Modeling Thermal Systems

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ME584

Agenda

- Basic Effects
- Circuit Analysis of Static Thermal Systems
- Circuit Analysis of Dynamic Thermal Systems

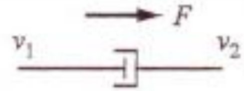
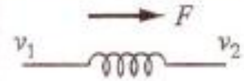
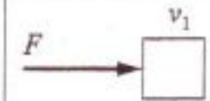
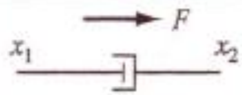
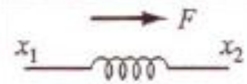
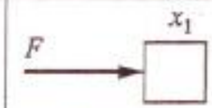
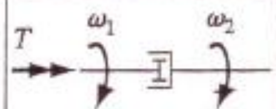
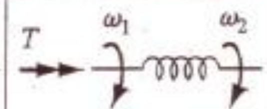
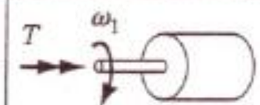
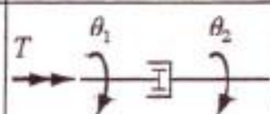
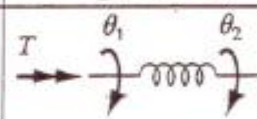
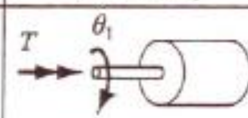
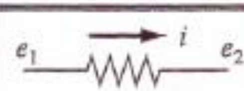
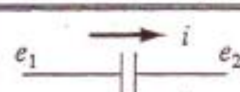
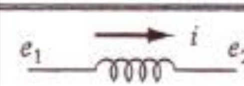
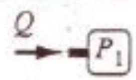
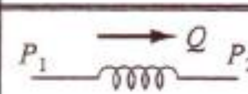
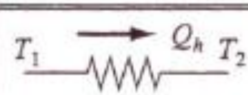
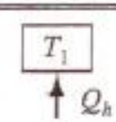
- Active Learning: Pair-share Exercises

Basic Effects

Thermal Systems

- Thermal Systems:
  - Energy is stored and transferred as heat
  - Exhibit static and dynamic behavior (resistance, capacitance, time constants. Thermal inductance does not exist.)
  - Nonlinear, variable-coefficient, distributed-parameter models
- Units:
  - Temperature T [^{0}C, K, ^{0}F, R]
  - Heat flow rate Q [J/s, BTU/hr]
- Thermal effects
  - Conduction
  - Convection
  - Radiation
  - Heat Storage Capacity

BASIC ELEMENTS IN ENGINEERING DISCIPLINES.

	Dissipative (Resistive)	Effort Storage (Capacitive)	Flow Storage (Inductive)
Mechanical Translation Effort = Force Flow = Velocity	 $F = b(v_1 - v_2)$	 $F = \frac{k}{D}(v_1 - v_2)$	 $F = mDv_1$
(Alternative Form) Effort = Force Flow = Position	 $F = bD(x_1 - x_2)$	 $F = k(x_1 - x_2)$	 $F = mD^2x_1$
Mechanical Rotation Effort = Torque Flow = Speed	 $T = b(\omega_1 - \omega_2)$	 $T = \frac{k}{D}(\omega_1 - \omega_2)$	 $T = JD\omega_1$
(Alternative Form) Effort = Torque Flow = Angle	 $T = bD(\theta_1 - \theta_2)$	 $T = k(\theta_1 - \theta_2)$	 $T = JD^2\theta_1$
Electrical Effort = Voltage Flow = Current	 $e_1 - e_2 = Ri$	 $e_1 - e_2 = \frac{1}{CD}i$	 $e_1 - e_2 = LDi$
Fluid Effort = Pressure Flow = Volume Flow Rate	 $P_1 - P_2 = RQ$	 $P_1 = \frac{1}{CD}Q$	 $P_1 - P_2 = LDQ$
Thermal Effort = Temperature Flow = Heat Flow	 $T_1 - T_2 = RQ_h$	 $T_1 = \frac{1}{CD}Q_h$	Does Not Exist

Thermal Conduction

- Conduction: Ability of materials to conduct heat
- Fourier's law of heat conduction

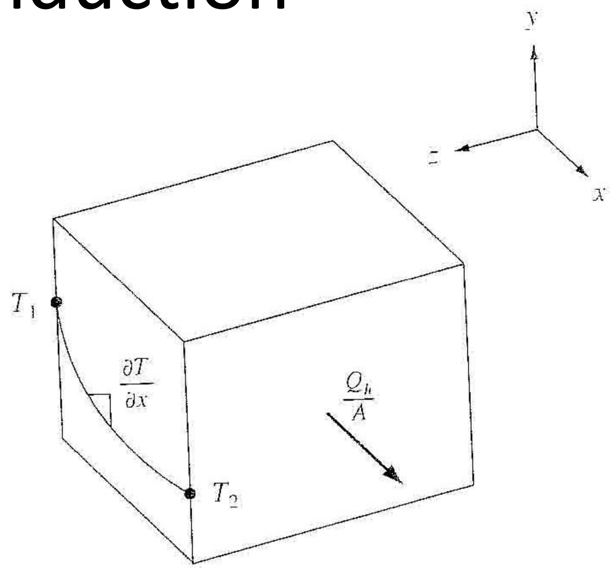
$$\frac{Q_h}{A} = -k_t \frac{dT}{dx}$$

Q_h : *heat transfer*

A : *surface area*

k_t : *thermal conductivity*

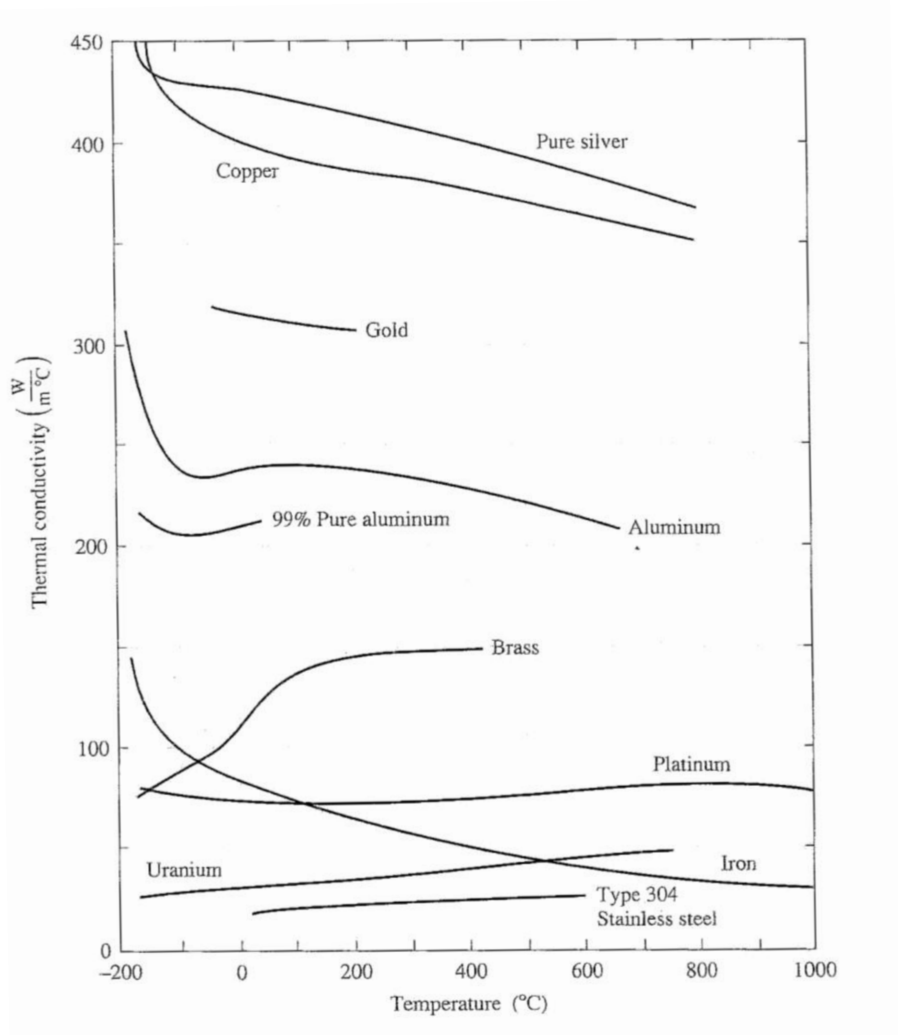
T : *Temperature*

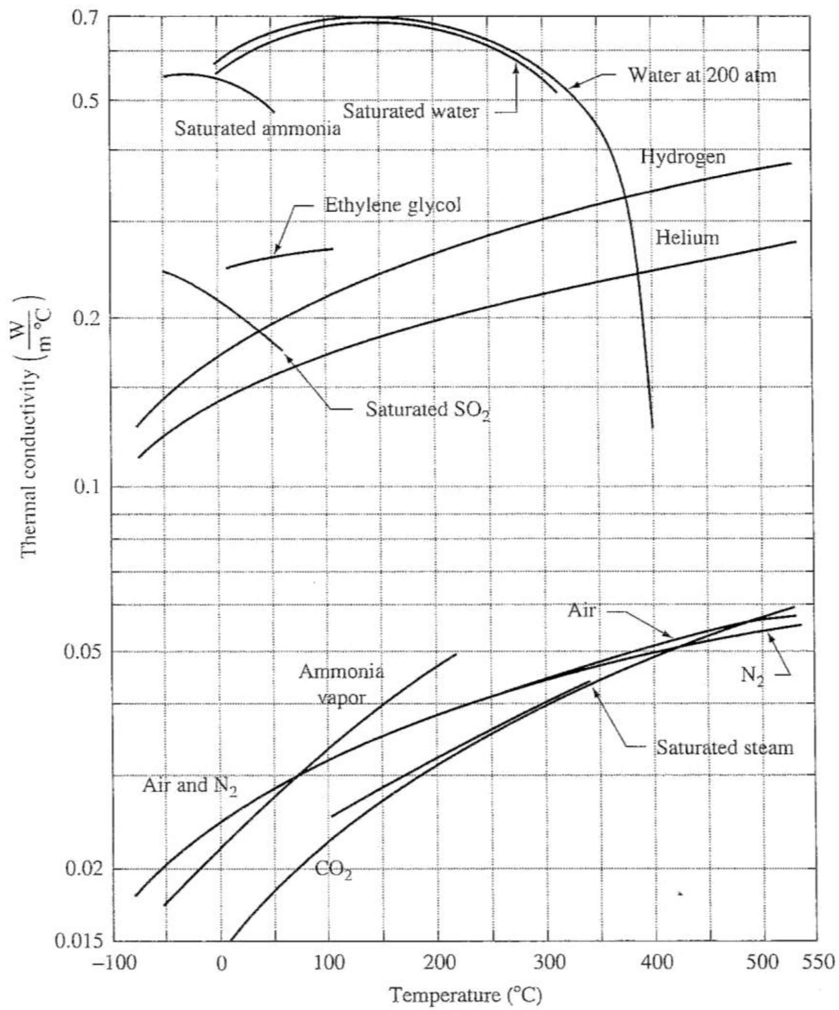


Minus sign indicates …

Temperature drops in direction of heat flow

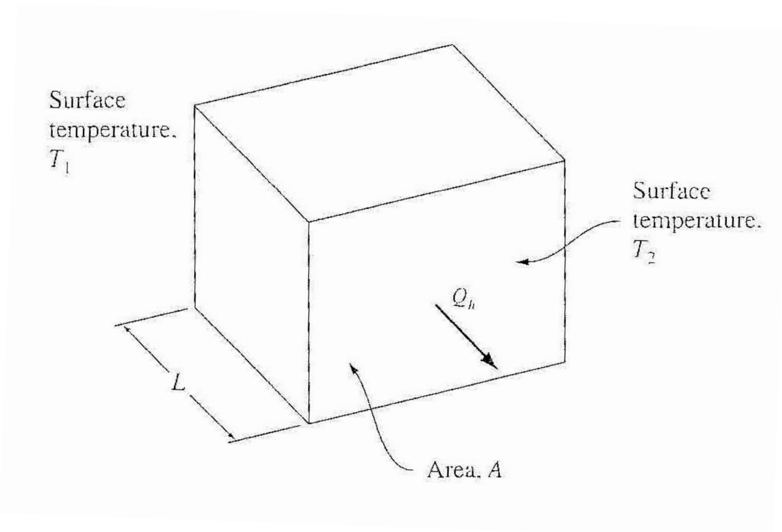
K_t: Metal V. Liquid/Gas





$K_t = K_t(T)$ => Nonlinear heat conduction equation

Conduction Through a Flat Plate

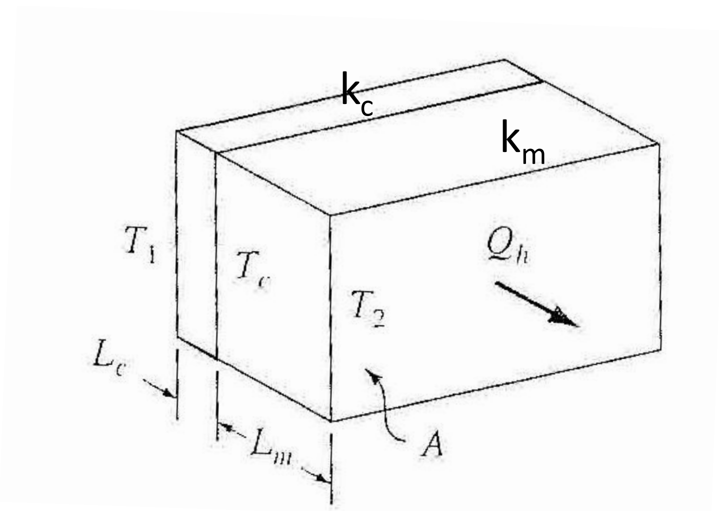


$$Q_h = -k_t A \frac{\partial T}{\partial x} \approx -k_t A \frac{\Delta T}{\Delta x} = -k_t A \frac{\delta T}{L}$$

$$where \quad \delta T = T_2 - T_1$$

Express as an impedance (similar to a resistor):

$$R = \frac{Q_h}{\delta T} = \frac{L}{A k_t}$$



What is R_{eq}?

$$R_{eq} = R_c + R_m = \frac{L_c}{A k_c} + \frac{L_m}{A k_m}$$

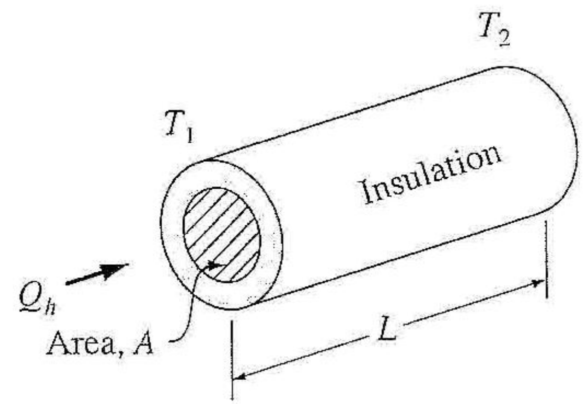
What is Q_h?

$$Q_h = \frac{T_2 - T_1}{R_{eq}}$$

Conduction in Various Shapes

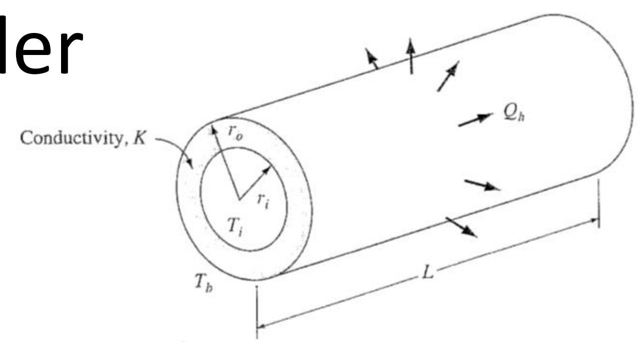
- Axial conduction in a rod

$$Q_h = \frac{k_t A}{L}(T_1 - T_2)$$



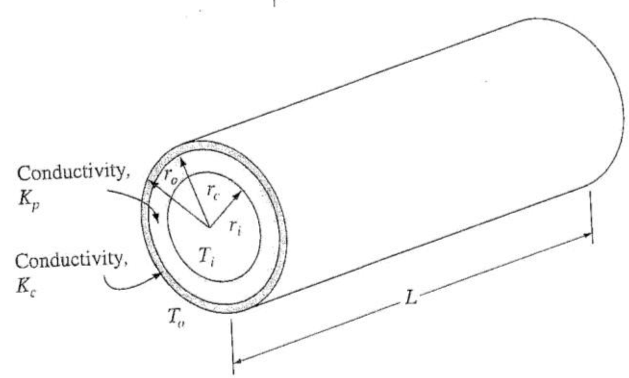
- Radial conduction in a cylinder

$$Q_h = \frac{(T_i - T_0)}{R_p + R_c} = \frac{(T_i - T_0)}{\frac{ln(r_c / r_i)}{2\pi k_p L} + \frac{ln(r_0 / r_c)}{2\pi k_c L}}$$



- With coated surface

$$Q_h = \frac{(T_i - T_0)}{R_p + R_c} = \frac{(T_i - T_0)}{\frac{ln(r_c / r_i)}{2\pi k_p L} + \frac{ln(r_0 / r_c)}{2\pi k_c L}}$$



Thermal Convection

- Convection: Heat transfer between a surface and a fluid exposed to surface via free or forced convection
- Newton's law of cooling:

$$\frac{Q_h}{A} = h(T_s - T_\infty)$$

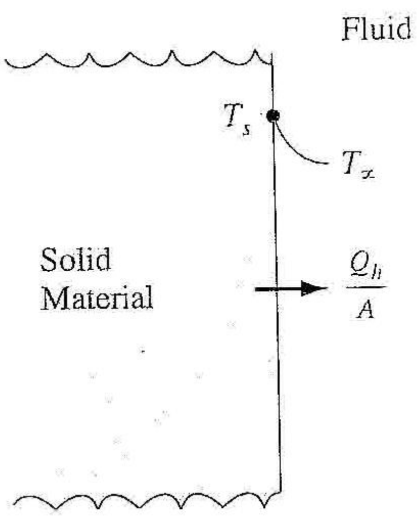
where

T_s = solid's surface temperature

T_∞ = temperature of free – stream fluid (constant)

h = convection coefficient, a function of

- properties of fluid (e.g., μ, ρ, k_t, C_p),
- solid's geometric configuration,
- free – stream fluid flow patterns



Thermal Resistance and Convection Coefficient

- Resistance

$$Q_h = hA\delta T \;\Rightarrow\; R = \frac{Q_h}{\delta T} = \frac{1}{hA}$$

- Typical Values for Convection Coefficient, h's

Fluid and condition	$\frac{\text{Btu}}{\text{hr ft}^2\,{}^\circ\text{F}} \times 5.698 =$	$\frac{\text{watt}}{\text{m}^2\,{}^\circ\text{C}}$
Air, free convection	1–5	6–30
Air, forced convection	5–100	30–600
Water, free convection	10–50	60–300
Water, forced convection	50–1000	300–6000
Water, boiling	500–10,000	3000–60,000
Superheated steam, free convection	1–5	6–30
Superheated steam, forced convection	5–50	30–300
Steam, condensing	1000–20,000	6,000–120,000
Oil, free convection	5–25	30–150
Oil, forced convection	25–500	150–3000
Oil, boiling	250–5000	1500–30,000

Thermal Radiation

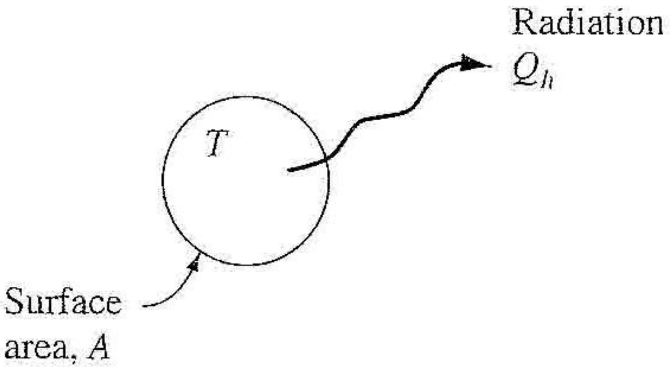
Radiation: heat transfer (when energy is high) without presence of surrounding medium

Heat transfer from ideal black body of surface area A:

$$\frac{Q_h}{A} = \sigma T^4$$

where σ = *Stefan – Boltzmann constant*

$$= 1.714 \times 10^{-9} \frac{Btu/hr}{ft^2 R^4} = 56.68 \times 10^{-9} \frac{Watt}{m^2 K^4}$$



Net radiant heat transfer between two bodies:

$$Q_h = F_e F_v \sigma A (T_H^4 - T_L^4)$$

where

F_e = *correction factor* (*emissivity less than black body*)

F_v = *view factor* (*accounts for lost radiation*)

T_H = *high temperature*

T_L = *low temperature*

Thermal Capacitance

- Capacitance: ability to store heat

$$Q_h = MC_p \frac{dT}{dt} \quad (assume\ no\ temperature\ gradient\ in\ mass)$$

M : mass of object

C_p : specific heat

- Specific heat of various substances

	Specific heat ratio, $k = C_p/C_v$	Specific heat, C_p kJ (kg°C)
Liquids		
ethylene glycol		2.36
octane		2.15
propane		2.41
salt water		3.93
water		4.18
Gases		
air	1.40	1.00
carbon dioxide	1.30	0.86
carbon monoxide	1.40	1.05
methane	1.31	2.26
propane	1.20	1.63

	Specific heat ratio, $k = C_p/C_v$	Specific heat, C_p kJ (kg°C)
Solids		
brick	—	0.92
concrete	—	0.96
earth	—	1.26
glass	—	0.84
paper	—	1.38
plastic	—	1.67
wood, white pine	—	2.51
Metals		
aluminum	—	0.90
copper	—	0.38
iron	—	0.45
lead	—	0.13
silver	—	0.24
zinc	—	0.39

Circuit Analysis of Static Thermal Systems

Example 1: Lowest Cost for Thermal Conductivity

Several metals are being considered for use in an application that conducts heat from one location to another through a solid rod of metal at room temperature. *Prepare* a table that lists the thermal conductivity of the following metals: aluminum, steel (use iron), copper, and silver. *Which* metal has the best thermal conductivity?

If the best conductor were used with a given rod diameter d^*, this would represent a reference thermal resistance per unit length. Now *calculate* what diameter d of rod would be required in the other metals to have the same thermal resistance per unit length as the best conductor. Based upon the density of each metal, *what* will be the weight per unit length for each? *Which* metal has the lowest weight for the given thermal resistance?

Obtain a rough estimate for the cost per pound of each of the preceding metals from industrial suppliers, and *calculate* the relative cost of each metal that is required to have the same resistance as discussed in the previous paragraph. *Which* metal has the lowest cost for the given thermal resistance?

Which metal would you use and why?

Example 1: Lowest Cost for Thermal Conductivity

For this case, we are looking for the diameter of a rod that will give the same thermal resistance as a reference material.

The thermal resistance of a rod is

$$R = \frac{L}{k\ A} = \frac{4\ L}{k\ \pi d^2}$$

For a rod made from a different material to have the same resistance per unit length as the reference material,

$$\frac{R}{R_r} = \frac{k_r\ d_r^2}{k\ d^2} = 1$$

Thus, the diameter ratio of the new material is

$$\frac{d}{d_r} = \sqrt{\frac{k_r}{k}}$$

The weight per unit length is $\frac{W}{L} = \frac{\pi}{4} d^2 \rho$ thus

$$\frac{W}{W_r} = \left(\frac{d}{d_r}\right)^2 \frac{\rho}{\rho_r}$$

Example 1: Lowest Cost for Thermal Conductivity

The cost per unit length C is the cost per weight C_w times the weight per length.

$$\frac{C}{L} = C_w \frac{W}{L} \qquad \text{thus} \qquad \frac{C}{C_r} = \frac{C_w}{C_{wr}} \frac{W}{W_r}$$

The spreadsheet shown below gives the conductivity, density, and cost of the candidate materials. The comparison factors are all based upon the fact that the thermal resistance is the same. They are the diameter ratio, the weight ratio, and the cost ratio. The best thermal conductivity is silver. The lowest weight is aluminum. The lowest cost is a tie between aluminum and steel. I would select aluminum because it is the lowest cost , it is the lowest weight, and has a relatively small diameter (compared to steel). Depending upon whether cost or weight is more important, copper might be the next best choice over steel since the weight and diameter are more desirable and the cost is not too much more; however, if cost is more important, steel would be the second best choice.

Material	conductivity $W/(m\,°K)$	density kg/m^3	cost/weight \$/kg	d/d_r	W/W_r	C/C_r
aluminum	236.0	2702	3.85	1.347	0.467	0.010
steel	83.5	7870	0.46	2.264	3.842	0.010
copper	401.0	8933	3.52	1.033	0.908	0.017
silver	428.0	10500	184.10	1.000	1.000	1.000

Example 2: Pair-Share: Heat Loss Through Window

- Calculate the amount of heat loss (in Watts) through an uncovered window in a home. The temperature difference between the air inside and outside of the house is 20^0C, with moderate convection on each side of the glass. The size of the glass is 0.75m by 1.2 m, and the thickness is 3 mm.

Example 2: Heat Loss Through Window

The thermal resistance of the glass due to its conductivity is

$$R_k = \frac{t}{k\ A}$$

using numbers

$$R_k = \frac{t}{k\ A} = \frac{0.003\ \text{m}}{0.81\ \frac{\text{watt}}{\text{m}\ {}^\circ\text{C}}\ 0.825\ \text{m}^2} = 0.004489\ \frac{{}^\circ\text{C}}{\text{watt}}$$

If the convection were perfect, the heat loss through the glass due to conductivity alone would be

$$Q_h = \frac{\Delta T}{R_k} = \frac{20\ {}^\circ\text{C}}{0.004489\ \frac{{}^\circ\text{C}}{\text{watt}}} = 4455\ \text{watts}$$

The total thermal resistance is composed of convection on the inside and outside of the glass, and the thermal conductivity resistance.

$$R_{Total} = \frac{1}{h_{in}\ A} + \frac{t}{k\ A} + \frac{1}{h_{out}\ A}$$

Example 2: Heat Loss Through Window

Using the following values:

$$h_{in} = 25 \frac{\text{watt}}{\text{m}^2\,°\text{C}} \qquad \text{(high free convection)}$$

$$h_{out} = 45 \frac{\text{watt}}{\text{m}^2\,°\text{C}} \qquad \text{(low forced convection)}$$

$$k = 0.81 \frac{\text{watt}}{\text{m}\,°\text{C}}$$

$$t = 3.0\ \text{mm}$$

$$A = 0.825\ \text{m}^2$$

the total resistance is

$$R_{Total} = \frac{1}{25 \frac{\text{watt}}{\text{m}^2\,°\text{C}} 0.825\,m^2} + \frac{0.003\ \text{m}}{0.81 \frac{\text{watt}}{\text{m}\,°\text{C}} 0.825\ \text{m}^2} + \frac{1}{45 \frac{\text{watt}}{\text{m}^2\,°\text{C}} 0.825\,m^2}$$

$$R_{Total} = [0.048485 + 0.004489 + 0.026936] \frac{°\text{C}}{\text{watt}} = 0.0799 \frac{°\text{C}}{\text{watt}}$$

With a 20 degree temperature difference considering the convection and conduction, the heat loss is

$$Q_h = \frac{\Delta T}{R_{Total}} = \frac{20\,°\text{C}}{0.0799 \frac{°\text{C}}{\text{watt}}} = 250\ \text{watts}$$

Example 3: Radiator Hose

- The radiator hose on an automobile is made of something similar to Buna™ rubber. The internal diameter of the hose is 50 mm, its length is 200mm, and its thickness is 6 mm. The water on the inside is maintained at 100^0C, and the outside air temperature is 50^0C. Calculate how much heat is conducted through the hose.

Example 3: Radiator Hose

The thermal resistance of the hose due to its conductivity is

$$R_k = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k\ L}$$

Eq. 6.10 (pg. 178)

using numbers

$$R_k = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi k\ L} = \frac{\ln\left(\frac{62}{50}\right)}{2\pi 0.465 \frac{\text{watt}}{\text{m}\,^\circ\text{C}} 0.200\ \text{m}} = 0.368 \frac{^\circ\text{C}}{\text{watt}}$$

If the convection were perfect, the heat loss through the hose due to conductivity alone would be

$$Q_h = \frac{\Delta T}{R_k} = \frac{50\,^\circ\text{C}}{0.368 \frac{^\circ\text{C}}{\text{watt}}} = 136\ \text{watts}$$

Example 4: Pair-Share: Iron Pipe

A ¾ inch pipe (ID = .824 inch, OD = 1.050 inches) is used to transmit hot water. The internal water temperature is 120^0F, and the ambient air temperature is 75^0F. Consider a pipe that is 5 feet in length.

- Calculate how much heat is transferred from the water to air.
- If a spongy rubber tape that is 0.1 inch thick is placed on the outside of the pipe, calculate how much heat is transferred.
- What is the reduction in heat loss with the use of the rubber insulation?

Example 4: Pair-Share: Iron Pipe

The total resistance is composed of three resistances in series.

$$R_k = \frac{\ln\left(\frac{d_o}{d_i}\right)}{2\pi k\ L} \qquad R_{hi} = \frac{1}{h_i\ A_i} = \frac{4}{\pi d_i^2\ h_i} \qquad R_{ho} = \frac{1}{h_o\ A_o} = \frac{4}{\pi d_o^2\ h_o}$$

assuming a low forced convection coefficient on the inside: $h_i = 200 \frac{\text{Btu}}{\text{hr ft}^2\ {}^\circ\text{F}}$

the conduction coefficient for steel: $k_{steel} = 30 \frac{\text{Btu}}{\text{hr ft}\ {}^\circ\text{F}}$

assuming a high free convection coefficient on the outside: $h_o = 5 \frac{\text{Btu}}{\text{hr ft}^2\ {}^\circ\text{F}}$

$$R_{hi} = \frac{4 \times 144 \frac{\text{in}^2}{\text{ft}^2}}{\pi 0.824^2\ \text{in}^2\ 200\left(\frac{\text{Btu}}{\text{hr ft}^2\ {}^\circ\text{F}}\right)} = 1.35 \frac{{}^\circ\text{F}}{\text{Btu/hr}}$$

$$R_k = \frac{\ln\left(\frac{1.050}{0.824}\right)}{2\pi 30\left(\frac{\text{Btu}}{\text{hr ft}\ {}^\circ\text{F}}\right) 5\ \text{ft}} = 0.0026 \frac{{}^\circ\text{F}}{\text{Btu/hr}}$$

Example 4: Pair-Share: Iron Pipe

$$R_{ho} = \frac{4 \times 144 \dfrac{\text{in}^2}{\text{ft}^2}}{\pi 1.050^2 \text{ in}^2\, 5\left(\dfrac{\text{Btu}}{\text{hr ft}^2\,{}^\circ\text{F}}\right)} = 33.26 \frac{{}^\circ\text{F}}{\text{Btu/hr}}$$

the total combined resistance:

$$R_{total} = (1.35 + 0.0026 + 33.26)\frac{{}^\circ\text{F}}{\text{Btu/hr}} = 34.61 \frac{{}^\circ\text{F}}{\text{Btu/hr}}$$

therefore, the heat loss is

$$Q = \frac{\Delta T}{R_{total}} = \frac{(120 - 75)\,{}^\circ\text{F}}{34.61 \dfrac{{}^\circ\text{F}}{\text{Btu/hr}}} = 1.30 \frac{\text{Btu}}{\text{hr}}$$

for spongy rubber,

$$k = 0.032 \frac{\text{Btu}}{\text{hr ft}\,{}^\circ\text{F}}$$

Example 4: Pair-Share: Iron Pipe

$$R_{sponge} = \frac{\ln\left(\frac{1.050+0.200}{0.824}\right)}{2\pi 0.032\left(\frac{\text{Btu}}{\text{hr ft}\,^\circ\text{F}}\right)5\text{ ft}} = 0.173\,\frac{^\circ\text{F}}{\text{Btu/hr}}$$

$$R_{total} = (34.61+0.17)\frac{^\circ\text{F}}{\text{Btu/hr}} = 34.78\,\frac{^\circ\text{F}}{\text{Btu/hr}}$$

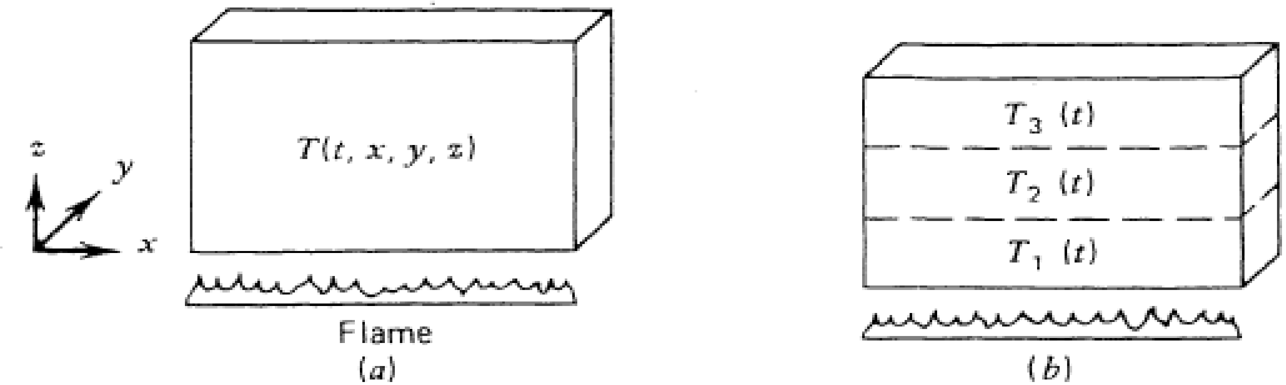
Therefore, the heat loss with the rubber is

$$Q = \frac{\Delta T}{R_{total}} = \frac{(120-75)\,^\circ\text{F}}{34.78\,\frac{^\circ\text{F}}{\text{Btu/hr}}} = 1.29\,\frac{\text{Btu}}{\text{hr}}$$

This represents a 1% reduction in heat flow.

Circuit Analysis of Dynamic Thermal Systems

Model Classification



Temperature distribution in a plate. (*a*) Distributed-parameter representation. (*b*) Lumped-parameter representation using three elements.

$$T = T(t, x, y, z)$$

$$f(T, \frac{\partial T}{\partial t}, \frac{\partial^2 T}{\partial x^2}, \frac{\partial^2 T}{\partial y^2}, \frac{\partial^2 T}{\partial z^2}) = 0$$

Example : Heat diffusion equation

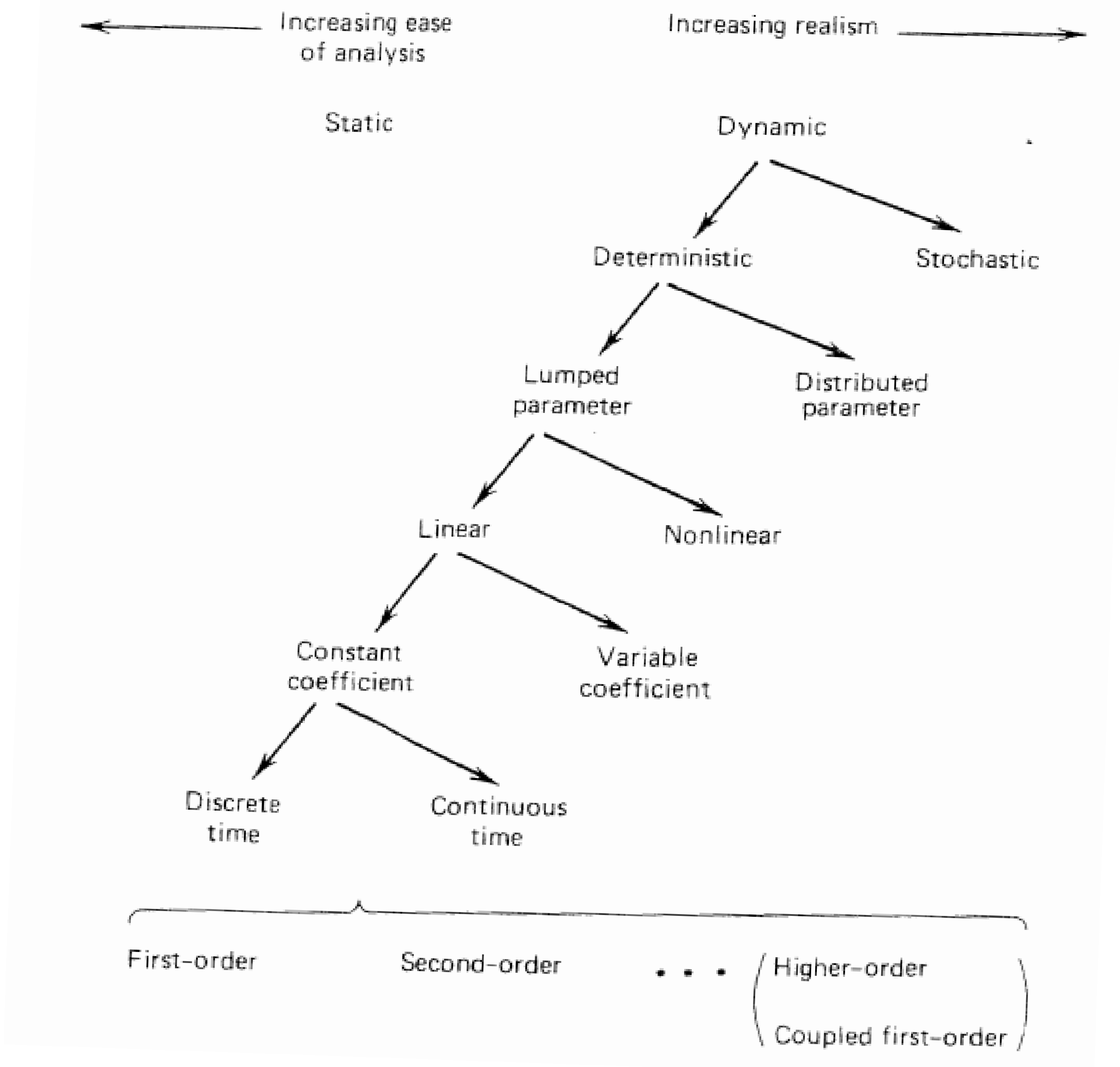
$$\frac{\partial T}{\partial t} - \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = 0$$

$$\alpha = thermal\ diffusivity = \frac{k}{\rho C_p}$$

$$T = T(t)$$

$$f(T, \frac{\partial T}{\partial t}) = 0$$

Model Classification Tree



Dynamic Thermal Systems

- Dynamic response: time rate of change of temperature during process due to large heat capacity or small time duration
- Diffusion equation, 2^{nd} order partial differential equation (PDE) in space and time, is used to model this process (beyond scope of class)
- Take lumped-parameter model approach **under certain conditions**

Lumped Parameter Model (LPM)

- LPM assumes that all properties of thermal resistance and capacitance are lumped at selected points in space and produces a set of ordinary differential equations (ODEs) in time
- Approximating PDEs in space with ODEs in time at certain locations in the system requires breaking system into lumps
- Choice of single-lumped or multiple-lumped capacitance model depends on convection and conduction resistances
- Example: Model the temperature in a house
    - Use a single temperature => a single ODE
    - Take a representative temperature for each room => an ODE for each room, leading to several equations
    - Either model is still more manageable than if lumping were not performed

Biot Number

Number of lumps required for accuracy depends on ratio of conduction and convection resistances, i.e., dimensionless Biot number N_b

$$N_b = \frac{R_{cond}}{R_{conv}} = \frac{hL_c}{k}$$

$$where$$

$$L_c = characteristic\ length = volume / surface\ area$$

$$L_c = thickness\ for\ a\ plate$$

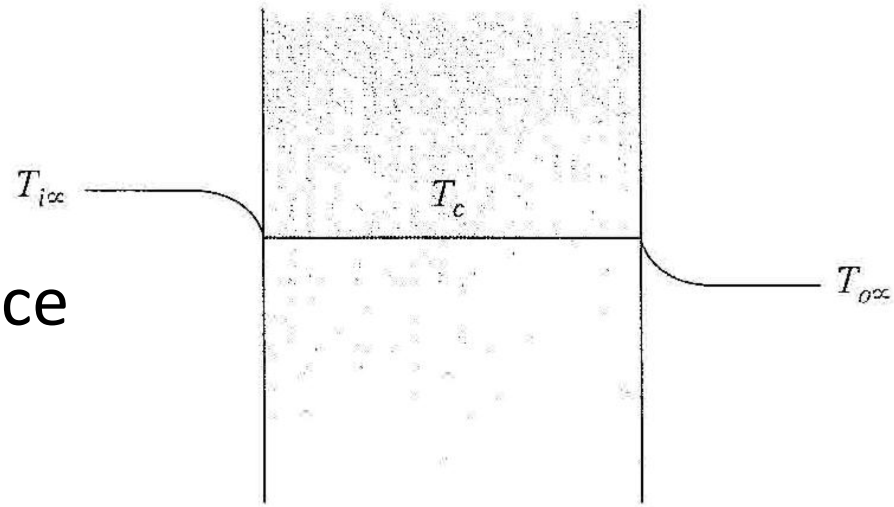
$$L_c = thickness / 2\ for\ a\ fin$$

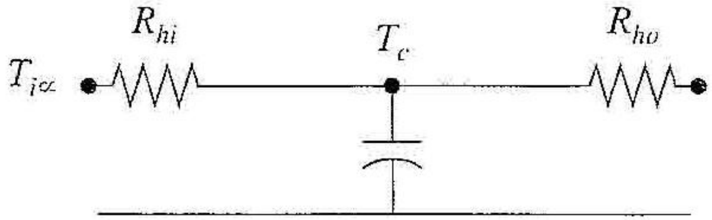
$$L_c = diameter / 4\ for\ a\ long\ cylinder$$

$$L_c = diameter / 6\ for\ a\ sphere$$

Small Biot Number (N_b < 0.1)

- $R_{cond} << R_{conv}$ → same temperature in solid
- Dominant temperature difference is between surface and free-stream fluid
- Model with single lumped capacitance, heat capacity of solid and convection heat transfer
- Ignore thermal conduction resistance in solid



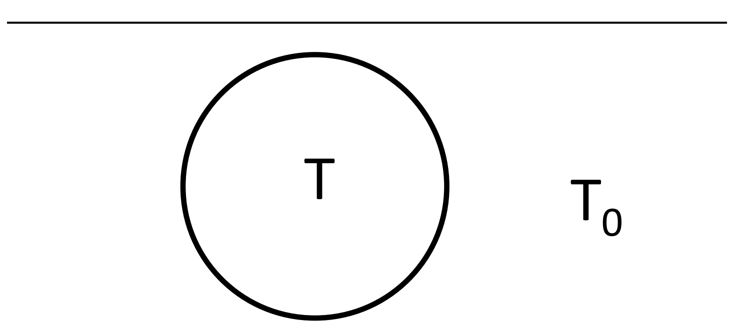


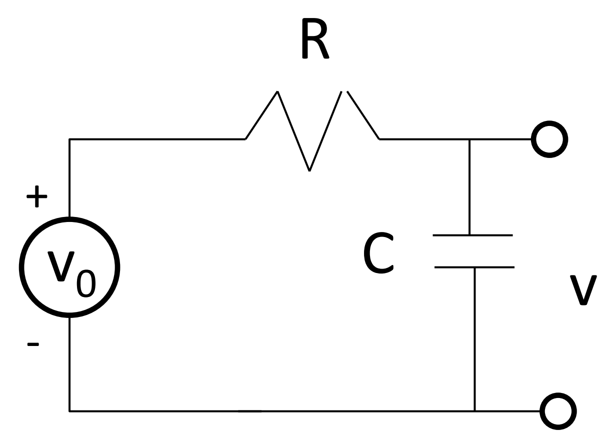
Large Biot Number ($N_b > 0.1$)

- $R_{cond} >> R_{conv} \rightarrow$ different temperatures in solid, and between solid and free-fluid stream
- Model with several lumps of capacitance and determine temperatures at several points inside solid as a function of time

Example 5: Single-Lumped Capacitance Model (SLCM): Quenching

- SLC model applies when heat capacity is significant (N_b is small)
- Example: Quenching is a process in which a heated object is placed into a liquid bath. The rapid cooling that occurs can improve certain properties such as hardness. Consider a copper sphere 1 in. in diameter with k = 212 BTU/hr-ft-^{0}F at 570^{0}F and immersed in a fluid such that h = 0.5 BTU/hr-ft^2 (See Figure below).
  - Show that its temperature can be considered uniform, and develop a model of the sphere's temperature as a function of the temperature T_0 of the surrounding fluid.
  - Show that the RC circuit (see below) is analogous to the thermal model of the sphere's quenching process.





Example 5: Single-Lumped Capacitance Model (SLCM): Quenching

- $A_s = 4\pi(1/24)^2 = 2.18\text{x}10^{-2}\ ft^2$, Volume = $(4/3)\pi(1/24)^3 = 3.03\text{x}10^{-4}\ ft^3$
- $L_c = V/A_s = 0.0139$, $N_b = hL_c/k = 3.28\text{x}10\text{-}5 \ll 1$
- Biot criterion: treat sphere as a single-lumped system with a single uniform T

$$(Q_h)_{conduction} = (Q_h)_{convection} \Rightarrow MC_p \frac{dT}{dt} = hA(T_0 - T) = \frac{1}{R}(T_0 - T)$$

$$MC_p = \rho V C_p = 17.3 \frac{slug}{ft^3}(3.03\times 10^{-4}\ ft^3)(2.93 \frac{BTU}{slug - {}^0F}) = 0.0154 \frac{BTU}{{}^0F};$$

$$R = \frac{1}{hA} = \frac{1}{0.5 \frac{BTU}{hr - ft^2 - {}^0F}(2.18\times 10^{-2}\ ft^2)} = 91.7 \frac{hr - {}^0F}{BTU}$$

$$Model\ is:\ \ 0.0154 \frac{dT}{dt} = \frac{1}{91.7}(T_0 - T)$$

$$\tau = RC = 91.7 \frac{hr - {}^0F}{BTU}(0.0154 \frac{BTU}{{}^0F}) = 1.4 hr$$

$$Circuit\ model\ is:\ C\frac{dv}{dt} = \frac{1}{R}(v_0 - v)$$

v plays the role of sphere's temperature T

If sphere is dropped into tank, T_0 acts like a step input,

sphere's temperature will reach T_0 in $4\tau = 5.6\ hr$

Multiple-Lumped Capacitance Model (MLCM)

- Models one-dimensional heat conduction with an equivalent thermal circuit with lumped capacitance and resistance
  - Capacitance at nodes connected by resistance elements, analogous to mechanical system's discrete masses connected by massless springs
  - Accuracy depends on number of lumps: large Bio number requires division into several nodes
  - Thermal properties are assumed homogeneous

Multiple-Lumped Capacitance Model (MLCM)

- Models one-dimension heat transfer in a flat plate with mass M and thickness L (<< length and width) by dividing into n elements

Resistance elements :

$$Q_h = \frac{1}{R_n}\delta T$$

$$R_n = \frac{L/n}{kA}$$

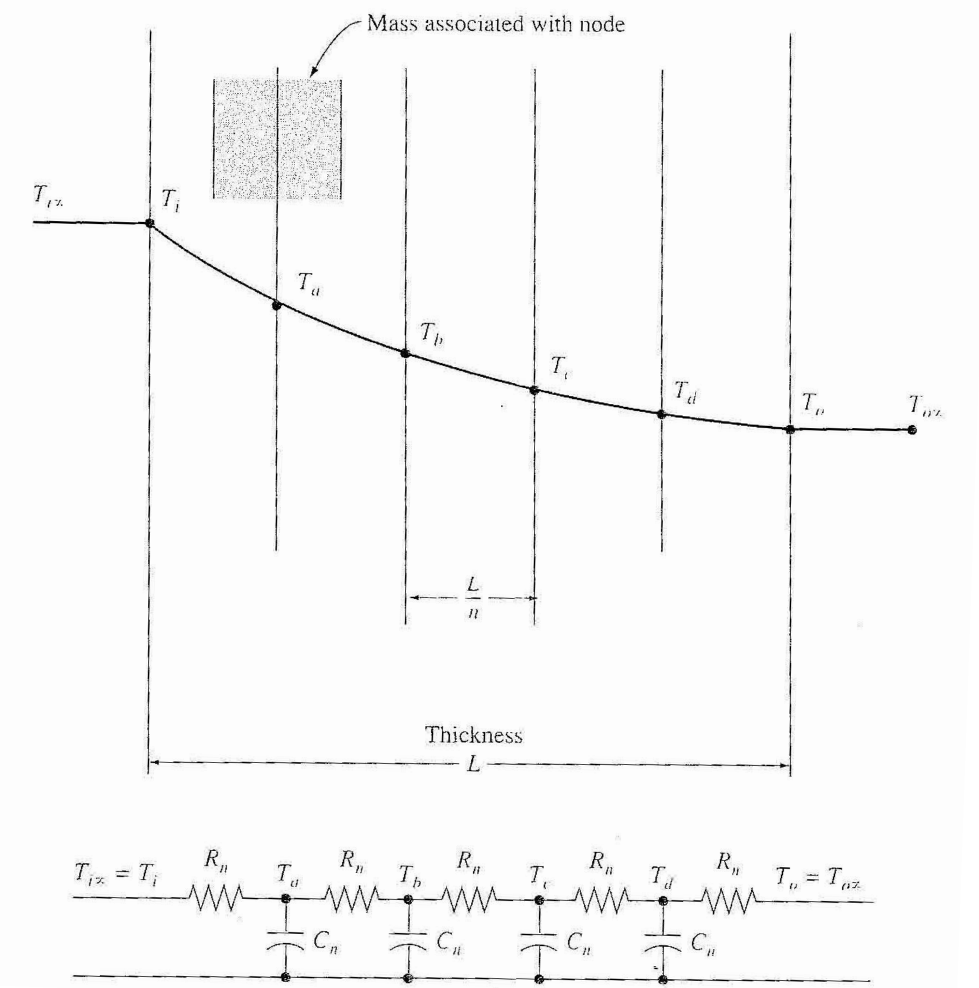
Capacitance elements:

$$Q_h = C_n \frac{dT}{dt}$$

$$C_n = C_p M/n$$

Constant Surface Temperature Sources

- When convection coefficients are high, convection resistance will be small, and surface temperature will equal free-stream temperature and acts as an ideal temperature source



Resistance between T_i *and* T_a :

$$Q_{ia} = \frac{1}{R_n}(T_i - T_a)$$

Capacitance at T_a :

$$Q_{ia} - Q_{ab} = C_p \frac{M}{n}\dot{T}_a$$

Resistance between T_a *and* T_b :

$$Q_{ab} = \frac{1}{R_n}(T_a - T_b)$$

Capacitance at T_b :

$$Q_{ab} - Q_{bc} = C_p \frac{M}{n}\dot{T}_b$$

Resistance between T_{n-1} *and* T_0 :

$$Q_{n-1,0} = \frac{1}{R_n}(T_{n-1} - T_0)$$

Constant Surface Temperature Sources

Substitute for Q and solve for $\dot{T}$ at each node:

$$\dot{T}_a = \frac{n}{C_p M R_n}\left[-2T_a + T_i + T_b\right]$$

$$\dot{T}_b = \frac{n}{C_p M R_n}\left[-2T_b + T_a + \underset{R_{h0}}{T_c}\right]$$

$$\vdots$$

$$\dot{T}_{n-1} = \frac{n}{C_p M R_n}\left[-2T_{n-1} + T_{n-2} + T_0\right]$$

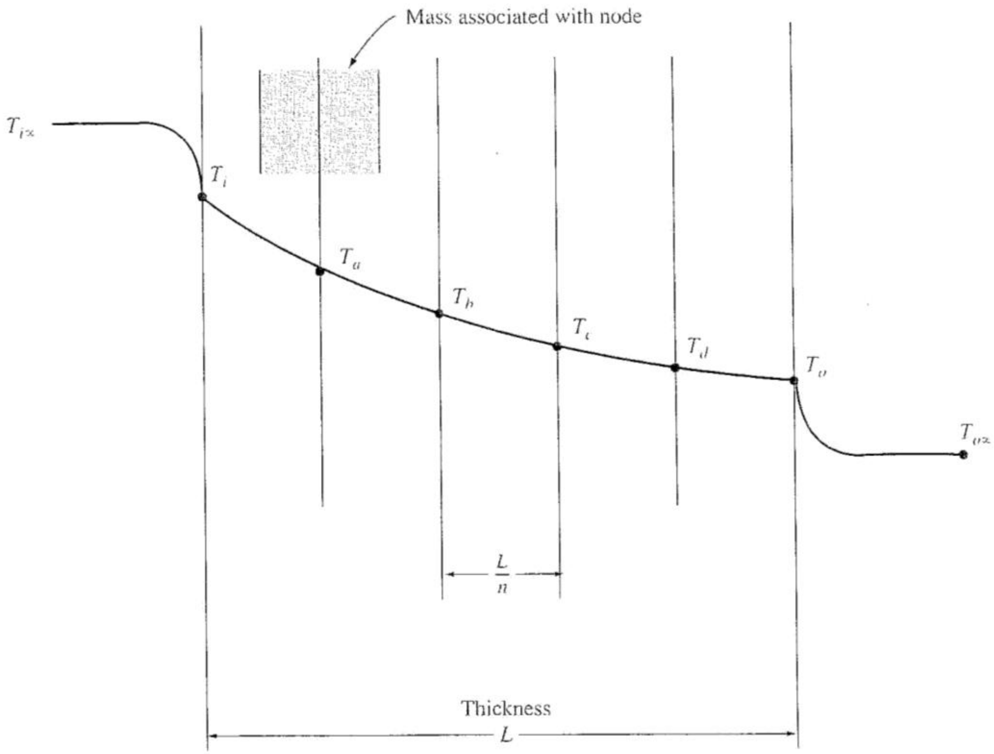
Use these equations to find heat transfer at any node,

e.g., heat going into inside wall

$$Q_i = \frac{1}{R_n}(T_i - T_a)$$

Convection Surface Temperature Sources

- When convection coefficients are not high, convection resistance will be large and there will be convection resistance between surface temperature and free-stream temperature, and significant temperature difference between them



$Resistance\ between\ T_{i\infty}\ and\ T_i:$

$$Q_i = \frac{1}{R_{hi}}(T_{i\infty} - T_i) \quad where \quad R_{hi} = \frac{1}{h_i A}$$

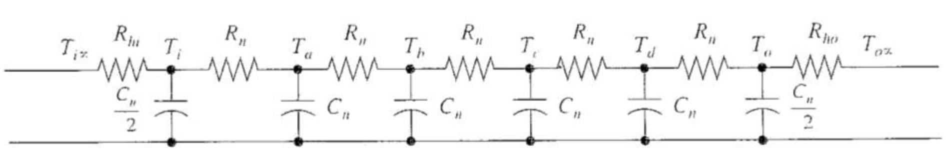
$Capacitance\ at\ T_i:$

$$Q_i - Q_{ia} = \frac{C_p}{2}\frac{M}{n}\dot{T}_i$$

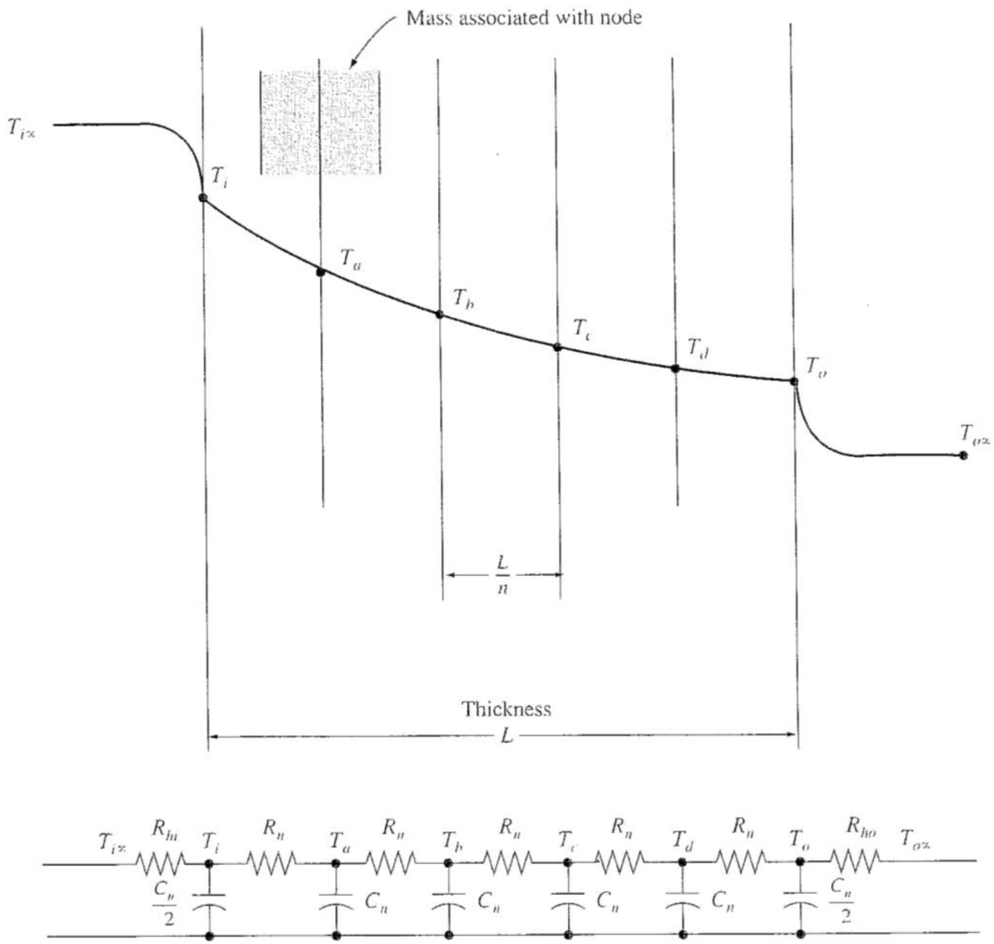
$Resistance\ between\ T_i\ and\ T_a:$

$$Q_{ia} = \frac{1}{R_n}(T_i - T_a)$$

(*continue on next page...*)



Convection Surface Temperature Sources



Capacitance at T_a :

$$Q_{ia} - Q_{ab} = C_p \frac{M}{n} \dot{T}_a$$

Resistance between T_{n-1} *and* T_0 :

$$Q_{n-1,0} = \frac{1}{R_n}(T_{n-1} - T_0)$$

Capacitance at T_0 :

$$Q_{n-1,0} - Q_0 = \frac{C_p}{2}\frac{M}{n}\dot{T}_0$$

Resistance between T_0 *and* T_{n-1} :

$$Q_0 = \frac{1}{R_{h0}}(T_0 - T_{0\infty}) \;\; where \;\; R_{h0} = \frac{1}{h_0 A}$$

Convection Surface Temperature Sources

$$Substitute\ \ for\ Q\ and\ solve\ for\ \dot{T}\ at\ each\ node:$$

$$\dot{T}_i = \frac{2n}{C_p M}\left[-\left(\frac{1}{R_{hi}} + \frac{1}{R_n}\right)T_i + \frac{T_{i\infty}}{R_{hi}} + \frac{T_a}{R_n}\right]$$

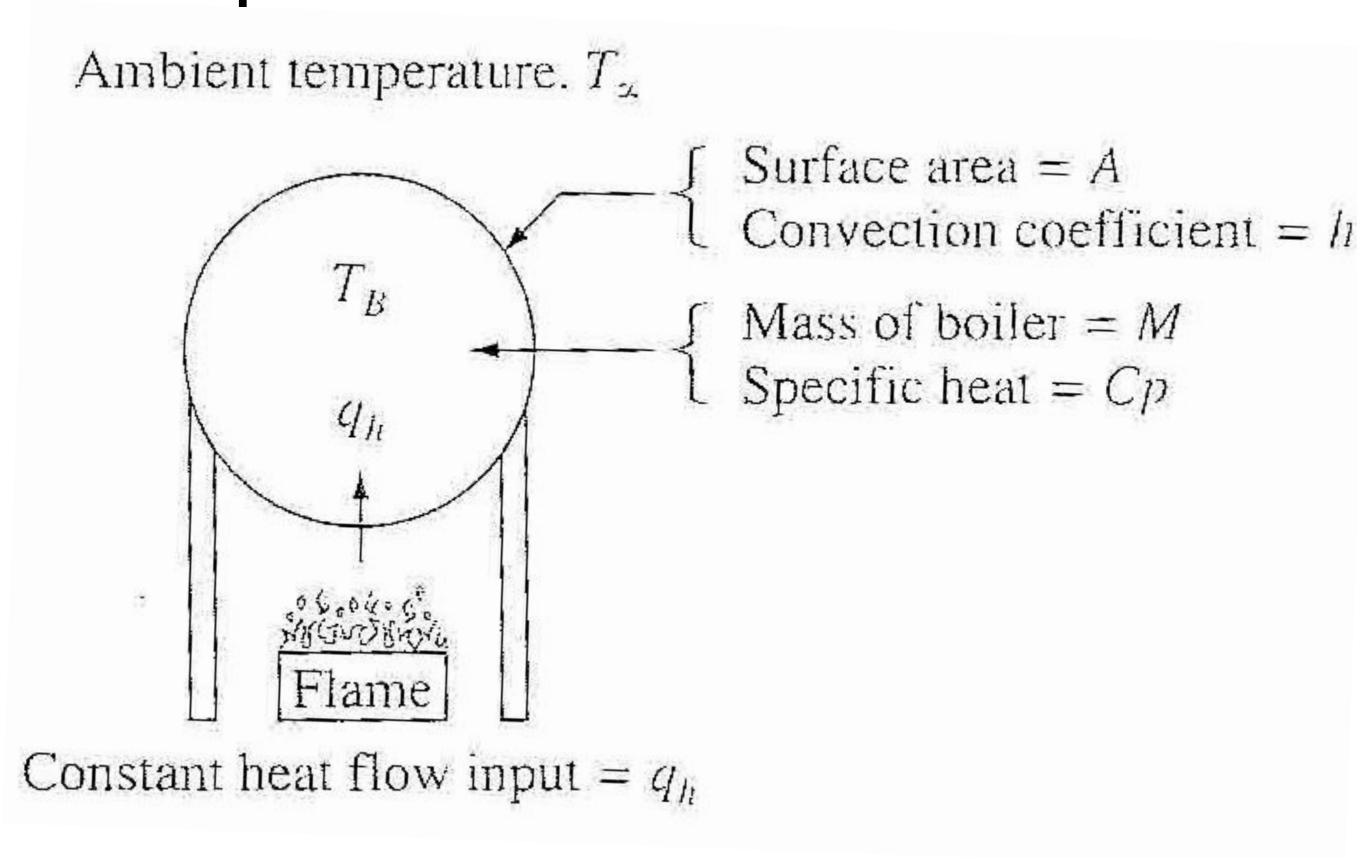
$$\dot{T}_a = \frac{n}{C_p M R_n}\left[-2T_a + T_i + T_b\right]$$

$$\vdots$$

$$\dot{T}_{n-1} = \frac{n}{C_p M R_n}\left[-2T_{n-1} + T_{n-2} + T_0\right]$$

$$\dot{T}_0 = \frac{2n}{C_p M}\left[-\left(\frac{1}{R_n} + \frac{1}{R_{h0}}\right)T_0 + \frac{T_{n-1}}{R_n} + \frac{T_{0\infty}}{R_{h0}}\right]$$

Example 6: Thermal Plant Boiler



Shown in the figure is a boiler used in a thermal energy plant. Write the modeling equations for the system, and derive a differential equation for the temperature inside the boiler as a function of the ambient temperature and the heat input. State the expressions for the time constant and the steady-state temperature

Example 6: Thermal Plant Boiler

The differential equation for this system is $q_h - q_{conv} = M\,C_p\,\dot{T}_B$

The convection heat transfer is $q_{conv} = h\,A\,(T_B - T_\infty)$

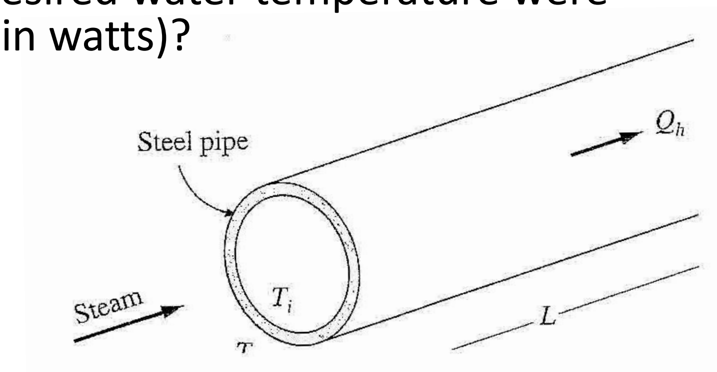
Rearranging, $\left[\frac{M\,C_p}{h\,A}D + 1\right]T_B = T_\infty + \frac{q_h}{h\,A}$

The time constant is $\tau = \frac{M\,C_p}{h\,A}$

The steady-state temperature is $T_B(\infty) = T_\infty + \frac{q_h}{h\,A}$

Example 7: PS: Pool Water Heater

- Shown below is a swimming pool operating in cool weather. The pool has a heater that provides a heat input of q_h watts. The water depth varies from 1.2m to 3m, and the pool is 5m wide by 10 m long. A pump runs continuously to keep the water thoroughly mixed.
  - Calculate the Biot number, and justify the assumption that the pool can be treated as a single-lumped capacitance model
  - Assuming that the significant heat transfer is the convection at the water surface and that no heat transfers out the sides and bottoms, write the modeling equations for the system using a single-lumped parameter capacitor model, and derive a differential equation for the temperature of the water as a function of the ambient temperature and the heat input.
  - Calculate the value of the time constant of the system. If a period of one time constant would be enough time to "take the chill off," would you recommend leaving the heater on all the time or turning it off overnight?
  - State the expression for the steady-state temperature of the water. If the ambient temperature were 20^0C and the desired water temperature were 25^0C, what size heater would be required (in watts)?



Example 7: PS: Swimming Pool Heater

Assuming a convection coefficient of high free convection, an average depth of 2.1 m, and using the thermal conductivity of water, we can calculate the Biot number for this system as follows.

$$N_b = \frac{h\,L_c}{k} = \frac{30\,\frac{\text{watt}}{\text{m}^2\,°\text{C}}\,2.1\,\text{m}}{600\,\frac{\text{watt}}{\text{m}\,°\text{C}}} = 0.105$$

Since the Biot number is about 0.1, we can treat the system as a lumped-parameter system.

The heat transfer and energy storage for this system can be stated as

$$C_p\ M\ \dot{T}_{water} = q_h - Q_{air} - Q_{ground}$$

The convection from the water surface to the air is

$$Q_{air} = h_{air}\ A_{surface}\left(T_{water} - T_{air}\right) \qquad \text{where} \quad A_{surface} = 50\,m^2$$

Example 7: PS: Swimming Pool Heater

If you calculate the thermal resistance of concrete, you would find that the conductivity resistance is very large relative to the convection resistance of the water on the bottom of the pool (by a factor of 35). Further, the resistance of the concrete is large relative to the convection resistance from the water to the air (by a factor of 7). Thus, the heat lost to the ground will be neglected.

Thus, the simplified differential equation can be stated as follows.

$$C_p \; M \; \dot{T}_{water} + h_{air} \; A_{surface} \left(T_{water} - T_{air} \right) = q_h$$

or

$$\left[\frac{C_p \; M}{h_{air} \; A_{surface}} D + 1 \right] T_{water} = T_{air} + \frac{q_h}{h_{air} \; A_{surface}}$$

From this differential equation, we can see that the time constant is

$$\tau = \frac{C_p \; M}{h_{air} \; A_{surface}}$$

Example 7: PS: Swimming Pool Heater

Using numbers, the time constant can be calculated.

$$\tau = \frac{4180 \dfrac{\text{J}}{\text{kg}\,°\text{C}}\, 105{,}000\ \text{kg}}{30 \dfrac{\text{J/s}}{\text{m}^2\,°\text{C}}\, 50\ \text{m}^2} = 292{,}600\ \text{s} = 81.3\ \text{hr}$$

Since this time constant is 3.87 days, turning the heater off overnight would not cause a large change in temperature; conversely, if the heater had been off for several days, turning on the heater overnight will not "take the chill off" very quickly, unless you have a high capacity heater.

The steady-state temperature of the water can be stated as follows.

$$T_{water} = T_{air} + \frac{q_h}{h_{air}\ A_{surface}}$$

Using the stated temperature differences and the assumed convection coefficient, we find the required heater power to maintain the temperature due to heat losses.

$$q_h = h_{air}\ A_{surface}\left(T_{water} - T_{air}\right) = 30 \frac{\text{watt}}{\text{m}^2\,°\text{C}}\, 50\ \text{m}^2\ 5\,°\text{C} = 7500\ \text{watts}$$

Example 8: Propane Tank

- A tank 75% (by volume) full of liquid propane is taken from a warehouse and placed in the direct sunlight on a hot summer day. We are concerned about the internal temperature (and therefore the internal pressure) of the tank as it sits in the sun. We want to know how long it takes to heat up and what is its final steady-state temperature.
  - Model the tank as a sphere 1 foot in diameter. The walls of the tank are steel of 0.100 of an inch thick. The temperature inside the warehouse is 90^0F, and the outside air temperature is 100^0F. (The tank is in Texas!) The solar insolation for this particular day is 900 watts/m^2 = 285 (Btu/hr)/ft^2. You should consider the sun as a constant heat flow source of energy over the exposed surface are of the sphere. The density of liquid propane is 20.8 lbm/ft^3. The density of steel is 490 lbm/ft^3. The specific heat of liquid propane is 0.58 Btu/(lbm ^{0}F). The specific heat of steel is 0.11 Btu/(lbm ^{0}F). The wind velocity is such that you should use a convection coefficient for high free convection or low forced convection.
  - Model this system, and derive a differential equation for the internal temperature of the propane as a function of the solar heat input and the ambient temperature (with a given initial temperature). Calculate the time constant for the system, and state how long it will take for the tank to be at its steady-state temperature. What will the steady-state temperature be for given conditions?

Example 8: Propane Tank

The differential equation for this system is $$M\,C_p\,\dot{T}_i = q_h - Q_{conv}$$

The convection from the tank is $$Q_{conv} = h\,A\left(T_i - T_{air}\right)$$

Rearranging, $$\left[\frac{M\,C_p}{h\,A}D + 1\right]T_i = T_{air} + \frac{q_h}{h\,A}$$

The time constant is $$\tau = \frac{M\,C_p}{h\,A}$$

The steady-state temperature is $$T_i(\infty) = T_{air} + \frac{q_h}{h\,A}$$

The surface area of a sphere is $$A = \pi\,d^2$$

The heat capacity of this system is composed of the heat capacity of the steel tank plus the heat capacity of the propane.

$$M\,C_p = M_t\,C_{pt} + M_p\,C_{pp}$$

Example 8: Propane Tank

The mass of the tank is $$M_t = \rho_{steel}\, A\, t$$

The mass of propane in the tank is $$M_p = 0.75\,\rho_p \frac{\pi d^3}{6}$$

The solar insolation is $285 \frac{\text{Btu / hr}}{\text{ft}^2}$, and the cross-sectional area that the tank is exposed to the sun is about 0.785 ft^2. Thus the total heat input is 224 Btu/hr.

Using the specified numbers: $$A = 3.142\ ft^2$$

$$M_t\, C_{pt} = 1.41 \frac{\text{Btu}}{{}^\circ\text{F}}$$

Example 8: Propane Tank

$$M_p C_{pp} = 7.02 \frac{\text{Btu}}{°\text{F}}$$

$$M C_p = 8.43 \frac{\text{Btu}}{°\text{F}}$$

$$h = 5 \frac{\text{Btu / hr}}{\text{ft}^2 \, °\text{F}}$$

$$\tau = 0.536 \text{ hr}$$

$$q_h = 224 \frac{\text{Btu}}{\text{hr}}$$

$$\frac{q_h}{h\,A} = 14.3\,°F$$

$$T_i(\infty) = 114.3\,°F$$

Thus, the time constant is 0.54 hr, and it will take 2.14 hours to come to a steady-state temperature. The solar heating adds 14.3 degrees above the air temperature.

Example 9: Pair-Share: Fish Tank

- A fish tank is fabricated using 4mm Plexiglas walls. The water on the inside of the tank is maintained at a constant temperature of 26^0C. The ambient room temperature is 22^0C. We are interested in modeling the heat transfer through the walls of the tank if there is a sudden change in room temperature. The inside of the tank has water that is circulated with a pump and therefore should have a convection coefficient on the lower end of forced convection with water. The outside of the tank has air that should have a convection coefficient on the upper end of free convection with air.
  - Calculate the Biot number for the inside and outside surfaces. Based upon these numbers, can lumped parameter modeling be used to model the temperature distribution in the Plexiglas? Can you assume that the convection heat transfer is so good in the water, that the water temperature on the inside surfaces is constant?
  - Based upon the Biot number, how many lumps should be considered in the lumped parameter analysis?

Example 9: Pair-Share: Fish Tank

The Biot number for the Plexiglas considering the inside convection is

$$N_b = \frac{h\,t}{k} = \frac{500 \frac{\text{watt}}{\text{m}^2\,{}^\circ\text{C}} 0.004 \text{ m}}{0.195 \frac{\text{watt}}{\text{m}\,{}^\circ\text{C}}} = 10.26$$

Since this Biot number is greater than 0.1, we should consider distributed-parameter effects. If we divide the thickness into 100 nodes, the Biot number for each node will be 0.1.

The Biot number for the Plexiglas considering the outside convection is

$$N_b = \frac{h\,t}{k} = \frac{30 \frac{\text{watt}}{\text{m}^2\,{}^\circ\text{C}} 0.004 \text{ m}}{0.195 \frac{\text{watt}}{\text{m}\,{}^\circ\text{C}}} = 0.62$$

Based upon this convection, we would be able to model the system with 6 lumps; however, the inside convection consideration dominates. One hundred nodes should be used.

Homework 6: Chapter 6

- 6.2
- 6.7
- 6.11
- 6.13
- 6.16
- 6.23

References

- Woods, R. L., and Lawrence, K., Modeling and Simulation of Dynamic Systems, Prentice Hall, 1997.
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