TOPICS IN SET THEORY: Example Sheet 4 [1]

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This Example Sheet involves questions that require significant extensions of the material covered in lectures. Most are optional; on the other hand, several are core results, which are chosen to illustrate the richness and flexibility of forcing and set-theoretic methods.

1 ABSOLUTENESS RESULTS

(i) Show that the following properties are expressible by Σ_1-formulas:
- (a) α is not a cardinal;
- (b) $cf(\alpha) \leq \beta$;
- (c) α is singular;
- (d) $|\, x \,| \leq |\, y \,|$.

(ii) Show that the following properties are expressible by Π_1-formulas:
- (a) α is a cardinal;
- (b) $cf(\alpha)$ is regular;
- (c) α is weakly inaccessible (i.e. $\alpha \neq \lambda^+$ for all λ);
- (d) $y = P(x)$;
- (e) the formula $V_x = y$.

2 WELL-FOUNDED RELATIONS AND ABSOLUTENESS OF $\mathbb{P}$-NAMES

An $L_\in$-formula $\varphi(\overline{v})$ is *downward absolute* (respectively, *upward absolute*) if whenever $\mathbb{M}$ is a transitive class, $(\forall \overline{x} \in M)(\varphi(\overline{x}) \to \varphi(\overline{x})^{\mathbb{M}})$ (respectively, $(\forall \overline{x} \in M)(\varphi(\overline{x})^{\mathbb{M}} \to \varphi(\overline{x}))$. A class $\mathbb{M}$ is an *inner model* of ZF if M is transitive, $Ord \subseteq M$, and for every axiom φ of ZF, $\mathbb{M} \models \varphi$ (i.e. $ZF \vdash \varphi^{\mathbb{M}}$).

(i) Show that Σ_1-formulas are upward absolute and Π_1-formulas are downward absolute.

(ii) Prove that if $V \models (\kappa$ is an inaccessible cardinal$)$, then $\mathbb{A} \models (\kappa$ is an inaccessible cardinal$)$ whenever $\mathbb{A}$ is an inner model of ZF.

(iii) Let R be a binary relation over X. Prove that R is well-founded if and only if there is an order-preserving function $\rho : X \to Ord : xRy \to \rho(x) < \rho(y)$. Deduce that the property "R is well-founded" is Δ_1 and hence absolute for standard transitive models of ZF.

(iv) Formulating carefully any relevant results about absoluteness of definitions by transfinite recursion, deduce that the property "τ is a $\mathbb{P}$-name" is absolute for transitive models of ZF. [HINT. Recast the definition of $\mathbb{P}$-name as a definition by transfinite recursion over a well-founded relation.]

3 MODELS OF FRAGMENTS OF ZFC; CLOSURE PROPERTIES

For a cardinal κ, let H_κ be the family of sets hereditarily of cardinality less than κ.

[1] Comments, improvements and corrections will be much appreciated; please send to ok261@cam.ac.uk; rev. 14/12/2014.

(i) Prove if $\kappa = cf(\kappa) > \aleph_0$, then $H_\kappa \models ZFC^-$, where ZFC^- is the theory whose axioms are those of ZFC without the Power Set axiom.

(ii) Deduce that no proof of the existence of $\mathbb{R}$ avoids some non-trivial use of the Power Set axiom. [HINT. Consider $H_{\aleph_1}$.]

(iii) Assume $\kappa = cf(\kappa) \geq \aleph_2$. Let $\mathbb{M}$ be a countable elementary submodel of $(H_\kappa, \in)$.

  (a) Suppose $\varphi(v)$ is an $L_\in$-formula with free variable v (possibly with parameters from M) such that in any model of ZFC^- there is exactly one element that satisfies $\varphi(v)$. Show that if $a \in H_\kappa$ and $H_\kappa \models \varphi[a]$, then $a \in M$.

  (b) The ordinals ω and ω_1 belong to M.

  (c) If $\{a, A, B, f\} \subseteq M, a \in A$ and $f : A \to B$ is a function (in V), then $f(a) \in M$.

  (d) If $X \in M$ is countable (in V), then $X \subseteq M$.

  (e) For every $\alpha \in \omega_1 \cup \{\omega_1\}$, $\alpha \cap M$ is an ordinal.

  (f) If $A = \{A_\alpha : \alpha < \omega_1\} \in M$, then $A_\alpha \in M$ for every $\alpha \in \omega_1 \cap M$.

4 Replacement Schema and Power Set Axiom in Generic Extensions

Suppose $G \subseteq P$ is generic over $\mathbb{M}$.

(i) Prove that for each $L_\in$-formula $\varphi, \mathbb{M}[G] \models \mathit{Replacement}_\varphi$. [HINT. Find a bounding ordinal $\delta \in M$ such that $V_\delta^{\mathbb{M}}$ contains a *set* of $\mathbb{P}$-names whose values are sufficient to provide all the required elements in $M[G]$.]

(ii) Prove that $\mathbb{M}[G] \models \mathit{PowerSet}$. [HINT. For $y \subseteq x, y \in M[G]$, let $\tau = \{(r, \rho) \in P \times range(\dot{x}) : r \Vdash (\rho \in \dot{y})\}$; show $\tau[G] = y$; let $\dot{u} = \{(p, \sigma) : \sigma \in M, \sigma \subseteq P \times range(\dot{x})\}$. Check that $P(x)^{\mathbb{M}[G]} \subseteq u$ and explain why this suffices.]

5 Concerning Forcings, Anti-chains and Generic Sets

Suppose that $\mathbb{P}$ is a forcing in a model $\mathbb{M}$ of ZFC.

(i) Prove that a set $G \subseteq P$ is generic over $\mathbb{M}$ if and only if for every maximal anti-chain $A \in M$ of $\mathbb{P}, \mid G \cap A \mid = 1$. [HINT. One direction uses AC.]

(ii) Assume that the forcing $\mathbb{P}$ has a least element $0_{\mathbb{P}}$. A set $D \subseteq P$ is:

  (a) *pre-dense above* $p \in P$ if $(\forall q \in P)(q \geq p \to (\exists d \in D)(d$ and q are compatible); D is *pre-dense* if D is pre-dense above $0_{\mathbb{M}}$;

  (b) *dense above* $p \in P$ if $(\forall q \in P)(q \geq p \to (\exists d \in D)(d \geq q)$. So D is dense in $\mathbb{P}$ if D is dense above $0_{\mathbb{M}}$.

  Suppose that E is pre-dense in $\mathbb{P}$ and $G \subseteq P$ is generic over $\mathbb{M}$. Show that $G \cap E \neq \emptyset$.

  Suppose that E is pre-dense above $q \in \mathbb{P}$ and $G \subseteq P$ is generic over $\mathbb{M}$. Show that if $q \in G$, then $G \cap E \neq \emptyset$.

(iii) Deduce that the following are equivalent for a directed downward closed set $G \subseteq P, \mathbb{P} \in M$ where $\mathbb{M}$ is a transitive model of ZFC.

  (a) G is generic in $\mathbb{P}$ over $\mathbb{M}$;

  (b) $G \cap D \neq \emptyset$ for every dense open set $D \subseteq P$ in $\mathbb{M}$;

  (c) $G \cap C \neq \emptyset$ for every dense subset $C \subseteq P$ in $\mathbb{M}$;

  (d) $G \cap B \neq \emptyset$ for every pre-dense subset $B \subseteq P$ in $\mathbb{M}$;

  (e) $G \cap A \neq \emptyset$ for every maximal anti-chain $A \subseteq P$ in $\mathbb{M}$.

(iv) Suppose that $\mathbb{M}$ is a CTM of ZFC, $\mathbb{P} \in \mathbb{M}$, $E \subseteq P$, $E \in M$, and G is generic over $\mathbb{M}$. Prove that either $G \cap E \neq \emptyset$ or $(\exists q \in G)(\forall r \in E)(r$ and q are incompatible). [HINT. Consider $\{p \in P : (\exists r \in E)(r \leq p)\} \cup \{q \in P : (\forall r \in E)(r$ and q are incompatible$)\} \in \mathbb{M}$.]

(v) Suppose $\mathbb{M}$ is a CTM of ZFC and $\mathbb{P} \in \mathbb{M}$ is a separative forcing. Prove that there are $2^{\aleph_0}$ generic sets in $\mathbb{P}$ over $\mathbb{M}$.

6 OPTIONAL. FORCING, CHAIN CONDITIONS, AND ELEMENTARY SUBMODELS

For a forcing $\mathbb{P}$, a cardinal κ is *large enough* (for $\mathbb{P}$) if $\kappa = cf(\kappa) > \aleph_1$ and the set of dense subsets of $\mathbb{P}$ belongs to H_κ (so in particular, $\mathbb{P}$, the conditions in $\mathbb{P}$ and every dense subset of $\mathbb{P}$ all belong to H_κ). For a set N, a condition $p \in P$ is called N*–generic* if for every $D \in N$ which is a dense subset of $\mathbb{P}$, $D \cap N$ is pre-dense above p.

Suppose that κ is large enough for $\mathbb{P}$. Prove the following are equivalent:

(i) $\mathbb{P}$ has the countable chain condition;

(ii) for every countable elementary submodel N of H_κ, $0_{\mathbb{P}}$ is N–generic;

(iii) every countable subset X of H_κ is contained in an elementary submodel N of H_κ such that $0_{\mathbb{P}}$ is N–generic.

[HINT. For $(i) \Rightarrow (ii)$, consider an $A \in N$ maximal relative to the property of being an anti-chain contained in D. For $(iii) \Rightarrow (i)$, show if $A \in N$ is a maximal anti-chain, then $\overline{A} = \{p \in P : (\exists q \in A)(q \leq_{\mathbb{P}} p)\} \in N$ is dense.]

7 THE FORCING RELATION

Suppose that $\mathbb{P}$ is a non-trivial forcing, $p, q \in P$, and φ is a formula in the vocabulary of ZFC which may contain $\mathbb{P}$–names. Show:

(i) if $p \Vdash_{\mathbb{P}} \varphi$ and $p \leq_{\mathbb{P}} q$, then $q \Vdash_{\mathbb{P}} \varphi$;

(ii) if $q \Vdash_{\mathbb{P}} \varphi$ for every $q \geq_{\mathbb{P}} p$ such that $p \neq q$, then $p \Vdash_{\mathbb{P}} \varphi$;

(iii) if $(\nexists r)(p \leq_{\mathbb{P}} r \wedge r \Vdash_{\mathbb{P}} \varphi)$, then $p \Vdash_{\mathbb{P}} \neg\varphi$;

(iv) $(\exists r)(p \leq_{\mathbb{P}} r)(r$ *decides* $\varphi)$, i.e. either $r \Vdash_{\mathbb{P}} \varphi$ or $r \Vdash_{\mathbb{P}} \neg\varphi$.

(v) if p does not decide φ, then $\bigwedge_{i=1,2}(\exists r_i)(p \leq_{\mathbb{P}} r_i)(r_1 \Vdash_{\mathbb{P}} \varphi) \wedge (r_2 \Vdash_{\mathbb{P}} \neg\varphi)$.

8 NAMES

Suppose $G \subseteq P$ is generic over $\mathbb{M}$.

(i) Suppose $\sigma, \tau \in \mathbb{M}^{\mathbb{P}}$. Show $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$.

(ii) Suppose $\tau \in \mathbb{M}^{\mathbb{P}}$ and $range(\tau) \subseteq \{\dot{n} : n \in \omega\}$. Let $\sigma = \{\langle p, \dot{n}\rangle : (\forall q \in P)(\langle q, \dot{n}\rangle \in \tau \to p \perp q)\}$. Show that $\sigma_G = \omega \setminus \tau_G$. [HINT. The set $\{r \in P : (\exists p \leq r)(\langle p, \dot{n}\rangle \in \sigma \vee \langle p, \dot{n}\rangle \in \tau)\}$ is dense.]

(iii) Suppose A is an anti–chain in $\mathbb{P}$ and for each $a \in A, \tau_a$ is a $\mathbb{P}$–name. Show there exists a $\mathbb{P}$–name τ such that for every $a \in A$, if $a \in G$, then $\tau[G] = \tau_a[G]$, and $\tau[G] = \emptyset$ if $G \cap A = \emptyset$. [HINT. Suppose $\tau_a = \{(q_{a,j}, \tau_{a,j}) : j < i_a\}$. Consider the $\mathbb{P}$–name $\tau = \{(r, \tau_{a,j}) : a \in A, j < i_a, r \geq q_{a,j}$, and $r \geq a\}$. This provides a useful way of constructing names from other names indexed by elements of anti–chains.]

9 Nice Names and Bounds for the Continuum

Suppose $\mathbb{P} \in \mathbb{M}$ and $G \subseteq P$ is generic over $\mathbb{M}$. A name $\tau \in \mathbb{M}^{\mathbb{P}}$ is a *nice* name for a subset x of $\sigma \in \mathbb{M}^{\mathbb{P}}$ if $\tau = \bigcup\{A_\pi \times \{\pi\} : \pi \in range(\sigma)\}$, where A_π is an anti–chain in $\mathbb{P}$.

(i) Prove that for all $\sigma, \rho \in \mathbb{M}^{\mathbb{P}}$ there exists a nice name τ such that $\Vdash_{\mathbb{P}} (\rho \subseteq \sigma \to \rho = \tau)$. [Hint. For $\pi \in range(\sigma)$, let A_π be maximal relative to the properties (1) $(\forall p \in A_\pi)(p \Vdash_{\mathbb{P}} \pi \in \rho)$ and (2) A_π is an anti–chain in $\mathbb{P}$; refer to a previous Question to check that τ as defined works.]

(ii) Suppose ($\mathbb{P}$ is a c.c.c. forcing and λ is a cardinal$)^{\mathbb{M}}$. Let $\kappa^* = (|\ \mathbb{P}\ |^\lambda)^{\mathbb{M}}$. Then $(2^\lambda \leq \kappa^*)^{\mathbb{M}[G]}$. [Hint. Count the number of nice names for the members of $P(\lambda)^{\mathbb{M}[G]}$, remembering that $\mathbb{P}$ has the countable chain condition.]

(iii) Deduce that if (λ is a cardinal and $\lambda^{\aleph_0} = \lambda)^{\mathbb{M}}$, then there is a generic extension $\mathbb{M}[H]$ such that $(2^{\aleph_0} = \lambda)^{\mathbb{M}[H]}$.

10 Forcing and the Independence of Diamond

(i) Prove that the theory $ZFC + \diamondsuit$ is relatively consistent. [Hint. It may be easier to verify $\diamondsuit$ in its functional form (see Example Sheet 3). Let $I = \{\langle \alpha, \zeta \rangle : \zeta < \alpha < \omega_1^{\mathbb{M}}\}$ and consider the forcing $\mathbb{Q} = Fn(I, 2, \aleph_1^{\mathbb{M}})$. Show that $\mathbb{Q}$ is countably complete, and that if G is generic over $\mathbb{M}$, then in $\mathbb{M}[G]$, a $\diamondsuit$–sequence is provided by $\langle (\bigcup G)_\alpha : \alpha < \omega_1 \rangle$. For this, noticing that $\mathbb{Q}$ adds no new ω–sequences and $\aleph_1^{\mathbb{M}} = \aleph_1^{\mathbb{M}[G]}$, define a sequence of ordinals and conditions forcing an arbitrary club to intersect the family of guesses for a function $f : \omega_1 \to \omega_1$. (Refer to Kunen, *Set Theory*, chapter VII, or Shelah, *Proper and Improper Forcing*, chapter 1, if difficulties arise. Remark: $\diamondsuit$ is true in L (as was explained in the talk by Professor Mathias); this was the earliest proof of its consistency, due to Jensen.)]

(ii) Deduce that $\diamondsuit$ is independent of ZFC.

(iii) Show if (λ is a cardinal and $\lambda^{\aleph_0} = \lambda$ and $\diamondsuit)^{\mathbb{M}}$, then there is a generic extension $\mathbb{M}[H]$ such that $\mathbb{M}[H] \models (2^{\aleph_0} = \lambda$ and there is a Suslin tree).

(iv) Suppose that $\mathbb{M} \models$ ($\mathbb{P}$ is a c.c.c. forcing and $|\ \mathbb{P}\ | \leq \aleph_1$ and $\diamondsuit$). Prove that for every $G \subseteq P$ generic over $\mathbb{M}$, $\mathbb{M}[G] \models \diamondsuit$. [Hint. In $\mathbb{M}$, use $\diamondsuit$ to guess nice names for subsets of ω_1.]

(v) Deduce that $\diamondsuit$ does not imply $V = L$.

(vi) Suppose that $\mathbb{M} \models$ ($\mathbb{P}$ is a c.c.c. forcing) and $\mathbb{M}[G] \models \diamondsuit$, where $G \subseteq P$ is generic over $\mathbb{M}$. Show that $\mathbb{M} \models \diamondsuit$. [Hint. Recall the equivalent characterisations of $\diamondsuit$ from Example Sheet 3 and the lemma about approximating functions in c.c.c. generic extensions.]

(vii) Optional. Prove that $\clubsuit$ is independent of ZFC.

11 Adding Cohen Reals and Suslin Trees

(i) A tree $\mathbb{T}$ is *ever-branching* if for every $s \in T$, the set $\{t \in T : s \leq_{\mathbb{T}} t\}$ is not linearly ordered. Let $\mathbb{M}$ be a CTM such that ($\mathbb{T}$ is an ever-branching Suslin tree$)^{\mathbb{M}}$. Suppose that $(\mathbb{P} = Fn(\lambda \times \omega, 2, \aleph_0) \wedge \lambda \geq \aleph_0)^{\mathbb{M}}$. Prove that for any set $G \subseteq P$ generic over $\mathbb{M}$, $\mathbb{M}[G] \models$ ($\mathbb{T}$ is a Suslin tree).

(ii) Deduce that there is a model of ZFC in which there is a Suslin tree but CH fails.

Remark: So the existence of Suslin trees does not imply CH (nor *a fortiori* $\diamondsuit$). The proof that CH does not imply Suslin trees is due to Jensen.

12 Optional. Cohen Reals add Suslin Trees.

Suppose that $\mathbb{P}$ is a separative countable forcing and $\mathbb{M}$ is a CTM. Prove Shelah's theorem: in every $\mathbb{P}$-generic extension of $\mathbb{M}$, there is a Suslin tree. In particular, the Cohen forcing adds a Suslin tree.

Remark: This result provides a direct proof by forcing of the relative consistency of the failure of Suslin's Hypothesis. Adding Cohen reals has $\diamondsuit$–like combinatorial consequences; it is far from innocuous.

13 The Levy Collapse

The *Levy collapse* is the forcing $Lv(\kappa)$ defined as follows:

$$\{p : p \text{ is a finite function, } dom(p) \subseteq \kappa \times \omega \text{, and } (\forall \langle \alpha, n\rangle \in dom(p))(p(\alpha, n) \in \alpha)\}$$

with the partial ordering $p \leq q$ iff $p \subseteq q$.

Suppose that κ is strongly inaccessible in the countable transitive model $\mathbb{M}$.

(i) What is the cardinality of a maximal anti-chain in $Lv(\kappa)$?

(ii) What is the largest cardinal λ for which $Lv(\kappa)$ is λ–closed?

(iii) Which cardinals are preserved and which are collapsed under forcing with $Lv(\kappa)$?

(iv) Let H be generic in $Lv(\kappa)$ over $\mathbb{M}$. Prove that $\kappa = \aleph_1^{\mathbb{M}[H]}$.

(v) Suppose κ is strongly inaccessible in the CTM M and let H be generic in $Lv(\kappa)$ over $\mathbb{M}$. Prove that KH holds in the generic extension $M[H]$. [Hint. Use the previous part and refer to Example Sheet 3.] Comments: $Con(ZFC + KH)$ can be proved from $Con(ZFC)$ but a more complicated forcing is involved (see Kunen, chapter VII, Exercise H19). However, $Con(ZFC + \neg KH)$, which requires an iterated forcing argument, *does* require the hypothesis of the existence of a strongly inaccessible cardinal, because $\neg KH$ implies $\aleph_2$ is inaccessible in L (see Kunen, Chapter VII, Exercise B9).

14 Zermelo, Strong Inaccessibility and Second-Order Set Theory

(i) Suppose $\kappa = cf(\kappa) > \aleph_0$. Prove that κ is a strongly inaccessible cardinal if and only if $V_\kappa = H_\kappa$.

(ii) Suppose $\kappa = cf(\kappa) > \aleph_0$. Show that if $(H_\kappa, \in)$ is a model of ZFC, then κ is a strongly inaccessible cardinal.

(iii) Let ZFC^2 denote the second-order axiomatic system obtained from ZFC in which the Replacement Schema is expressed as the following single axiom with a universal second-order quantifier: $(\forall C)(\forall z \forall u \forall v)(\langle z, v\rangle \in C \wedge \langle z, u\rangle \in C \Rightarrow v = u) \Rightarrow \exists x \forall y(y \in x \Leftrightarrow \exists z(z \in a \wedge \langle z, y\rangle \in C))$, i.e. if C is a functional class and a is a set, then $\{C(z) : z \in a\}$, the image of a under C, is also a set. The intended interpretation of the second-order variables ranges over arbitrary subsets of the domain. Notice that models of ZFC^2 are well-founded.
Show that if κ is a cardinal such that $V_\kappa \models ZFC^2$, then κ is strongly inaccessible.

(iv) For each strongly inaccessible cardinal κ, there is up to isomorphism exactly one model of ZFC^2 with the set of ordinals of order type κ, namely, $(V_\kappa, \in)$. [Hint. Mostowski Collapse; induction on rank.]

15 The Generalized Δ-System Lemma

(i) Suppose that $\lambda < \kappa = cf(\kappa)$ and $\bigwedge_{\alpha<\kappa} \alpha^\lambda < \kappa$. Prove that if $\mid A \mid = \kappa$ and $x \in A$ implies $\mid x \mid < \lambda$, then there exists $B \subseteq A$ such that $\mid B \mid = \kappa$ and $(\exists r)(\forall x \in B)(x \neq y \to x \cap y = r)$. [Hint. This is a standard result. WLOG, $\bigcup A \subseteq \kappa$ and some fixed $\rho < \lambda$ is the order type of every $x = \langle x(\xi) : \xi < \rho \rangle \in A$. Using $\bigwedge_{\alpha<\kappa} \alpha^\lambda < \kappa$ and $\kappa = cf(\kappa)$, let ξ_0 be the minimal ξ such that $\{x(\xi) : \xi \in A\}$ is cofinal in κ; let $\sigma = sup\{x(\eta) + 1 : x \in A \wedge \eta < \xi_0\}$, so (*) $x \restriction \xi_0 \subseteq \sigma < \kappa$; now define by induction $\{x(\alpha) : \alpha < \kappa\}$ such that $x_\alpha(\xi_0) > max\{\sigma, sup\{x_\beta(\eta) : \beta < \alpha \wedge \eta < \rho\}\}$. Use (*) and $\sigma^\lambda < \kappa$ to refine $\{x(\alpha) : \alpha < \kappa\}$ and extract a root $r \subseteq \sigma$.]

(ii) Find a family of $\aleph_\omega$ finite sets such that no subfamily of size $\aleph_\omega$ has a root.

16 Optional. Moderately Large Cardinals Do Not Decide CH

A cardinal κ is called *Ramsey* if $\kappa \to (\kappa)^{<\omega}_2$, i.e. every colouring $f : [\kappa]^{<\omega} \to 2$ of all the finite subsets of κ by 2 colours has a monochromatic subset of cardinality κ. In 1961, Scott proved that the existence of large enough cardinals contradicts the axiom $V = L$. However, the Continuum Hypothesis appears to behave differently.

(i) Show that the set ${}^\lambda 2$ with the lexicographic order $\preceq_{lex}$ contains no increasing or decreasing sequences of length λ^+. [Hint. Otherwise suppose H is e.g. $\preceq_{lex}$-increasing of size λ^+; WLOG, $H = \{h_\alpha : \alpha < \lambda^+\}$ and for some least $\gamma \leq \lambda, \forall g, h \in H, g \restriction \gamma \neq h \restriction \gamma$. Now find $\xi^* < \gamma$ such that $\{h_\alpha \restriction \xi^* : \alpha < \lambda^+\}$ has cardinality λ^+.]

(ii) Prove that $2^\lambda \not\to (\lambda^+)^2_2$. [Hint. Otherwise, consider a homogeneous set Y of size λ^+ for the 2-colouring F of $[{}^\lambda 2]^2$ given by $F(\{f, g\}) = 0$ if and only if $f \preceq_{lex} g$. This is due to Sierpiński and Kurepa independently.]

(iii) Prove that if κ is a Ramsey cardinal, then κ is strongly inaccessible. [Hint. Regularity is easy: use the colouring $c(\{\alpha, \beta\}) = 0 \Leftrightarrow (\exists \xi)(\{\alpha, \beta\} \subseteq X_\xi)$ where $\kappa = \bigcup_{\zeta<\gamma} X_\zeta, \mid X_\zeta \mid < \kappa$; for strong inaccessibility, use the previous part.]

(iv) Prove that κ is a Ramsey cardinal if and only if for all $\beta < \kappa, \kappa \to (\kappa)^{<\omega}_\beta$. [Hint. For the hard direction, if $f : [\kappa]^{<\omega} \to \beta$, define a 2-colouring g as follows: $g(\{\xi_1, \ldots, \xi_n\}) = 0 \Leftrightarrow n = 2m \wedge f(\{\xi_1, \ldots, \xi_m\}) = f(\{\xi_{m+1}, \ldots, \xi_{2m}\})$. Notice if H is homogeneous for g and of cardinality κ, then $g \restriction [H]^n \equiv 0$ and so H is homogeneous for f also.]

(v) Suppose that $\mathbb{M} \models (\kappa$ is a Ramsey cardinal and $\mathbb{P}$ is a forcing of cardinality less than $\kappa)$. Let G be $\mathbb{P}$-generic over $\mathbb{M}$. Prove that $\mathbb{M}[G] \models (\kappa$ is a Ramsey cardinal). [Hint. If $\mathbb{M}[G] \models (\tau_G$ is a colouring of $[\kappa]^{<\omega})$, and this statement is forced by some condition $p \in G$, consider the colouring $g : [\kappa]^{<\omega} \to P(\mathbb{P} \times 2)$ defined by $g(a) = \{\langle q, x \rangle : p \leq_{\mathbb{P}} q \wedge q \Vdash (\tau(\dot{a}) = \dot{x})\}$. Note that $g \in M$ and $P(\mathbb{P} \times 2)$ has cardinality β for some $\beta < \kappa$ (by strong inaccessibility). Use the previous part, in $\mathbb{M}$, to obtain a homogeneous set $Y \in M$; show $p \Vdash (Y$ is homogeneous for $\tau)$.]

(vi) Deduce that CH is independent of $ZFC + (\exists \kappa)(\kappa$ is a Ramsey cardinal).

REMARK. This type of result in very general form is due to Levy and Solovay (1967); see A. Kanamori, *The Higher Infinite*, Springer, 2009. The power of large cardinals to decide a statement φ is thus circumscribed by the existence of forcings of relatively small size, if such forcings can be used to establish the relative independence of φ from ZFC.

17 OPTIONAL. EMBEDDINGS AND ELEMENTARY EMBEDDINGS OF V

An uncountable cardinal κ is *measurable* if there exists a κ complete non-principal ultrafilter U over κ. Suppose that $\mathbb{M}$ is a (transitive) model of ZFC. While the product structure $\mathbb{M}^\kappa$ does not satisfy much of ZFC, the ultrapower $\mathbb{M}^\kappa/U$ of $\mathbb{M}$ relative to U is a model of ZFC. Its elements are the equivalence classes $(f)_U$ of the relation $\equiv_U$ defined by $g \equiv_U h$ if $\{\alpha < \kappa : g(\alpha) = h(\alpha)\} \in U$ for $g, h \in M^\kappa$. (Care is needed in the case where M is a proper class; one uses the restricted equivalence classes consisting of elements of least rank, a stratagem called Scott's trick.) The mapping $\iota_U : M \to M^\kappa/U$ defined by $\iota_U(a) = (a^*)_U$ where $a^*(\alpha) = a$ for all $\alpha < \kappa$ is an elementary embedding, and so $\mathbb{M}^\kappa/U$ is a model of ZFC. If U is $\aleph_1$-complete, then $\mathbb{M}^\kappa/U$ is also well-founded and satisfies the hypotheses of Mostowski's Lemma. Its transitive collapse under the Mostowski collapsing map π is denoted $\mathbb{M}_U$. Taking $M = V$, the associated mapping $j_U : V \to V_U$, defined by $j_U(a) = \pi(\iota_U(a^*))$ is an elementary embedding of V into V_U.

(i) SCOTT'S THEOREM; 1961

Show that if there is a measurable cardinal, then $V \neq L$. [HINT. Suppose that κ is a measurable cardinal; what is the first ordinal moved by j_U? If κ is the smallest measurable cardinal, what will it be in the ultrapower?]

(ii) KUNEN'S THEOREM; 1971

Prove that no non-trivial elementary embedding exists from V into V. [HINT. Otherwise, use question 7(iv) from Example Sheet 3 to derive a contradiction; for details, see A. Kanamori, *The Higher Infinite*, Springer, 2009, pages 319–320.]

REMARK. Joel Hamkins has shown very recently that every countable model of set theory $(M, \in^M)$, including every well-founded model, is isomorphic to a submodel of its own constructible universe $(L^M, \in^L)$. In terms of embeddings (i.e. injective homomorphisms), there is an embedding $j : (M, \in^M) \to (L^M, \in^L)$ that is elementary for quantifier-free assertions in the language of set theory. See: Hamkins, J.D., J. Math. Log., 13, 1350006 (2013) [27 pages].

18 OPTIONAL. FORCING AND PARTIAL ISOMORPHISMS

Suppose that A and B are τ–structures in a vocabulary τ. Say that A and B are *partially isomorphic*, denoted $A \simeq_p B$ if some non-empty family $F \subseteq \text{PART}(A, B)$ of the partial isomorphisms from A to B is a *back-and-forth* set for A and B:

$$(\forall f \in F)(\forall a \in A)(\exists g \in F)(f \subseteq g \wedge a \in dom(g)) \text{ and}$$
$$(\forall f \in F)(\forall b \in B)(\exists g \in F)(f \subseteq g \wedge b \in range(g)).$$

(i) Prove that if τ, A, B are countable and $A \simeq_p B$, then $A \simeq B$.

(ii) Show that the converse of (i) fails. [HINT. Consider the linear orders $\mathbb{Q}$ and $\mathbb{R}$.]

(iii) Prove that if two structures A and B are partially isomorphic, then there is a forcing extension in which they are isomorphic.

REMARK. Partial isomorphism yields a characterization of elementary equivalence in the infinitary language $L_{\infty,\omega}$. For a recent introduction to these ideas, see J. Väänänen, *Models and Games*, Cambridge University Press, 2011.

19 Optional. Martin's Maximum

A forcing $\mathbb{P}$ is called *stationary-preserving* if $\mathbb{P}$ does not destroy stationary subsets of ω_1: if $\mathbb{M} \models (S$ is a stationary subset of $\omega_1)$, then $\mathbb{M}[G] \models (S$ is a stationary subset of $\omega_1)$, whenever G is $\mathbb{P}$–generic over $\mathbb{M}$. *Martin's Maximum* is the statement MM: for every stationary-preserving forcing $\mathbb{P}$, if $(\forall \alpha < \omega_1)(D_\alpha$ is dense open in $\mathbb{P})$, then there exists a $\{D_\alpha : \alpha < \omega_1\}$–generic set G in $\mathbb{P}$.

(i) Show that if $\mathbb{P}$ has the countable chain condition, then $\mathbb{P}$ is stationary-preserving.

(ii) Give an example of a stationary-preserving forcing that has an uncountable anti-chain.

(iii) Prove MM implies $MA_{\aleph_1}$: for every c.c.c. forcing $\mathbb{P}$, if $\{D_\alpha : \alpha < \omega_1\}$ is a family of dense sets in $\mathbb{P}$, then there exists a $\{D_\alpha : \alpha < \omega_1\}$–generic set G in $\mathbb{P}$.

(iv) Show $MA_{\aleph_1}$ implies Suslin's Hypothesis.

REMARK. The relative consistency strength of $ZFC + MM$ is far stronger than that of $ZFC + MA$ which is equiconsistent with ZFC. MM requires a large cardinal axiom for its consistency.

20 Optional. Normal Functions and Mahlo Cardinals

A (class) function $G : Ord \to Ord$ is called *normal* if G is (strictly) increasing ($\alpha < \beta \to G(\alpha) < G(\beta)$) and continuous (for all limit $\delta \in Ord, G(\delta) = \bigcup_{\alpha<\delta} G(\alpha)$). A (strongly) inaccessible cardinal κ is called *(strongly) Mahlo* if $\{\alpha < \kappa : \alpha = cf(\alpha)\}$ is stationary in κ.

(i) (a) Prove in ZFC that every normal function G has a fixed point: there is $\delta \in Ord$ such that $G(\delta) = \delta$.

(b) Call the statement "every normal function has a regular fixed point" the *regular fixed point axiom* $RFPA$. Show that $RFPA$ is not provable in ZFC.

(ii) (a) Suppose κ is strongly Mahlo. Show $\{\alpha < \kappa : \alpha$ is strongly inaccessible$\}$ is stationary in κ

(b) Prove that if κ is strongly Mahlo, then $V_\kappa \models RFPA$.

21 The Axiom of Choice in L

(i) Define by transfinite recursion a Δ_1 well-ordering $<_\alpha$ of L_α for $\alpha \in Ord$ and a Σ_1 well-ordering $<_L$ of L. [Hint. The case $<_\delta$ for limit δ is immediate (take unions). For the successor case $<_{\beta+1}$, recall that the codes of formulas without parameters can be identified with elements of L_ω and can be well-ordered using the lexicographic order; since the parameters of a set in $L_{\beta+1}$ arise already in L_β, they can be well-ordered by $<_\beta$. So the predicate $x <_{\beta+1} \Leftrightarrow (x \in L_\beta \wedge y \in L_\beta \wedge x <_\beta y) \bigvee (x \in L_\beta \wedge y \in (L_{\beta+1} \setminus L_\beta)) \bigvee (x \in (L_{\beta+1} \setminus L_\beta) \wedge y \in (L_{\beta+1} \setminus L_\beta) \wedge$ the first formula defining x over L_β precedes the first formula defining y over $L_\beta)$. For the global case, note that the predicate $x <_L y \Leftrightarrow (\exists \alpha)(x \in L_\alpha \wedge x <_\alpha y)$ has the required complexity.]

(ii) Deduce that $V = L$ implies AC.

(iii) Conclude that L is a model of ZFC.

22 Optional. Dilworth's Theorem and Galvin's Conjecture

(i) Prove *Dilworth's theorem*: If a partial order $\mathbb{P}$ has at least n^2+1 elements for some $n<\omega$, then it has either a chain of size $n+1$ or an anti-chain of size $n+1$. Equivalently, if the largest anti-chain in $\mathbb{P}$ has size n, then P can be decomposed into n chains.

(ii) Suggest and prove a generalization of Dilworth's theorem for infinite cardinals. Give examples to support the alleged optimality of your result.

(iii) Galvin's Conjecture

Let κ_D be the least cardinal κ, if it exists, such that for every partial order $\mathbb{P}$, if every suborder of $\mathbb{P}$ of size less than κ can be decomposed into countably many chains, then $\mathbb{P}$ can also be decomposed into countably many chains. Can you eliminate $\aleph_0$ and $\aleph_1$ as candidate values for κ_D? *Galvin's Conjecture* states that $\aleph_2$ is a possible value for κ_D. See S. Todorcevic, *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic, 17 (2011), 1-72.

REMARK. After the classic works of Gödel and Cohen, the following are accessible and list many further suggestions for reading and research.

Woodin, W.H., *The Continuum Hypothesis, part I*, Notices Amer. Math. Soc. 48 (2001), 567-576.

Woodin, W.H., *The Continuum Hypothesis, part II*, Notices Amer. Math. Soc. 48 (2001), 681-690.

Woodin, W.H., *Correction to: The Continuum Hypothesis. Part II*, Notices Amer. Math. Soc. 49 (2002), 46.

Dehornoy, P., *Recent progress on the Continuum Hypothesis (after Woodin)*;

`http://www.math.unicaen.fr/~dehornoy/Surveys/DgtUS.pdf`
`http://www.math.unicaen.fr/~dehornoy/Surveys/Dgt.pdf`

Koellner, P., *The Continuum Hypothesis*, Stanford Encyclopaedia of Philosophy, September 2011;

`http://www.logic.harvard.edu/EFI_CH.pdf`

Steprans, J., *History of the Continuum in the Twentieth Century*, to appear, Vol. 6 History of Logic;

`http://www.math.yorku.ca/~steprans/Research/PDFSOfArticles/hoc2INDEXED.pdf`

REMARK. For research problems in set theory, go to the sources; there are some treasure houses.

Shelah, S., *On what I do not understand (and have something to say): Part I*, Fund. Math. 166 (2000), 1-82;

`http://matwbn.icm.edu.pl/ksiazki/fm/fm166/fm16612.pdf`

Shelah, S., *On what I do not understand (and have something to say): Part II*, Math. Japonica 51 (2000), 329-377;

http://shelah.logic.at/files/702.pdf

Fremlin, D.H., *Problems*;

https://www.essex.ac.uk/maths/people/fremlin/problems.pdf

Miller, A.W., *Some interesting problems*;

http://www.math.wisc.edu/~miller/res/problems.pdf

Todorcevic, S., *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic, 17 (2011), 1-72.