Relative P-th Order of Entire Functions of Two Complex Variables

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Abstract— **In this paper we introduce the idea of relative p-th order of entire functions of two complex variables. After proving some basic results, we observe that the relative p-th order of a transcendental entire function with respect to an entire function is the same as that of its partial derivatives. Further we study the equality of relative p-th order of two functions when they are asymptotically equivalent.**

Index Terms— **Entire functions, polydisc, relative order, relative p-th order, several complex variables.**

I. Introduction, Definitions and Notation

Let f and g be two non-constant entire functions and

$$F(r) = \max\{|f(z)| : |z| = r\},$$
$$G(r) = \max\{|g(z)| : |z| = r\},$$

be the maximum modulus functions of f and g respectively. Then $F(r)$ is strictly increasing and continuous function of r and its inverse

$F^{-1}: (|f(0)|, \infty) \to (0, \infty)$ exists and $\lim_{R\to\infty} F^{-1}(R) = \infty$.

Bernal [3] introduced the definition of relative order of f with respect to g as

$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^\mu)$ for all $r > r_0(\mu) > 0\}$.

In [8] Lahiri and Banerjee considered a more general definition of order as follows:

Definition 1.1. If $p \geq 1$ is a positive integer, then the p-th generalized relative order of f with respect to g, denoted by $\rho_g^p(f)$ is defined by

$\rho_g^p(f) = \inf\{\mu > 0 : F(r) < G\left(exp^{[p-1]} r^\mu\right)$ for all $r > r_0(\mu) > 0\}$.

Note 1.2. If $p = 1$ then $\rho_g^p(f) = \rho_g(f)$. If $p = 1$, $g(z) = \exp z$, then $\rho_g^p(f) = \rho(f)$, the classical order of f.

During the past decades, several authors made close investigations on the properties of entire functions related to relative order. In the case of relative order, it therefore seems reasonable to define suitably the relative order of entire functions of two complex variables and to investigate its basic properties, which we attempts in this paper. In this regards we first need the following definition of order of entire functions.

Let $f(z_1, z_2)$ be a non-constant entire function of two complex variables z_1 and z_2, holomorphic in the closed poly disc

$\{(z_1, z_2): |z_j| \leq r_j, j = 1,2$ for all $r_1 \geq 0, r_2 \geq 0\}$.

Let $F(r_1, r_2) = \max\{|f(z_1, z_2)|: |z_j| \leq r_j, j = 1,2\}$.

Then by the Hartogs theorem and maximum principle {[4], p-2, p-51} $F(r_1, r_2)$ is increasing function of r_1, r_2. The order $\rho = \rho(f)$ of $f(z_1, z_2)$ is defined {[4], p-338} as the infimum of all positive numbers μ for which

$$F(r_1, r_2) < \exp[(r_1 r_2)^\mu] \qquad \ldots (1.1)$$

holds for all sufficiently large values of r_1 and r_2.

In other words

$\rho(f) = \inf\{\mu > 0 : F(r_1, r_2) < \exp[(r_1 r_2)^\mu]$ for all $r_1 \geq R(\mu), r_2 \geq R(\mu)\}$.

Equivalent formula for $\rho(f)$ is $\{[4], p - 339\ (\textit{see also}\ [1])\}$

$$\rho(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log\log F(r_1, r_2)}{\log(r_1 r_2)}.$$

A more general approach to the problem of relative order of entire functions has been demonstrated by Kiselman [7].

Let h and k be two functions defined on $\mathbb{R}$ such that $h, k\colon \mathbb{R} \to [-\infty, \infty]$. The order of h relative to k is

$order(h{:}k) = \inf\ [a > 0 : \exists\, c_a \in \mathbb{R}, \forall\, x \in \mathbb{R}, f(x) \leq a^{-1} g(ax) + c_a]$.

If H be an entire function then the growth function of H is defined by

$$h(t) = \sup[log|H(z)|, |z| \leq e^t],\ t \in \mathbb{R}.$$

If H and K are two entire functions then the order of H relative to K is now defined by

$$order\ (H{:}K) = order(h{:}k).$$

As observed by Kiselman [6], the expression $a^{-1}g(ax) + c_a$ may be replaced by $g(ax) + c_a$ if $g(t) = e^t$ because then the infimum in the cases coincide. Taking $c_a = 0$ in the above definition, one may easily verify that

$$order(H : K) = \rho_K(H)$$

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i.e., the order $(H : K)$ coincides with the Bernal's definition of relative order.

Further if $K = exp\, z$ then order $(H : K)$ coincides with the classical order of H.

The papers [5], [6] and [7] made detailed investigations on entire functions and relative order$(H : K)$ but our analysis of relative order, generated from Bernal's relative order, made in the present paper have little relevance to the studies made in the above papers by Kiselman and others.

In 2007 Banerjee and Dutta [2] introduced the definition of relative order of an entire function $f(z_1, z_2)$ with respect to an entire function $g(z_1, z_2)$ as follows:

Definition 1.3. Let $g(z_1, z_2)$ be an entire function holomorphic in the closed polydisc $\{(z_1, z_2) : |z_j| \leq r_j; j = 1, 2\}$ and let

$$G(r_1, r_2) = \max\{|g(z_1, z_2)| : |z_j| \leq r_j \,;\, j = 1;\, 2\}.$$

The relative order of f with respect to g, denoted by $\rho_g(f)$ and is defined by

$\rho_g(f) = \inf\,\{\mu > 0 : F(r_1, r_2) < G(r_1^{\mu}, r_2^{\mu});$ for $r_1 \geq R(\mu), r_2 \geq R(\mu)\}$.

The definition coincides with that of classical (1.1) if $(z_1, z_2) = e^{z_1 z_2}$.

Notation 1.4. [9] $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m,
$log^{[m]}x = log\left(log^{[m-1]}x\right)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

In this paper we introduce the idea of relative p-th order of entire functions of two complex variables.

Definition 1.5. Let $f(z_1, z_2)$ and $g(z_1, z_2)$ be two entire functions of two complex variables z_1, z_2 with maximum modulus functions $F(r_1, r_2)$ and $G(r_1, r_2)$ respectively then relative p-th order of f with respect to g, denoted by $\rho_g^p(f)$ and is defined by

$\rho_g^p(f) = \inf\{\mu > 0: F(r_1, r_2) < G(exp^{[p-1]}r_1^{\mu}, exp^{[p-1]}r_2^{\mu})$ $;for\ r_i \geq R(\mu); i = 1,2\}$, where $p \geq 1$ is a positive integer.

Note 1.6. If we consider p = 1 then Definition 1.5 coincide with Definition 1.3.

Definition 1.7. The function $g(z_1, z_2)$ is said to have the property (A) if for any $\sigma > 1$ and for all large r_1, r_2, $[G(r_1, r_2)]^2 < G(r_1^{\sigma}, r_2^{\sigma})$.

The definition follows from Definition 1.1 in [2].

The function $g(z_1, z_2) = e^{z_1 z_2}$ has the property (A) but the function $g(z_1, z_2) = z_1 z_2$ does not have the property (A).

Throughout we shall assume f, g, h etc. are non-constant entire functions of two complex variables and $F(r_1, r_2), G(r_1, r_2), H(r_1, r_2)$ etc. denotes respectively their maximum modulus in the polydisc $\{(z_1, z_2): |z_j| \leq r_j, j = 1, 2\}$.

II. Lemmas

The following lemmas will be required.

Lemma 2.1. [2] Let g has the property (A). Then for any positive integer n and for all $\sigma > 1$,

$$[G(r_1; r_2)]^n < G(r_1{}^{\sigma}, r_2{}^{\sigma})$$

holds for all large r_1, r_2.

Lemma 2.2. [2] Let $f(z_1, z_2)$ be non-constant entire function and $\alpha > 1$; $0 < \beta < \alpha$. Then
$F(\alpha r_1, \alpha r_2) > \beta F(r_1, r_2)$ for all large r_1, r_2.

Lemma 2.3. [2] Let $f(z_1, z_2)$ be non-constant entire function, $s > 1$; $0 < \mu < \lambda$ and n is a positive integer. Then

(a) $\exists\, K = K(s; f) > 0$ such that $[F(r_1, r_2)]^s \leq KF(r_1^s, r_2^s)$ for $r_1, r_2 > 0$;

(b) $\lim_{r_1, r_2 \to \infty} \frac{F(r_1^s, r_2^s)}{F(r_1, r_2)} = \infty = \lim_{r_1, r_2 \to \infty} \frac{F(r_1^{\lambda}, r_2^{\lambda})}{F(r_1^{\mu}, r_2^{\mu})}$.

Lemma 2.4. Let $f(z_1, z_2)$ be a transcendental entire function then

$\frac{F(r_1, r_2)}{r_1} \leq \bar{F}(r_1, r_2) \leq \frac{F(2r_1, r_2)}{r_1} \leq F(2r_1, r_2)$ for $r_1, r_2 \geq 1$,
where $\bar{F}(r_1, r_2) = \max_{|z_j| = r_j,\ j=1,2} \left|\frac{\partial f(z_1, z_2)}{\partial z_1}\right|$.

Lemma 2.4 follows from Theorem 5.1 in [2].

III. Preliminary Theorem

Theorem 3.1. Let f, g, h be entire functions of two complex variables. Then

(a) if f is a polynomial and g is transcendental entire, then $\rho_g^p(f) = 0$;

(b) if $F(r_1, r_2) \leq H(r_1, r_2)$ for all large r_1, r_2 then $\rho_g^p(f) \leq \rho_g^p(h)$.

***Proof*.**

(a) If f is a polynomial and g is transcendental entire, then there exists a positive integer n such that

$$F(r_1, r_2) \leq Mr_1^n r_2^n$$

and

$$G(r_1, r_2) > Kr_1^m r_2^m$$

for all large r_1, r_2, where M and K are constant and $m > 0$ may be any real number. We have then for all large r_1, r_2 and $\mu > 0$,

$$G\left(exp^{[p-1]}r_1^{\mu}, exp^{[p-1]}r_2^{\mu}\right) > K(exp^{[p-1]}r_1^{\mu}.exp^{[p-1]}r_2^{\mu})^m$$

$$> K(r_1^{\mu}r_2^{\mu})^m$$

$$> Mr_1^n r_2^n \text{ by choosing } m \text{ suitably}$$

$$\geq F(r_1, r_2).$$

Thus for all large r_1, r_2 and $\mu > 0$,

$$F(r_1, r_2) < G\left(exp^{[p-1]}r_1^{\mu}, exp^{[p-1]}r_2^{\mu}\right).$$

Since $\mu > 0$ is arbitrary, we must have

$$\rho_g^p(f) \leq 0 \quad i.e., \quad \rho_g^p(f) = 0.$$

(b) Let $\epsilon > 0$ be arbitrary then from the definition of relative order, we have

$$H(r_1, r_2) < G\left(exp^{[p-1]}r_1^{\rho_g^p(h)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(h)+\epsilon}\right).$$

So for all large r_1, r_2,

$$F(r_1, r_2) \leq H(r_1, r_2)$$
$$< G\left(exp^{[p-1]}r_1^{\rho_g^p(h)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(h)+\epsilon}\right)$$

So,

$$\rho_g^p(f) \leq \rho_g^p(h) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\rho_g^p(f) \leq \rho_g^p(h).$$

This completes the proof.

IV. Sum and Product Theorems

Theorem 4.1. Let f_1 and f_2 be entire functions of two complex variables having relative p −th orders $\rho_g^p(f_1)$ and $\rho_g^p(f_2)$ respectively. Then

(i) $\rho_g^p(f_1 \pm f_2) \leq \max\{\rho_g^p(f_1), \ \rho_g^p(f_2)\}$

and

(ii) $\rho_g^p(f_1.f_2) \leq \max\{\rho_g^p(f_1), \ \rho_g^p(f_2)\}$,

provided g has the property (A).

The equality holds in (i) if $\rho_g^p(f_1) \neq \rho_g^p(f_2)$.

Proof. First suppose that relative p −th order of f_1 and f_2 are finite, if one of them or both are infinite then the theorem is trivial. Let $f = f_1 + f_2$, $\rho = \rho_g^p(f)$, $\rho_i = \rho_g^p(f_i), i = 1,2$ and $\rho_1 \leq \rho_2$. Therefor for any $\epsilon > 0$ and for all large r_1, r_2

$$F_1(r_1, r_2) < G\left(exp^{[p-1]}r_1^{\rho_1+\epsilon}, exp^{[p-1]}r_2^{\rho_1+\epsilon}\right)$$
$$\leq G\left(exp^{[p-1]}r_1^{\rho_2+\epsilon}, exp^{[p-1]}r_2^{\rho_2+\epsilon}\right)$$

and

$$F_2(r_1, r_2) < G\left(exp^{[p-1]}r_1^{\rho_2+\epsilon}, exp^{[p-1]}r_2^{\rho_2+\epsilon}\right)$$

hold.

So for all large r_1, r_2,

$$F(r_1, r_2) \leq F_1(r_1, r_2) + F_2(r_1, r_2)$$
$$< 2G\left(exp^{[p-1]}r_1^{\rho_2+\epsilon}, exp^{[p-1]}r_2^{\rho_2+\epsilon}\right)$$
$$< G\left(3exp^{[p-1]}r_1^{\rho_2+\epsilon}, 3exp^{[p-1]}r_2^{\rho_2+\epsilon}\right), \text{ by Lemma 2.2}$$
$$< G\left(exp^{[p-1]}r_1^{\rho_2+3\epsilon}, exp^{[p-1]}r_2^{\rho_2+3\epsilon}\right)$$

$$\therefore \quad \rho \leq \rho_2 + 3\epsilon$$

Since $\epsilon > 0$ arbitrary,

$$\rho \leq \rho_2 \qquad \text{... (4.1)}$$

Next let $\rho_1 < \rho_2$ and suppose $\rho_1 < \mu < \lambda < \rho_2$. Then for all large r_1, r_2

$$F_1(r_1, r_2) < G\left(exp^{[p-1]}r_1^{\mu}, exp^{[p-1]}r_2^{\mu}\right) \qquad \text{... (4.2)}$$

and there exists non-decreasing sequence $\{r_{ik}\}, r_{ik} \to \infty; i = 1,2$; as $k \to \infty$ such that

$$F_2(r_{1k}, r_{2k}) > G\left(exp^{[p-1]}r_{1k}^{\lambda}, exp^{[p-1]}r_{2k}^{\lambda}\right) \quad \text{for} \quad k = 1,2, \ldots \qquad \text{... (4.3)}$$

Using Lemma 2.3, we see that

$$G(r_1^{\lambda}, r_2^{\lambda}) > 2G(r_1^{\mu}, r_2^{\mu}) \text{ for all large } r_1, r_2. \qquad \text{... (4.4)}$$

So from (4.2), (4.3) and (4.4),

$$F_2(r_{1k}, r_{2k}) > 2F_1(r_{1k}, r_{2k}) \text{ for } k = 1,2, \ldots$$

Therefore

$$F(r_{1k}, r_{2k}) \geq F_2(r_{1k}, r_{2k}) - F_1(r_{1k}, r_{2k})$$

$$> \frac{1}{2}F_2(r_{1k}, r_{2k})$$

$$> \frac{1}{2}G\left(exp^{[p-1]}r_{1k}^{\lambda}, exp^{[p-1]}r_{2k}^{\lambda}\right), \text{ from (4.3)}$$

$$> G\left(\frac{1}{3}exp^{[p-1]}r_{1k}^{\lambda}, \frac{1}{3}exp^{[p-1]}r_{2k}^{\lambda}\right),$$

for all large k and by Lemma2.2

$$> G\left(exp^{[p-1]}r_{1k}^{\lambda-\epsilon}, exp^{[p-1]}r_{2k}^{\lambda-\epsilon}\right),$$

where $\epsilon > 0$ is arbitrary.

This gives $\rho \geq \lambda - \epsilon$ and since $\lambda \in (\rho_1, \rho_2)$ and $\epsilon > 0$ is arbitrary,

we have

$$\rho \geq \rho_2 \qquad \text{... (4.5)}$$

Combining (4.1) and (4.5),

$$\rho_g^p(f_1 + f_2) = \rho_g^p(f_2) = \max\{\rho_g^p(f_1), \rho_g^p(f_2)\}.$$

For the second part, we let $f = f_1.f_2$, $\rho = \rho_g^p(f)$ and $\rho_g^p(f_1) \leq \rho_g^p(f_2)$.

Then for arbitrary $\epsilon > 0$,

$$F(r_1,r_2) \leq F_1(r_1,r_2).F_2(r_1,r_2)$$

$$< G\left(exp^{[p-1]}r_1^{\rho_1+\epsilon}, exp^{[p-1]}r_2^{\rho_1+\epsilon}\right). G\left(exp^{[p-1]}r_1^{\rho_2+\epsilon}, exp^{[p-1]}r_2^{\rho_2+\epsilon}\right)$$
$$\leq [G\left(exp^{[p-1]}r_1^{\rho_2+\epsilon}, exp^{[p-1]}r_2^{\rho_2+\epsilon}\right)]^2$$

$$< G\left(exp^{[p-1]}r_1^{\sigma(\rho_2+\epsilon)}, exp^{[p-1]}r_2^{\sigma(\rho_2+\epsilon)}\right),$$ for any $\sigma > 1$ since g has the property (A).

So

$$\rho \leq \sigma(\rho_2 + \epsilon).$$

Now letting $\epsilon \to 0$ and $\sigma \to 1_+$, we have

$$\rho \leq \rho_2.$$

Therefore

$$\rho_g^p(f_1.f_2) \leq \rho_g^p(f_2) = \max\{\rho_g^p(f_1), \rho_g^p(f_2)\}.$$

This completes the proof.

V. Relative Order of the Partial Derivatives

Regarding the relative order of f and its partial derivatives $\frac{\partial f}{\partial z_1}$, $\frac{\partial f}{\partial z_2}$ with respect to g and $\frac{\partial g}{\partial z_1}$, $\frac{\partial g}{\partial z_2}$, we prove the following theorem.

Theorem 5.1. If f and g are transcendental entire functions of two complex variables and g has the property (A) then

$$\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) = \rho_g^p(f) = \rho_{\frac{\partial g}{\partial z_1}}^p(f).$$

Proof. From the definition of $\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)$, we have for any $\epsilon > 0$,

$$\bar{F}(r_1,r_2) < G\left(exp^{[p-1]}r_1^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}\right)$$

for $r_1, r_2 \geq r_0(\epsilon)$.

Hence from Lemma 2.4,

$$F(r_1,r_2) < r_1 G\left(exp^{[p-1]}r_1^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}\right)$$
$$\leq \left[G\left(exp^{[p-1]}r_1^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}\right)\right]^2$$
$$\leq G\left(exp^{[p-1]}r_1^{\sigma[\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon]}, exp^{[p-1]}r_2^{\sigma[\rho_g^p\left(\frac{\partial f}{\partial z_1}\right)+\epsilon]}\right)$$

for every $\sigma > 1$, by Lemma 2.1.since g has the property (A). So,

$$\rho_g^p(f) \leq [\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) + \epsilon]\sigma.$$

Letting$\sigma \to 1_+$, since $\epsilon > 0$ is arbitrary, we have

$$\rho_g^p(f) \leq \rho_g^p\left(\frac{\partial f}{\partial z_1}\right). \qquad \text{... (5.1)}$$

Similarly from $\bar{F}(r_1,r_2) \leq F(2r_1,r_2)$ of Lemma 2.4 gives

$$\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) \leq \rho_g^p(f). \qquad \text{... (5.2)}$$

So from (5.1) and (5.2)

$$\rho_g^p\left(\frac{\partial f}{\partial z_1}\right) = \rho_g^p(f).$$

This proves first part of the theorem.

For the second part we see that under the hypothesis of Lemma 2.4, we obtain

$$\frac{G(r_1,r_2)}{r_1} \leq \bar{G}(r_1,r_2) \leq G(2r_1,r_2). \qquad \text{... (5.3)}$$

Now by the definition of $\rho_{\frac{\partial g}{\partial z_1}}^p(f)$, for given $\epsilon > 0$

$$F(r_1,r_2) < \bar{G}\left(exp^{[p-1]}r_1^{\rho_{\frac{\partial g}{\partial z_1}}^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_{\frac{\partial g}{\partial z_1}}^p(f)+\epsilon}\right)$$
$$\leq G\left(2exp^{[p-1]}r_1^{\rho_{\frac{\partial g}{\partial z_1}}^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_{\frac{\partial g}{\partial z_1}}^p(f)+\epsilon}\right),$$ using (5.3)
$$< G\left(exp^{[p-1]}r_1^{\rho_{\frac{\partial g}{\partial z_1}}^p(f)+2\epsilon}, exp^{[p-1]}r_2^{\rho_{\frac{\partial g}{\partial z_1}}^p(f)+2\epsilon}\right).$$

So

$$\rho_g^p(f) \leq \rho_{\frac{\partial g}{\partial z_1}}^p(f) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this gives

$$\rho_g^p(f) \leq \rho_{\frac{\partial g}{\partial z_1}}^p(f).$$

Again from (5.3)

$$F(r_1,r_2) < G\left(exp^{[p-1]}r_1^{\rho_g^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(f)+\epsilon}\right)$$
$$\leq r_1.\bar{G}\left(exp^{[p-1]}r_1^{\rho_g^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(f)+\epsilon}\right)$$
$$\leq \left[\bar{G}\left(exp^{[p-1]}r_1^{\rho_g^p(f)+\epsilon}, exp^{[p-1]}r_2^{\rho_g^p(f)+\epsilon}\right)\right]^2$$
$$\leq \bar{G}\left(exp^{[p-1]}r_1^{\sigma[\rho_g^p(f)+\epsilon]}, exp^{[p-1]}r_2^{\sigma[\rho_g^p(f)+\epsilon]}\right),$$ for any $\sigma > 1$.

So

$$\rho_{\frac{\partial g}{\partial z_1}}^p(f) \leq \sigma[\rho_g^p(f) + \epsilon].$$

Now letting $\sigma \to 1_+$, since $\epsilon > 0$ is arbitrary

$$\rho^p_{\frac{\partial g}{\partial z_1}}(f) \leq \rho^p_g(f)$$

and so

$$\rho^p_{\frac{\partial g}{\partial z_1}}(f) = \rho^p_g(f).$$

Consequently,

$$\rho^p_g\left(\frac{\partial f}{\partial z_1}\right) = \rho^p_g(f) = \rho^p_{\frac{\partial g}{\partial z_1}}(f).$$

This proves the theorem.

Note 5.2. Similar results holds for other partial derivatives.

VI. ASYMPTOTIC BEHAVIOUR

Definition 6.1. [2] Two entire functions g_1 and $g_2$are said to be asymptotically equivalent if there exists $l, 0 < l < \infty$ such that

$$\frac{G_1(r_1,r_2)}{G_2(r_1,r_2)} \to l \text{ as } r_1, r_2 \to \infty,$$

and in this case we write $g_1 \sim g_2$.

If $g_1 \sim g_2$ then clearly $g_2 \sim g_1$.

Theorem 6.2. If $g_1 \sim g_2$ and if f is an entire function of two complex variables then

$$\rho^p_{g_1}(f) = \rho^p_{g_2}(f).$$

Proof. Let $\epsilon > 0$, then from Lemma 2.2 and for all large r_1, r_2

$$G_1(r_1,r_2) < (l+\epsilon)G_2(r_1,r_2) < G_2(\alpha r_1, \alpha r_2) \qquad \ldots (6.1)$$

where $\alpha > 1$ is such that $l + \epsilon < \alpha$ Now,

$$F(r_1,r_2) < G_1\left(exp^{[p-1]}r_1^{\rho^p_{g_1}(f)+\epsilon}, exp^{[p-1]}r_2^{\rho^p_{g_1}(f)+\epsilon}\right)$$

$$< G_2\left(exp^{[p-1]}r_1^{\rho^p_{g_1}(f)+\epsilon}, exp^{[p-1]}r_2^{\rho^p_{g_1}(f)+\epsilon}\right),$$

using (6.1)

Since $\epsilon > 0$ is arbitrary, we have for all large r_1, r_2

$$\rho^p_{g_2}(f) \leq \rho^p_{g_1}(f).$$

The reverse inequality is clear because $g_2 \sim g_1$ and so

$$\rho^p_{g_1}(f) = \rho^p_{g_2}(f).$$

Theorem 6.3. Let f_1, f_2, g be entire functions of two complex variables and $f_1 \sim f_2$. Then

$$\rho^p_g(f_1) = \rho^p_g(f_2).$$

The proof is similar as Theorem 6.2.

REFERENCES

[1] A. K. Agarwal, On the properties of an entire function of two complex variables, Canadian Journal of Mathematics, 20 (1968), pp. 51-57.
[2] D. Banerjee and R. K. Dutta, Relative order of entire functions of two complex variables, International J. of Math. Sci. & Engg. Appls., 1(1) (2007), pp. 141-154.
[3] L. Bernal, Orden relative de crecimiento de funciones enteras, Collect. Math. 39 (1988), pp. 209-229.
[4] B. A. Fuks, Theory of analytic functions of several complex variables, Moscow, 1963.
[5] S. Halvarsson, Growth properties of entire functions depending on a parameter, Annales Polonici Mathe-matici, 14(1) (1996), pp. 71-96.
[6] C. O. Kiselman, Order and type as measures of growth for convex or entire functions, Proc. Lond. Math.Soc., 66(3) (1993), pp. 152-186.
[7] C. O. Kiselman, Plurisubharmonic functions and potential theory in several complex variables, a contribution to the book project, Development of Mathematics 1950-2000, edited by jean-Paul Pier.
[8] B. K. Lahiri and Dibyendu Banerjee, Generalized relative order of entire functions, Proc. Nat. Acad. Sci. India, 72(A), IV (2002), pp. 351-371.
[9] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., 69 (1963) , pp. 411-414.

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