Compact convex sets where all continuous convex functions have continuous envelopes and some results on split faces.

Åsvald Lima

Abstract

It is well known that a compact convex set K is a Bauer simplex if and only if for every continuous convex function f on K, the upper envelope $\hat{f}$ is continuous and affine [4]. In this paper we shall study compact convex sets with the property that $\hat{f}$ is merely continuous for every continuous convex function f and we shall see how they are related to Bauer simplexes. Furthermore we shall generalize some results of E.M. Alfsen and T.B Andersen [3] (cf. also [2]) and M. Rogalski [15] to obtain new characterizations of Bauer simplexes by faces.

My Theorem 5 is based on a recent result of J. Vesterstrøm (Theorem 2.1 in [17]). I am indebted to J. Vesterstrøm who kindly communicated to me a preliminary version of [17] during the preparation of this paper. I also want to thank E. Alfsen and T. B. Andersen for helpful comments.

1. Preliminaries and notation.

Let K be a compact convex set in a real locally convex Hausdorff space E.

We shall use the following symbols:

$\partial_e K$:	the set of extreme points in K.
$C(K)$:	the Banach space of continuous real-valued functions on K.
$A(K)$:	the Banach space of continuous affine real-valued functions on K.
$P(K)$:	the uniformly closed convex cone of continuous convex real-valued functions on K.

If X is a compact subset of K, then we shall denote by $M(X)$ the Banach space of all signed (Radon-)measures on X, and by $M_1^+(X)$ the w^*-compact convex set of normalized positive (Radon-) measures on X.

A signed measure μ on K is said to be a <u>boundary measure</u> if $|\mu|$ is maximal in Choquet's ordering of positive measures. Cf. [6] or [2]. The linear subspace of $M(K)$ of all (signed) boundary measures is denoted by Q, and $Q_1 = Q \cap M_1^+(K)$.

If $x \in K$, then

$$M_x = \{\mu \in M_1^+(K) : \int_K f\, d\mu = f(x), \quad \text{all} \quad f \in A(K)\}$$

and $Q_x = M_x \cap Q_1$. M_x is a w^*-compact convex set and Q_x is a face in M_x, just as Q_1 is a face in $M_1^+(K)$. See e.g. [2].

If $\mu \in M_1^+(K)$, then the <u>barycenter</u> of μ is the unique point $x \in K$, such that $\mu \in M_x$, and we shall write $x = r(\mu)$. See e.g. [2]. The map $r: M_1^+(K) \to K$ defined by $\mu \rightsquigarrow r(\mu)$ is <u>continuous</u> and affine. See e.g. [13]. Clearly this map is surjective since $r(\varepsilon_x) = x$ for all $x \in K$. Note, however, that the restricted map from Q_1 to K is also surjective by virtue of the Choquet-Bishop-de Leeuw theorem [5],[6]. In particular it follows that the restriction r_e of the barycenter map to the set $M_1^+(\overline{\partial_e K})$ containing Q_1 will also be surjective.

If $f,g: K \to \mathbb{R}$, then $f < g$ means that $f(x) < g(x)$ for all $x \in K$.

If $f: K \to \mathbb{R}$ is bounded, we define

$$\hat{f}(x) = \inf\{a(x): a \in A(K), \quad a > f\},$$
$$\check{f}(x) = \sup\{a(x): a \in A(K), \quad a < f\}.$$

The function $\hat{f}$ is the smallest upper semi-continuous (u.s.c.) concave function majorizing f. Dually $\check{f}$ is the greatest lower semi-continuous (l.s.c.) convex function majorized by f.

If S is a subset of K, then $co(S)$ is the convex hull of S and $\overline{co}(S)$ is the closed convex hull of S.

2. Continuous convex extension of functions defined on $\partial_e K$.

Our first lemma can be deduced from a general theorem of Edwards [7], but for the sake of completeness we have included a proof.

Lemma 1: Suppose $X \subseteq \partial_e K$ is compact and let $f \in C(X)$. Then there exists a $g \in P(K)$ such that $g|X = f$.

Our method of proof is based on an approximation technique used in [16].

<u>Proof:</u> We may suppose $0 < f \leq 1$. Let $0 < \varepsilon < 1$, and let the restriction map $g \rightsquigarrow g|X$ of $P(K)$ into $C(X)$ be denoted by T.

Define functions f_1 and g_1 by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in X \\ \sup\{f(y): y \in X\} & \text{if } x \in K \setminus X, \end{cases}$$

and

$$g_1(x) = \begin{cases} f(x) & \text{if } x \in X \\ \inf\{f(y): y \in X\} & \text{if } x \in K \setminus X. \end{cases}$$

Now $f_1 \geq g_1$ and f_1, $-g_1$ is l.s.c. and concave. Since f_1 is l.s.c. we have $f_1|\partial_e K = \check{f}_1|\partial_e K$ and hence $f_1 \geq \check{f}_1 \geq g_1$. Let $g_1' = \max(g_1 - \varepsilon, 0)$. Then we have $\check{f}_1 > g_1'$ and g_1' is u.s.c. For each $x \in K$ we can find a $g_x \in A(K) \subseteq P(K)$ such that $g_x < f_1$ and $g_1'(x) < g_x(x)$. Since g_1' is u.s.c., $V_x = \{y \in K: g_1'(y) - g_x(y) < 0\}$ is open and $x \in V_x$. By compactness we can find $x_1, \cdots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} V_{x_i}$. Define $k_1 = \max(g_{x_1}, \cdots, g_{x_n})$. Then $k_1 \in P(K)$, $g_1' < k_1 < \check{f}_1$, hence $T(k_1) < f$ and $\|f - T(k_1)\| < \varepsilon$. Furthermore $0 < k_1 < 1$ and

$$0 < \varepsilon^{-1}(f - T(k_1)) \leq 1.$$

Suppose for induction that we have found $k_1, \cdots, k_n \in P(K)$ such that for $i = 1, \cdots, n$:

(2.1) $\quad 0 < k_i < \varepsilon^{i-1}$

(2.2) $\quad T(k_1 + \cdots + k_i) < f$

(2.3) $\quad \|f - T(k_1 + \cdots + k_i)\| < \varepsilon^i$

Then we have

$$(2.4)\qquad 0 < \varepsilon^{-n}(f-T(k_1+\cdots+k_n)) < 1,$$

and we can repeat the argument above to get $k'_{n+1} \in P(K)$ such that

$$0 < k'_{n+1} < 1$$

$$T(k'_{n+1}) < \varepsilon^{-n}(f-T(k_1+\cdots+k_n))$$

$$\|\varepsilon^{-n}(f-T(k_1+\cdots+k_n)) - k'_{n+1}\| < \varepsilon.$$

Defining $k_{n+1} = \varepsilon^n k'_{n+1}$, we see that (2.1), (2.2) and (2.3) are fulfilled with $n+1$ in place of n. Hence there exists a sequence $\{k_i\}_{i=1}^{\infty} \subseteq P(K)$ such that (2.1), (2.2) and (2.3) are fulfilled for every i.

Defining $g = \sum_{i=1}^{\infty} k_i$, we have $g \in P(K)$ and $g|X = f$, and the proof is complete.

<u>Corollary 2:</u> The following statements are equivalent :

(i) $\partial_e K$ is closed

(ii) There exists for every $f \in C(\partial_e K)$ a $g \in P(K)$ such that $g|\partial_e K = f$.

(iii) There exists for every $f \in P(K)$ a $g \in -P(K)$ such that $g|\partial_e K = f|\partial_e K$.

<u>Proof:</u> (i) => (ii) follows from Lemma 1.

(ii) => (iii) is obvious.

(iii) => (i) For $f \in C(K)$ we define

$$B_f = \{x \in K : f(x) = \hat{f}(x)\}.$$

It is well known that $\partial_e K = \bigcap\{B_f : f \in P(K)\}$ [13]. Now it follows from (iii) that $\overline{\partial_e K} \subseteq B_f$ for $f \in P(K)$, hence $\overline{\partial_e K} \subseteq \bigcap \{B_f: f \in P(K)\} = \partial_e K$. The proof is complete.

Remark 3: If $\partial_e K$ is closed, then by Corollary 2 every $f \in A(M_1^+(\overline{\partial_e K}))$ is of the form $\widehat{g \circ r_e}$ for some $g \in P(K)$. (If $\partial_e K$ is closed then $\partial_e K$ and $\partial_e M_1^+(\overline{\partial_e K})$ are homeomorphic by r_e.)

3. Continuous convex functions with continuous envelopes.

Let X be a compact convex set in a Hausdorff locally convex space and let $p: X \to K$ be a continuous, surjective and affine map.

Proposition 4: Let K, X and p be as above. If $f: X \to \mathbb{R}$ is u.s.c. and concave, then for each $y \in K$ we have

$$\sup\{f(x): x \in p^{-1}(y)\} = \inf\{g(y): g \in A(K),\ g \circ p > f\}.$$

Definition: If $f: X \to \mathbb{R}$ is u.s.c. we define $\hat{f}^p: K \to \mathbb{R}$ by

$$\hat{f}^p(y) = \sup\{f(x): x \in p^{-1}(y)\}$$

for each $y \in K$.

Proof: Let $\alpha \in \mathbb{R}$. To each $y \in K$ there exists a $x \in p^{-1}(y)$ such that $\hat{f}^p(y) = f(x)$. Hence we have

$$\{y \in K: \hat{f}^p(y) \geq \alpha\} = p(\{x \in X: f(x) \geq \alpha\})$$

such that $\hat{f}^p$ is u.s.c.

Let $(y_1, y_2, \lambda) \in K \times K \times [0,1]$ and let $x_1, x_2 \in X$ be such that $p(x_i) = y_i$ and $\hat{f}^p(y_i) = f(x_i)$. Then we have:

$$\hat{f}^p(\lambda y_1 + (1-\lambda)y_2) \geq$$
$$f(\lambda x_1 + (1-\lambda)x_2) \geq$$
$$\lambda f(x_1) + (1-\lambda)f(x_2) =$$
$$\lambda \hat{f}^p(y_1) + (1-\lambda)\hat{f}^p(y_2)$$

such that $\hat{f}^p$ is concave.

It follows that

$$\hat{f}^p(y) = \inf\{g(y) : g \in A(K), \quad g > \hat{f}^p\}$$

and since for every $g \in A(K)$, $g > \hat{f}^p$ if and only if $g \circ p > f$, we have

$$\hat{f}^p(y) = \inf\{g(y) : g \in A(K), \quad g \circ p > f\}.$$

The proof is complete.

<u>Observation:</u> Let K, X and p be as above and let $f \in C(K)$. If we define $g = \widehat{f \circ p}$, then we have $\hat{f} = \hat{g}^p$. Suppose $k \in -P(K)$ and $k \geq f$. Then we have $f \circ p \leq k \circ p$, hence $f \circ p \leq g = \widehat{f \circ p} \leq k \circ p$ and $f \leq \hat{g}^p \leq k$. Thus we have $\hat{f} = \hat{g}^p$.

<u>Theorem 5:</u> Let K, X and p be as above. The following statements are equivalent:

(i) p is open

(ii) $\hat{f}^p \in C(K)$ for every $f \in C(X)$

(iii) $\hat{f}^p \in -P(K)$ for every $f \in -P(X)$

(iv) $\hat{f}^p \in -P(K)$ for every $f \in A(X)$

(v) $p(\{x \in X : f(x) > 0\})$ is open in K for every $f \in A(X)$.

Proof: (i) => (ii). In the proof of Proposition 4 we showed that $\hat{f}^p$ is u.s.c. if $f \in C(X)$. Let $\alpha \in \mathbb{R}$ and observe that $p(\{x \in X: f(x) > \alpha\}) = \{y \in K: \hat{f}^p(y) > \alpha\}$. Thus we have that f^p is l.s.c.

(ii) => (iii) => (iv) is obvious since it follows as in the proof of Proposition 4 that $\hat{f}^p$ is concave when f is concave.

(iv) <=> (v) is obvious.

(iv) <=> (i) follows from Proposition 4 and Theorem 2.1 in [17], and the proof is complete.

Remark 6: The deep part of Theorem 5, (iv) => (i) is due to J. Vesterstrøm [17].

It is easy to give a direct proof of (iii) => (i).

Definition: We shall say that K is a CE-compact convex set if $\hat{f}$ is continuous for every $f \in P(K)$.

Observation: Suppose K is a CE-compact convex set and let F be a closed face in K. Then F is a CE-compact convex set.

Proof: If $g \in P(F)$, then by Corollary 2 and Tietze's theorem there exists a $f \in P(K)$ such that $f|\partial_e F = g|\partial_e F$. Now we have $\hat{g} = \widehat{f|F} = \hat{f}|F$, and the proof is complete.

Theorem 7: The statements (i) - (v) below are related as follows: (i) => (ii) <=> (iii) <=> (iv) => (v).

(i) K is a Bauer simplex.

(ii) There exists a CE-compact convex set X and an open, continuous, surjective and affine map $p: X \to K$.

(iii) $r_e\colon M_1^+(\overline{\partial_e K}) \to K$ is open

(iv) K is a CE-compact convex set.

(v) $\partial_e K$ is closed.

Proof:

(i) => (ii). Let $X = M_1^+(\overline{\partial_e K})$ and let $p = r_e$. If K is a Bauer simplex, then r_e is a homeomorphism by a theorem of Bauer [4].

(ii) => (iv). If $f \in P(K)$, then by Theorem 5 and the observation before Theorem 5, $\hat{f}$ is continuous.

(iv) => (v). Follows from Corollary 2.

(iv) => (iii). By Remark 3 and the observation before Theorem 5, if $f \in A(M_1^+(\overline{\partial_e K}))$, then $\hat{f}^p = \hat{g}$ for some $g \in P(K)$. (iii) now follows from Theorem 5.

(iii) => (ii) is obvious, and the proof is complete.

Remark 8: We will later give two examples where (v) in Theorem 7 is satisfied but not (iv).

If K is a square in $\mathbb{R}^2$, then obvioulsy (iv) in Theorem 7 is satisfied but not (i).

If K is a CE-compact convex set, then since $\partial_e K$ is closed we have $M_1^+(\overline{\partial_e K}) = M_1^+(\partial_e K) = Q_1$.

J. Vesterstrøm proved in [17] that (iii) <=> (iv), but his proof is quite different from that of mine.

Definition: A subset S of K is called a σ-face if S is a union of faces in K.

The term σ-face was introduced by Goullet dé Rugy in [10]. Closed σ-faces were also studied by Alfsen in [1] under the name stable subsets.

If $f \in C(K)$ we define

$$\Lambda(f) = \{\mu \in M_1^+(\overline{\partial_e K}): \ \mu(f) = \hat{f}(r_e(\mu))\}.$$

We have that <u>$\Lambda(f)$ is a σ-face in $M_1^+(\overline{\partial_e K})$</u>, in fact if $\mu = \lambda\nu_1 + (1-\lambda)\nu_2 \in \Lambda(f)$ with $\lambda \in [0,1]$ and $\nu_1, \nu_2 \in M_1^+(\overline{\partial_e K})$, then

$$\lambda\hat{f}(r_e(\nu_1)) + (1-\lambda)\hat{f}(r_e(\nu_2)) \geq$$

$$\lambda\nu_1(f) + (1-\lambda)\nu_2(f) = \mu(f) = \hat{f}(r_e(\mu)) =$$

$$\hat{f}(\lambda r_e(\nu_1)) + (1-\lambda)r_e(\nu_2)) \geq \lambda\hat{f}(r_e(\nu_1)) + (1-\lambda)\ \hat{f}(r_e(\nu_2))$$

such that $\hat{f}(r_e(\nu_1)) = \nu_1(f)$, i.e. $\nu_1 \in \Lambda(f)$.

<u>Proposition 9:</u> The following are equivalent :

(i) K is a CE-compact convex set.

(ii) If $f \in C(\partial_e K)$ then $\hat{f}$ is continuous and $\hat{f}|\partial_e K = f$.

(iii) $\Lambda(f)$ is a w^*-closed σ-face in $M_1^+(\overline{\partial_e K})$ for every $f \in P(K)$.

<u>Proof:</u>

(i) => (ii). Suppose (i) is fulfilled and let $f \in C(\partial_e K)$. By Theorem 7 and Corollary 2, $f = g|\partial_e K$ for some $g \in P(K)$, and by Theorem 7 $\hat{f} = \hat{g}$ is continuous.

(ii) => (i) is trivial.

(i) => (iii). If $f \in P(K)$ we define ϕ_f: $M_1^+(\overline{\partial_e K}) \to \mathbb{R}$ by by $\phi_f(\mu) = \hat{f}(r_e(\mu)) - \mu(f)$. Then $\phi_f(\mu) \geq 0$ for every μ (see e.g. [2]) and $\Lambda(f) = \phi_f^{-1}(0)$. Hence $\Lambda(f)$ is closed since f is continuous.

(iii) => (i). Let $f \in P(K)$ and let $\alpha \in \mathbb{R}$. We only need to show that $\hat{f}$ is l.s.c. The set $\{\mu \in M_1^+(\overline{\partial_e K}) : \mu(f) \leq \alpha\}$ is closed and hence $A_\alpha = \{\mu \in \Lambda(f) : \mu(f) = \hat{f}(r_e(\mu)) \leq \alpha\}$ is compact. Thus $r_e(A_\alpha)$ is compact. Since for $f \in P(K)$, $r_e(\Lambda(f)) = K$ [2], it easily follows that $r_e(A_\alpha) = \{x \in K : \hat{f}(x) \leq \alpha\}$. Thus $\hat{f}$ is l.s.c. and the proof is complete.

<u>Theorem 10:</u> Let K be a metrizable CE-compact convex set and suppose that X is a simplex and that $p: X \to K$ is a continuous, surjective and affine map. Then there exists a continuous, surjective and affine map $\phi: X \to M_1^+(\overline{\partial_e K}) = Q_1$ such that $p = r_e \circ \phi$.

<u>Proof:</u> We define a multivalued map $\psi: X \to 2^{M_1^+(\overline{\partial_e K})}$ by $\psi(x) = r_e^{-1}(p(x))$. Since p and r_e is affine it is easily seen that ψ is convex, i.e.

$$\lambda\psi(x) + (1-\lambda)\psi(y) \subseteq \psi(\lambda x+(1-\lambda)y)$$

when $\quad (x,y,\lambda) \in X \times X \times [0,1]$.

If $U \subseteq M_1^+(\overline{\partial_e K})$ is open, then $p^{-1}(r_e(U))$ is open in X since r_e is an open map by Theorem 7. The statement $x \in p^{-1}(r_e(U))$ is equivalent to $r_e^{-1}(p(x)) \subseteq r_e^{-1}(r_e(U))$, which in turn is equivalent to $\psi(x) \cap U \neq \emptyset$. This shows that

$$\{x \in X: \psi(x) \cap U \neq \emptyset\} = p^{-1}(r_e(U)).$$

Hence ψ is l.s.c.

Now it follows by Lazar's selection theorem [12] (cf. [18] or [10] for a simple proof) that there exists a continuous affine function $\phi: X \to M_1^+(\overline{\partial_e K})$ such that $\phi(x) \in r_e^{-1}(p(x))$ for all $x \in X$. Obviously $\partial_e M_1^+(\overline{\partial_e K}) \subseteq \phi(x)$, hence ϕ is surjective. Now we have $p = r_e \circ \phi$ and the proof is complete.

<u>Theorem 11:</u> Let K be a CE-compact convex set. If $F \subseteq K$ is a face, then $\overline{F}$ is a face.

<u>Proof:</u> If $S = \partial_e K \cap \overline{F}$, then S is a closed σ-face and we have $\overline{co}(S) \subseteq \overline{F}$. Let $x \in F$. Every discrete measure on K representing x is supported by F, and since the set of discrete measures in M_x is dense in M_x(cf. [2]) it follows that every representing measure for x is supported by $\overline{F}$. If $\mu \in Q_x$, then μ is supported by S, and hence $x \in \overline{co}(S)$. This shows that $F \subseteq \overline{co}(S)$ and $\overline{F} = \overline{co}(S)$.

Let $G \subseteq K$ be a closed σ-face. Then we have that χ_G (the characteristic function to G) is u.s.c. and convex, and by Proposition 5.6 in [10] we have for all $x \in K$:

$$(3.1) \qquad \hat{\chi}_G(x) = \sup_{\mu \in M_x} \mu(\chi_G) = \sup_{\mu \in Q_x} \mu(\chi_G)$$

From this it follows that $\overline{co}(G) = \hat{\chi}_G^{-1}(1)$ and in particular $\overline{F} = \overline{co}(S) = \hat{\chi}_S^{-1}(1)$.

Define

$$P = \{f \in C(\partial_e K) : 0 \leq f \leq 1 \text{ and } f|S = 1\}.$$

Then P is a convex set, and $\{f\}_{f \in P}$ converges at every point $x \in \partial_e K$ to $\chi_S(x) = \hat{\chi}_S(x)$.

By Proposition 9 $\hat{f}$ and $\check{f}$ are continuous for every $f \in P$. If we define for every $f \in P$ a set F_f by

$$F_f = \{x \in K : \hat{f}(x) = \check{f}(x) = 1\}.$$

then F_f is closed for every $f \in P$.

Let $f \in P$ and let $x, y, z \in K$ and $\lambda \in [0,1]$ be such that $x = \lambda y + (1-\lambda)z \in F_f$. Then we have that

$$1 = \hat{f}(x) = \hat{f}(\lambda y + (1-\lambda)z) \geq \lambda\hat{f}(y) + (1-\lambda)\hat{f}(z) \geq$$
$$\lambda\check{f}(y) + (1-\lambda)\check{f}(z) \geq \check{f}(\lambda y + (1-\lambda)z) = \check{f}(x) = 1$$

Hence

$$1 = \hat{f}(y) = \check{f}(y) = \hat{f}(z) = \check{f}(z)$$

so $y,z \in F_f$. This shows that each F_f is a closed σ-face, and hence $\bigcap\{F_f\colon f\in P\}$ is a closed σ-face.

By the known formulas (see e.g. [2])

$$(3.2) \qquad \hat{f}(x) = \sup_{\mu\in Q_x} \mu(f)$$

and

$$(3.3) \qquad \check{f}(x) = \inf_{\mu\in Q_x} \mu(f)$$

which hold for every $f\in P$ and all $x\in K$, and by the density of F in $\overline{F}$, we have $\overline{F} \subseteq \bigcap\{F_f\colon f\in P\}$.

Suppose $x\in K\setminus\overline{F}$. Then $\hat{\chi}_s(x) < 1$ and by (3.1), (3.2), (3.3) and Theorem 7.1 in [9] (or Lemma 5.4 in [10]) we get that:

$$1 > \hat{\chi}_s(x) = \sup_{\mu\in Q_x} \mu(\chi_s) = \sup_{\mu\in Q_x} \inf_{f\in P} \mu(f)$$

$$= \inf_{f\in P} \sup_{\mu\in Q_x} \mu(f) = \inf_{f\in P} \hat{f}(x)$$

Hence we can find a $f\in P$ such that $x\notin F_f$. Thus we have that $\overline{F} = \bigcap\{F_f\colon f\in P\}$, and $\overline{F}$ must be a closed σ-face. But since $\overline{F}$ is convex, $\overline{F}$ is a closed face (see e.g. [1]), and the proof is complete.

4. Split faces and Bauer simplexes.

If F is a non-empty subset of K then $F' = \cup\{G: G$ is a face in K and $G \cap F = \emptyset\}$ is called the complementary set of F.

A complementary set is a σ-face and it is a face if and only if it is convex (see e.g. [10] [2]).

If F is a proper closed face in K, then for every $x \in K$ there exists a convex combination

$$(4.1) \qquad x = \lambda y + (1-\lambda)z \quad \text{where} \quad y \in F,\ z \in F',\ \lambda \leq \hat{\chi}_F(x)$$

(see e.g. [2]). The face F is said to be a split face if F' is a face and if for every $x \in K \setminus (F \cup F')$ y and λ in the above decomposition (4.1) are uniquely determined. The face F is said to be a parallel face if F' is a face and if for every $x \in K \setminus (F \cup F')$, λ in the above decomposition (4.1) is uniquely determined.

For results on split and parallel faces see [2], [3], [11], [14] and [15]. Every split face is a parallel face and a closed face F in K is parallel if and only if $\hat{\chi}_F$ is affine.

In [3] it is proved that the collection of all split faces is closed under finite convex hulls and arbitrary intersections. Thus the collection of all sets $F \cap \partial_e K$ where F is a split face, satisfies the axioms of closed sets for a topology, which is called the facial topology on $\partial_e K$. The facial topology is compact and it is Hausdorff if and only if K is a Bauer simplex.

If $x \in K$, then the smallest face of K containing x will be denoted by face (x).

Remark 12: It is easy to see that if $x_1, \cdots, x_n \in \partial_x K$ and all $\{x_i\}$ are split faces, then the set co $(x_1, \cdots, x_n)$ is a face in K and that this set is a Bauer simplex. In particular if $x \in co(x_1, \cdots, x_n)$, then x has a unique maximal representing measure on K.

Proposition 13: Let K be a CE-compact convex set. Suppose that $B \subseteq \partial_e K$ and that $\{x\}$ is a split face for every $x \in B$. Then the set $\overline{co}(B)$ is a face in K and this set is a Bauer simplex.

Proof: co(B) is a convex σ-face and hence a face. By Theorem 11 $\overline{co}(B)$ is a face.

We have $\overline{B} \subseteq \partial_e K \cap \overline{co}(B)$, $\overline{co}(\overline{B}) = \overline{co}(B)$ and by Milman's theorem it follows that $\partial_e \overline{co}(B) = \overline{B}$. Let $g \in C(\overline{B})$ and let $f \in C(\partial_e K)$ be an extension of g to $\partial_e K$. By Proposition 9 $\hat{f}$ is continuous, so $g' = \hat{f}|F$ is a continuous concave extension on g to $\overline{co}(B)$. By formula (3.2) and Remark 12 g' is affine on co(B). Thus by continuity, g' is affine on $\overline{co}(B)$. This proves that $\overline{co}(B)$ is a Bauer simplex, and the proof is complete.

Corollary 14: The following are equivalent:

(i) K is a Bauer simplex.

(ii) K is a CE-compact convex set and the set $SF(K) = \{x \in \partial_e K : \{x\}$ is a split face$\}$ is dense in $\partial_e K$.

Remark 15: There exist compact convex sets K_1 and K_2 such that

(i) $\partial_e K_1$ is closed, K_1 is an α-polytope but no CE-compact convex set.

(ii) $\partial_e K_2$ is closed, K_2 is no α-polytope and no CE-compact convex set.

The compact convex sets in Proposition 20 in [15] and in Theorem 6.4 in [3] satisfy the assertions.

Remark 16: The compact convex set K_1 in Remark 15 constructed by M. Rogalski [15] has the property that $\partial_e K_1$ is homeomorphic to $[0,1]$. Rogalski showed that every irrational number in $[0,1]$ is a split face (Corollary 25) and he left it as an open problem whether the rational numbers are split faces. By Theorem 2.12 in [11] it follows that the rational numbers in $[0,1]$ are not split faces.

Proposition 17: The following statements hold in a compact convex set K.

(a) A subset $S \subseteq K$ is a closed σ-face if and only if χ_S is u.s.c. and convex.

(b) The collection of all closed σ-faces in K is closed under finite unions and arbitrary intersections. Hence the collection of all sets of the form $S \cap \partial_e K$, where S is a closed σ-face, satisfies the axioms of closed sets for a topology on $\partial_e K$.

(c) There is for each $x \in K$ a smallest σ-face, $S(x)$, containing x.

(d) Each face is a σ-face and if F is a face in K and S is a σ-face in F, then S is a σ-face in K.

(e) If $S \subseteq K$ is a closed subset, then S is a σ-face if and only if for every $x \in S$ and every $\mu \in M_x$, μ is supported by S.

(f) If $S \subseteq K$ is a closed σ-face, then $S \cap \partial_e K \neq \emptyset$ and if $S \subsetneq \partial_e K$, then $S \cap \partial_e K$ consists of more than one point.

<u>Proof:</u> (a), (b), (c) and (d) are easy to prove.

(e) is proved in [1].

It only remains to prove (f).

Let $S \subseteq K$ be a closed σ-face. Let $\{S_\alpha\}_{\alpha \in I}$ be all closed σ-faces in K such that $S \cap S_\alpha \neq \emptyset$ for each $\alpha \in I$. Then by Zorn's lemma the family $\{S \cap S_\alpha\}_{\alpha \in I}$ has a minimal element S_o. Suppose $x,y \in S_o$ and $x \neq y$. Let $f \in A(K)$ and $f(x) < f(y)$. Then $\{z \in S_o\colon\ f(z) = \sup_{v \in S_o} f(v)\}$ is a non-empty closed σ-face by (e) and this set is properly contained in S_o. Since S_o is minimal, S_o can not contain more than one element, and this element must be extreme in K, since χ_{S_o} is convex. Hence we have $S \cap \partial_e K \neq \emptyset$.

Suppose $S \nsubseteq \partial_e K$. Let $x \in S \cap \partial_e K$ and let $y \in S \setminus \partial_e K$. There exists a $f \in A(K)$ such that $f(x) < f(y)$. Hence the set

$$S_1 = \{z \in S\colon\ f(z) = \sup_{v \in S} f(v)\}$$

is a closed σ-face by (e) and $x \notin S_1$. Let $z \in S_1 \cap \partial_e K$. Then $z \in S \cap \partial_e K$ and $z \neq x$, and the proof is complete.

<u>Definition:</u> The topology on $\partial_e K$ described in (b) above will be called the <u>σ-face topology</u> and it will be denoted by the letter σ.

<u>Proposition 18:</u> $\partial_e K$ with the topology σ is a compact T_1 space, and σ is Hausdorff if and only if $\partial_e K$ is closed.

<u>Proof:</u> Trivially σ is T_1. It is also easily seen that $\partial_e K$ is compact in the topology σ. (The proof is the same as that of

Proposition 4.2 in [3]).

Obviously the identity map $\Pi:(\partial_e K, \text{rel.top.}) \to (\partial_e K,\sigma)$ is continuous and bijective, and hence if $\partial_e K$ is compact then Π is a homeomorphism. Thus if $\partial_e K$ is closed, then σ is Hausdorff.

Suppose now that $\partial_e K$ is not closed, and let $x \in \overline{\partial_e K} \setminus \partial_e K$. Then by (f) $S(x) \cap \partial_e K$ will consist of more than one point. Let $\{x_\alpha\} \subseteq \partial_e K$ be a net that converges to x, and let $z \in S(x) \cap \partial_e K$. Let S be a closed σ-face such that $z \in \partial_e K \setminus S$. If $x \in S$, then $S(x) \subseteq S$ and hence $z \in S$. Thus $x \notin S$, so $K \setminus S$ is an open neighbourhood of x, and hence there exists an α_o such that $x_\alpha \in \partial_e K \setminus S$ for all $\alpha \geq \alpha_o$. This shows that $x_\alpha \to z$ in the σ-face topology for all $z \in S(x) \cap \partial_e K$, so σ can not be Hausdorff, and the proof is complete.

<u>Remark 19:</u> The idea to the proof of Proposition 18 has been taken from [8]. We also could have proved the proposition as Lemma 6.1 in [3] was proved.

<u>Definition:</u> Following [2] we shall say that K satisfies <u>Størmer's axiom</u> if for every family $\{F_\alpha\}$ of split faces in K, the set $\overline{\text{co}}(\bigcup_\alpha F_\alpha)$ is a split face in K.

We will now prove a generalization of Theorem II.7.19 in [2] and of Corollary 38 in [15].

<u>Theorem 20:</u> The following statements are equivalent:

(i) K is a Bauer simplex.

(ii) If F is any face in K, then $\overline{F}$ is a split face.

(iii) K satisfies Størmer's axiom and every extreme point in K is a split face.

(iv) If $B \subseteq \partial_e K$, then $\overline{co}(B)$ is a split face.

(v) If $B \subseteq \partial_e K$, then $\overline{co}(B)$ is a parallel face.

(vi) The facial topology on $\partial_e K$ is Hausdorff.

(vii) If $f \in A(K)$, then there exists a split face F such that $F \cap \partial_e K = \partial_e K \cap \{x \in K: f(x) \leq 0\}$.

<u>Proof:</u> (i) => (ii) => (iii) is proved in [2] (Theorem II.7.19 and Theorem II.6.22).

(iii) => (iv) => (v) is trivial.

(iv) => (vi). Suppose $B \subseteq \partial_e K$ is relatively closed. Then $\overline{B} \cap \partial_e K = B$. By (iv), $F = \overline{co}(B) = \overline{co}(\overline{B})$ is a split face. By Milman's theorem we have that $\partial_e F \subseteq \overline{B}$ and since F is a face, we have that $\partial_e F \subseteq \partial_e K \cap \overline{B} = B$. Hence $\partial_e F = B$. Thus the facial topology on $\partial_e K$ equals the relative topology on $\partial_e K$.

(vi) => (iv). $\partial_e K$ is Hausdorff in the topology σ since σ is a finer topology than the facial topology. By Proposition 18 $\partial_e K$ is closed. Let $B \subseteq \partial_e K$. Then we have that $\overline{B} \subseteq \partial_e K$ and $\overline{co}(B) = \overline{co}(\overline{B})$. $\overline{B}$ is closed in the facial topology, and hence there exists a split face F such that $\partial_e F = F \cap \partial_e K = \overline{B}$. Thus we have $F = \overline{co}(B)$.

(v) => (i). Just as in the proof of (iv) => (vi) we get that if $B \subseteq \partial_e K$ is relatively closed, then B is of the form $B = F \cap \partial_e K = \partial_e F$ where F is a parallel face. Thus σ is Hausdorff and, by Proposition 18, $\partial_e K$ is closed.

Let $x \in K$ and let $\mu, \nu \in Q_x$. Since we can view μ and ν as positive regular Borel measures, if $\mu(X) = \nu(X)$ for each compact set $X \subseteq \partial_e K$, then we have $\mu = \nu$. Hence x has a unique maximal representing measure.

Suppose $X \subseteq \partial_e K$ is compact. Then by (v), $F = \overline{co}(X)$ is a parallel face. Since $\hat{X}_F$ is affine, the set of functions $\{a_\alpha\} = \{a \in A(K): a > X_F\}$ is directed downwards and $\{a_\alpha\}$ converges pointwise to $\hat{X}_F$. Hence we have

$$\mu(X) = \mu(F) = \mu(X_F) = \mu(\hat{X}_F) = \lim_\alpha \mu(a_\alpha) = \lim_\alpha \nu(a_\alpha)$$
$$= \nu(\hat{X}_F) = \nu(X_F) = \nu(F) = \nu(X).$$

(iv) => (vii). Let $f \in A(K)$ and define $B = \{x \in K: f(x) \leq 0\}$. $B \cap \partial_e K$ is relatively closed and $F = \overline{co}(B \cap \partial_e K)$ is a split face by (iv) such that $F \cap \partial_e K = \partial_e F = B \cap \partial_e K$.

(vii) => (vi). Let $x, y \in \partial_e K$ and $x \neq y$. Then we can find a $f \in A(K)$ such that $f(x) < 0 < f(y)$. Define sets

$$B = \{z \in K: f(x) \leq 0\}.$$

and

$$C = \{z \in K: f(z) \geq 0\}.$$

Let F_x and F_y be split faces such that $F_x \cap \partial_e K = B \cap \partial_e K$ and $F_y \cap \partial_e K = C \cap \partial_e K$. Now we have that $K = B \cup C$ such that $\partial_e K = \partial_e K \cap (F_x \cup F_y) = \partial_e F_x \cup \partial_e F_y$. This shows that the facial topology is Hausdorff, and the proof is complete.

Remark 21: The equivalence of (i) and (vi) was proved by E. Alfsen and T.B. Andersen in [3]. In [3] (vi) => (i) was proved by showing that (vi) implies that every $f \in C(\partial_e K)$ has a continuous affine extention to K.

In [15] M. Rogalski proved the equivalence of (i) and (iii) for a large class of compact convex sets.

<u>Proposition 22:</u> Let F be a closed face in a compact convex set K. The following statements are equivalent :

(i) F is a split face.

(ii) If G is any face in K, then $co(F \cup G)$ is a face in K.

(iii) For all $z \in F'$, $co(F \cup face(z))$ is a face in K.

<u>Proof:</u>

(i) => (ii). Let G be a face in K and let $u, v \in K$ and $\alpha \in \langle 0,1 \rangle$ be such that

$$z = \alpha u + (1-\alpha)v \in co(F \cup G).$$

If $z \in F \cup G$, then $u, v \in F \cup G$, so we will suppose that $z \in co(F \cup G) \setminus (F \cup G)$. Then z has a decomposition

$$z = \lambda x + (1-\lambda)y$$

where $x \in F$, $y \in G$ and $\lambda \in \langle 0,1 \rangle$. By (4.1),

$$y = \gamma y_1 + (1-\gamma)y_2$$

where $y_1 \in F$, $y_2 \in F'$ and $\gamma \in [0,1\rangle$, and hence

$$z = \lambda x + (1-\lambda)\gamma y_1 + (1-\lambda)(1-\gamma)y_2$$

where $(\lambda + (1-\lambda)\gamma)^{-1}(\lambda x + (1-\lambda)\gamma y_1) \in F$.

By (4.1)

$$u = \beta u_1 + (1-\beta)u_2,$$

$$v = \delta v_1 + (1-\delta)v_2$$

where $u_1, v_1 \in F$, $u_2, v_2 \in F'$ and $\beta, \delta \in [0,1]$.

Hence we have that

$$z = \alpha\beta u_1 + (1-\alpha)\delta v_1 + \alpha(1-\beta)u_2 + (1-\alpha)(1-\delta)v_2$$

where $(\alpha\beta + (1-\alpha)\delta)^{-1}(\alpha\beta u_1 + (1-\alpha)\delta v_1) \in F$

and $(1-\alpha\beta-(1-\alpha)\delta)^{-1}(\alpha(1-\beta)u_2 + (1-\alpha)(1-\delta)v_2) \in F'$.

If $\beta = \delta = 0$, then $u, v \in F'$, and hence $z \in F'$. Since $\lambda \neq 0$, this is impossible, so not both β and δ are zero. By the uniqueness of the decomposition of z after F and F', we find that

$$y_2 = (1-\alpha\beta-(1-\alpha)\delta)^{-1}(\alpha(1-\beta)u_2 + (1-\alpha)(1-\delta)v_2),$$

and since $y_2 \in F' \cap G$ we have that $u_2, v_2 \in G$. Thus we have that $u, v \in co(F \cup G)$, and hence $co(F \cup G)$ is a face.

(ii) => (iii) is trivial.

(iii) => (i). Without loss of generality, we can suppose that for some $f_o \in E^*$, $f_o \neq 0$, we have that $K \subseteq f_o^{-1}(1)$.

First we want to show that if $z \in F'$, then

$$F' \cap co(F \cup face(z)) = face(z).$$

Suppose $u \in F' \cap co(F \cup face(z))$. Then

$$u = \alpha u_1 + (1-\alpha)u_2$$

where $u_1 \in F$, $u_2 \in face(z)$ and $\alpha \in [0,1]$. Since $u \in F'$, we have that $u = u_2 \in face(z)$ and hence $F' \cap co(F \cup face(z)) \subseteq face(z)$. The other inclusion is trivial.

Next we want to show that F' is a face. We only need to show that F' is convex. Suppose $z_1, z_2 \in F'$ and $\lambda \in \langle 0,1 \rangle$ and let

$$x = \lambda z_1 + (1-\lambda)z_2.$$

If $x \notin F'$, then $x \in K \setminus (F \cup F')$ and by (4.1)

$$x = \delta y + (1-\delta)z$$

where $y \in F$, $z \in F'$ and $\delta \in \langle 0,1 \rangle$. Since $co(F \cup face(z))$ is a face and $x \in co(F \cup face(z))$, we have that $z_1, z_2 \in F' \cap co(F \cup face(z)) = face(z)$.

Hence $\quad x = \lambda z_1 + (1-\lambda)z_2 \in \mathrm{face}(z) \subseteq F'$.

This contradiction shows that F' is a face.

Let $x \in K \setminus (F \cup F')$ and suppose for $i = 1,2$ that

$$x = \lambda_i y_i + (1-\lambda_i)u_i$$

where $\quad y_i \in F, \quad u_i \in F' \quad$ and $\quad \lambda_i \in \langle 0,1\rangle$.

Since $\mathrm{co}(F \cup \mathrm{face}(u_1))$ is a face, we have that $y_2, u_2 \in \mathrm{co}(F \cup \mathrm{face}(u_1))$, and hence $u_2 \in \mathrm{face}(u_1)$. Thus we have (see e.g.[1])

$$u_1 = \beta u_2 + (1-\beta)z'$$

where $\quad z' \in K \quad$ and $\quad \beta \in \langle 0,1]$, and hence

$$x = \lambda_1 y_1 + (1-\lambda_1)\beta u_2 + (1-\lambda_1)(1-\beta)z'.$$

Let $f \in E^*$ such that $f(u_2) = 0$. Then we have

$$f(\lambda_2 y_2) = f(\lambda_1 y_1 + (1-\lambda_1)(1-\beta)z')$$

and since these f's separate points in E, we have

$$\lambda_2 y_2 = \lambda_1 y_1 + (1-\lambda_1)(1-\beta)z'.$$

Now $\quad K \subseteq f_0^{-1}(1)$ implies that

$$\lambda_2 = \lambda_1 + (1-\lambda_1)(1-\beta)$$

so $\quad \lambda_2 \geq \lambda_1$

By a dual argument, we find that $u_1 \in \mathrm{face}(u_2)$ and $\lambda_1 \geq \lambda_2$. Hence we have $\lambda_1 = \lambda_2$ and $\beta = 1$ such that $u_1 = u_2$ and $y_1 = y_2$, and the proof is complete.

<u>Remark 23:</u> (i) => (ii) in Proposition 22 was pointed out to me by T. B. Andersen.

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