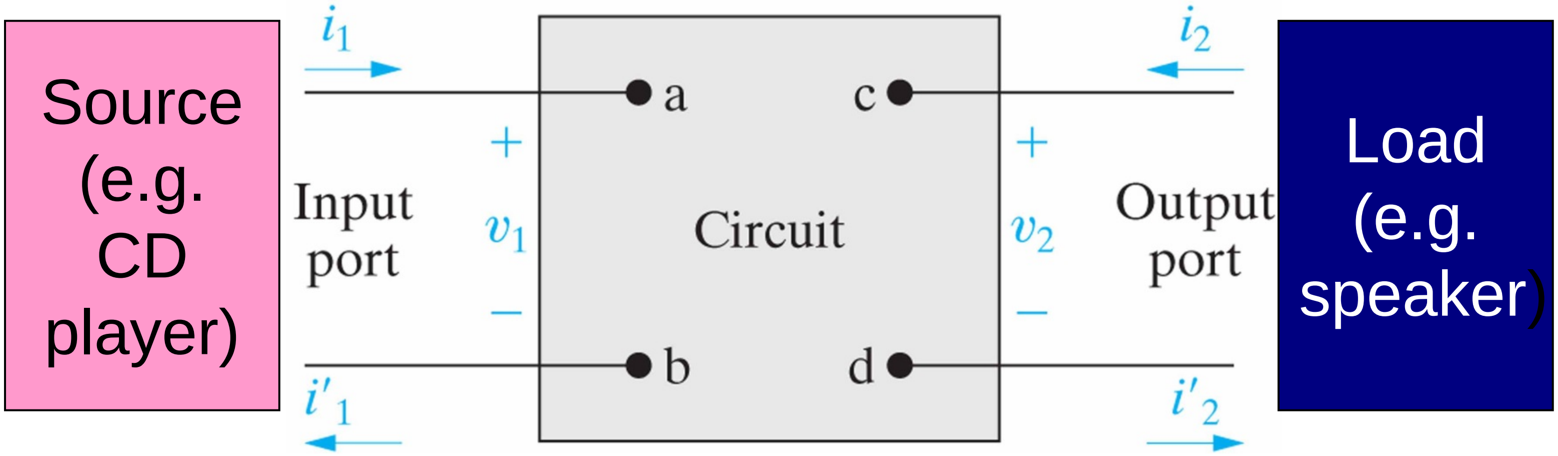
Chapter 18
Two-Port Circuits

Motivation

- Thévenin and Norton equivalent circuits are used in representing the contribution of a circuit to one specific pair of terminals.
- Usually, a signal is fed into one pair of terminals (input port), processed by the system, then extracted at a second pair of terminals (output port). It would be convenient to relate the v/i at one port to the v/i at the other port without knowing the element values and how they are connected inside the "black box".

How to model the "black box"?



- We will see that a two-port circuit can be modeled by a 2×2 matrix to relate the *v*/*i* variables, where the four matrix elements can be obtained by performing 2 experiments.

Restrictions of the model

- No energy stored within the circuit.
- No independent source.
- Each port is not a current source or sink, i.e. $i_1 = i_1',\ i_2 = i_2'.$
- No inter-port connection, i.e. between ac, ad, bc, bd.

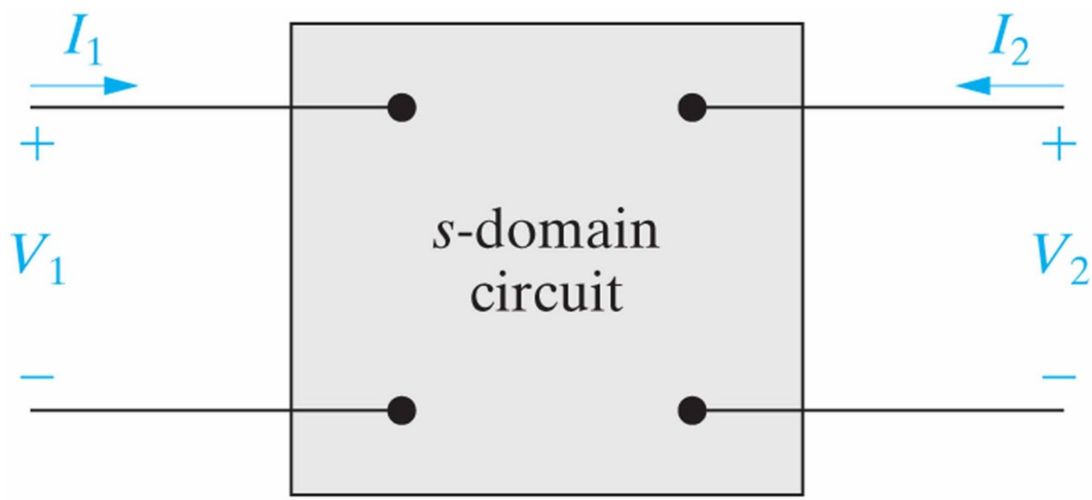
Key points

- How to calculate the 6 possible 2×2 matrices of a two-port circuit?
- How to find the 4 simultaneous equations in solving a terminated two-port circuit?
- How to find the total 2×2 matrix of a circuit consisting of interconnected two-port circuits?

Section 18.1
The Terminal Equations

s-domain model

- The most general description of a two-port circuit is carried out in the s-domain.



- Any 2 out of the 4 variables $\{V_1, I_1, V_2, I_2\}$ can be determined by the other 2 variables and 2 simultaneous equations.

Six possible sets of terminal equations (1)

$$\begin{cases} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}; \ [Z] \text{ is the impedance matrix;} \\ \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}; \ [Y] = [Z]^{-1} \text{ is the admittance matrix;} \end{cases}$$

$$\begin{cases} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}; [A] \text{ is a transmission matrix;} \\ \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{12} \\ b_{21} & -b_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}; \ [B] = [A]^{-1} \text{ is a transmission matrix;} \end{cases}$$

Six possible sets of terminal equations (2)

$$\begin{cases} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}; [H] \text{ is a hybrid matrix;} \\ \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}; \ [G] = [H]^{-1} \text{ is a hybrid matrix;} \end{cases}$$

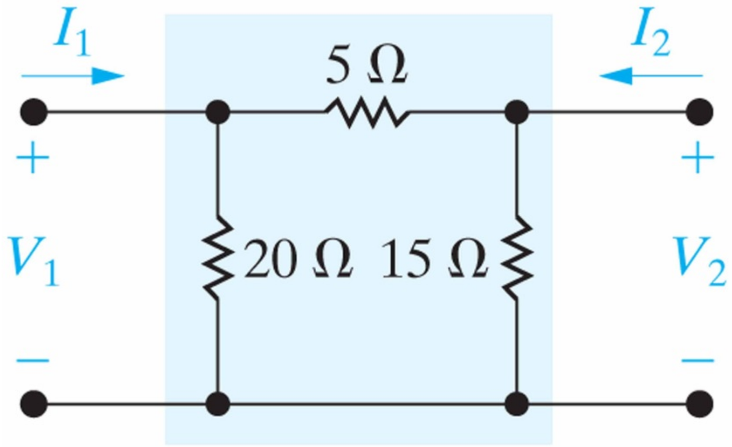
- Which set is chosen depends on which variables are given. E.g. If the source voltage and current $\{V_1, I_1\}$ are given, choosing transmission matrix $[B]$ in the analysis.

Section 18.2
The Two-Port Parameters

1. Calculation of matrix $[Z]$
2. Relations among 6 matrixes

Example 18.1: Finding [Z] (1)

- Q: Find the impedance matrix [Z] for a given resistive circuit (not a "black box"):



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

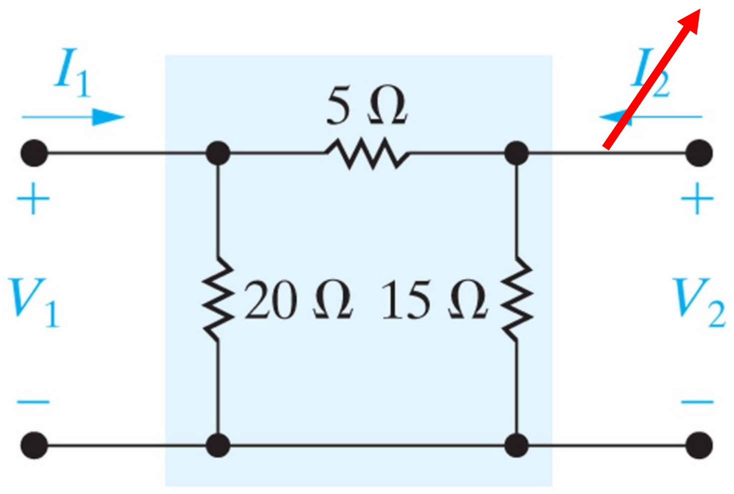
- By definition, $z_{11} = (V_1/I_1)$ when $I_2 = 0$, i.e. the input impedance when port 2 is open. $\Rightarrow z_{11} = (20\ \Omega)//(20\ \Omega) = 10\ \Omega$.

Example 18.1: (2)

- By definition, $z_{21}=(V_2/I_1)$ when $I_2=0$, i.e. the transfer impedance when port 2 is open.
- When port 2 is open:

$$\begin{cases} V_2 = \dfrac{15\,\Omega}{5\,\Omega+15\,\Omega}V_1 = 0.75V_1, \\ \dfrac{V_1}{I_1} = z_{11} = 10\,\Omega, \Rightarrow I_1 = \dfrac{V_1}{10\,\Omega}, \end{cases}$$

$$\Rightarrow z_{21} = \frac{V_2}{I_1} = \frac{0.75V_1}{V_1/(10\,\Omega)} = 7.5\,\Omega.$$

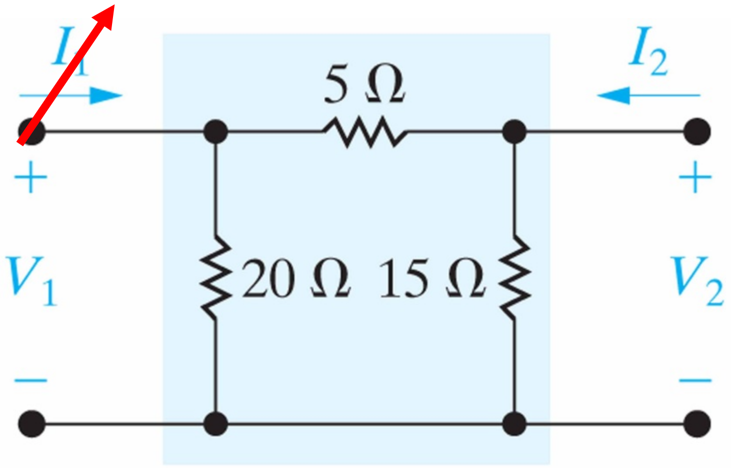


Example 18.1: (3)

- By definition, $z_{22} = (V_2/I_2)$ when $I_1 = 0$, i.e. the output impedance when port 1 is open. $\Rightarrow z_{22} = (15\ \Omega)//(25\ \Omega) = 9.375\ \Omega$.

- $z_{12} = (V_1/I_2)$ when $I_1 = 0$, $\Rightarrow$

$$\begin{cases} V_1 = \dfrac{20\,\Omega}{20\,\Omega + 5\,\Omega} V_2 = 0.8V_2, \\ \dfrac{V_2}{I_2} = z_{22} = 9.375\,\Omega, \Rightarrow I_2 = \dfrac{V_2}{9.375\,\Omega}, \end{cases}$$



$$\Rightarrow z_{12} = \frac{V_1}{I_2} = \frac{0.8V_2}{V_2/(9.375\,\Omega)} = 7.5\,\Omega.$$

Comments

- When the circuit is well known, calculation of $[Z]$ by circuit analysis methods shows the physical meaning of each matrix element.
- When the circuit is a "black box", we can perform 2 test experiments to get $[Z]$: (1) Open port 2, apply a current I_1 to port 1, measure the input voltage V_1 and output voltage V_2. (2) Open port 1, apply a current I_2 to port 2, measure the terminal voltages V_1 and V_2.

Relations among the 6 matrixes

- If we know one matrix, we can derive all the others analytically (Table 18.1).
- $[Y]=[Z]^{-1}$, $[B]=[A]^{-1}$, $[G]=[H]^{-1}$, elements between mutually inverse matrixes can be easily related.
- E.g.

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta y}\begin{bmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{bmatrix},$$

where $\Delta y \equiv \det[Y] = y_{11}y_{22} - y_{12}y_{21}$.

Represent $[Z]$ by elements of $[A]$ (1)

- $[Z]$ and $[A]$ are not mutually inverse, relation between their elements are less explicit.
- By definitions of $[Z]$ and $[A]$,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix},$$

  the independent variables of $[Z]$ and $[A]$ are $\{I_1, I_2\}$ and $\{V_2, I_2\}$, respectively.
- Key of matrix transformation: Representing the distinct independent variable V_2 by $\{I_1, I_2\}$.

Represent $[Z]$ by elements of $[A]$ (2)

- By definitions of $[A]$ and $[Z]$,

$$\begin{cases} V_1 = a_{11}V_2 - a_{12}I_2 \cdots (1) \\ I_1 = a_{21}V_2 - a_{22}I_2 \cdots (2) \end{cases}$$

$$(2) \Rightarrow V_2 = \frac{1}{a_{21}} I_1 + \frac{a_{22}}{a_{21}} I_2 = z_{21} I_1 + z_{22} I_2 \cdots (3),$$

$$(1),(3) \Rightarrow V_1 = a_{11}\left(\frac{1}{a_{21}} I_1 + \frac{a_{22}}{a_{21}} I_2 \right) - a_{12} I_2$$

$$= \frac{a_{11}}{a_{21}} I_1 + \left(\frac{a_{11}a_{22}}{a_{21}} - a_{12} \right) I_2 = z_{11} I_1 + z_{12} I_2 \cdots (4)$$

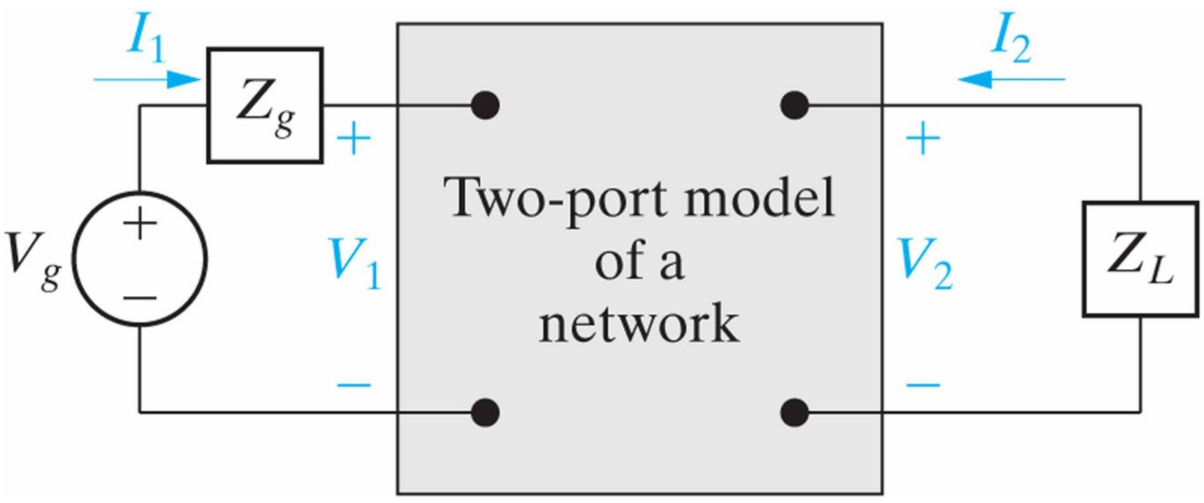
$$\Rightarrow \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \frac{1}{a_{21}} \begin{bmatrix} a_{11} & \Delta a \\ 1 & a_{22} \end{bmatrix}, \text{ where } \Delta a \equiv \det[A].$$

Section 18.3
Analysis of the Terminated Two-Port Circuit

1. Analysis in terms of $[Z]$
2. Analysis in terms of $[T] \neq [Z]$

Model of the terminated two-port circuit

- A two-port circuit is typically driven at port 1 and loaded at port 2, which can be modeled as:



- The goal is to solve $\{V_1, I_1, V_2, I_2\}$ as functions of given parameters V_g, Z_g, Z_L, and matrix elements of the two-port circuit.

Analysis in terms of $[Z]$

- Four equations are needed to solve the four unknowns $\{V_1, I_1, V_2, I_2\}$.

$$\begin{cases} V_1 = z_{11}I_1 + z_{12}I_2 \cdots (1) \\ V_2 = z_{21}I_1 + z_{22}I_2 \cdots (2) \end{cases} \cdots \text{two-port equations}$$

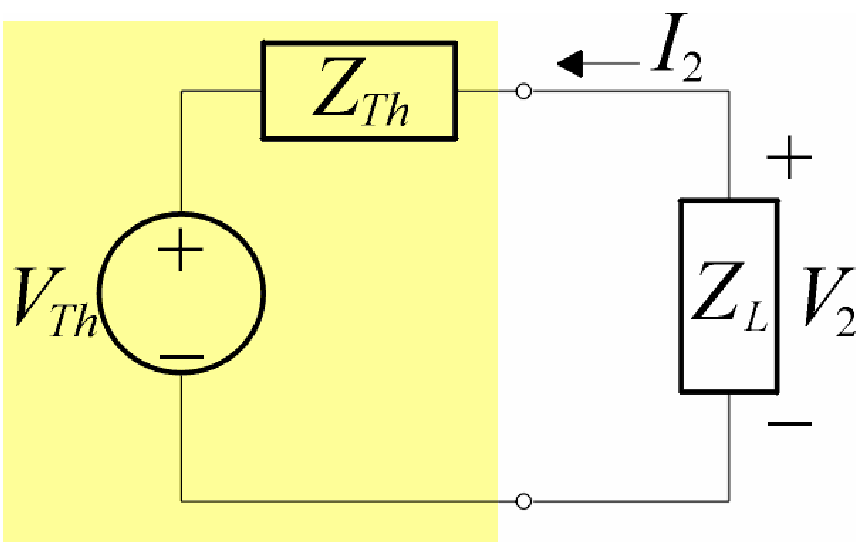
$$\begin{cases} V_1 = V_g - I_1 Z_g \cdots (3) \\ V_2 = -I_2 Z_L \cdots (4) \end{cases} \cdots \text{constraint equations due to terminations}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & z_{11} & z_{12} \\ 0 & -1 & z_{21} & z_{22} \\ 1 & 0 & Z_g & 0 \\ 0 & 1 & 0 & Z_L \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_g \\ 0 \end{bmatrix},$$

$\{V_1, I_1, V_2, I_2\}$ are derived by inverse matrix method.

Thévenin equivalent circuit with respect to port 2

- Once $\{V_1, I_1, V_2, I_2\}$ are solved, $\{V_{Th}, Z_{Th}\}$ can be determined by Z_L and $\{V_2, I_2\}$:



$$\begin{cases} V_2 = \dfrac{Z_L}{Z_{Th} + Z_L} V_{Th} \cdots (1) \\ I_2 = \dfrac{V_2 - V_{Th}}{Z_{Th}} \cdots (2) \end{cases}$$

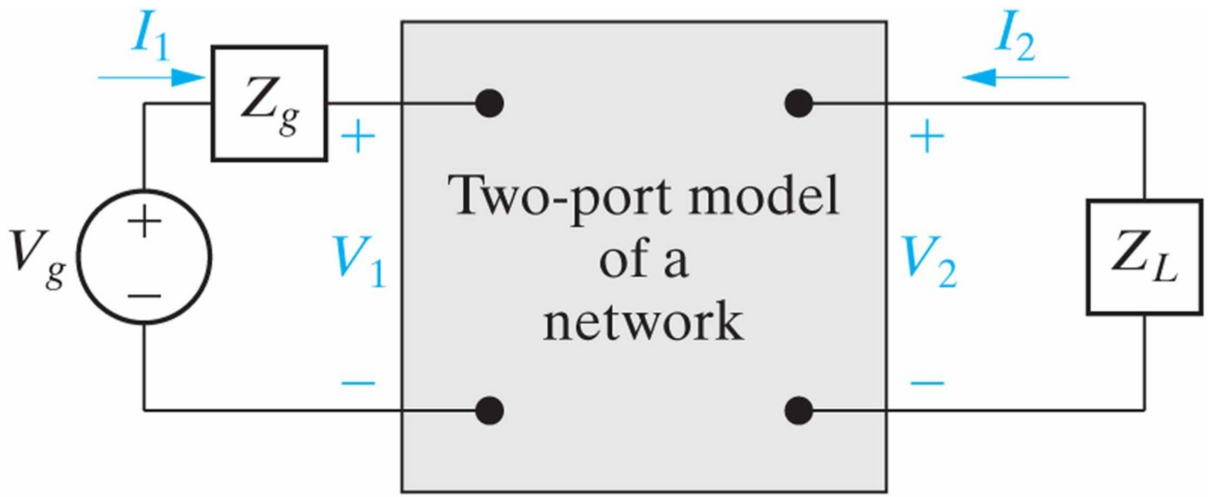
$$\Rightarrow \begin{bmatrix} Z_L & -V_2 \\ 1 & I_2 \end{bmatrix} \times \begin{bmatrix} V_{Th} \\ Z_{Th} \end{bmatrix} = \begin{bmatrix} V_2 Z_L \\ V_2 \end{bmatrix}; \quad \begin{bmatrix} V_{Th} \\ Z_{Th} \end{bmatrix} = \begin{bmatrix} Z_L & -V_2 \\ 1 & I_2 \end{bmatrix}^{-1} \times \begin{bmatrix} V_2 Z_L \\ V_2 \end{bmatrix}.$$

Terminal behavior (1)

- The terminal behavior of the circuit can be described by manipulations of $\{V_1, I_1, V_2, I_2\}$:
- Input impedance: $Z_{in} \equiv \dfrac{V_1}{I_1} = z_{11} - \dfrac{z_{12}z_{21}}{z_{22} + Z_L};$
- Output current: $I_2 = \dfrac{-z_{21}V_g}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}};$
- Current gain: $\dfrac{I_2}{I_1} = -\dfrac{z_{21}}{z_{22} + Z_L};$
- Voltage gains: $\begin{cases} \dfrac{V_2}{V_1} = \dfrac{z_{21}Z_L}{z_{11}Z_L + \Delta z}; \\ \dfrac{V_2}{V_g} = \dfrac{z_{21}Z_L}{(z_{11} + Z_g)(z_{22} + Z_L) - z_{12}z_{21}}; \end{cases}$

Terminal behavior (2)

- Thévenin voltage: $V_{Th} = \frac{z_{21}}{z_{11} + Z_g} V_g$;
- Thévenin impedance: $Z_{Th} = z_{22} - \frac{z_{12} z_{21}}{z_{11} + Z_g}$;

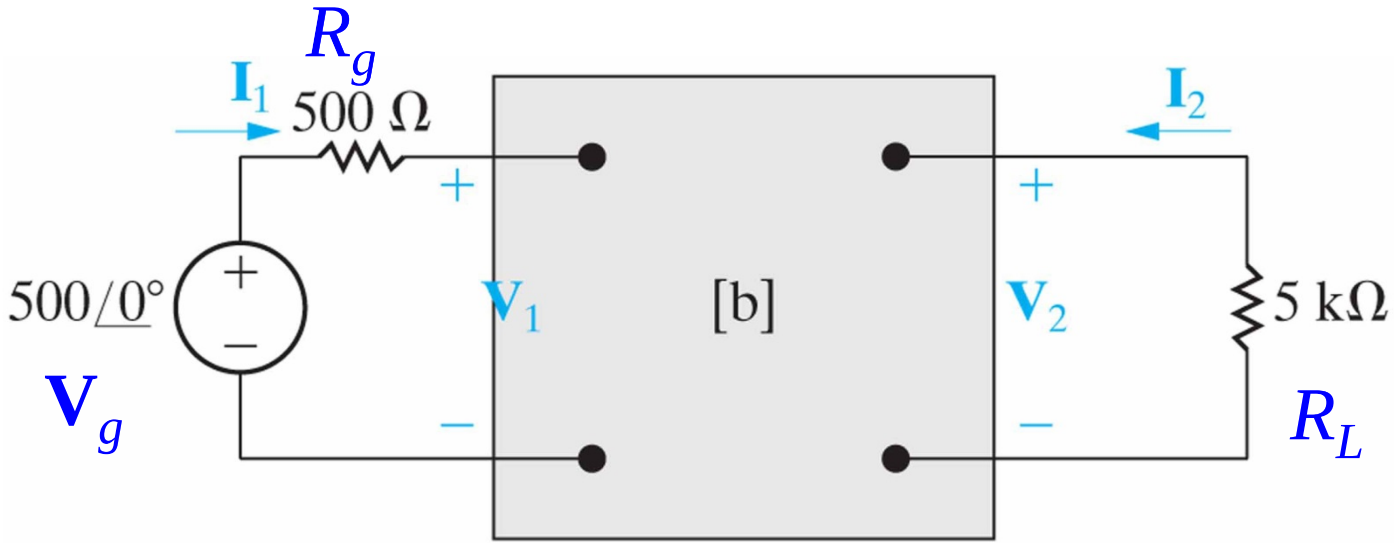


Analysis in term of a two-port matrix $[T]\neq[Z]$

- If the two-port circuit is modeled by $[T]\neq[Z]$, $T=\{Y, A, B, H, G\}$, the terminal behavior can be determined by two methods:
- Use the 2 two-port equations of $[T]$ to get a new 4×4 matrix in solving $\{V_1, I_1, V_2, I_2\}$ (Table 18.2);
- Transform $[T]$ into $[Z]$ by Table 18.1, borrow the formulas derived by analysis in terms of $[Z]$.

Example 18.4: Analysis in terms of $[B]$ (1)

- Q: Find (1) output voltage V_2, (2,3) average powers delivered to the load P_2 and input port P_1, for a terminated two-port circuit with known $[B]$.



$$[B] = \begin{bmatrix} -20 & 3\,\text{k}\Omega \\ -2\,\text{mS} & 0.2 \end{bmatrix} \quad \begin{matrix} -b_{12} \\ -b_{22} \end{matrix}$$

Example 18.4 (2)

- Use the voltage gain formula of Table 18.2:

$$\frac{V_2}{V_g} = \frac{\Delta b Z_L}{b_{12} + b_{11}Z_g + b_{22}Z_L + b_{21}Z_g Z_L};$$

$$\Delta b = b_{11}b_{22} - b_{12}b_{21} = (-20)(-0.2) - (-3\,\text{k}\Omega)(-2\,\text{mS}) = 4 - 6 = -2,$$

$$\Rightarrow \frac{V_2}{V_g} = \frac{(-2)(5\,\text{k}\Omega)}{(-3\,\text{k}\Omega) + (-20)(0.5\,\text{k}\Omega) + (-0.2)(5\,\text{k}\Omega) + \ldots} = \frac{10}{19},$$

$$\Rightarrow V_2 = \frac{10}{19} 500\angle 0^\circ = 263.16\angle 0^\circ \text{ V}.$$

Example 18.4 (3)

- The average power of the load is formulated by

$$P_2 = \frac{1}{2}\frac{|V_2|^2}{R_L} = \frac{1}{2}\frac{|263.16\angle 0^\circ\ \text{V}|^2}{(5\ \text{k}\Omega)} = 6.93\ \text{W}.$$

- The average power delivered to port 1 is formulated by $P_1 = \frac{1}{2}|I_1|^2\,\text{Re}(Z_{in}).$

$$Z_{in} \equiv \frac{V_1}{I_1} = \frac{b_{22}Z_L + b_{12}}{b_{21}Z_L + b_{11}} = \frac{(-0.2)(5\ \text{k}\Omega) - (3\ \text{k}\Omega)}{(-2\ \text{mS})(5\ \text{k}\Omega) - 20} = 133.33\ \Omega;$$

$$I_1 = \frac{V_g}{Z_g + Z_{in}} = \frac{500\angle 0^\circ\ \text{V}}{(500\ \Omega) + (133.33\ \Omega)} = 0.789\angle 0^\circ\ \text{A},$$

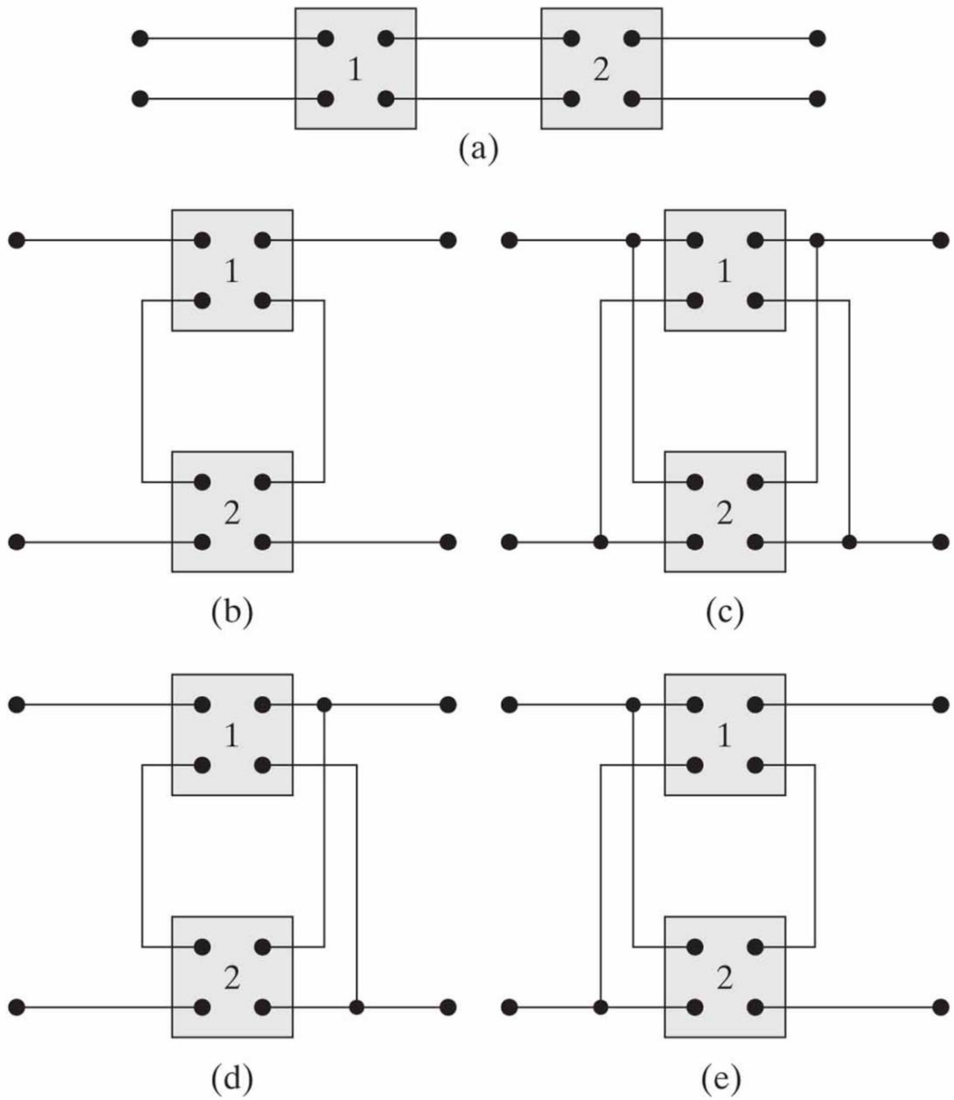
$$\Rightarrow P_1 = \frac{1}{2}(0.789)^2(133.33) = 41.55\ \text{W}.$$

Section 18.4 Interconnected Two-Port Circuits

Why interconnected?

- Design of a large system is simplified by first designing subsections (usually modeled by two-port circuits), then interconnecting these units to complete the system.

Five types of interconnections of two-port circuits



a. Cascade: Better use $[A]$.

b. Series: $[Z]$

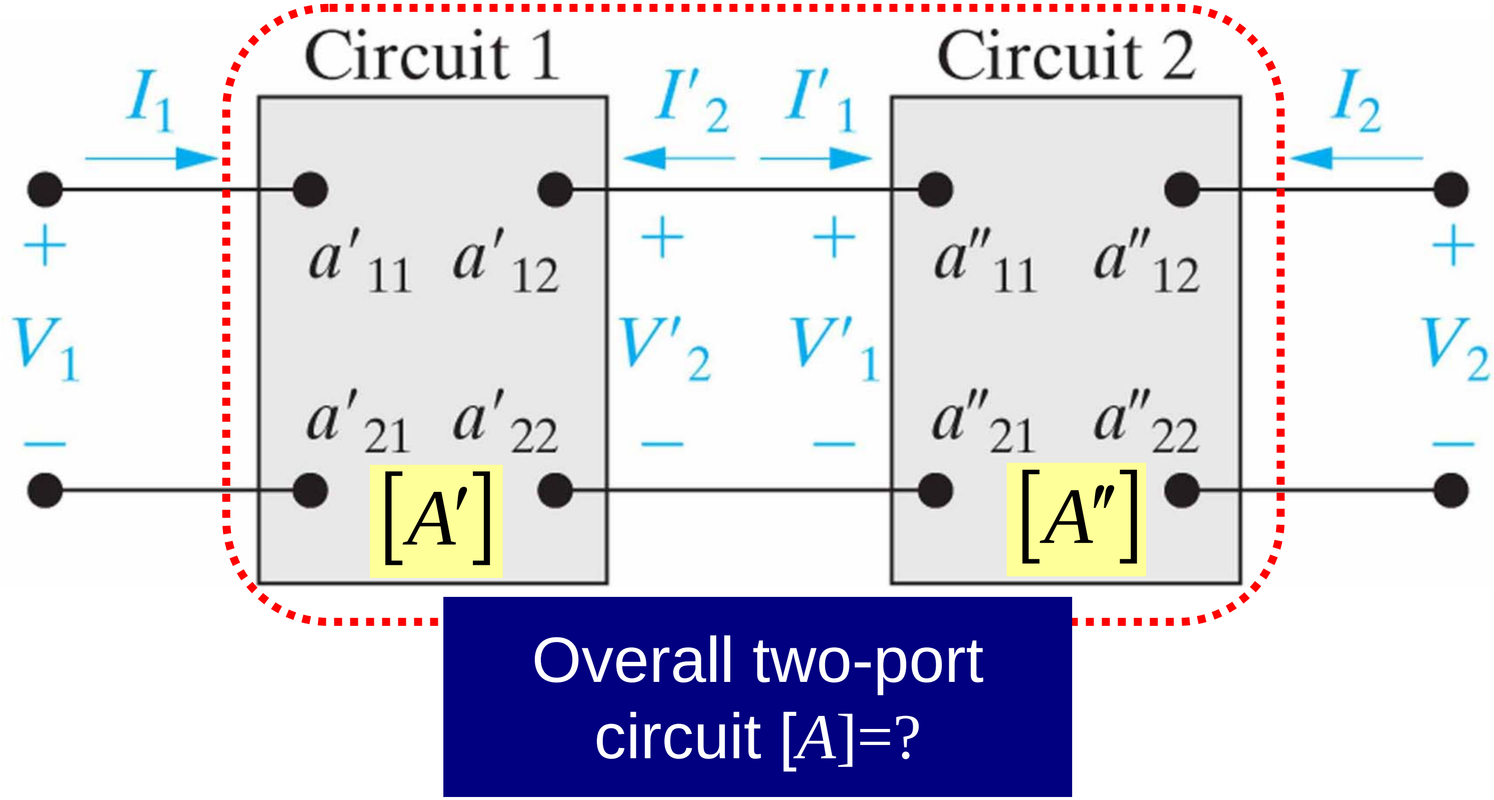
c. Parallel: $[Y]$

d. Series-parallel: $[H]$.

e. Parallel-series: $[G]$.

Analysis of cascade connection (1)

- Goal: Derive the overall matrix $[A]$ of two cascaded two-port circuits with known transmission matrixes $[A']$ and $[A'']$.



Analysis of cascade connection (2)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = [A'] \times \begin{bmatrix} V_2' \\ I_2' \end{bmatrix} = [A'] \times \begin{bmatrix} V_1' \\ -I_1' \end{bmatrix} \cdots (1)$$

$$\begin{bmatrix} V_1' \\ I_1' \end{bmatrix} = [A''] \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} a''_{11} & -a''_{12} \\ a''_{21} & -a''_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} V_1' \\ -I_1' \end{bmatrix} = \begin{bmatrix} a''_{11} & -a''_{12} \\ -a''_{21} & a''_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = [A''_{-}] \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \cdots (2)$$

$$\text{By } (1),(2),\quad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = [A'] \times [A''_{-}] \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = [A] \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix},$$

$$\Rightarrow [A] = [A'] \times [A''_{-}],\quad \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} = \begin{bmatrix} a'_{11}a''_{11} + a'_{12}a''_{21} & -(a'_{11}a''_{12} + a'_{12}a''_{22}) \\ a'_{21}a''_{11} + a'_{22}a''_{21} & -(a'_{21}a''_{12} + a'_{22}a''_{22}) \end{bmatrix}.$$

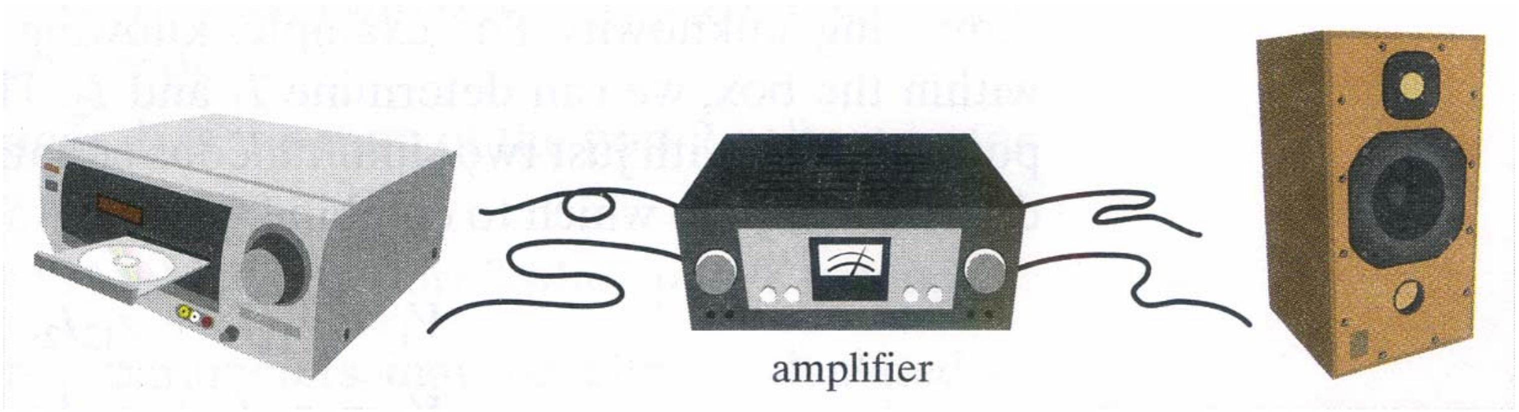
Key points

- How to calculate the 6 possible 2$\times$2 matrices of a two-port circuit?
- How to find the 4 simultaneous equations in solving a terminated two-port circuit?
- How to find the total 2$\times$2 matrix of a circuit consisting of interconnected two-port circuits?

Practical Perspective
Audio Amplifier

Application of two-port circuits

- Q: Whether it would be safe to use a given audio amplifier to connect a music player modeled by $\{V_g = 2\ \text{V (rms)}, Z_g = 100\ \Omega\}$ to a speaker modeled by a load resistor $Z_L = 32\ \Omega$ with a power rating of $100\ \text{W}$?



Find the [H] by 2 test experiments (1)

- Definition of hybrid matrix [H]: $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$;
- Test 1:

$I_1 = 2.5$ mA (rms)

$V_1 = 1.25$ V (rms)

$V_2 = 0$ (short)

$I_2 = 3.75$ A (rms)

$$V_1 = h_{11} I_1, \Rightarrow h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2 = 0} = \frac{1.25\text{ V}}{2.5\text{ mA}} = 500\,\Omega.$$

Input impedance

$$I_2 = h_{21} I_1, \Rightarrow h_{21} = \left. \frac{I_2}{I_1} \right|_{V_2 = 0} = \frac{3.75\text{ A}}{2.5\text{ mA}} = 1500.$$

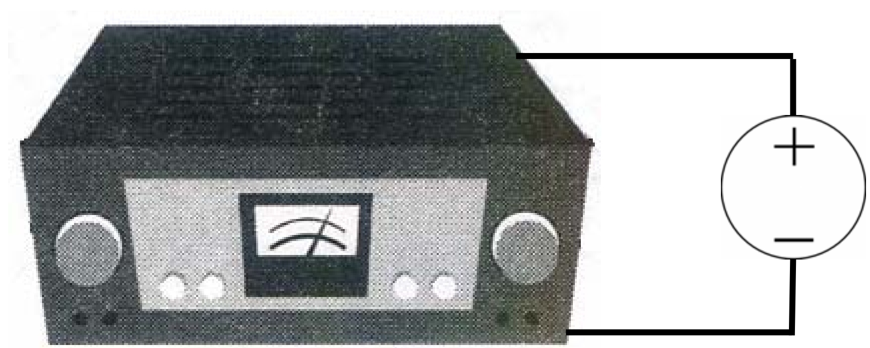
Current gain

Find the [H] by 2 test experiments (2)

- Definition of hybrid matrix [H]: $\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$;
- Test 2:

$I_1 = 0$ (open)

$V_1 = 50$ mV (rms)



$V_2 = 50$ V (rms)

$I_2 = 2.5$ A (rms)

$$V_1 = h_{12}V_2, \Rightarrow h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{50\text{ mV}}{50\text{ V}} = 10^{-3}.$$

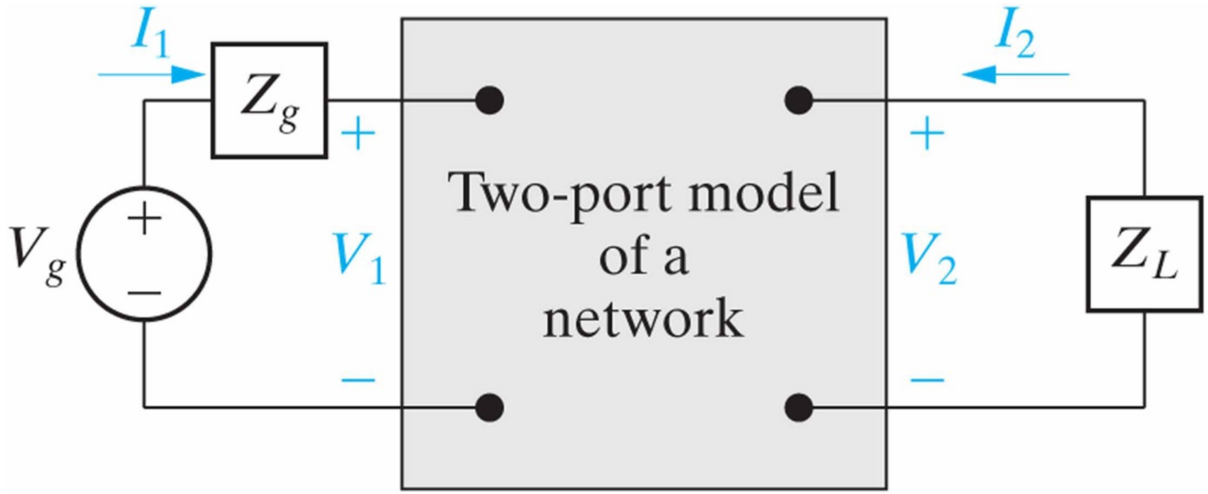
Voltage gain

$$I_2 = h_{22}V_2, \Rightarrow h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{2.5\text{ A}}{50\text{ V}} = (20\ \Omega)^{-1}.$$

Output admittance

Find the power dissipation on the load

- For a terminated two-port circuit:



the power dissipated on Z_L is

$$P_L = \mathrm{Re}\{-V_2 I_2^*\} = \mathrm{Re}\{-(-I_2 Z_L) I_2^*\} = |I_2|^2 \,\mathrm{Re}\{Z_L\},$$

where I_2 is the rms output current phasor.

Method 1: Use terminated 2-port eqs for $[H]$

- By looking at Table 18.2:

$$I_2 = \frac{h_{21}V_g}{(1+h_{22}Z_L)(h_{11}+Z_g)-h_{12}h_{21}Z_L} = 1.98\ \text{A (rms)},$$

$$\text{where} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 500\,\Omega & 10^{-3} \\ 1500 & (20\,\Omega)^{-1} \end{bmatrix};$$

$$V_g = 2\ \text{V (rms)},\ Z_g = 100\,\Omega,\ Z_L = 32\,\Omega.$$

Not safe!

$$\Rightarrow P_L = |I_2|^2\,\text{Re}\{Z_L\} = (1.98)^2(32) = \underline{126\ \text{W} > 100\ \text{W}}.$$

Method 2: Use system of terminated eqs of [Z]

- Transform [H] to [Z] (Table 18.1):

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \frac{1}{h_{22}} \begin{bmatrix} \Delta h & h_{12} \\ -h_{21} & 1 \end{bmatrix} = \begin{bmatrix} 470 & 0.02 \\ -30{,}000 & 20 \end{bmatrix} \Omega.$$

- By system of terminated equations:

$$\begin{bmatrix} V_1 \\ V_2 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & z_{11} & z_{12} \\ 0 & -1 & z_{21} & z_{22} \\ 1 & 0 & Z_g & 0 \\ 0 & 1 & 0 & Z_L \end{bmatrix}^{-1} \times \begin{bmatrix} 0 \\ 0 \\ V_g \\ 0 \end{bmatrix} = \begin{bmatrix} 1.66\text{ V} \\ -63.5\text{ V} \\ 3.4\text{ mA} \\ 1.98\text{ A} \end{bmatrix}.$$

$$\Rightarrow P_L = |I_2|^2 \operatorname{Re}\{Z_L\} = (1.98)^2 (32) = 126\text{ W} > 100\text{ W}.$$