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Discussion

Modified log-wake law for turbulent flow in smooth pipes

By JUNKE GUO and PIERRE Y. JULIEN, Journal of Hydraulic Research, Volume 41, 2003, Issue 5, pp. 493–501

Discusser:
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The Authors have found a clever solution to the discontinuity of the velocity gradient at the axis of uniform turbulent pipe flow in the log-wake model by means of a modification with a cubic correction function. Their modified log-wake law leads to a very good replication of the velocity profiles of a large series of smooth pipe flow experiments (see Fig. 5). Moreover, it leads to a correct eddy viscosity profile in the centre of the pipe that is not obtained with the original log-wake law. This is important for the determination of the diffusion of different quantities in the centre region of the pipe. However the physical reasoning by the Authors is not always convincing.

1 Effect of the wall shear stress

The Authors split the shear stress in two parts (Eq. 8), a wall shear stress effect and an effect of the pressure gradient. The first term is then the shear stress that would apply in a uniform flow if no pressure gradient were present. However, the uniform flow is a result of the balance between a pressure gradient and the wall shear stress and hence absence of a pressure gradient would implicate also no wall shear stress. The wall shear stress effect defined here becomes very large far from the wall (see the dotted line in Fig. 10). It tends to infinity at the pipe axis, whereas the real shear stress decreases linearly to become zero at the axis. The second term in Eq. (8) then obviously tends to minus infinity there. A simple physical interpretation of the total, linear shear stress is obvious. This shear stress is required in a steady, uniform flow to balance the pressure gradient. But the splitting of the shear stress in two terms that both go to infinity at the axis to yield zero together does not appear to have a sound physical basis.

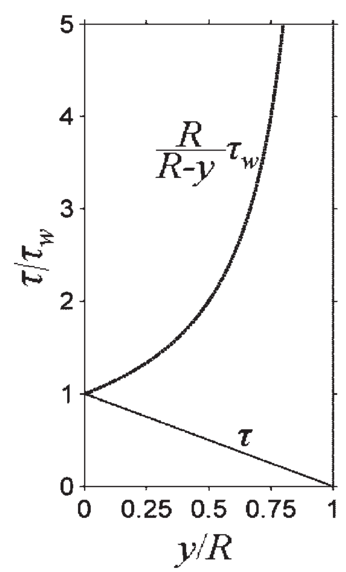

Figure 10 Terms in Eq. (8).

2 Modified log-wake law

The Authors go on assuming that the integral of the first part of the shear stress, "the effect of the wall shear stress", yields the logarithmic wall velocity function with a small cubic correction function F_1. However, the axial boundary condition requires a zero velocity gradient at the axis and hence also a zero gradient of this last velocity contribution (see Section 2.3). But the combination of the very large shear stress component and the very small velocity gradient for the contribution of the "effect of the wall shear stress" requires an unrealistic large eddy viscosity in the core region, even tending to infinity at the axis.

In the usual derivation of the law of the wall, the presence of a single length scale, which in the wall region is the distance to the wall, is required for the logarithmic velocity profile to apply. Hence a constant shear stress is needed, which is approximated in the near wall region. The deviation from the constant shear stress in pipe flow shows that, farther from the wall, another length scale, the radius R, is involved. This will lead to a correction on the logarithmic velocity, which for sufficiently large Reynolds number is only dependent on $\xi = y/R$: the wake function $F_2(\xi)$. Usually the Coles wake function is assumed, which is just a simple function that satisfies the constraints given in Eqs (18a)–(18c). However, obviously condition (18c) does not fulfil the desired condition of a zero velocity gradient at the pipe axis.

This desired boundary condition leads at the axis to (see Eq. 23)

$$F_2'(\xi = 1) = -\frac{1}{\kappa} \tag{43}$$

One of the simplest functions that satisfies the conditions (18a), (18b) and (43), is the combination of Coles wake function and the cubic correction function F_1 found by the authors, i.e. the modified log-wake law. Hence, the tricks to arrive at this modified log-wake law are not needed.

The good replication of the measured velocity profiles with this simple choice of modified log-wake law is very attractive. However, the profiles as compared in Fig. 5 are not optimal for checking the used wake function. To check and maybe improve the choice of a wake function, profiles obtained by subtraction of the logarithmic wall velocity function would presumably serve better.

Eddy viscosity profiles are obtained from the known linear shear stress distribution and the measured or computed velocity profile. Hence, a correct velocity profile leads to a correct eddy viscosity profile. The Authors owe both the correct maximum at $\xi \approx 0.3$ and the constant eddy viscosity near the axis to the modified log-wake law. However, the first result can also be obtained with the classic log-wake law. The combination of a logarithmic velocity profile and the linear shear stress profile leads to a parabolic eddy viscosity profile with its maximum at $\xi \approx 0.5$ and the Coles wake function drives this maximum to lower ξ-values. On the other hand, the correct non-zero eddy viscosity near the axis is an important new result, which was obtained by the Authors by means of the modification of the log-wake law with the cubic correction term.

3 u_{max} and friction factor

The modified log-wake law leads directly to an expression for the maximum velocity as a function of the Reynolds number (Eq. 29). However starting in Section 3 a different expression is used, a combination of power laws obtained by curve-fitting on experimental data. This last expression lacks the theoretical background, which leads to the logarithmic law of the wall and the Reynolds-independence of the wake functions. Moreover, it fits the experimental data hardly better than Eq. (29) as can be seen in Fig. 11, which is Fig. 4 with the curve following expression (29) inserted (with $\kappa = 0.42$ and $A = 5.72$). Moreover, the precision of the data in Figs 4 and 11 appears to be slightly exaggerated. In both the measurements of Nikuradze and Zagarola the values of u_* and u_{max} vary over a factor 10^3 and the pressure gradient hence even over 10^6. It is not likely that over these large ranges the value of u_{max}/u_* can be obtained with a (systematic) error of less than about 2%.

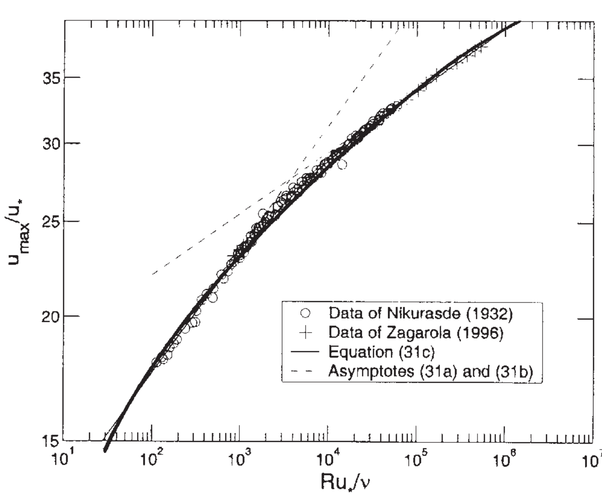

Figure 11 This is Fig. 4 with Eq. (29) added (the bold drawn line).

4 Conclusions

The modified log-wake law yields an important improvement of the reproduction of the velocity profiles and eddy viscosity profiles in the core region of pipe flows. It may be worthwhile to use the same method for uniform free surface flow and uniform channel flow.

The physical interpretation of the Authors to arrive at the modified log-wake law is not convincing. The same result can be obtained directly by means of a wake function that satisfies the desired boundary conditions.

The use of the modified log-wake law is to be preferred to a power law based on curve-fitting to obtain the maximum velocity in pipe flow.

Reply by the Authors

The Authors would like to thank the Discusser for his thoughtful critique. The discussion centers around the following key elements addressed in sequence: (1) the physical reasoning is not convincing in the light of the indetermination of Eq. (8) at $y = R$; (2) the discussion suggests solving Eqs (18a), (18b) and (43) instead of (18a)–(18c) to define the wake function; (3) Fig. 5 is apparently not optimal for checking the wake function; and (4) the proposed relationship for the maximum velocity lacks theoretical background.

1 Physical reasoning

The Discusser indicates that Eq. (8) does not provide a clear physical reasoning because of the indetermination at $y = R$. In the paper, Eq. (8) clearly dissociated the effects of wall shear stress from the pressure gradient. It is indeed a good point that the wall shear stress and pressure gradient in uniform pipe flows cannot be separated. We concur with the Discusser that as long as the flow is steady and uniform, the wall shear stress and pressure gradient are linked according to Eq. (9).

We nevertheless maintain that there is clear physical reasoning behind the formulation proposed in the paper. To clarify our viewpoint, we would like to present Eq. (8) in a slightly modified form to isolate the singularity at the pipe axis. Indeed, Eq. (8) can be simply rewritten as follows:

$$\begin{aligned}\tau &= \frac{R\tau_w}{R-y} + \frac{R}{2}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{(2R-y)y}{(R-y)R} \\ &= \tau_w + \frac{R}{2}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{y}{R} + \frac{y}{R-y}\left(\tau_w + \frac{R}{2}\frac{\mathrm{d}p}{\mathrm{d}x}\right)\end{aligned} \tag{8a}$$

In this modified form of Eq. (8a), one clearly notices the presence of three distinct terms.

The first term of Eq. (8a) is the constant wall shear stress. The physical reasoning behind this first term is that a constant boundary shear stress near the pipe wall leads to the well-known logarithmic velocity profile.

The second term is the contribution of the pressure gradient, which is zero at the pipe wall and increases to the pipe centerline. This second term leads to the symmetrical wake function suggested by Coles and Eq. (20) of our paper. The physical reasoning behind the second term is clearly linked to the effect of the pressure gradient in pipes. A perfectly symmetrical velocity profile suggests that the velocity and its gradient be zero at the pipe wall, as well as the gradient at the pipe axis. This leads to the sine-square wake function defined by Hinze (1975, p. 698) based on Coles' data.

The third term in the parenthesis vanishes because of Eq. (9), except at $y = R$ because of the indetermination of the type 0/0. This indetermination caused by the third term can be lifted with a correct analysis of the axial boundary condition as suggested in Section 2.3 of the paper. Specifically, we recommended Eq. (25) with $n = 3$. Based on this, we maintain our claim that there is indeed physical reasoning behind the formulation presented in the paper. Accordingly, the velocity profile in pipes can be determined from three distinct components: (1) the logarithmic velocity profile stems from the constant wall shear stress; (2) the wake function stems from the pressure gradient; and (3) the modification proposed in the paper stems from the boundary condition along the pipe axis.

2 Modified log-wake law

The Discusser is keen to state that based on the sine-square function, one can directly add the cubic correction by forcing it to satisfy the required conditions. This approach seems natural and was examined by Guo (1998) in the earlier developments of the modified log-wake law. The ideas behind both possible cases seem equally valid at first glance. However, we now prefer to think that the axial boundary condition is intrinsically different from the pressure gradient for the reasons stated above under item 1. The reader may also consider that a perfectly symmetrical wake function (e.g. sine-square function) offers definite advantages. For instance, it can easily be used to approximate a velocity profile near the separation and reattachment points in the case of a sudden expansion or sudden contraction, as shown in Fig. 12. It is clear in this case that the wall shear stress is zero at the separation or reattachment point. A modified wake function that includes Eq. (43) as suggested by the Discusser would be inappropriate, because it would impose a velocity gradient at the pipe centerline. The Authors, thus still prefer the concept of separating the axial boundary condition from a symmetrical wake function.

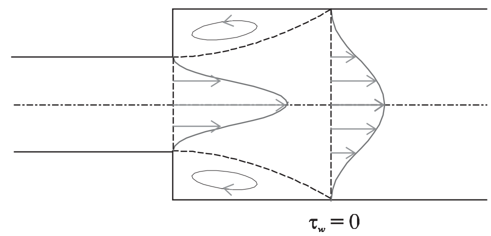

Figure 12 The sine-square law at separation and reattachment points.

3 Plotting the wake functions

As per the Discusser's suggestion, a comparison of the modified wake law (28) with experimental data is shown in Fig. 13. The data are extracted from the modified log-wake law (27) after considering $A = 5.72$, as suggested by the Discusser. The figure shows that the modified wake law proposed in the paper (the solid line) really improves the wake function near pipe axis. Although the data do not fall into a single curve, the error of the universal function (28) relative to the value of u/u_* is acceptable. Furthermore, the measured data profiles (the dash-dotted lines) look fairly parallel to each other. It can also be noticed that the measured profiles move downward as the Reynolds number increases. This implies that the constant A might be slightly dependent on the Reynolds number. This conjecture has been confirmed in a recent study of the modified log-wake law in zero-pressure-gradient boundary layers (Guo *et al.*, submitted for publication).

The Discusser also correctly points out that the classic log-wake law can also produce the maximum eddy viscosity near $\xi = 0.3$. This result was expected owing to the fact that the axial boundary correction primarily affects the velocity profile near the pipe axis, as shown in Fig. 13. The proposed modified log-wake law nevertheless defines a non-zero eddy viscosity at the pipe axis, which is in agreement with the laboratory measurements. This is viewed as an improvement over existing methods.

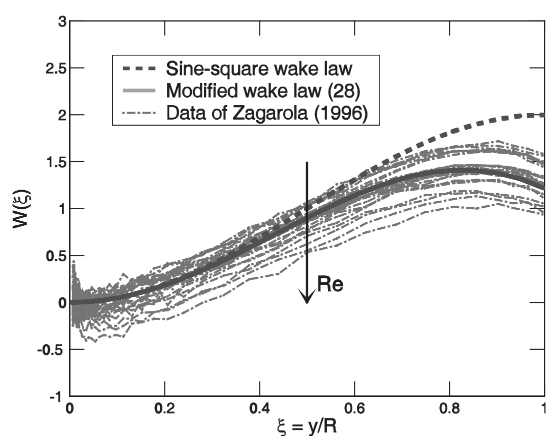

Figure 13 Comparison of the modified wake law (28) with Zagarola's (1996) data.

4 Maximum velocity u_{max}

The Discusser correctly points out that starting in Section 3 power laws are used for the maximum velocity and the friction factor. Although this seems to detract from the logarithmic version presented earlier in the paper, it is difficult to ascertain whether a logarithmic formulation is definitely better than the power formulation. We are pleased to notice that the Discusser obtains nice graphical agreement in Fig. 11 using the logarithmic law (29) with a constant $A = 5.72$. We concur with him that the logarithmic law (29) is accurate enough in practice. However, as shown in Fig. 13, the reader has to keep in mind that the constant A is slightly dependent on the Reynolds number. Another slight advantage of using the power law proposed in the paper is that the friction factor equation (38e) yields a direct solution for the friction factor. The logarithmic formulation that the Discusser prefers leads to a friction factor equation that blends power and logarithm in a way that requires an iterative procedure to define the friction factor. Aside from these subtleties, the implicit formulation suggested by the Discusser is equally fine.

5 Conclusions

1. The singularities raised in the discussion can be circumvented and we maintain that there is clear physical reasoning behind the proposed formulation. Indeed, the logarithmic component stems from the pipe wall shear stress, the wake function stems from the pressure gradient and the modified log-wake formulation that we propose stems from the axial boundary condition, which is the source of the indetermination suggested by the Discusser.
2. We prefer to separate the axial boundary condition from the wake function for the reasons explained under item 1 above. Also, a symmetrical wake function offers definite advantages, including proper modeling of the reattachment point of sudden expansions. It is clear in this case that the velocity profile may be approximated as a symmetrical function, like the sine-square function. In this case, the wall shear stress is zero and a wake function that includes (43) as suggested by the Discusser would not yield a satisfactory velocity profile at the pipe axis.
3. The modified wake law is compared with measurements in Fig. 13. It is clearly shown that the proposed modified wake law indeed improves the wake function near pipe axis. This leads to improvements in the determination of the eddy viscosity near the pipe axis.
4. The Discusser correctly shows that the logarithmic law is also a good approximation for the maximum velocity. The power formulation proposed in the paper offers a slight advantage in avoiding an iterative procedure for the computation of the friction factor. Again, we thank the Discusser for his valuable comments.