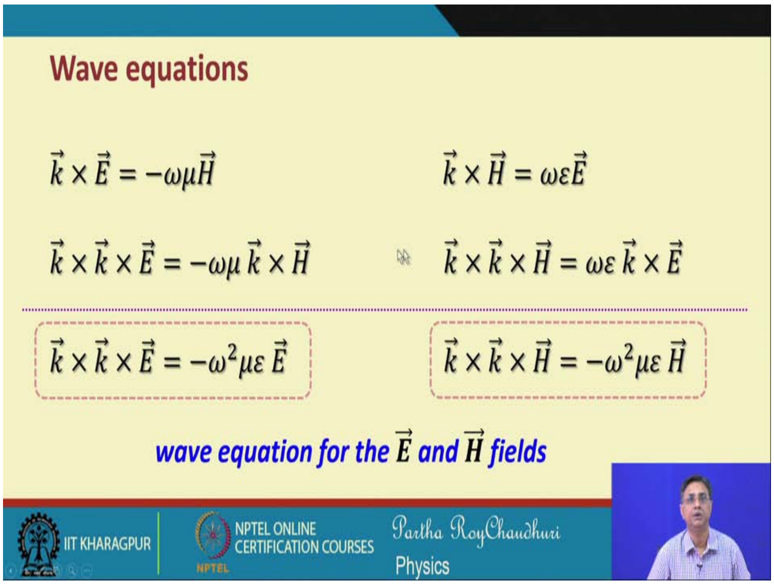

So, if we organize this equation k cross E equal to minus omega mu H then you can take k cross k cross E, which will be equal to this quantity. And this actually represents the wave equation, which is purely in terms of the electric field. And in the same way if we write this equation starting from k cross h equal to omega epsilon E and we replace E by and we replace this by k cross E, then k cross k cross H will be equal to omega square mu epsilon H. So, these 2 are the wave questions pair of a set of wave equation which will represent the electromagnetic waves in an isotropic medium for both the electric field and also for the magnetic field.

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So, in this section what we have discussed is the relation between the displacement field and the electric field incident electric field, there connection in terms of the direction and also in terms of the magnitude, which is actually connected by the permittivity values. Then we discuss the Permittivity tensor for the anisotropic medium. From there we have seen in a very special situation that for a for a for a set of orthogonal coordinate axes characteristic of the medium. We could look at the principal axes system and the values of the permittivity in that Principal axes system that also have seen. Then we have made a small comparison about the Isotropic Uniaxial and Biaxial medium then we talked about the wave equation in Anisotropic medium.

In the next section will continue with the wave equations in the anisotropic medium, and we will see the various aspects of the waves; the relation of the general propagation vector, the electric field, the displacement vector, the pointing vector the magnetic field all these things will be will be connected. And we will see how the propagation of the electromagnetic waves gives rise to various phenomena in general Anisotropic medium.

Thank you.

Lecture 2 : Propagation in anisotropic media

- Linear anisotropic media

- Characteristic surfaces:
  - Index ellipsoïd
  - Index surface
  - Radial velocity surface or Ray surface

For more details about the calculations in these lectures:

Optical waves in crystals, Yariv & Yeh (more details about some of the calculations done in this chapter, but does not mention the radial velocity surface)

Optics, Born&Wolf (details of the calculation of the ray surface/ radial velocity surface from page 792 of the 7th edition)

Review about isotropic media

- **Maxwell's Equations** (in a medium without sources) $div\,\vec{D} = 0 \;,\; div\,\vec{B} = 0 \;,\quad rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \;,\; rot\,\vec{H} = \frac{\partial \vec{D}}{\partial t}$
- **Polarization :** when it travels through a medium, the EM wave induces a polarization that adds up in phase to the one in vacuum:

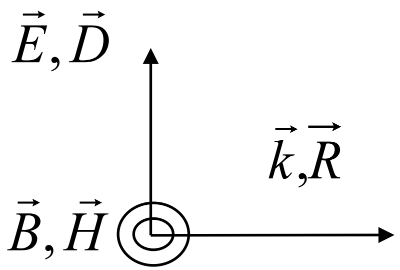
$$\vec{P}^0 = \varepsilon_0 \vec{E} \qquad \vec{P}^1 = \varepsilon_0 \chi\, \vec{E} \qquad \Rightarrow \; \vec{D} = \varepsilon_0 (1+\chi) \vec{E} = \varepsilon_0 \varepsilon\, \vec{E}$$

(the medium is non-magnetic: $\mu = \mu_0$)

- We try to find a solution to Maxwell's equations as a **plane wave** : $\vec{E} = \vec{E}_0\; e^{-i(\omega t - \vec{k}.\vec{r})}$

(other fields: same type of expression)

- We get : $\vec{k}.\vec{D} = 0 \;,\; \vec{k}.\vec{B} = 0 \;,\; \vec{k} \times \vec{E} = \omega \vec{B} \;,\; \vec{k} \times \vec{H} = -\omega \vec{D}$



- *E and D are parallel and transverse*
- *B and H are perpend to them, also transverse*
- *The Poynting vector* $\vec{R} \propto \vec{E} \times \vec{H}$ *is parallel to* $\vec{k}$

Review about isotropic media

Which EM waves can propagate in the medium ?

- **From Maxwell's equations**

$$\vec{D} = -\frac{1}{\omega}\vec{k}\times\vec{H} = -\frac{1}{\mu_0\omega^2}\vec{k}\times\left(\vec{k}\times\vec{E}\right) = \frac{k^2}{\mu_0\omega^2}\vec{E} \quad \text{because} \quad \vec{E}\perp\vec{k}$$

- On the other hand, $\vec{D} = \varepsilon_0\,\varepsilon\,\vec{E}$

- So : - All transverse polarizations are possible

  - Dispersion relationship: $\boxed{k^2 = \mu_0\varepsilon_0\,\omega^2\,\varepsilon}$

- In general, we define the index of refraction: $\varepsilon = n^2 \qquad c = 1/\sqrt{\mu_0\varepsilon_0}$

- We get : $\boxed{k = \frac{\omega}{c}n}$ • Phase velocity : $\boxed{v_\phi = \frac{\omega}{k} = \frac{c}{n}}$

Linear anisotropic media

• In an anisotropic medium, the susceptibility, and thus the permittivity, are no longer scalar, but **tensors** (3x3 matrices):

$$\vec{P}^{1} = \varepsilon_0 [\chi] \vec{E} \;\Rightarrow\; \vec{D} = \varepsilon_0 \left([I] + [\chi] \right) \vec{E} = \varepsilon_0 [\varepsilon] \vec{E}$$

-> the medium is still linear, but no longer isotropic.

• If there is **no absorption**, it can be shown that the permittivity tensor is **symmetric**

$$\boxed{\varepsilon_{ij} = \varepsilon_{ji}}$$

• It thus has **three real eigenvalues**, corresponding to orthogonal eigenvectors.
In the basis of these eigenvectors, the matrix can be written as :

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

• By analogy with the isotropic medium, we define indices:

$$\boxed{\varepsilon_i = n_i^2}$$

The indices n_i are called ***the principal indices of refraction****.*

⇨ **In an anisotropic medium, the index « seen » by an EM wave depends on its polarization state.**

Linear anisotropic media

Classification of materials depending on their permittivity tensor

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_x & 0 \\ 0 & 0 & \varepsilon_x \end{bmatrix} \Rightarrow n^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Isotropic medium with index n

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_x & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \Rightarrow \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix}$$

Uniaxial anisotropic medium, with optic axis Oz

$$\begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \Rightarrow \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$$

Biaxial anisotropic medium

Linear anisotropic media

What is the structure of an EM wave propagating in an anisotropic medium ?

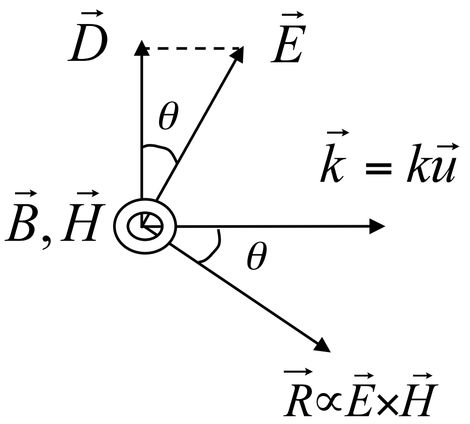
$\vec{D} = \varepsilon_0 [\varepsilon] \vec{E}$ ***-> D is no longer parallel to E***

$\vec{k}.\vec{D} = 0\ ,\ \vec{k}.\vec{B} = 0$ -> **D** and **B** are still orthogonal to **k**

$\vec{k} \times \vec{H} = -\omega \vec{D}$ -> **D** and **H** are orthogonal

$\vec{k} \times \vec{E} = \omega \vec{B}$ -> **E** and **B** are orthogonal

D, E and k are in the same plane, orthogonal to B



$$\vec{D} = -\frac{1}{\mu_0 \omega^2} \vec{k} \times \left(\vec{k} \times \vec{E}\right) = \frac{k^2}{\mu_0 \omega^2} \left[\vec{E} - \left(\vec{u}.\vec{E}\right) \vec{u}\right]$$

- **D** is the projection of **E** in the plane orthogonal to **u** (to within a mult. factor)

- The **Poynting vector** (direction of « light ray») **is no longer parallel to the wave vector** (direction of propagation of the phase)

Linear anisotropic media

Which waves can propagate in a given direction?

• Definitions: $k_0 = \omega\sqrt{\varepsilon_0\mu_0} = \dfrac{\omega}{c}$ and $n^2 = \dfrac{k^2}{k_0^2}$ *n is the index «seen» by the wave with direction **u***

• From Maxwell: $\vec{D} = \varepsilon_0 n^2\left[\vec{E} - P^{\vec{u}}(\vec{E})\right] = \varepsilon_0 n^2 P_\perp^{\vec{u}}\,\vec{E}$ $P_\perp^{\vec{u}}$ *is the projector on the plane perpendicular to **u***

• Also: $\vec{D} = \varepsilon_0\left[\varepsilon\right]\vec{E} \Rightarrow \vec{E} = \dfrac{1}{\varepsilon_0}\left[\eta\right]\vec{D}$

$\left[\eta\right] = \left[\varepsilon\right]^{-1}$ ***: Permeability tensor.*** *In the eigenvectors basis,*

$$\left[\eta\right] = \begin{bmatrix} 1/n_x^2 & 0 & 0 \\ 0 & 1/n_y^2 & 0 \\ 0 & 0 & 1/n_z^2 \end{bmatrix}$$

• **Condition on *D*:**

$$\boxed{\frac{\vec{D}}{n^2} = P_\perp^{\vec{u}}\left[\eta\right]\vec{D} = \left[A_{\vec{u}}\right]\vec{D}}$$

⇨ **The solutions for *D* are eigenvectors of matrix A_u**

Linear anisotropic media

$$\boxed{\frac{\vec{D}}{n^2} = P_{\perp}^{\vec{u}}\left[\eta\right]\vec{D} = \left[A_{\vec{u}}\right]\vec{D}}$$

• **Eigenvectors :** polarization states (vector **D**) that can propagate in the medium without changing their shape

Different from the isotropic case: not all polarization states can propagate

• **Eigenvalues :** corresponding indices of refraction $1/n^2$ (and phase velocities c/n)

⇨ The polarization states and the corresponding indices of refraction depend on the direction of propagation **u**

Linear anisotropic media

Can we show some specific properties of these eigenstates?

$$\boxed{\frac{\vec{D}}{n^2} = P_\perp^{\vec{u}}\left[\eta\right]\vec{D} = \left[A_{\vec{u}}\right]\vec{D}}$$

*$P_\perp^{\vec{\mathbf{u}}}$ is the projector on the plane perpendicular to **u**, i.e. on the phase plane.*

• In a basis consisting of 2 vectors in the phase plane plus the vector **u**, the matrix can be written as:

$$\left[A_{\vec{\mathbf{u}}}\right] = P_\perp^{\vec{u}}\left[\eta\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{12} & \eta_{22} & \eta_{23} \\ \eta_{13} & \eta_{23} & \eta_{33} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} \eta_{11} & \eta_{12} \\ \eta_{12} & \eta_{22} \end{matrix}} & \begin{matrix} \eta_{13} \\ \eta_{23} \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix}$$

[η] is not diagonal in this basis but it remains symmetric (like [ε])

Only the restriction of $[A_u]$ in the phase plane plays a role. We thus have to find the two eigenvectors and eigenvalues of a 2x2 **symmetric matrix**

⇒ 2 **real** eigenvalues $1/n'^2$ and $1/n''^2$(we will show later that they must be positive)

⇒ **orthogonal** eigenvectors **D'⊥D''**

Since the matrix is also real, **D'** and **D''** are real (linear polarizations)

Linear anisotropic media

Can we calculate more precisely the eigenvalues?

• The two eigenvalues $1/n^2$ depend on the direction of **u** (direction of propagation) :

$$n'(\vec{u}) = n'(\alpha,\beta,\gamma) \qquad \vec{u} = \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$
$$n''(\vec{u}) = n''(\alpha,\beta,\gamma)$$

• To calculate them, we need to solve the equation:

$$\det\left\{[A_{\vec{u}}] - \frac{1}{n^2} I\right\} = 0$$

• We have to calculate the determinant of a 3x3 matrix. We do this calculation in the (x,y,z) frame where [ε] and [η] are diagonal. After some tedious calculations, we get **Fresnel's equation for the indices of refraction**:

$$\boxed{\frac{\alpha^2 n_x^2}{n^2 - n_x^2} + \frac{\beta^2 n_y^2}{n^2 - n_y^2} + \frac{\gamma^2 n_z^2}{n^2 - n_z^2} = 0}$$

This equation can turn into an equation of the type $n^4 - Sn^2 + P = 0$:
⇒ we get two solutions n'^2 and n''^2 (we showed before that they must be real)
⇒ we can show now that they are positive ($S>0$ and $P>0$):
⇒n' and n'' are real numbers (propagation without absorption)

Linear anisotropic media

We will now show an easier, graphical way to represent the solutions to Fresnel's equation and the corresponding eigen polarizations

Several characteristic surfaces

- Index ellipsoïd
- Index surface
- Radial velocity surface or Ray surface

Characteristic surface 1: **Index ellipsoïd**

Definition

Let us recall the wave equation in terms of **D** and **E**:

$$\frac{\mathbf{D}}{n^2}=\varepsilon_0\left[\mathbf{E}-(\mathbf{u}.\mathbf{E})\mathbf{u}\right]$$

Scalar-multiplying it by **D** we obtain :

$$\frac{\mathbf{D}^2}{n^2}=\varepsilon_0\mathbf{D}.\mathbf{E}=\frac{\mathbf{D}_x^2}{n_x^2}+\frac{\mathbf{D}_y^2}{n_y^2}+\frac{\mathbf{D}_z^2}{n_z^2}$$

For a given D vector, there is **only one** index n of propagation, given by the above equation.

The surface we obtain when we report the value of the index n in the corresponding direction of D is called **the index ellipsoïd** and its equation is:

$$\frac{X^2}{n_x^2}+\frac{Y^2}{n_y^2}+\frac{Z^2}{n_z^2}=1$$

Index Ellipsoïd – directions of E and B

For a given direction of D corresponding to a point M on the ellipsoïd:

1) E is normal to the ellipsoïd at point M

Proof:

Vector normal to the index ellipsoïd at point (X_0,Y_0,Z_0): $\begin{bmatrix} X_0/n_x^2 \\ Y_0/n_y^2 \\ Z_0/n_z^2 \end{bmatrix}$

Vector E corresponding to this direction of D (X_0,Y_0,Z_0):

$$\frac{1}{\varepsilon_0}\begin{bmatrix} D_x/n_x^2 \\ D_y/n_y^2 \\ D_z/n_z^2 \end{bmatrix} = \frac{D}{\varepsilon_0 n}\begin{bmatrix} X_0/n_x^2 \\ Y_0/n_y^2 \\ Z_0/n_z^2 \end{bmatrix}$$

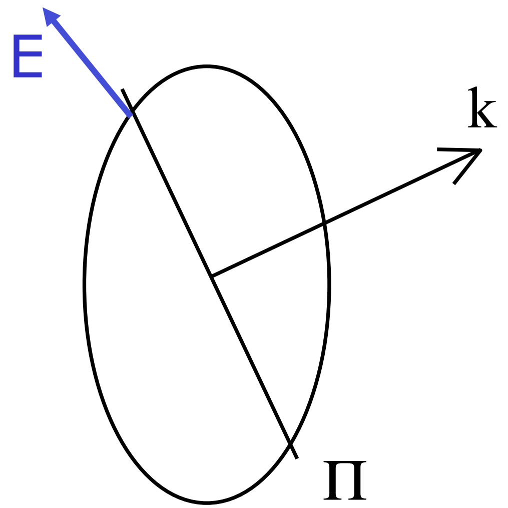
2) B, orthogonal to E, is thus tangent to the ellipsoïd at point M

Index Ellipsoïd – graphical determination of eigenpolarizations

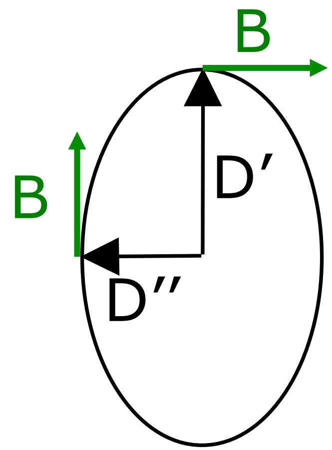
For a given direction of the wave vector k:

•D is in the phase plane (orthogonal to k) thus on the intersection of the index ellipsoïd with the plane orthogonal to k which is an ellipse

•D is orthogonal to B, which is tangent to the ellipsoïd, thus the only possibilities for D′ and D″ are the two principal axes of the ellipse



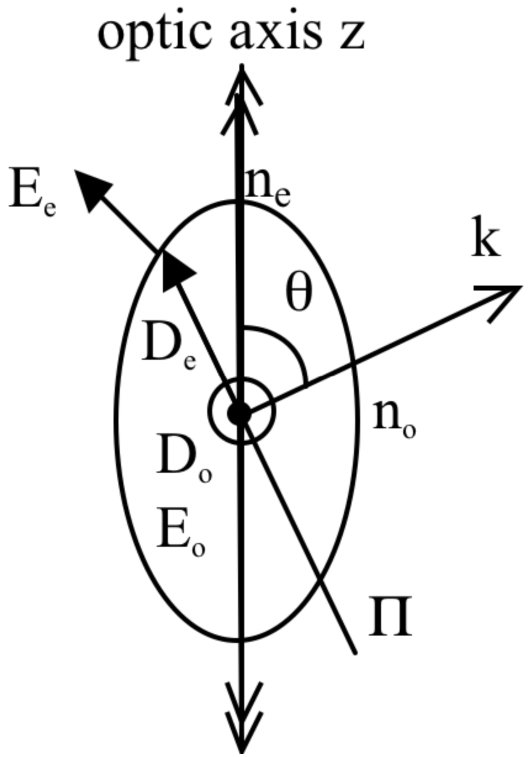
Intersection of the ellipsoïd by the phase plane Π



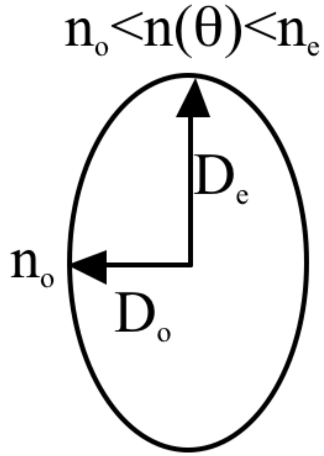
Index Ellipsoïd in the case of a uniaxial medium

Uniaxial medium: $n_x = n_y = n_o$ $n_z = n_e$ optic axis z
Positive uniaxial $n_e > n_o$

Ellipsoïd invariant under rotation about the z axis: $\frac{X^2}{n_o^2} + \frac{Y^2}{n_o^2} + \frac{Z^2}{n_e^2} = 1$



Projection of the ellipsoïd in plane Π



$D_o \perp$ k & optic axis

$D_e \perp$ k & D_o

or D_e projection of the optic axis on the phase plane Π

E_o // D_o

E_e normal to the ellipsoïd

Characteristic surface 2: **Index surface**

Definition

We have shown that for a fixed direction of propagation k, there are two possible values for index of refraction n’ and n’’

The index surface represents the two values of the index in the direction of k (locus of points N satisfying **ON**=n(**u**)**u**). It thus has two shells.

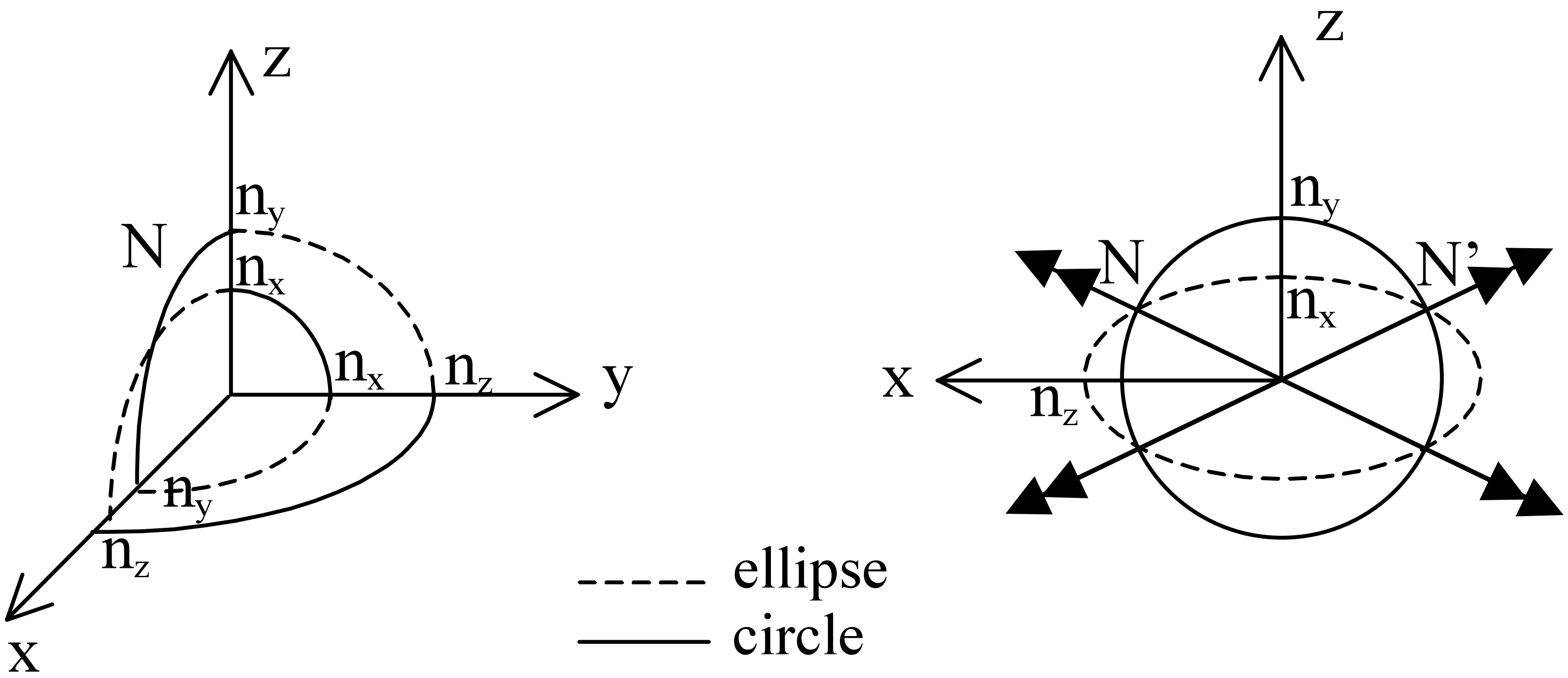
Writing ON=(X=nα,Y=nβ,Z=nγ) where $X^2+Y^2+Z^2=n^2$, we can write Fresnel’s equation relative to the wave normals as a function of X, Y and Z. We get a complicated equation from which we can imagine the shape of the normal surface by choosing $n_x<n_y<n_z$ and studying its intersection with each of the coordinate planes x, y or z=0. We obtain for example in the xz plane:

a circle with radius n_y

an ellipse of equation $x^2/n_z^2+z^2/n_x^2=1$

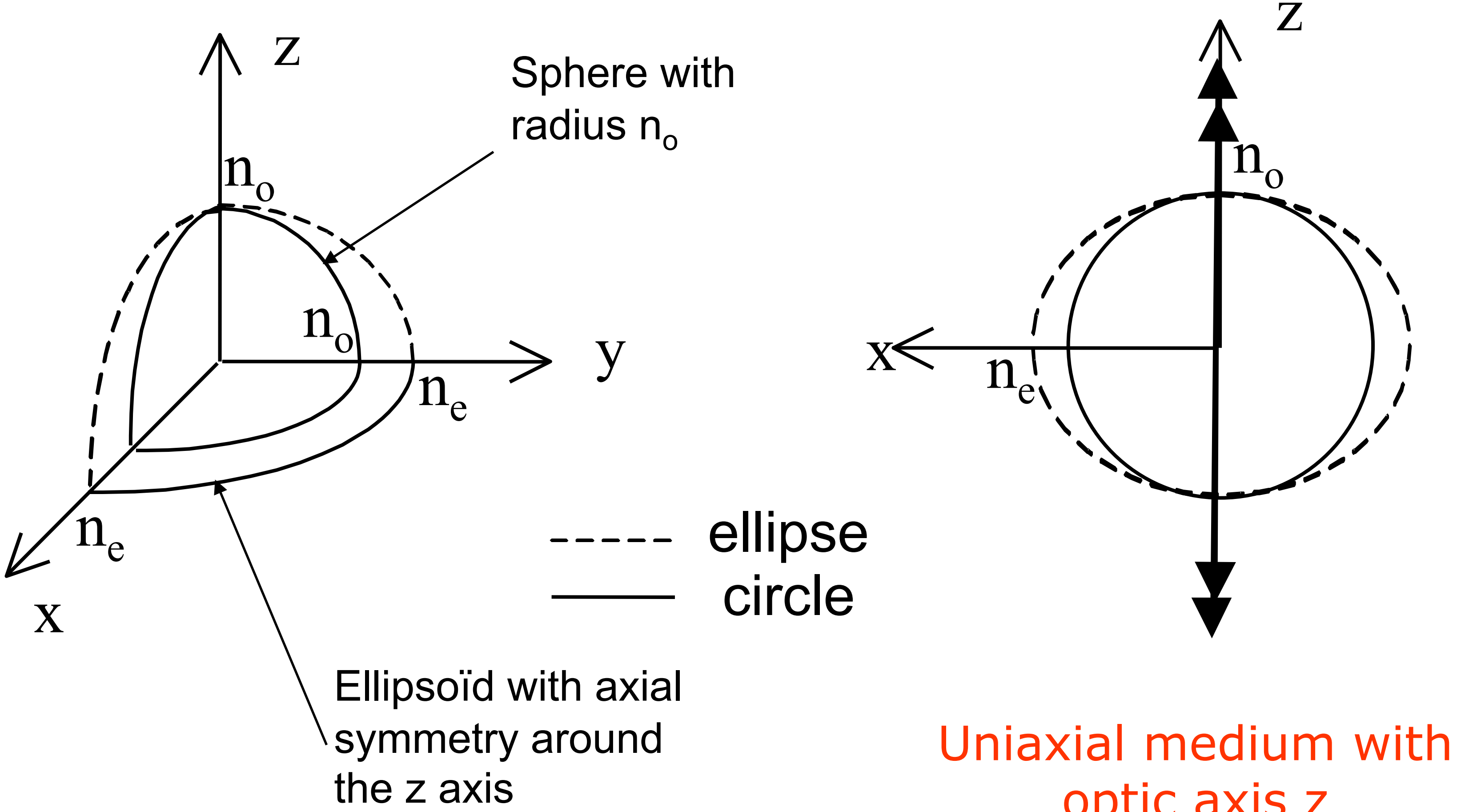
Using circular permutations we can obtain the three intersections.

Shape of the index surface when $n_x<n_y<n_z$



There are two directions ON and ON′ for k that satisfy n′=n′′: this is why the medium is called BIAXIAL.

Shape of the index surface when $n_x = n_y = n_o \quad n_z = n_e$

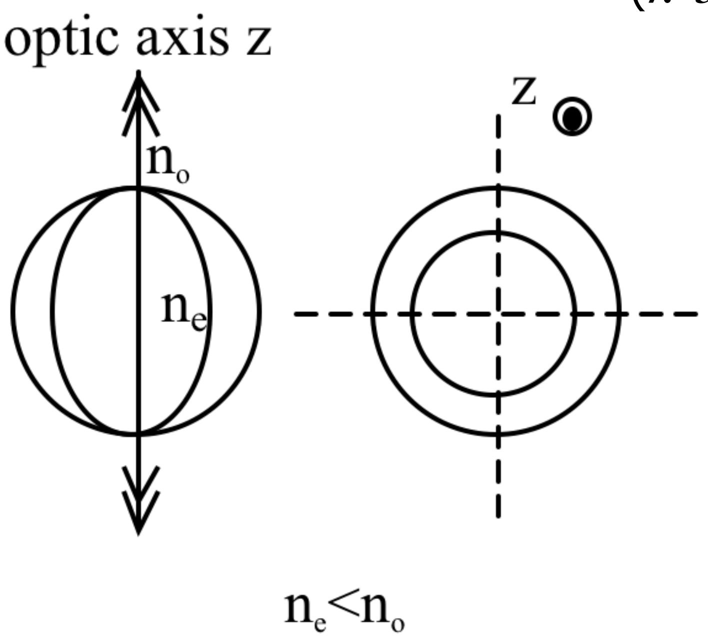


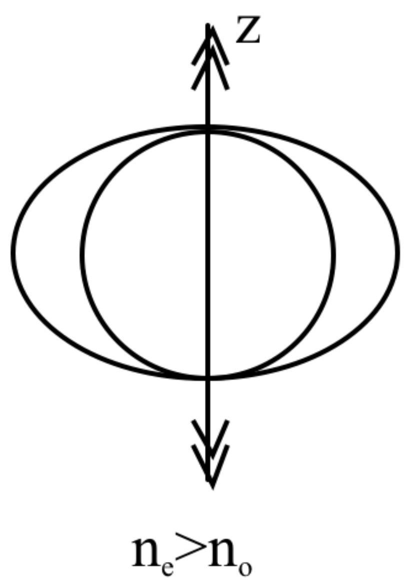
Uniaxial medium with
optic axis z

Positive and negative uniaxial crystals

Negative uniaxial $n_e < n_o$	Positive uniaxial $n_e > n_o$
calcite	Quartz
n_o= 1,658 n_e=1,486	n_o= 1,544 n_e=1,553

(λ=589nm)





2. Propagation in anisotropic media

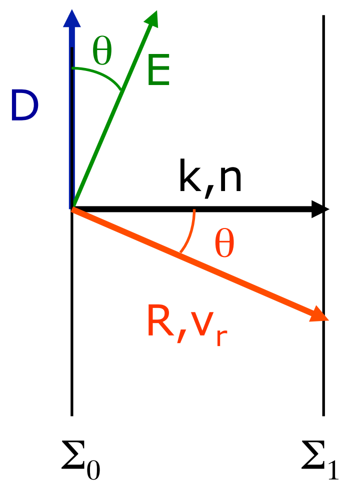
Characteristic surface 3:

Radial velocity surface or Ray surface

For each EM wave defined by (k,D) propagating with index n, there is a similar description where:

(k,D) is replaced by (R, E)

Index n is replaced by the « apparent » radial velocity defined by:



$$v_r = \frac{c}{n\cos\theta}$$

Radial velocity surface

Definition

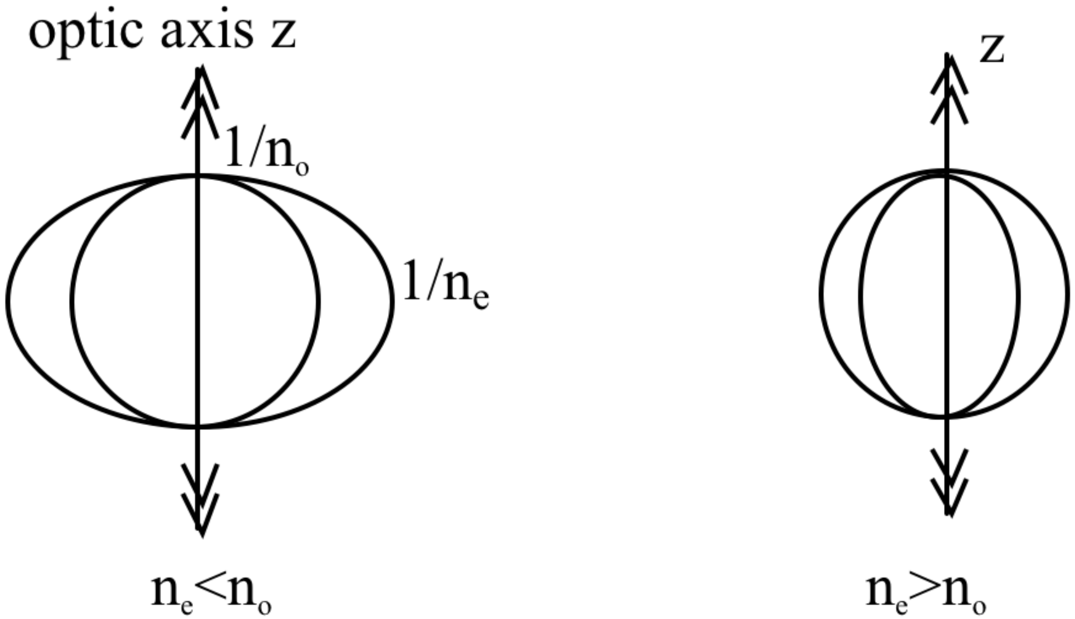
In a very similar way as we defined the indices surface, we define a radial velocity surface by reporting in the direction of the ray (Poynting vector R) the value of the radial velocity $v_r = v/\cos\theta$ (where θ is the angle between k and S).

The shape of this surface is determined from Fresnel's equation relative to the radial velocities which has a similar expression as the one relative to the wave normals:

$$\frac{\alpha^2 v_x^2}{v^2 - v_x^2} + \frac{\beta^2 v_y^2}{v^2 - v_y^2} + \frac{\gamma^2 v_z^2}{v^2 - v_z^2} = 0 \quad \text{where } v_i = c/n_i$$

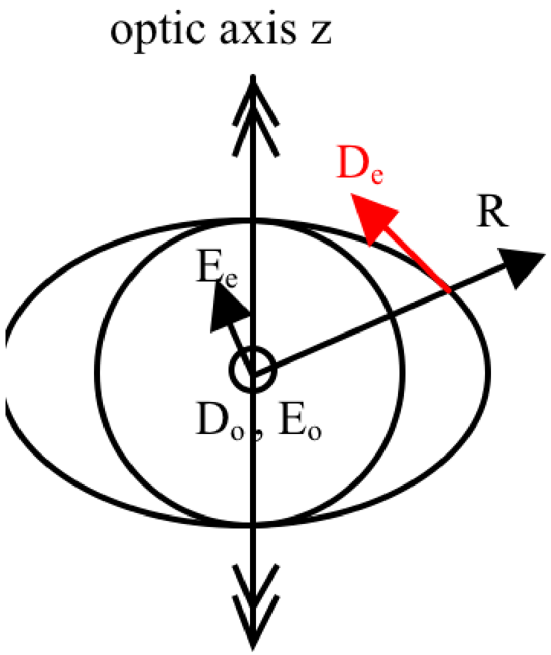
We get a surface with two shells very similar to the normal surface, but with its characteristic length along the directions of the axes x,y,z equal to c/n_i or rather $1/n_i$.

Radial velocity surface for a uniaxial medium



Property of the radial velocity surface

The radial velocity surface gives access, for a given direction of the Poynting vector **R** (light ray), to the corresponding directions of D_o , E_o , D_e , E_e :



For the ordinary wave: $\mathbf{E_o} \perp \mathbf{z}$ & **R**

$\mathbf{D_o} // \mathbf{E_o}$

For the extraordinary wave:

$\mathbf{E_e} \perp \mathbf{E_o}$ & **R**,

$\mathbf{D_e}$ tangent to the extraordinary shell of the radial velocity surface (see demonstration in the next slide)

Finding the direction of D_e from the velocity surface

Relationship between D_e and E_e :
$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_0 \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Coordinates of the Poynting vector in the (x,y,z) reference frame:
$$\mathbf{R} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Extraordinary shell of the velocity surface:
$$n_e^2 x^2 + n_e^2 y^2 + n_o^2 z^2 = 1$$

Vector normal to that shell in the direction of R:
$$\mathbf{N_e} = \begin{bmatrix} \alpha n_e^2 \\ \beta n_e^2 \\ \gamma n_o^2 \end{bmatrix}$$

Finally:
$$\mathbf{N_e} \cdot \mathbf{D_e} = \varepsilon_0 n_o^2 n_e^2 (\alpha E_x + \beta E_y + \gamma E_z) = \varepsilon_0 n_o^2 n_e^2 \mathbf{R} \cdot \mathbf{E_e} = 0$$

⇨ **D_e is tangent to the extraordinary shell of the radial velocity surface**

Summary about the characteristic surfaces

•Index ellipsoïd:

We plot in the direction of D (linear polarisation state that can propagate in the medium) the corresponding value of the index.

It is the simplest surface (only one shell) to represent the dielectric properties of the medium

•Radial velocity surface:

We plot in the direction of the ray R (direction of propagation of energy, direction of the Poynting vector) the two possible values of the radial velocities corresponding to the two eigen polarization states that can propagate with this direction of ray.

It is a more complex surface, with two shells, useful to construct the refracted or reflected rays in an anisotropic medium

•Index surface:

We plot in the direction of the wave vector k (perpendicular to the wave surface) the two possible values of the index of refraction corresponding to the two eigen polarization states that can propagate in that direction of k.

It also has two shells, it will be useful later to calculate more easily the path or phase differences when propagating through an anisotropic medium

Coupled Mode Analysis

Chapter 4

Physics 208, Electro-optics

Peter Beyersdorf

Class Outline

- The dielectric tensor
- Plane wave propagation in anisotropic media
- The index ellipsoid
- Phase velocity, group velocity and energy
- Crystal types
- Propagation in uniaxial crystals
- Propagation in biaxial crystals
- Optical activity and Faraday rotation
- Coupled mode analysis of wave propagation

The Dielectric Tensor

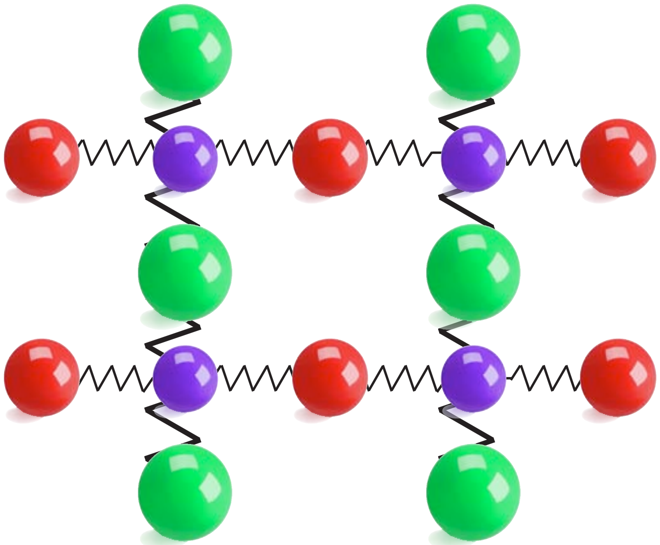
- ϵ relates the electric field to the electric displacement by

$$\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$$

- In anisotropic materials the polarization may not be in the same direction as the driving electric field. Why?

Anisotropic Media

Charges in a material are the source of polarization. They are bound to neighboring nuclei like masses on springs.

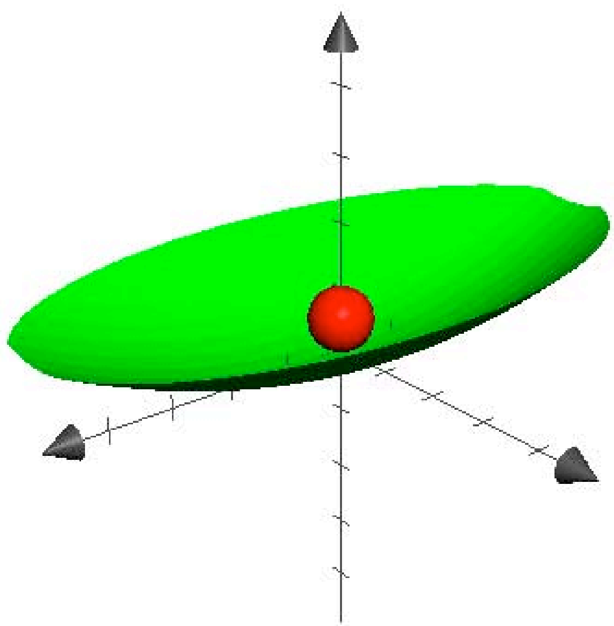


In an anisotropic material the stiffness of the springs is different depending on the orientation.

The wells of the electrostatic potential that the charges sit in are not symmetric and therefore the material response (material polarization) is not necessarily in the direction of the driving field.

Analogies

What every-day phenomena can have a response in a direction different than the driving force?



The Dielectric Tensor

The susceptibility tensor χ relates the polarization of the material to the driving electric field by

$$\vec{P} = \epsilon_0 \bar{\chi} \vec{E}$$

or

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \epsilon_0 \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Thus the material permittivity ϵ in $\vec{D} = \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$

is also a tensor $\bar{\epsilon} = \epsilon_0 \left(\bar{I} + \bar{\chi} \right)$

Tensor Notation

For brevity, we often express tensor equations such as

$$\vec{D} = \epsilon_0 \left(\bar{I} + \bar{\chi} \right) \vec{E}$$

in the form

$$D_i = \epsilon_0 (\delta_{ij} + \chi_{ij}) E_j$$

where summation over repeated indices is assumed, so that this is equivalent to

$$D_i = \sum_j \epsilon_0 (\delta_{ij} + \chi_{ij}) E_j$$

Dielectric Tensor Properties

The dielectric tensor is Hermetian such that

$$\epsilon_{ij} = \epsilon_{ji}^*$$

In a lossless material all elements of ϵ are real so the tensor is symmetric $\epsilon_{ij} = \epsilon_{ji}$ and can be described by 6 (rather than 9) elements

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

Plane Wave Propagation in Anisotropic Media

For a given propagation direction in a crystal (or other anisotropic material) the potential wells for the charges will be ellipsoidal, and so there will be two directions for the driving field for which the material response is in the same direction.

These two polarization states each have a (different) phase velocity associated with them. Light polarized at either of these angles will propagate through the material with the polarization unchanged.

Wave Equation in Anisotropic Materials

From Maxwell's equations in a dielectric (ρ=J=0), using d/dt→iω and ∇→i$\vec{k}$ we have

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$i\vec{k} \times \vec{E} + i\omega\vec{B} = 0 \qquad i\vec{k} \times \vec{H} - i\omega\vec{D} = \vec{J}$$

$$\vec{k} \times \vec{E} = -\omega\mu\vec{H} \qquad \vec{k} \times \vec{H} = \omega\epsilon\vec{E}$$

$$\vec{k} \times \vec{k} \times \vec{E} + \omega^2\mu\epsilon\vec{E} = 0$$

Which is the wave equation for a plane wave, most easily analyzed in the **principle coordinate system** where ϵ is diagonal (i.e. a coordinate system aligned to the crystal axes).

Wave Equation in Anisotropic Materials

$$\vec{k} \times \vec{k} \times \vec{E} + \omega^2 \mu \epsilon \vec{E} \quad = \quad 0$$

or

$$\begin{bmatrix} \omega^2 \mu \epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \mu \epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \mu \epsilon_z - k_x^2 - k_y^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

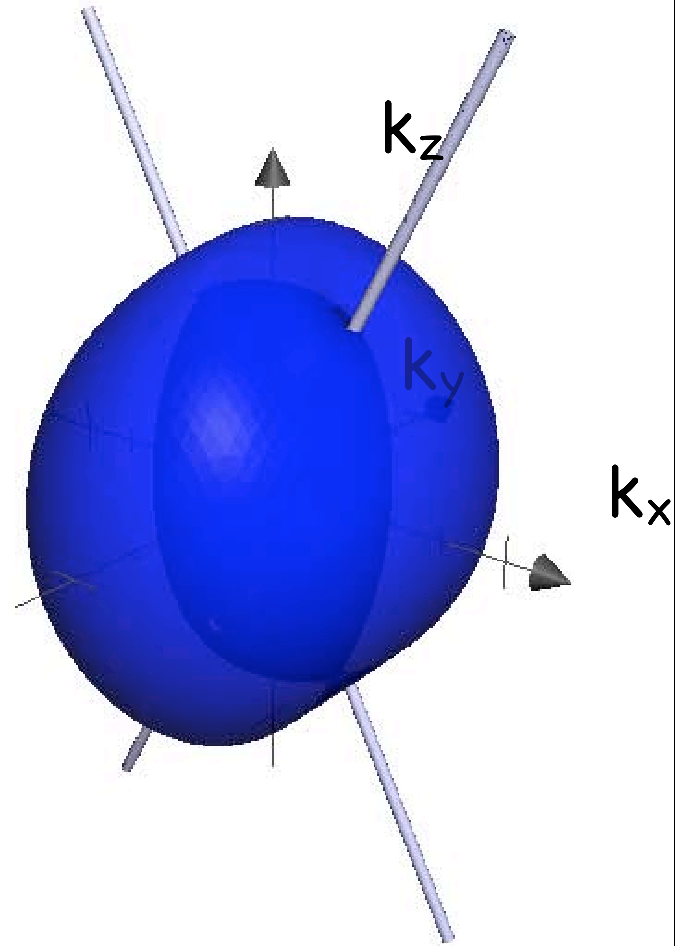
for which the determinant of the matrix must be zero for solutions (other than k=ω=0) to exist

$$\begin{vmatrix} \omega^2 \mu \epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \mu \epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \mu \epsilon_z - k_x^2 - k_y^2 \end{vmatrix} = 0$$

This defines two surfaces in k-space called the normal shells.

Normal Shells

In a given direction, going out from the origin, a line intersect this surface at two points, corresponding to the magnitude of the two k-vectors (and hence the two values of the phase velocity) wave in this direction can have.



The two directions where the surfaces meet are called the optical axes. Waves propagating in these directions will have only one possible phase velocity.

Wave Equation in Anisotropic Materials

$$\vec{k} \times \vec{k} \times \vec{E} + \omega^2 \mu \epsilon \vec{E} = 0$$

or

$$\begin{bmatrix} \omega^2 \mu \epsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2 \mu \epsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2 \mu \epsilon_z - k_x^2 - k_y^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = E_0 \begin{bmatrix} \frac{k_x}{k^2 - \omega^2 \mu \epsilon_x} \\ \frac{k_y}{k^2 - \omega^2 \mu \epsilon_y} \\ \frac{k_z}{k^2 - \omega^2 \mu \epsilon_z} \end{bmatrix}$$

where $E_0^2 = E_x^2 + E_y^2 + E_z^2$

Example

Find the possible phase velocities ($v_p = \omega / k$) for a wave propagating along the x-axis in a crystal, and the associated polarization directions.

L'Hopital's Rule

$$v_p = \frac{1}{\sqrt{\mu \epsilon_y}} \quad \text{or} \quad v_p = \frac{1}{\sqrt{\mu \epsilon_z}}$$

L'Hopital's rule can be applied to limit problems if the following conditions are met:

(1) The limit is written as a quotient AND
(2) The quotient is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

polarized in the y or z direction respectively

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

L'Hopital's rule can be applied several times, as long as the quotient is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Wave Propagation in Anisotropic Materials

Gauss' law $\vec{\nabla} \cdot \vec{D} = \rho$
in a dielectric (ρ=0) can be written as

$$i\vec{k} \cdot \vec{D} = 0$$

implying the propagation direction $\hat{k}$ is orthogonal to the displacement vector $\vec{D}$.

But the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ which gives the direction of energy flow is orthogonal to the electric field $\vec{E}$, not the displacement vector. Thus energy does not flow in the direction of the wave's propagation if the polarization of the wave ($\hat{E}$) is not an eigen-vector of the material's dielectric tensor!

Spatial Walkoff

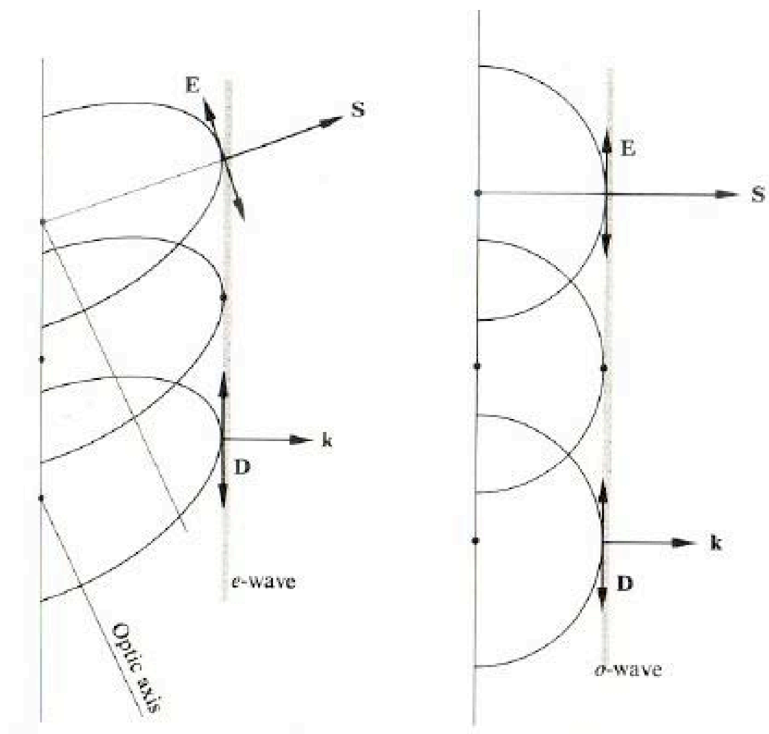


Image 8.26 from Hecht

Eigenstates of a material

For any direction of propagation there exists two polarization directions in which the displacement vector is parallel to the transverse component of the electric field. These are the eigenstates of the material.

Requiring $\vec{D}$ and $\vec{E}$ be parallel in the matrix equation

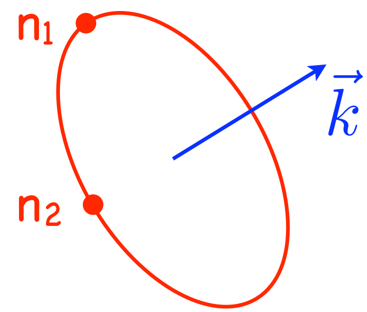
$$\vec{D} = \bar{\epsilon}\vec{E}$$

is equivalent to finding the eigenvalues and eigenvectors of ϵ satisfying

$$\lambda_i \vec{u_i} = \bar{\epsilon}\vec{u_i}$$

Index Ellipsoid

Index of refraction as a function of polarization angle

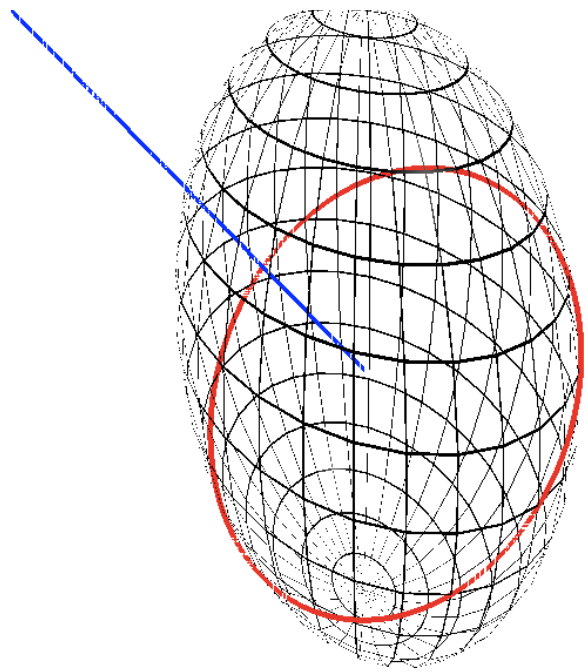


The polarization directions that have a max and min index of refraction form the major and minor axes of an ellipse defining n(θ) the index fora wave with the electric displacement vector at an angle of θ in the transverse plane.

Index Ellipsoid

The sum of all index ellipses plotted in three dimensions is the index ellipsoid, defined by

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$



Example

Find the angle of the direction of the optical axis for Lithium Niobate

Example

Find the angle of the direction of the optical axis for Topaz

Crystal Types

The index ellipsoid is determined by the three principle indices of refraction n_x, n_y and n_z.

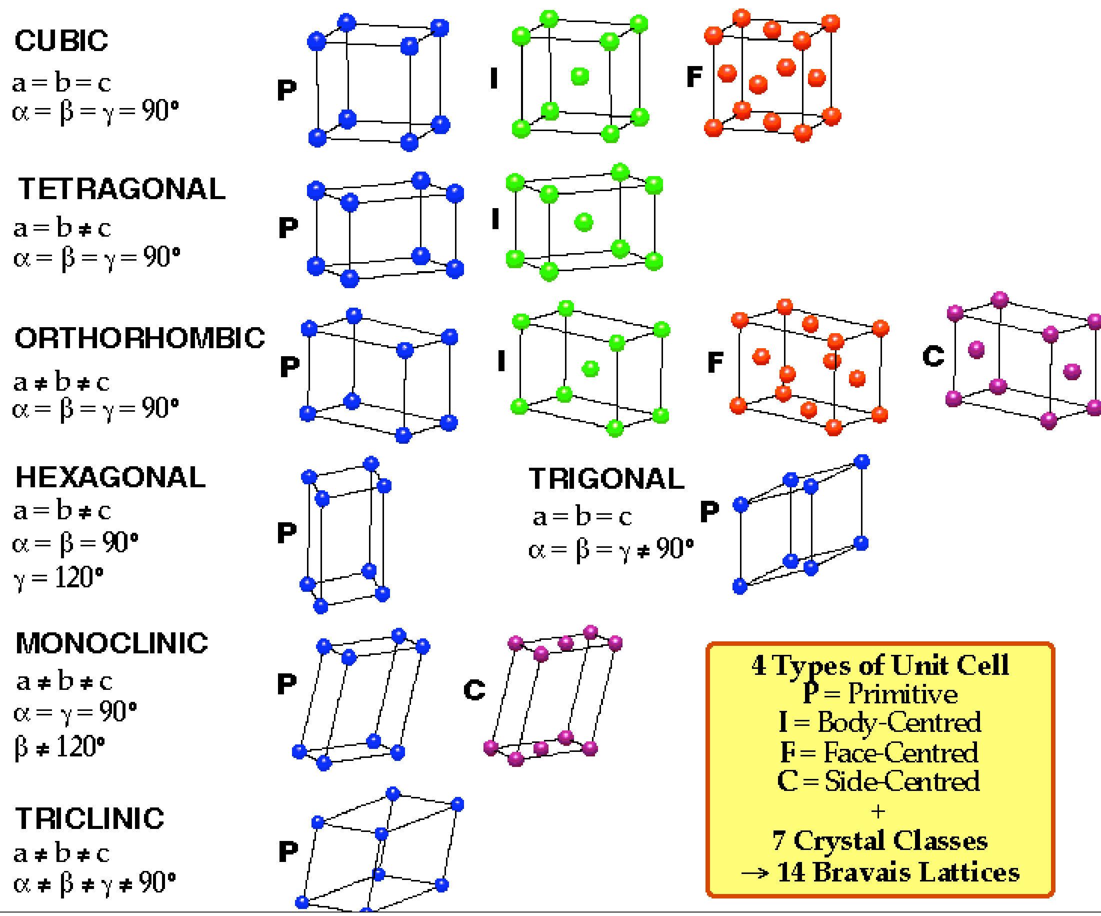
A crystal with $n_x \neq n_y \neq n_z$ will have _____ optical axes

A crystal with $n_x = n_y \neq n_z$ will have _____ optical axes

A crystal with $n_x = n_y = n_z$ will have ______ optical axes

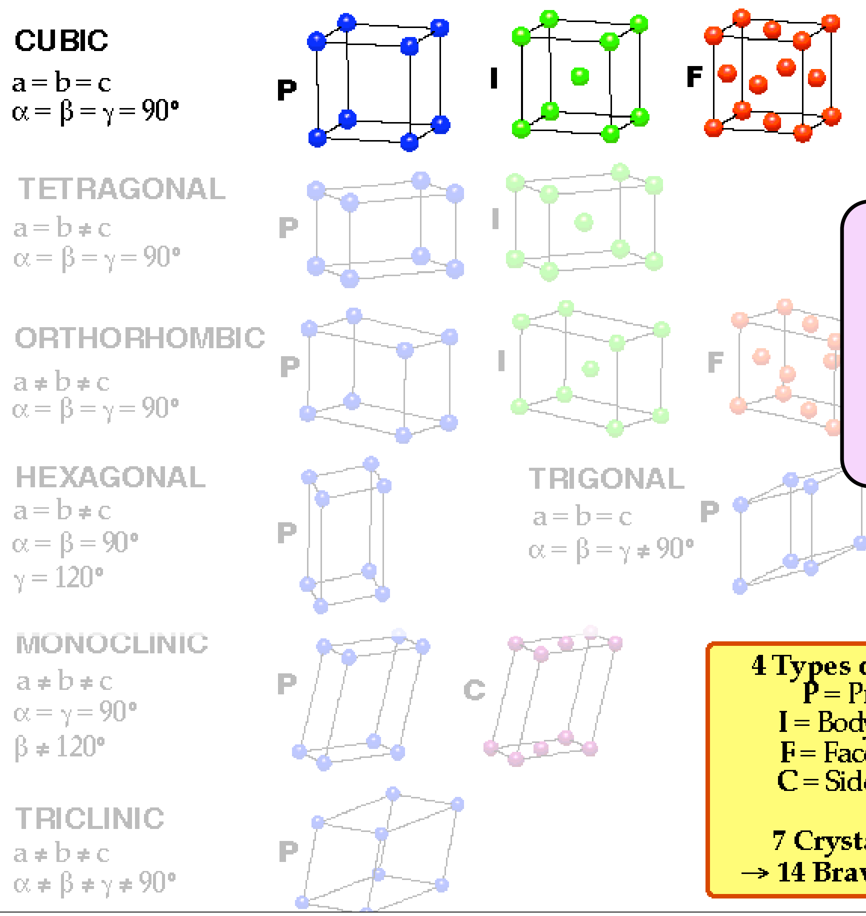
Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor



Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor



Isotropic

$$\epsilon = \epsilon_0 \begin{bmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

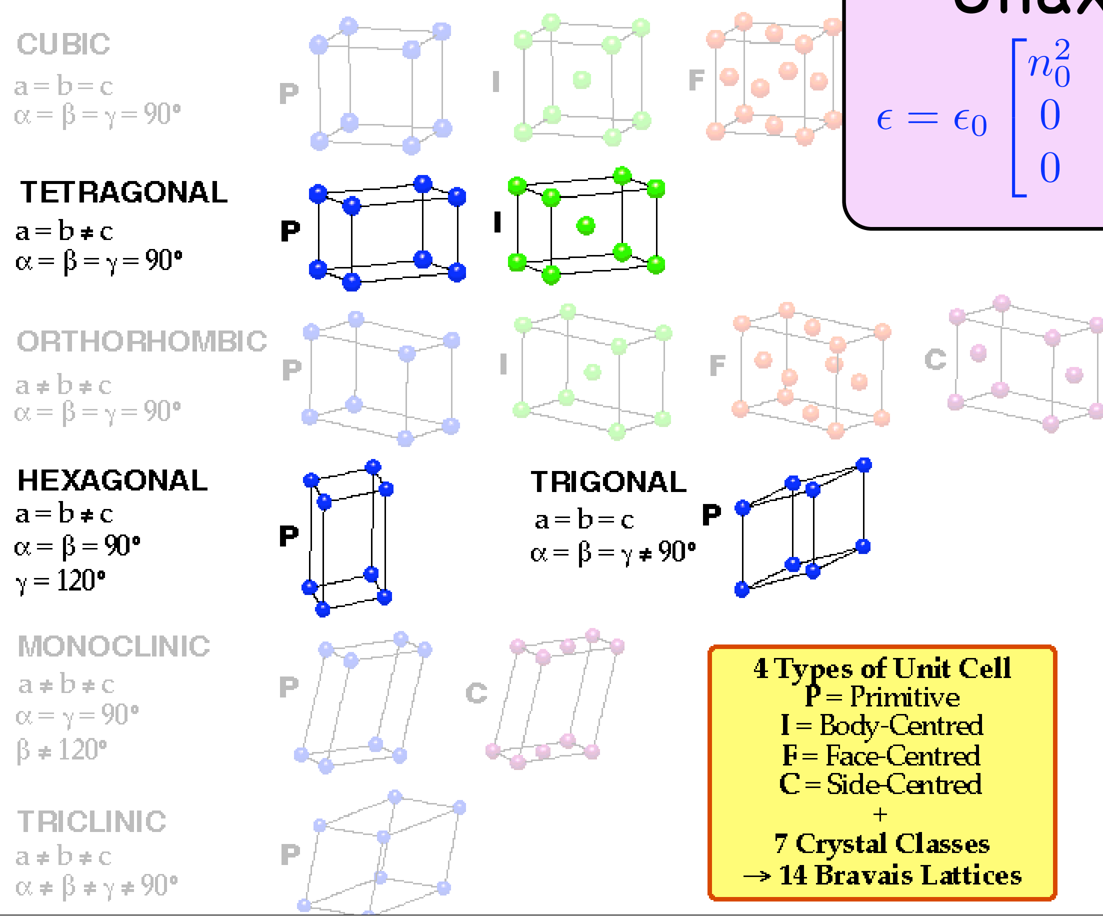
4 Types of Unit Cell
P = Primitive
I = Body-Centred
F = Face-Centred
C = Side-Centred
+
7 Crystal Classes
→ 14 Bravais Lattices

Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor

Unaxial

$$\epsilon = \epsilon_0 \begin{bmatrix} n_0^2 & 0 & 0 \\ 0 & n_0^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix}$$

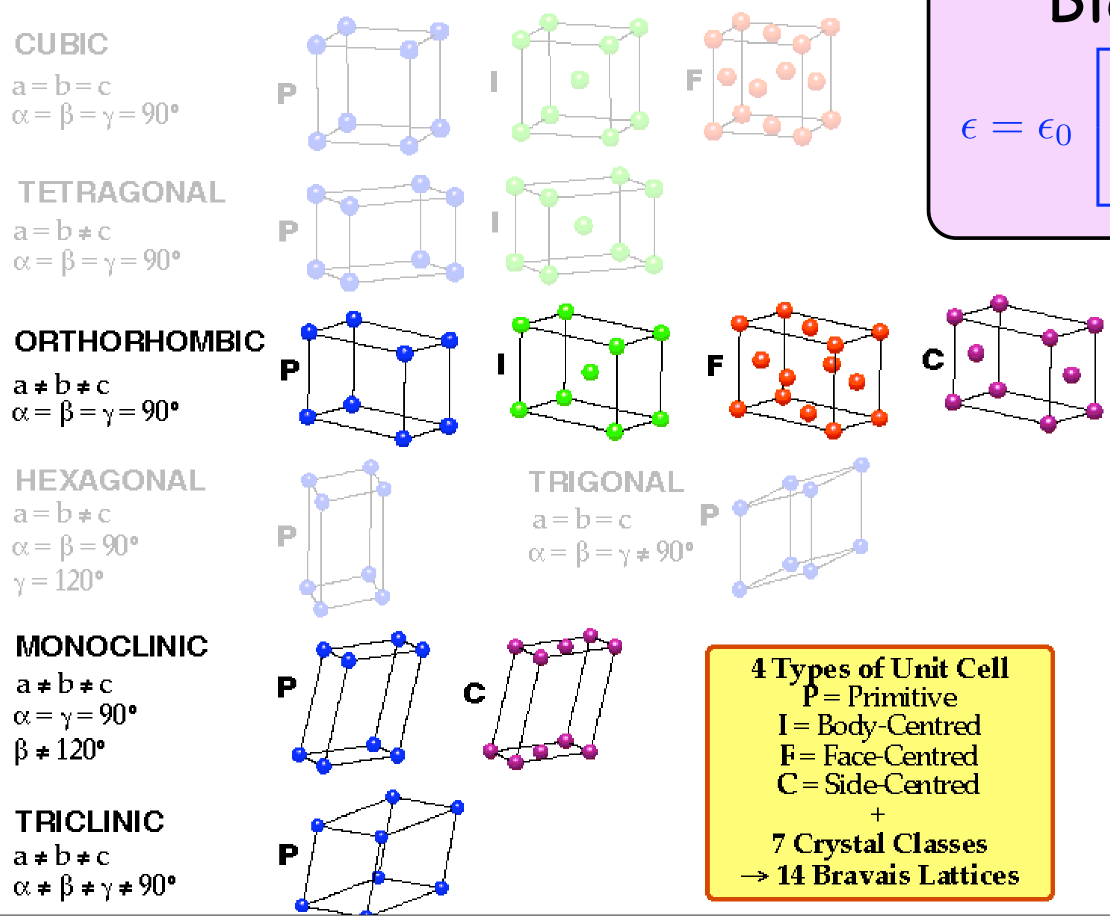


Crystal Properties

The geometry of a crystal determines the form of the dielectric tensor

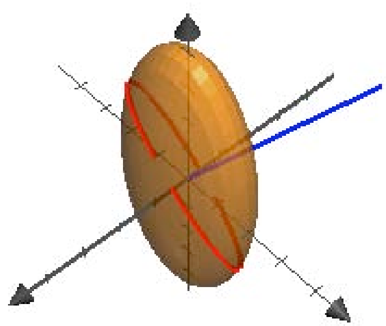
Biaxial

$$\epsilon = \epsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$$



Biaxial Crystals

By convention $n_x<n_y<n_z$ so the optical axis is always in the xz plane

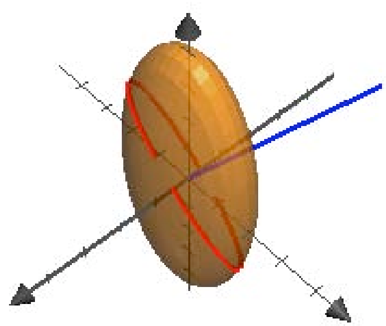


Uniaxial Crystals

By convention $n_o \equiv n_x = n_y$ and $n_e \equiv n_z$

The optical axis in a uniaxial crystal is always in the z-direction

A negative uniaxial crystal is one in which $n_e < n_o$ while a positive uniaxial crystal has $n_e > n_o$.



Example

A wave is propagating in the [1,1,1] direction in Mica, what are the two principle indices of refraction and in which direction are the eigenpolarizations?

Example

A wave is propagating in the [1,1,1] direction in Mica, what are the two principle indices of refraction and in which polarization directions do these correspond to?

plane of polarization is given by $\vec{k} \cdot \vec{r} = 0$ so x+y+z=0.
The intersection of this plane and the index ellipsoid, given by

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

is,

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{(x+y)^2}{n_z^2} = 1$$

Find extremes of $r^2=x^2+y^2+z^2$ subject to the preceding two constraints to find directions of transverse eigenpolarizations. Plug directions back into $r=(x^2+y^2+z^2)^{1/2}$ to find index for each polarization

Example Solution in Mathematica

```
nx = 1.552;
ny = 1.582;
nz = 1.588;
```

In[148]:=
```
eq1 = (x / nx) ^ 2 + (y / ny) ^ 2 + (z / nz) ^ 2 == 1;
eq2 = x + y + z == 0;
eq3 = r == Sqrt[x ^ 2 + y ^ 2 + z ^ 2];
```

In[151]:=
```
rsol = r /. Solve[{eq1, eq2, eq3}, {x, y, r}][[1]][[1]]
```

Out[151]= $\sqrt{\left(2.45481 + 0.\ z + 0.0269076\, z^2 - 0.0328601 \sqrt{1.29082 - 1.\ z}\ z \sqrt{1.29082 + z}\right)}$

In[208]:=
```
rsolmax = Maximize[rsol, z]
zmax = z /. rsolmax[[2]][[1]];
rmax = rsolmax[[1]];
rsolmin = Minimize[rsol, z]
zmin = z /. rsolmin[[2]][[1]];
rmin = rsolmin[[1]];
```

Out[208]= {1.58295, {z → -0.936293}}

Out[211]= {1.56375, {z → 0.280416}}

In[220]:=
```
solmax = Solve[{eq1, eq2, eq3}, {x, y, r}] /. z → zmax
xmax = x /. solmax[[1]];
ymax = y /. solmax[[1]];
solmin = Solve[{eq1, eq2, eq3}, {x, y, r}] /. z → zmin
xmin = x /. solmin[[1]];
ymin = y /. solmin[[1]];
```

Out[220]= {{r → 1.58295, x → -0.303467, y → 1.23976},
 {r → 1.56559, x → 1.22184, y → -0.285544}}

Out[223]= {{r → 1.56375, x → -1.21895, y → 0.938534},
 {r → 1.57116, x → 0.943902, y → -1.22432}}

In[228]:=
```
p1 =
 {rmax, {xmax, ymax, zmax} /
   Sqrt[xmax ^ 2 + ymax ^ 2 + zmax ^ 2]}
p2 =
 {rmin, {xmin, ymin, zmin} /
   Sqrt[xmin ^ 2 + ymin ^ 2 + zmin ^ 2]}
```

Out[228]= {1.58295, {-0.19171, 0.783194, -0.591485}}

Out[229]= {1.56375, {-0.779504, 0.600181, 0.179323}}

Index of extraordinary ray

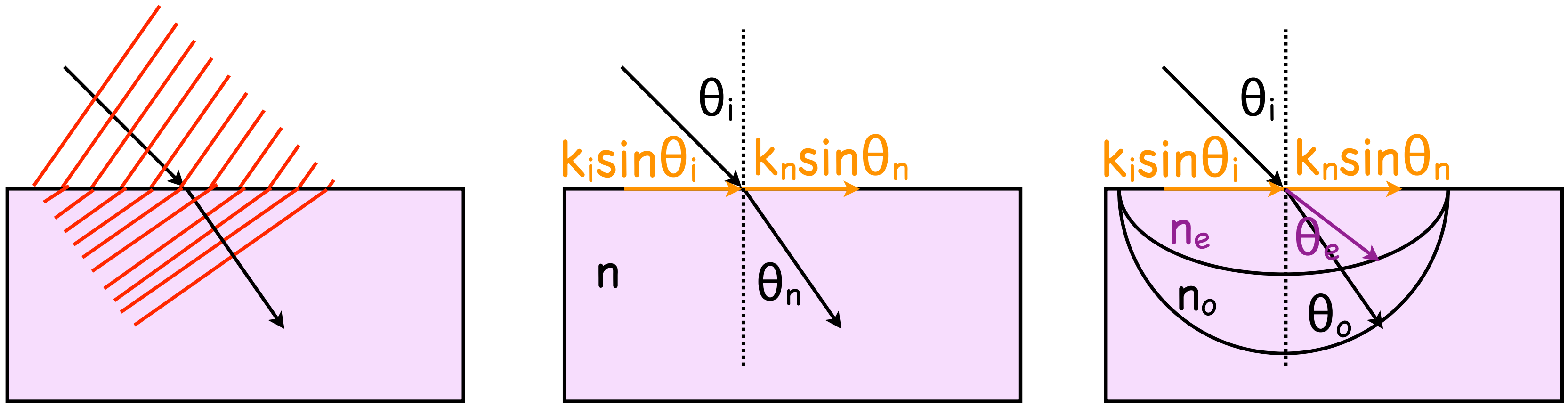
The index of refraction of the ordinary ray is always equal to the index seen when the ray propagates along the optical axis

The extra-ordinary ray sees an index of refraction that is a function of the ray propagation angle from the optical axis. In a uniaxial crystal

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

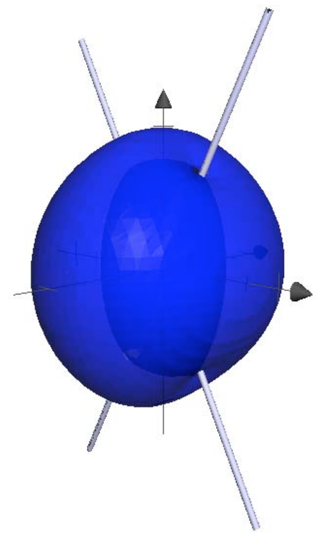
Double Refraction

Snell's law requires the tangential component of the k-vector be continuos across a boundary



Conical Refraction

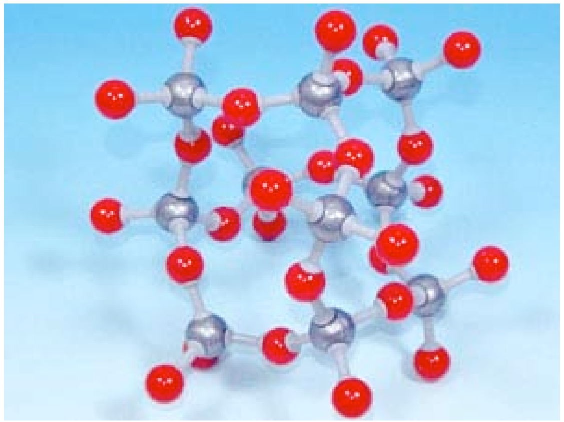
The normal surface is a surface of constant ω in k-space. Group velocity $v_g = \nabla_k \omega$ is perpendicular to the normal surfaces, but in a biaxial crystal the normal surfaces are singular along the optical axis.

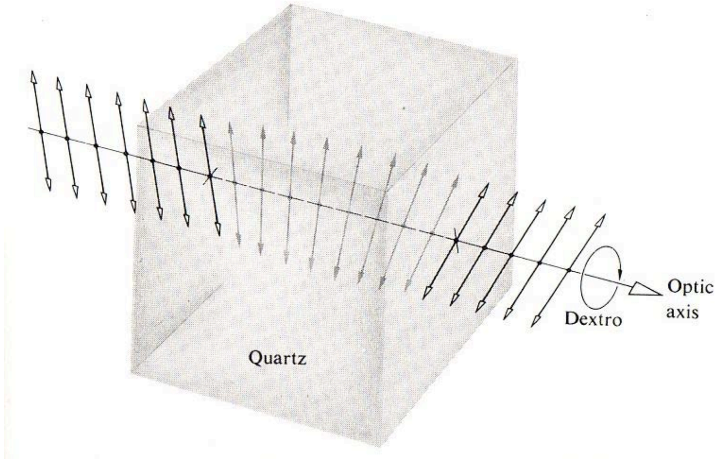


The surrounding region is conical, thus a wave along the optical axis will spread out conically

Optical Activity

Quartz and other materials with a helical molecular structure exhibit “Optical Activity”, a rotation of the plane of polarization when passing through the crystal.





Optical Activity

Optically active materials can be thought of as birefringent, having a different index of refraction for right-circular polarization and for left-circular polarization.

The specific rotary power describes how much rotation there is per unit length

$$\rho = \frac{\pi}{\lambda}(n_l - n_r)$$

The source of this optical activity is the induced dipole moment of the molecule formed by changing magnetic flux through the helical molecule

Optical Activity

The displacement vector thus has an additional contribution in a direction perpendicular to the electric field

$$\vec{D} = \epsilon \vec{E} + i\epsilon_0 \vec{G} \times \vec{E}$$

where $\vec{G} = (g_{ij}k_i k_j / k_0^2)\hat{k}$ is called the gyration vector and

$$\vec{G} \times \vec{E} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ G_x & G_y & G_z \\ E_x & E_y & E_z \end{bmatrix} = \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = [G]\vec{E}$$

so we can define an effective dielectric tensor

$$\epsilon' = \epsilon + i\epsilon_0[G] \quad \text{so that} \quad \vec{D} = \epsilon' \vec{E}$$

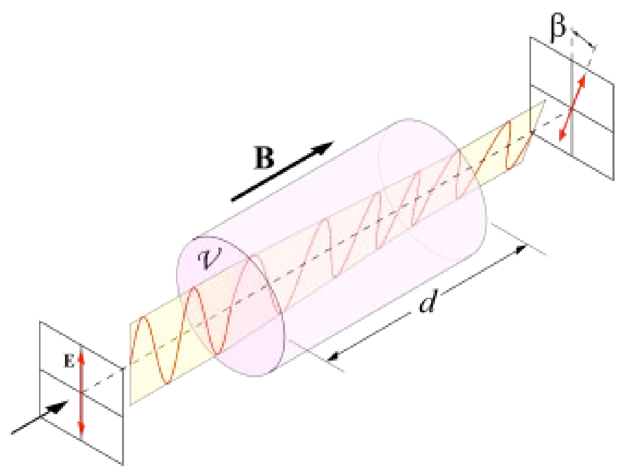
allowing the wave equation to be solved for eigenpolarizations of propagation and two corresponding indices of refraction that depend on [G]

Faraday Rotation

A form of optical activity induced by an external magnetic field.

$$\rho = VB$$

where V is called the Verdet constant. The sense of rotation depends on the direction of propagation relative to the direction of the magnetic field



Faraday Rotation

The motion of the charges in a material driven by the electric field feel a Lorentz force $q\vec{v} \times \vec{B}$ resulting in an induced dipole with a term proportional to $\vec{B} \times \vec{E}$ such that

$$\vec{D} = \epsilon\vec{E} + i\epsilon_0\gamma\vec{B} \times \vec{E}$$

where $\gamma = -V n_0 \lambda_0 / \pi$.

For Faraday rotation the imaginary term is proportional to B, while for optical activity it is proportional to k.

Thus a wave double passing (in opposite directions) a material will net zero polarization rotation from optical activity, but twice the one-way rotation due to Faraday rotation.

Exercise

Given an isotropic material that is perturbed such that

$$\vec{D} = \epsilon\vec{E} + i\Delta\epsilon\vec{E}$$

where $\Delta\epsilon$ is

$$\Delta\epsilon = \epsilon_o \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix}$$

find the eigenpolarizations for the electric field and associated indices of refraction. If a linearly polarized plane wave were to propagate along the direction of G through a thickness d of such material, what would the output wave look like?

Exercise

```
In[1]:= M = ({{ϵ, 0, 0}, {0, ϵ, 0}, {0, 0, ϵ}} -
          I ϵ0 {{0 , -Gz, Gy}, {Gz, 0, -Gx}, {-Gy, Gx, 0}})
        MatrixForm[M]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} \epsilon & i\,Gz\,\epsilon 0 & -i\,Gy\,\epsilon 0 \\ -i\,Gz\,\epsilon 0 & \epsilon & i\,Gx\,\epsilon 0 \\ i\,Gy\,\epsilon 0 & -i\,Gx\,\epsilon 0 & \epsilon \end{pmatrix}$$

```
In[38]:= NumericalValues =
          {ϵ → 1.5^2, Gx → 0, Gy → 0 , Gz → 0.001,  ϵ0 → 1}

Out[38]= {ϵ → 2.25, Gx → 0, Gy → 0, Gz → 0.001, ϵ0 → 1}

In[5]:= Simplify[Eigensystem[M]]
```

Out[5]=

$$\left\{\left\{\epsilon,\ \epsilon - \sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2},\ \epsilon + \sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}\right\},\right.$$

$$\left\{\left\{\frac{Gx}{Gz}, \frac{Gy}{Gz}, 1\right\}, \left\{\frac{Gx\,Gy\,\epsilon 0 + i\,Gz\sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}}{Gy\,Gz\,\epsilon 0 - i\,Gx\sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}},\right.\right.$$

$$\left. -\frac{i\,(Gx^2+Gz^2)\ \epsilon 0}{i\,Gy\,Gz\,\epsilon 0 + Gx\sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}},\ 1\right\},$$

$$\left\{\frac{Gx\,Gy\,\epsilon 0 - i\,Gz\sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}}{Gy\,Gz\,\epsilon 0 + i\,Gx\sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}},\right.$$

$$\left.\left.\left.\frac{i\,(Gx^2+Gz^2)\ \epsilon 0}{-i\,Gy\,Gz\,\epsilon 0 + Gx\sqrt{(Gx^2+Gy^2+Gz^2)\ \epsilon 0^2}},\ 1\right\}\right\}\right\}$$

```
In[39]:= IndexofRefraction =
          Sqrt[Eigenvalues[M /. NumericalValues]]
         val1 =
           (Eigensystem[M /. NumericalValues])[[2]][[1]][[1]];

Out[39]= {1.50033, 1.5, 1.49967}

In[41]:= EigenPolarization =
          Chop[Eigenvectors[M /. NumericalValues] / val1]

Out[41]= {{1., -1. ⅈ, 0}, {0, 0, 1.41421 ⅈ}, {1., 1. ⅈ, 0}}
```

Coupled Mode Analysis

When the eigenstates of an unperturbed system are known, a small perturbation can be treated as a coupling between these states.

Rather than rediagonalizing the matrix for ϵ one could solve the wave equation with the perturbed matrix $\epsilon+\Delta\epsilon$

$$E = A_1 e^{i(k_1 z - \omega t)}\hat{e}_1 + A_2 e^{i(k_2 z - \omega t)}\hat{e}_2$$

with ϵ

$$\downarrow$$

$$E = A_3 e^{i(k_3 z - \omega t)}\hat{e}_3 + A_4 e^{i(k_4 z - \omega t)}\hat{e}_4$$

or

$$E = A_1(z) e^{i(k_1 z - \omega t)}\hat{e}_1 + A_2(z) e^{i(k_2 z - \omega t)}\hat{e}_2$$

with $\epsilon+\Delta\epsilon$

Exercise

Given an isotropic material that is perturbed such that

$$\vec{D} = \epsilon\vec{E} + i\Delta\epsilon\vec{E}$$

where $\Delta\epsilon$ is

$$\Delta\epsilon = \begin{bmatrix} 0 & -G_z & G_y \\ G_z & 0 & -G_x \\ -G_y & G_x & 0 \end{bmatrix}$$

find the coupled mode expression for the electric field for a linearly polarized plane wave propagating along the direction of G.

From

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} - \omega^2 \mu (\epsilon + \Delta\epsilon) \vec{E} = 0$$

Find solutions of the form

$$E = E_x(z) e^{i(kz - \omega t)} \hat{\imath} + E_y(z) e^{i(kz - \omega t)} \hat{\jmath}$$

where $\frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$ i.e we are finding slowly varying amplitudes $E_x(z)$ and $E_y(z)$ to the unperturbed solutions.

Since E varies only along z $\vec{\nabla} \to \hat{k} \frac{d}{dz}$

use $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\nabla^2 E + \vec{\nabla}(\vec{\nabla} \cdot \vec{E})$

$$-\frac{d^2}{dz^2} \vec{E} + \hat{k} \frac{d}{dz} \left(\hat{k} \frac{d}{dz} \cdot \vec{E} \right) - \omega^2 \mu (\epsilon + \Delta\epsilon) \vec{E} = 0$$

$$-\frac{d^2}{dz^2} \underbrace{\left[\vec{E} - \hat{k} (\hat{k} \cdot E) \right]}_{(E_x \hat{\imath} + E_y \hat{\jmath} + E_z \hat{k}) - \hat{k} E_z} - \omega^2 \mu (\epsilon + \Delta\epsilon) \vec{E} = 0$$

$(E_x \hat{\imath} + E_y \hat{\jmath} + E_z \hat{k}) - \hat{k} E_z$ is transverse component of $\vec{E}$, $\vec{E}_T$

for $\vec{E} \cdot \vec{k} = 0$ (assume isotropic medium) $\vec{E}_T = \vec{E}$

$$-\frac{d^2}{dz^2} \vec{E} + \omega^2 \mu (\epsilon + \Delta\epsilon) \vec{E} = 0$$

← The 1-dimensional wave equation in isotropic media

$$-\frac{d^2}{dz^2}\left(E_x(z)\,e^{i(kz-\omega t)}\right)+\omega^2\mu(\epsilon+\Delta\epsilon)\,E_x(z)\,e^{i(kz-\omega t)}\ \hat{i}$$

$$-\frac{d}{dz^2}\left(E_y(z)\,e^{i(kz-\omega t)}\right)+\omega^2\mu(\epsilon+\Delta\epsilon)\,E_y(z)\,e^{i(kz-\omega t)}\ \hat{j}\quad=0$$

using chain rule to evaluate terms like

$$-\frac{d^2}{dz^2}\left(E_x(z)\,e^{i(kz-\omega t)}\right)=-\frac{d}{dz}\left[E_x(z)\,ik\,e^{i(kz-\omega t)}+e^{i(kz-\omega t)}\frac{d}{dz}E_x(z)\right]$$

$$=+E_x(z)\,k^2e^{i(kz-\omega t)}-ik\,e^{i(kz-\omega t)}\frac{d}{dz}E_x(z)$$

$$-ik\,e^{i(kz-\omega t)}\frac{d}{dz}E_x(z)-e^{i(kz-\omega t)}\frac{d^2}{dz^2}E_x(z)$$

$$=\left[-\frac{d^2}{dz^2}E_x(z)-2ik\frac{d}{dz}E_x(z)+k^2E_x(z)\right]e^{i(kz-\omega t)}$$

using $k^2=\omega^2\mu\epsilon$ we get

This is from wave equation solutions to the unperturbed material

$$\left[\frac{d^2}{dz^2}E_x(z)+2ik\frac{d}{dz}E_x(z)+\omega^2\mu\Delta\epsilon\,E_x(z)\right]\hat{i}$$

$$+\left[\frac{d^2}{dz^2}E_y(z)+2ik\frac{d}{dz}E_y(z)+\omega^2\mu\Delta\epsilon\,E_y(z)\right]\hat{j}\ =0$$

for slow variations $\frac{d^2}{dz^2}E << k\frac{d}{dz}E$ we can neglect $\frac{d^2}{dz^2}$ terms

Envelope varies slowly compared to optical frequency

$$\left[2ik\frac{d}{dz}E_x(z)+\omega^2\mu\Delta\epsilon\, E_x(z)\right]\hat{i}$$
$$+\left[2ik\frac{d}{dz}E_y(z)+\omega^2\mu\Delta\epsilon\, E_y(z)\right]\hat{j} \approx 0$$

for $G_x = G_y = 0;\ G_z = G$ we have $\Delta\epsilon = i\epsilon_0\begin{bmatrix} 0 & -G & 0 \\ G & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

giving

$$\left[2ik\frac{d}{dz}E_x(z)-\omega^2\mu(i\epsilon_0 G)\,E_y(z)\right]\hat{i}$$
$$+\left[2ik\frac{d}{dz}E_y(z)+\omega^2\mu(i\epsilon_0 G)\,E_x(z)\right]\hat{j} = 0$$

These coupled differential equations reduce to coupled equations that can be solved using linear algebra when we use phasors to represent the fields

let $E_x(z) = \mathrm{Re}\left[\tilde{E}_x(z)\right]$ where $\tilde{E}_x(z) = \tilde{E}_x\, e^{i(\kappa z+\alpha_x)}$

$E_y(z) = \mathrm{Re}\left[\tilde{E}_y(z)\right]$ $\qquad \tilde{E}_y(z) = \tilde{E}_y\, e^{i(\kappa z+\alpha_y)}$

then with $\frac{d}{dz}\tilde{E}(z) = i\kappa\tilde{E}(z)$ we have

$$-2k\kappa\tilde{E}_x - i\omega^2\mu\epsilon_0 G\tilde{E}_y = 0$$

and $$-2k\kappa\tilde{E}_y + i\omega^2\mu\epsilon_0 G\tilde{E}_x = 0$$

or with $k^2 = \omega^2\mu\epsilon$

$$\begin{bmatrix} -2k\kappa & -ik^2G \\ ik^2G & -2k\kappa \end{bmatrix}\begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = 0$$

$$\begin{bmatrix} -2k\chi & -ik^2G \\ ik^2G & -2k\chi \end{bmatrix} \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{bmatrix} = 0$$

requiring $\begin{vmatrix} -2k\chi & -ik^2G \\ ik^2G & -2k\chi \end{vmatrix} = 0$ for non trivial solutions to $\tilde{E}_x$ and $\tilde{E}_y$ to exist.

This gives $\chi = \pm \frac{1}{2} k G$

Assuming input field is polarized along x and has amplitude E_0

$$\mathrm{Re}[\tilde{E}_x(0)] = E_0 \quad \text{and} \quad \mathrm{Re}[\tilde{E}_y(0)] = 0$$

thus $\alpha_x = 0$

$$E_x(z) = E_0 \cos\left(\tfrac{1}{2} k G z\right)$$

Solving for $\tilde{E}_y$ gives $\tilde{E}_y(z) = i \frac{2\chi}{k} \frac{E_0}{G} e^{i\chi z}$

So $E_y(z) = -E_0 \sin\left(\tfrac{1}{2} k G z\right)$

Diagonalization

Consider the same problem where

$$\epsilon = \epsilon_0 \begin{bmatrix} n^2 & -iG & 0 \\ i\,G & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

Find the normal modes of propagation by finding the eigenvectors of the dielectric tensor

Diagonalization

Consider the same problem where

$$\epsilon = \begin{bmatrix} n^2 & -i\epsilon_0 G & 0 \\ i\epsilon_0 G & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

Find the normal modes of propagation by finding the eigenvectors of the dielectric tensor

$\lambda=n^2,\ n^2-\epsilon G, n^2+\epsilon G$

$\vec{u}=[0,0,1],\ [i,1,0],\ [-i,1,0]$

Summary

- Material polarization is not necessarily parallel to electric field in anisotropic materials
  - permittivity of material (ϵ) is a tensor relating E to D
  - Direction of wave vector and energy flow can be different
- Normal shells and index ellipsoid are geometrical constructions to visualize the propagation parameters of a wave

References

- Yariv & Yeh "Optical Waves in Crystals" chapter 4
- Hecht "Optics" section 8.4
- Fowles "Modern Optics" section 6.7