Examples 1

1. see Prop (2.2). If $f, g : X \to Y$ are continuous and Y Hausdorff then $\{x \in X | f(x) = g(x)\}$ is closed in X.

2. Let G be a group and $\mathcal{L}$ a non-empty family of normal subgroups s.t. if $K_1, K_2 \in \mathcal{L}$ and K_3 is a normal subgroup containing $K_1 \cap K_2$ then $K_3 \in \mathcal{L}$. Let $\mathcal{T}$ be the family of all unions of sets of cosets Kg with $K \in \mathcal{L}, g \in G$. Show that $\mathcal{T}$ is a topology on G and that G is a topological group with respect to this topology. Show also that $\mathcal{L}$ is the set of open normal subgroups of G with respect to this topology.

3. Lemma (1.1)(c). G a topological group. To prove:
Every open subgroup of G is closed.
Every closed subgroup of finite index is open.
If G compact, every open subgroup of G has finite index.

4. Prop (1.3)(b). G a compact, totally disconnected topological group. To prove:
A subset of G is both closed and open if and only if it is a union of finitely many cosets of open normal subgroups.

5. Recall if $S \subseteq G$ then the centralizer of S is

$$C_G(S) = \{g \in G | gs = sg \ \ \forall s \in S\}$$

and the normalizer of a subgroup H is

$$N_G(H) = \{g \in G | g^{-1}hg \in H \ \& \ ghg^{-1} \in H \ \ \forall h \in H\}.$$

Suppose G is a Hausdorff topological group.
(a) Prove that centralizers of subsets and normalizers of closed subgroups are closed.
(b) Prove that each closed abelian subgroup A is contained in a maximal closed abelian subgroup (i.e. the family of closed abelian subgroups containing A, partially ordered with respect to inclusion, has a maximal element).
(c) Prove that if there is a series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that G_i/G_{i+1} is abelian for each i, then there is such a series consisting of closed subgroups. (Consider closures).

6. G a profinite group and $X \leq G$, then

$$\overline{X} = \cap\{K | X \leq K \leq_0 G\}.$$

7. Let X be a dense subgroup of a topological group G (i.e. $\overline{X} = G$). Prove that $N = \overline{N \cap X}$ for each open subgroup N of G.

8. If G is finite what is the profinite completion of G?

9. Give an example of a group which has trivial pro-p completions for each p but a non-trivial profinite completion (or describe the properties such a group should have).