# Canonicity and representable relation algebras

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## What am I going to do?

In the 1960s, Monk proved that the variety RRA of representable relation algebras is *canonical.*

This is an instance of general results on canonical varieties due to Fine, van Benthem, and Goldblatt (1975–89).

Recent work has shown that RRA only just manages to be canonical:

- it has no canonical axiomatisation,
- any first-order axiomatisation of it has infinitely many non-canonical axioms.

This is connected to a recent counterexample to the converse of the Fine–van Benthem–Goldblatt theorems.



## Outline

1. Relation algebras; representations
2. Duality: atom structures, complex algebras
3. Canonical extensions; canonicity of RRA
4. RRA has no canonical axiomatisation
5. More



## 1. Relation algebras

These are algebras

$$\mathcal{A} = (A,\ +,\ -,\ 0,\ 1,\ 1',\ \breve{},\ ;)$$

satisfying certain equations laid down by Tarski in 1941.
(We will only need that $(A, +, -, 0, 1)$ is a boolean algebra.)

RA denotes the class (variety) of relation algebras.

Relation algebras are examples of *Boolean algebras with operators* (BAOs).

## Representations; RRA

Intended examples of relation algebras:

$$\begin{aligned}
\mathfrak{Re}(U) &= (\wp(U \times U),\ \cup,\ \setminus,\ \emptyset,\ U \times U,\ Id_U,\ -^{-1},\ |\ ) \\
\text{where}\quad Id_U &= \{(x,x) : x \in U\}, \\
R^{-1} &= \{(y,x) : (x,y) \in R\}, \\
R|S &= \{(x,y) : \exists z((x,z) \in R \wedge (z,y) \in S)\}.
\end{aligned}$$

A *representation* of a relation algebra $\mathcal{A}$ is an embedding $h : \mathcal{A} \to \prod_{i \in I} \mathfrak{Re}(U_i)$ for some sets $U_i$ (assumed pairwise disjoint).

$h$ provides a 'representation' of each $a \in \mathcal{A}$ as a *binary relation* $h(a)$ on the base set $\bigcup_{i \in I} U_i$.

$\mathcal{A}$ is said to be *representable* if it has a representation.
RRA denotes the class of representable relation algebras.



## Some facts about RRA

Lyndon (1950): RRA $\subset$ RA.

Tarski (1955): RRA is a variety — equationally axiomatised.

Monk (1964): RRA is not finitely axiomatisable.

Generally, RRA is rather wild.
Most questions about it seem to have the answer 'no'.



## 2. Duality: atom structures and complex algebras

Consider relation symbols $R_{1'}$ (unary), $R_{\smile}$ (binary), $R_{;}$ (ternary).
An *atom structure* is a relational structure of this type.

Given an atom structure $S = (A, R_{1'}, R_{\smile}, R_{;})$, we can form its *complex algebra* $\mathrm{Cm}\, S = (\wp(A),\ \cup,\ \setminus,\ \emptyset,\ A,\ 1',\ \breve{\ },\ ;)$, where

$$\begin{aligned}
1' &= \{x \in A : S \models R_{1'}(x)\}, \\
\breve{X} &= \{y \in A : \exists x \in X (S \models R_{\smile}(x,y))\}. \\
X\,;Y &= \{z \in A : \exists x \in X\ \exists y \in Y (S \models R_{;}(x,y,z))\}.
\end{aligned}$$

Under certain conditions on $S$, $\mathrm{Cm}\, S$ will be a relation algebra.

Any atomic relation algebra $\mathcal{A}$ has an atom structure, $\mathrm{At}\,\mathcal{A}$.
If $\mathcal{A}$ is complete atomic, $\mathrm{Cm}\,\mathrm{At}\,\mathcal{A} \cong \mathcal{A}$.

This can all be done for BAOs too.
E.g., we can form $\mathrm{Cm}\, S$ from an atom structure ('frame') $S$.



## 3. Canonical extensions (Jónsson–Tarski, 1951)

An *ultrafilter* of a relation algebra $\mathcal{A}$ is an uf. of its boolean reduct.

The set of ultrafilters of $\mathcal{A}$ forms an atom structure $\mathcal{A}_+$.
Operations: e.g., $\mathcal{A}_+ \models R_{;}(\alpha, \beta, \gamma)$ iff $a \in \alpha,\ b \in \beta \Rightarrow a\,;b \in \gamma$.

The *canonical extension* of $\mathcal{A}$ is $\mathcal{A}^\sigma = \mathrm{Cm}(\mathcal{A}_+)$.
($\mathcal{A}$ embeds canonically into $\mathcal{A}^\sigma$, so this is an extension of $\mathcal{A}$.)

Again, this can be done for BAOs.
For BAOs $(A, +, -, 0, 1, \Diamond)$ ($\Diamond$ a unary function), it's close to the famous *canonical model* construction in modal logic (Lemmon, 1966)
— useful in proving completeness of axioms over a class of frames.

## Canonicity of RRA

A class $\mathcal{K}$ of relation algebras/BAOs is *canonical* if $\mathcal{A} \in \mathcal{K} \Rightarrow \mathcal{A}^\sigma \in \mathcal{K}$.

**Theorem 1 (Monk, 1960s)** RRA *is canonical.*

Monk's proof was reported in McKenzie's Ph.D. thesis (1966).

We can now prove it using general results of Fine, van Benthem, & Goldblatt (1975–89) on canonicity of modal logics/varieties of BAOs.

**Fact 2 (Goldblatt)** *If $\mathcal{K}$ is an elementary class of atom structures, then the variety* $\mathsf{HSP}\{\mathrm{Cm}\, S : S \in \mathcal{K}\}$ *generated by $\mathcal{K}$ is canonical.*

**Proof. [of theorem 1]** By definition, RRA is generated by $\mathcal{K} = \{\mathrm{At}\,\mathfrak{Re}(U) : U \text{ a set}\}$. But $\mathcal{K}$ is elementary (exercise). ■



## Why is canonicity of RRA interesting?

- One of the few positive properties of RRA.
- Easier to work with $\mathcal{A}_+$ than $\mathcal{A}$.
- If $\mathcal{A} \in$ RRA then $\mathcal{A}^\sigma$ is 'completely representable' — it has a representation respecting all infs and sups in $\mathcal{A}^\sigma$.

  So canonical extensions of RRAs are rather well behaved.
- But, unlike in modal logic, knowing RRA is canonical doesn't seem to help to axiomatise it.
- It turns out that RRA *is only barely canonical.*

  The connection between $\mathcal{A}$ being representable and $\mathcal{A}^\sigma$ being representable is rather loose.

  The rest of the talk is mainly about this.



## 4. Canonical axioms for RRA?

An axiom $\tau$ of the signature of relation algebras is *canonical* if $\mathcal{A} \models \tau \Rightarrow \mathcal{A}^\sigma \models \tau$.

Any canonically axiomatised variety is canonical.

But the converse may not hold. Canonicity might come about 'in the limit' — an 'emergent property' of a set of non-canonical axioms.

**Question 3 (Venema, ∼1995)** Does RRA have a canonical axiomatisation – each individual axiom in it is canonical?

Bit different from usual questions about unpleasantness of axiomatisations of RRA.
It is undecidable whether an equation is canonical.



## Sahlqvist axioms

After 1997, Venema conjectured 'no', because of:

**Fact 4 (Venema, 1997)** RRA *has no Sahlqvist axiomatisation.*

Sahlqvist equations are defined syntactically. They are paradigm examples of canonical axioms. E,g., all positive equations are Sahlqvist.

Venema knew Sahlqvist equations are preserved under (Monk) completions of relation algebras. So fact 4 follows from

**Fact 5 (IH, 1997)** RRA *is not closed under completions.*

(This answered an implicit question of Monk (1970).)

But arbitrary canonical equations might not be preserved under completions. So how can we confirm Venema's conjecture?

## Key idea

Take a fixed axiomatisation $\Gamma$ of RRA.

Measure representability of a relation algebra by how much of $\Gamma$ it satisfies.

Canonicity of RRA: if $\mathcal{A}$ is fully representable ($\mathcal{A} \in$ RRA), so is $\mathcal{A}^\sigma$.

By first-order compactness, if there is a canonical axiomatisation of RRA, then *$\mathcal{A}^\sigma$ can be made as representable as we like, by making $\mathcal{A}$ sufficiently representable.*

So we construct relation algebras $\mathcal{A}$ of *unbounded representability,* whilst the representability of $\mathcal{A}^\sigma$ is *bounded.*
This shows RRA has no canonical axiomatisation.

The construction uses graphs of large chromatic number.



## Graphs

Here, graphs are undirected and loop-free: $G = (V, E)$ where $E \subseteq V \times V$ is irreflexive and symmetric.

For $k \geq 3$, a *cycle of length $k$* in $G$ is a sequence $v_1, \ldots, v_k \in V$ of distinct nodes with $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1) \in E$.

A subset $X \subseteq V$ is *independent* if $E \cap (X \times X) = \emptyset$.

For $k < \omega$, a *$k$-colouring* of $G$ is a partition of $V$ into $\leq k$ independent sets. The *chromatic number $\chi(G)$* of $G$ is the least $k < \omega$ such that $G$ has a $k$-colouring, and $\infty$ if there is no such $k$.

**Fact 6** A graph has a 2-colouring iff it has no cycles of odd length.



## Relation algebras from graphs

Given a graph $G = (V, E)$, we can form an atom structure $\alpha(G)$:

- The atoms are $1', \mathsf{r}_x, \mathsf{b}_x, \mathsf{g}_x$ (for $x \in V$).
- $\alpha(G) \models R_{1'}(a)$ iff $a = 1'$,
- $\alpha(G) \models R_{\smile}(a, b)$ iff $a = \breve{b}$,
- $\alpha(G) \models R_{;}(a, b, c)$ for all atoms $a, b, c$ except where:
  - one of $a, b, c$ is $1'$ and the other two are distinct,
  - $a = \mathsf{r}_x,\ \ b = \mathsf{r}_y,\ \ c = \mathsf{r}_z,$ where $\{x, y, z\}$ is independent,
  - similarly for $\mathsf{b}, \mathsf{g}$.

(We won't use these details.)

We write $\mathcal{A}(G) = \mathrm{Cm}\,\alpha(G)$ — a kind of 'Monk algebra'.

**Lemma 7** *For any graph $G$, $\mathcal{A}(G)$ is a relation algebra.*

What about representability. . . ?



## Chromatic number and axioms

**Theorem 8 (Hirsch–IH)** *For infinite $G$, $\mathcal{A}(G) \in$ RRA iff $\chi(G) = \infty$.*

In fact, *the higher $\chi(G)$ is, the more representable $\mathcal{A}(G)$ is*:

**Proposition 9** *There is a special equational axiomatisation $\{\gamma_n : n < \omega\}$ of RRA, obtained by games, such that*

1. *$\gamma_{n+1} \vdash \gamma_n$ for all $n$ (the axioms get stronger),*
2. *For all $n < \omega$ there is $k < \omega$ such that for all graphs $G$, $\chi(G) \geq k \Rightarrow \mathcal{A}(G) \models \gamma_n$,*
3. *For all $k < \omega$ there is $n < \omega$ such that for all infinite $G$, $\mathcal{A}(G) \models \gamma_n \Rightarrow \chi(G) \geq k$.*

Proof uses Ramsey-type arguments.

*The axioms pull the chromatic number along, and vice versa.*

## The example: Erdős graphs

Erdős (1959, probabilistic): for any finite $k$, there is a finite graph of chromatic number $> k$ and with no cycles of length $< k$.

Using a variant of Erdős's argument, it can be shown that

**Proposition 10** *For any $2 \leq k < \omega$, there is an inverse system*

$$\mathcal{S}: \quad G_0 \xleftarrow{\pi_0} G_1 \xleftarrow{\pi_1} \cdots$$

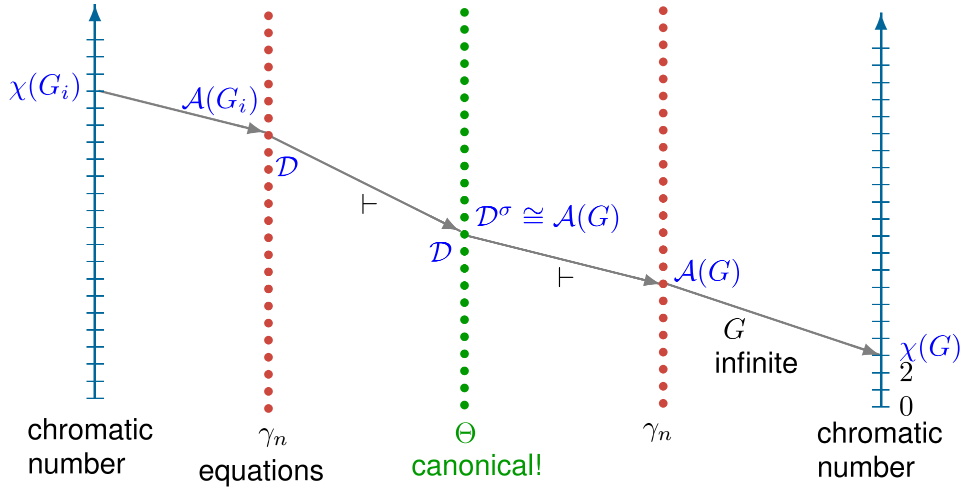
*of finite graphs $G_i$ with $\chi(G_i) = k$, and surjective 'p-morphisms' $\pi_i : G_{i+1} \twoheadrightarrow G_i$, such that $G = \varprojlim \mathcal{S}$ is infinite and $\chi(G) = 2$.*

Duality gives a direct system $\mathcal{A}(G_0) \to \mathcal{A}(G_1) \to \cdots$ of finite relation algebras and embeddings. Let $\mathcal{D} = \varinjlim \langle \mathcal{A}(G_i) : i < \omega \rangle$.

**Lemma 11 (essentially Goldblatt)** $\mathcal{D}^\sigma \cong \mathcal{A}(G)$.



**Theorem 12 (IH–Venema, 2002)** *There is no canonical first-order axiomatisation of* RRA*.*






## Worse...

**Corollary 13 (Venema, 1997)** RRA *has no Sahlqvist axiomatisation.*

Can strengthen theorem 12 to show

**Theorem 14** RRA *has no axiomatisation with only finitely many non-canonical first-order sentences.*

The proof uses an inverse system of finite graphs of chromatic number $k$, with inverse limit of chromatic number $l$ (for arbitrary $2 \leq l \leq k < \omega$).

**Corollary 15 (Monk, 1964)** RRA *is not finitely axiomatisable.*



## 5. More Erdős graphs!

These methods have been used in other applications:

[Hirsch–IH, 2002] *The class $\{S : \mathrm{Cm}\, S \in \mathsf{RRA}\}$ is not elementary.* This answers a question of Maddux (1982).

[Goldblatt–IH–Venema, 2003] *There are canonical varieties not generated by any elementary class of atom structures.* Modal version of this answers a question of Fine (1975).

Can make a counterexample with relation algebras. Suppose $\chi(G) > f(k) \Rightarrow \mathcal{A}(G) \models \gamma_k$. Let $G_k$ be a finite (Erdős) graph with $\chi(G) > f(k)$ and with no cycles of length $< k$. Put

$$\mathcal{K} = \{\mathcal{A} \in \mathsf{RA} : \forall k < \omega(|\mathcal{A}| \geq |\mathcal{A}(G_k)| \Rightarrow \mathcal{A} \models \gamma_k)\}.$$

Ex: the variety $\mathsf{HSP}\mathcal{K}$ is canonical but not elementarily generated.

## Conclusion

Monk showed RRA is canonical.

We have seen it's only barely canonical.

So even this rare silver lining of RRA has its cloud.

But the proof method applies to other interesting results.

Probabilistic constructions may find other applications in algebraic logic in future.



## References

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