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Linear birth/immigration-death process with binomial catastrophes

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Abstract

In this paper, we study a birth/immigration-death processes under mild (binomial) catastrophes. We obtain explicit expressions for both the time-dependent (transient) and the limiting (equilibrium) factorial moments, which are then used to construct the transient and equilibrium distribution of the population size. We demonstrate that our approach is also applicable to multidimensional systems such as stochastic processes operating under a random environment and other variations of the model at hand. We also obtain various stochastic order results for the number of individuals with respect to the system parameters, as well as the relaxation time.

Keywords: birth-death processes; catastrophes; time-dependent and equilibrium moments; equilibrium distribution; relaxation time

AMS 2000 *Subject Classification*: 60K25, 90B22

1 Introduction

1.1 Literature overview and motivation

Birth-death processes have a rich history in probabilistic modeling, including applications in ecology, genetics, and evolution, see [18, 52, 59, 73] and the references therein. Moreover, populations can suffer dramatic declines from disease or food shortage but, perhaps surprisingly, such populations can survive for long periods of time and, although they may eventually become extinct, they can exhibit an apparently stationary regime. This behavior has been successfully modeled using the birth-death process with catastrophes [4, 62, 63]. In particular, metapopulation models, epidemics, and migratory flows provide practical examples of populations subject to disasters (e.g., habitat destruction, environmental catastrophes). Birth-death processes with catastrophes have been extensively studied during the past two decades (see e.g. [8, 12, 13, 37, 53, 56]). These papers deal with the computation of important measures such as the equilibrium probabilities, the extinction probability and the mean time to extinction for the underlying models. Recently, several authors considered the problem of computing the transient distribution of processes influenced by Poisson generated catastrophes. Various methods have been developed for specific models. These methods include among others: a) the direct solution of the Chapman-Kolmogorov equations, see [70] for a solution to the simple immigration-catastrophe process, [51] for the M/M/1 queue with catastrophes, and [16, 71] for the birth/immigration-death process with catastrophes, b) a lattice-path combinatorics approach, see [15, 49, 50], c) a probabilistic approach using the age distribution of a Poisson process, see [21, 54, 67]. These ideas have been further exploited and applied to various models, see [32, 33, 68, 69, 72]. In [22] the authors review the different methodologies for the derivation of the transient distribution and discuss the pros and cons.

The catastrophe mechanism, in its simplest form, instantaneously removes the whole population (total catastrophe case) whenever a catastrophic event occurs. More specifically, the total catastrophe model assumes that the individuals are exposed to a catastrophic effect that massively affects the entire population and leads to extinction (see, e.g., [3, 53, 55, 67, 68, 71] and the references therein). However, in most practical situations, that is not

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the case, and for this reason mild (binomial) catastrophes are considered to capture better real life circumstances. It is natural to assume that a catastrophe concurrently eliminates part of the population leading to the selection of the binomial catastrophe scenario as the best fitting model (see [5, 8, 13]). Under the assumption that the individuals of a population are exposed simultaneously to a catastrophic effect, the appropriate model is that of binomial catastrophes. More specifically, the binomial catastrophe model assumes that the individuals are exposed to a catastrophic effect simultaneously and every individual survives a catastrophe with probability $p \in (0,1)$, independently of anything else. Moreover, in the Markovian setting, it is assumed that the intercatastrophe intervals are exponentially distributed at rate γ and, given the precatastrophe population size i, the number of surviving individuals after the catastrophe follows a binomial distribution with parameters i and p, thus, the rate of down-jumps from state i to state j, due to catastrophes, is

$$\binom{i}{j} p^j (1-p)^{i-j}\gamma, \;\; j = 0, 1, 2, \dots, i, \;\; i \geq 0.$$

In contrast to the total catastrophe rule, binomial catastrophes are considered to capture better the behavior of catastrophic events having mild influence on the population. Moreover, considering an appropriate limit ($p \to 0^+$) the model of binomial catastrophes is reduced to that of total catastrophes, see Section 5.3 for more details on this topic.

Models with binomial catastrophes were introduced in [13]. In particular, the authors in [13] consider population growth Markov models (birth/immigration models) subject to catastrophes, where the rate of the catastrophe depends on the population size. They distinguish three types of catastrophes: 1) Geometric catastrophes, 2) Binomial catastrophes, and 3) Uniform catastrophes. In the case of binomial catastrophes, the authors assume that the catastrophes are generated by an independent Poisson process, at the epochs of which the members of the population survive with probability p, independently of each other. In [13] the authors obtain an expression for the equilibrium expected population size. In [22], for a different birth model still subject to binomial catastrophes, the authors discuss possible extensions of the existing methods in order to calculate the transient distribution of the system at hand. Several other papers deal with the equilibrium analysis of birth models subject to binomial catastrophes (see, e.g. [2, 5, 20, 22, 23, 24, 25, 38, 40] and the references therein).

We present in this manuscript a methodology for the time-dependent (transient) and limiting (equilibrium) analysis of the birth/immigration-death process with binomial catastrophes. Our analysis includes the derivation of the factorial moments, the numerical reconstruction of the distribution of the number of individuals in the system and the relaxation time.

1.2 Model description

In this manuscript we analyze the linear birth/immigration-death model subject to binomial catastrophes. In particular, we consider a continuous time Markov chain (CTMC) $\{N(t) : t \geq 0\}$, with state space the non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots, \}$ and generator $Q = (q_{ij})_{i,j \in \mathbb{N}_0}$, with

$$q_{ij} = \begin{cases} i\lambda + \nu, & \text{if } j = i+1, \, i \geq 0 \\ \binom{i}{j} p^j (1-p)^{i-j}\gamma + i\mu \, \delta_{j,i-1}, & \text{if } j = 0, 1, \dots, i, \, i \geq 0, \end{cases}$$

where $\delta_{j,i-1}$ denotes the Kronecker delta, taking value 1, when $j = i-1$, and value 0, otherwise. Due to the particular form of the rates q_{ij} ($i \neq j$) at which the stochastic process $N(t)$ jumps from state i to state j, we may refer to the process $N(t)$ as a birth/immigration-death and catastrophe model: The rate $q_{i,i+1} = i\lambda + \nu$ describes the individual up-jumps associated with an immigration Poisson process at rate ν and the individual births occurring at rate λ. Moreover, we assume that there exist two sources of population decrease: single deaths and a random catastrophe mechanism (epidemics, human interventions, natural disasters). Thus, there exist two types of down-jumps; the first one $q_{i,i-1} = i\mu$ describes the death mechanism according to which the individual lifetime span is terminated after an exponentially distributed random time at rate μ. The second downward type of jump occurs at rate $q_{ij} = \binom{i}{j} p^j (1-p)^{i-j}\gamma$, $j = 0, 1, \dots, i$, and is due to the occurrence of a catastrophic event. Catastrophes occur according to a Poisson process at rate γ. At the epochs of the catastrophe process the present population size is thinned. In particular, we assume that a random number of the members of the population is affected at a catastrophe epoch, and that each member of the population survives the catastrophe with probability p, independently of the other members. Taking into account that all aforementioned rates are positive, we can immediately deduce that the stochastic process $N(t)$ is irreducible and regular. The transition diagram of the process $N(t)$ is depicted in Figure 1.

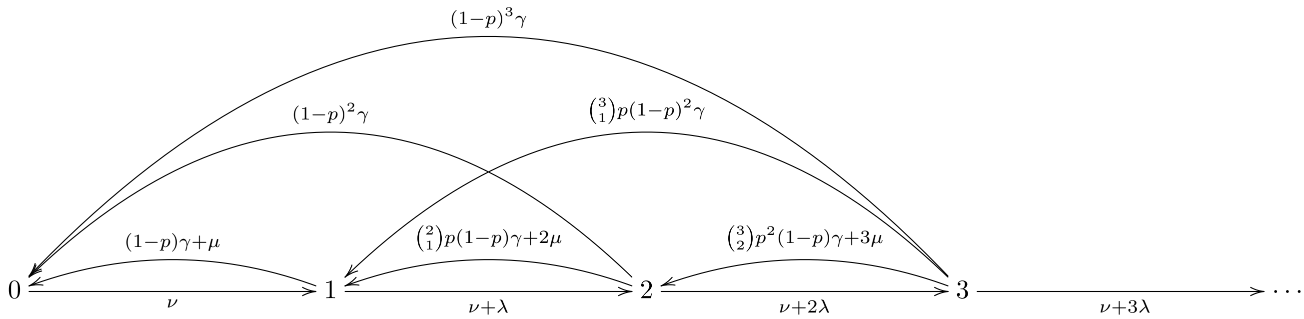


Figure 1: Transition rate diagram of $\{N(t) : t \geq 0\}$.

As briefly stated in the literature overview section the birth/immigration-death model subject to binomial catastrophes has vast applications in the study of biological populations. Moreover, this model can also accurately capture in a marketing context the random changes in the number of customers of a certain mobile phone provider. In the mobile phone industry, there are several companies that provide a certain mobile phone service. All companies try to increase their number of customers, to this they regularly promote new products and make offers regarding their service costs. When a company performs such a campaign, customers join its service abandoning their previous provider. In particular, it seems reasonable to assume that new customers join the company either due to the advertising promotion (batch immigration) or due to a mouth to mouth advertisement (births), moreover every existing customer may decide to abandon the company due to either dissatisfaction (death) or due to the promotional campaign of a rival company (binomial catastrophes). In the latter case, we can without loss of generality assume that at the opportunity of such a campaign a customer decides whether to join a rival company with probability $1-p$ or to stay with its current provider with probability p. The model with single immigrations is analysed in Sections 3 to 6, while the batch immigration model is treated in Section 7.2.

1.3 Probabilistic quantities of interest

In the study of birth-death processes the following probabilistic quantities are of primary interest:

i) The factorial (transient and equilibrium) moments of the stochastic process: Obtaining the transient distribution for the general birth-death process is extremely difficult and usually researchers resort to easier analysis obtaining the factorial (transient and equilibrium) moments, say $m_{(k)}(t) = \mathbb{E}[N(t)(N(t)-1)\cdots(N(t)-k+1)]$, $k = 1, 2, \ldots$ and $m_{(k)} = \lim_{t\to\infty} m_{(k)}(t)$, respectively, and other tractable quantities of interest. An approach for obtaining the transient factorial moments that does not require to first obtain the transient distribution, lies in the use of a time-dependent version of Little's law, which is discussed in [30, 31].

   In this manuscript we present a stable iterative procedure for the derivation of the transient factorial moments. In particular for the case of the birth/immigration-death model with binomial catastrophes, the iterative scheme yields closed-form expressions for the transient factorial moments, see Section 2.

ii) The transient distribution: For the general birth-death process, Karlin and McGregor, in [42], present a study on the existence, uniqueness and analytic properties of the matrix $\mathbf{\Pi}(t) = (\pi_{ij}(t))_{i,j\in\mathbb{N}_0}$, with $\pi_{ij}(t) = \mathbb{P}[N(t) = j|N(0) = i]$, $t \geq 0$, as well as certain auxiliary conditions. In a sequel paper Karlin and McGregor [41], establish the equivalence between properties of the stochastic process and the properties of the sequences of the birth and death rates. They, also, evaluate in terms of the birth and death rates several equilibrium and transient probabilistic quantities associated with the process $\{N(t) : t \geq 0\}$. In [39, 42] the authors establish a general correspondence between continuous time birth-death processes and continued fractions of the Stieltjes-Jacobi type together with their associated orthogonal polynomials. While, in [27] the authors revise this fundamental correspondence using the relation between weighted lattice paths and continued fractions.

In [77] the authors present the solution to the transient distribution of the linear birth-death model. In particular, they consider the probability generating function (PGF) of the transient distribution, defined as $\Pi(t,z) = \sum_{n=0}^{\infty} \pi_n(t) z^n$, $|z| \leq 1$, with $\pi_n(t) = \mathbb{P}[N(t) = n]$, which satisfies a partial differential equation (PDE) of the form

$$z\frac{\partial \Pi(t,z)}{\partial t} + A(z)\Pi(t,z) + B(z)\frac{\partial \Pi(t,z)}{\partial z} = C(z)\Pi(t,0). \tag{1.1}$$

Using the characteristic equation of the PDE they derive the PGF of the transient distribution, see [77, Theorem 1, page 146]. Using a similar approach Chao and Zheng in [16] solve the functional equation of the PGF of the transient distribution

$$z\frac{\partial \Pi(t,z)}{\partial t} + A(z)\Pi(t,z) + B(z)\frac{\partial \Pi(t,z)}{\partial z} = D(z) \tag{1.2}$$

for the birth/immigration-death model under total catastrophes.

In [22] the authors discuss the extension of the existing methods for the transient analysis of a population process subject to binomial catastrophes. They carry out the time-dependent analysis for the compound Poisson immigration process subject to binomial catastrophes. In particular they assume that the catastrophes are generated by a second independent Poisson process and at the epochs of such a catastrophe the members of the population survive with probability p, independently of each other.

In this manuscript, we study the transient and equilibrium behavior of a special type of model, the linear birth/immigration-death process subject to binomial catastrophes.More specifically, our analysis deals with the calculation of the transient and equilibrium factorial moments, the derivation of the transient and equilibrium distributions, as well as the relaxation time. In particular, the PGF of the transient distribution, say $\Pi(t,z)$, of a Markovian model combining linear and binomial transition rates satisfies a partial q-difference differential equation of the following form

$$z\frac{\partial \Pi(t,z)}{\partial t} + A(z)\Pi(t,z) + B(z)\frac{\partial \Pi(t,z)}{\partial z} = C(z)\Pi(t,0) + D(z)\Pi(t, 1-p+pz), \tag{1.3}$$

where $A(z), B(z), C(z)$ and $D(z)$ are known polynomials in z depending on the model parameters. Note that the functional equation for the PGF of the transient distribution, (1.3), involves a difference-differential equation.

iii) The equilibrium distribution of the stochastic process and the stability condition: For the general birth-death Markovian process, with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_1, \mu_2, \ldots$, the equilibrium distribution (see, e.g. [36, 43, 65]), $\pi_n = \lim_{t\to\infty} \mathbb{P}[N(t) = n]$, $n = 0, 1, 2, \ldots$, is of the form

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}}, \tag{1.4}$$

$$\pi_n = \pi_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}, \; n = 1, 2, \ldots. \tag{1.5}$$

The foregoing equations also show us what condition is necessary for these limiting probabilities to exist. Namely, it is necessary that

$$\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} < \infty.$$

This condition (stability condition) may be shown to also be sufficient. The stability condition guarantees the ergodicity of the birth-death process.

In the special case of linear birth and death rates, i.e. $\lambda_{i-1} = (i-1)\lambda + \nu$ and $\mu_i = i\mu + \theta$, $i = 1, 2, \ldots$, we would like to report that the stability condition assumes the form $\lambda < \mu$. Under the stability condition one can derive the equilibrium distribution in terms of hypergeometric series defined as

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \prod_{i=1}^{n} \frac{(a+i-1)(b+i-1)}{(c+i-1)}, \tag{1.6} $$

where we assume that $c + i - 1 \neq 0$ for all $i = 1, 2, 3 \ldots$ and $|z| < 1$ (see [61]). In particular, equations (1.4) and (1.5) assume the form

$$ \begin{aligned} \pi_0 &= \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{(i-1)\lambda+\nu}{i\mu+\theta}} = \left({}_2F_1(\nu/\lambda, 1; \theta/\mu + 1; \lambda/\mu) \right)^{-1}, \\ \pi_n &= \pi_0 \prod_{i=1}^{n} \frac{(i-1)\lambda + \nu}{i\mu + \theta}, \; n = 1, 2, \ldots. \end{aligned} $$

In [16] the authors also present the solution to the equilibrium distribution of the linear birth/immigration-death model with total catastrophes. In particular, they consider the PGF of the equilibrium distribution, defined as $\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$, $|z| \leq 1$, which satisfies an ordinary differential equation (ODE) of the form

$$ A(z)\Pi(z) + B(z)\frac{d\Pi(z)}{dz} = D(z), \tag{1.7} $$

where the functions $A(z), B(z)$ and $D(z)$ are known polynomials in z depending on the model parameters.

In this paper we derive the equilibrium distribution by solving the corresponding PGF, say $\Pi(z)$. In the case of a Markovian model combining linear and binomial transition rates the PGF satisfies an ordinary q-difference differential equation of the following form

$$ A(z)\Pi(z) + B(z)\frac{d\,\Pi(z)}{dz} = C(z)\Pi(0) + D(z)\Pi(1 - p + pz), \tag{1.8} $$

where $A(z), B(z), C(z)$ and $D(z)$ are known polynomials in z. Note that the functional equation for the PGF of the equilibrium distribution, (1.8), involves a difference-differential equation. Bellman and Cooke in [9] have set the foundations for the study of such equations when the functions $A(z), B(z), C(z)$ and $D(z)$ are constant. Unfortunately, this theory can not be directly extended in our case. In this manuscript we propose to solve these type of difference-differential equations by using a factorial moment generating function approach that extends the idea of the Frobenius method, see e.g. [78, page 233]. Furthermore, we demonstrate the effectiveness of the approach on a class of models combining linear and binomial rates. We also illustrate that this technique works equally well when we have multiple types of binomial rates, i.e. with different probabilities, and also under a multi-dimensional setting.

iv) The relaxation time: The speed of convergence to stationarity of the process $\{N(t) : t \geq 0\}$ is usually characterized by the decay parameter

$$ \mathscr{A} := \sup\{a \geq 0 : \pi_n - \pi_n(t) = O(e^{-at}) \text{ as } t \to \infty \text{ for all } n \in \mathbb{N}_0\}, $$

or its reciprocal $\mathscr{R} := 1/\mathscr{A}$, the relaxation time of the stochastic process, see e.g. [17]. If $m_{(1)} = \lim_{t\to\infty} \mathbb{E}[N(t)] < \infty$ we also have

$$ \mathscr{R} := \inf\{r > 0 : m_{(1)} - \mathbb{E}[N(t)] = O(e^{-t/r}) \text{ as } t \to \infty\}. $$

The relaxation times of many specific birth-death processes are known. In particular, in [47] the authors study the relaxation time of the truncated birth-death process and derive analytic lower bounds. They also perform a sensitivity analysis on the relaxation time with respect to the birth and death rates. Kartashov in [44] showed that the rate of convergence to stability of an ergodic birth-death processes on the non-negative integers with birth rates λ_i and death rates μ_{i+1}, $i = 0, 1, 2, \ldots$, is bounded by the quantity

$$\sup_{\boldsymbol{u}} \inf_{i\geq 0} \{\lambda_i + \mu_{i+1} - \frac{\lambda_i}{u_i} - \mu_{i+1} u_{i+1}\}, \tag{1.9}$$

where $\boldsymbol{u} = (u_0, u_1, \ldots)$ is a sequence of positive numbers. Subsequently, van Doorn [19] pointed out that actually the rate of convergence equals (1.9).

1.4 Structure of the manuscript

The remainder of the paper is organized as follows. In Section 2 we look at the transient behavior of the system and in particular we derive the transient factorial moments. Section 3 is devoted to the equilibrium analysis of the model. In Section 4 we derive the relaxation time and in Section 5 we present several ordering and limiting results. In Section 6 we present numerical results for the birth/immigration-death process under mild (binomial) catastrophes. Finally, in Section 7 we discuss several extensions of the model at hand.

2 Transient analysis

In this section we demonstrate a methodology for the derivation of the transient distribution and the transient factorial moments of $\{N(t) : t \geq 0\}$. Let $\pi_n(t)$ denote the probability that there are n individuals present at time t, i.e. $\pi_n(t) = \mathbb{P}[N(t) = n]$. The Kolmogorov equations for the model at hand read as follows:

$$\frac{d\pi_0(t)}{dt} + (\nu + \gamma)\pi_0(t) = \mu\pi_1(t) + \gamma \sum_{k=0}^{\infty} (1-p)^k \pi_k(t), \tag{2.1}$$

$$\frac{d\pi_n(t)}{dt} + (\nu + n\lambda + \gamma + n\mu)\pi_n(t) = (\nu + (n-1)\lambda)\pi_{n-1}(t) + (n+1)\mu\pi_{n+1}(t) + \gamma \sum_{k=n}^{\infty} \binom{k}{n} p^n (1-p)^{k-n} \pi_k(t), \; n \geq 1. \tag{2.2}$$

Let $\Pi(t, z) = \sum_{n=0}^{\infty} \pi_n(t) z^n$ denote the time-dependent (transient) PGF. Multiplying equations (2.1) and (2.2) with z^0 and z^n, respectively, and summing over all $n = 0, 1, \ldots$, yields

$$\frac{\partial \Pi(t, z)}{\partial t} + [\nu(1-z) + \gamma]\Pi(t, z) + (z-1)(\mu - \lambda z)\frac{\partial \Pi(t, z)}{\partial z} = \gamma \Pi(t, 1-p+pz). \tag{2.3}$$

We show in the sequel how to derive the transient factorial moments, as well as the transient distribution, based on the functional equation for the PGF given in (2.3). The approach consists of the following steps:

Step 1 We first define the transient factorial moments

$$m_{(k)}(t) = \mathbb{E}[N(t)(N(t)-1)(N(t)-2)\cdots(N(t)-k+1)], \; k = 1, 2, \ldots, \; t \geq 0.$$

Step 2 Define the factorial moment generating function (FMGF)

$$M(t, z) := \sum_{k=0}^{\infty} m_{(k)}(t) \frac{z^k}{k!} = \mathbb{E}[\sum_{k=0}^{\infty} \binom{N(t)}{k} z^k] = \mathbb{E}[(1+z)^{N(t)}] = \Pi(t, z+1).$$

Note that the FMGF and the PGF are connected via the relationship $M(t, z) = \Pi(t, z+1)$. Hence, if we are able to obtain all factorial moments or equivalently the FMGF, we can then obtain the PGF. A similar approach is performed in [35] for the birth/immigration-death model.
Remember that the PGF satisfies the functional equation (2.3), hence after a change of variable, $z := z+1$, the FMGF satisfies the following functional equation

$$\frac{\partial M(t, z)}{\partial t} + [\gamma - \nu z]M(t, z) + z(\mu - \lambda - \lambda z)\frac{\partial M(t, z)}{\partial z} = \gamma M(t, pz). \tag{2.4}$$

In what follows we seek a solution to this difference-differential equation without addressing for the time being any problems of convergence. For this reason, we treat the FMGF as a formal power series, see [75].

Step 3 Determine the factorial moments.

At a first glance the two functional equations (2.3) and (2.4) seem to be similar, however, as it is demonstrated in the proof of Lemma 2.1, equating the coefficients of z^k, $k = 0, 1, 2, \ldots$, in equation (2.4) produces an iterative scheme for the calculation of the factorial moments.

Lemma 2.1 *The transient factorial moments* $m_{(k)}(t) = \mathbb{E}[N(t)(N(t)-1)(N(t)-2)\cdots(N(t)-k+1)]$ *of the number of individuals for the general model described in Section 1 are given as follows:*

$$m_{(1)}(t) = \frac{\nu}{\mu-\lambda+\gamma(1-p)}\left(1-e^{-(\mu-\lambda+\gamma(1-p))t}\right) + m_{(1)}(0)e^{-(\mu-\lambda+\gamma(1-p))t}, \tag{2.5}$$

$$\begin{aligned} m_{(k)}(t) &= k!\prod_{i=1}^{k}\frac{(i-1)\lambda+\nu}{i(\mu-\lambda)+\gamma(1-p^i)} \\ &\quad -\sum_{i=1}^{k}\frac{1}{(i(\mu-\lambda)+\gamma(1-p^i))}\frac{k!\prod_{r=1}^{k}((r-1)\lambda+\nu)}{\prod_{\substack{r=1\\ r\neq i}}^{k}((r-i)(\mu-\lambda)+\gamma(p^i-p^r))}e^{-(i(\mu-\lambda)+\gamma(1-p^i))t} \\ &\quad +\sum_{j=1}^{k}\frac{m_{(j)}(0)}{j((j-1)\lambda+\nu)}\sum_{i=j}^{k}\frac{k!\prod_{r=j}^{k}((r-1)\lambda+\nu)}{\prod_{\substack{r=j\\ r\neq i}}^{k}((r-i)(\mu-\lambda)+\gamma(p^i-p^r))}e^{-(i(\mu-\lambda)+\gamma(1-p^i))t},\ k>1. \end{aligned} \tag{2.6}$$

Proof. Equating the coefficients of z^k on both sides of equation (2.4) yields

$$m'_{(0)}(t) = 0 \tag{2.7}$$

$$m'_{(k)}(t) + (k(\mu-\lambda)+\gamma(1-p^k))m_{(k)}(t) = k((k-1)\lambda+\nu)\, m_{(k-1)}(t),\ k\geq 1. \tag{2.8}$$

Equivalently, (2.7) can be replaced by the normalizing condition

$$m_{(0)}(t) = 1.$$

Remark that the system of equations (2.7) and (2.8) can be directly obtained upon differentiating equation (2.3) with respect to z and setting $z = 1$, cf. [13].

Then, in order to solve the difference-differential equation (2.8) we define the Laplace transform

$$\tilde{m}_{(k)}(s) = \int_0^\infty m_{(k)}(t)e^{-st}dt.$$

Multiplying both sides of equation (2.8) by e^{-st} and integrating from 0 to infinity yields

$$\int_0^\infty m'_{(k)}(t)e^{-st}dt + (k(\mu-\lambda)+\gamma(1-p^k))\tilde{m}_{(k)}(s) = k((k-1)\lambda+\nu)\,\tilde{m}_{(k-1)}(s),\ k\geq 1. \tag{2.9}$$

Integrating by parts we have that

$$\int_0^\infty m'_{(k)}(t)e^{-st}dt = s\,\tilde{m}_{(k)}(s) - m_{(k)}(0).$$

Then, substituting this last expression into (2.9) yields

$$(k(\mu-\lambda)+\gamma(1-p^k)+s)\tilde{m}_{(k)}(s) = k((k-1)\lambda+\nu)\,\tilde{m}_{(k-1)}(s) + m_{(k)}(0),\ k\geq 1. \tag{2.10}$$

Observe that $m_{(k)}(0)$ is the k-th factorial moment of the initial distribution, i.e. $m_{(k)}(0) = E[N(0)(N(0)-1)\ldots(N(0)-k+1)]$, $k\geq 1$. Solving recursively equation (2.10) we obtain the following result

$$\begin{aligned} \tilde{m}_{(k)}(s) &= \tilde{m}_{(0)}(s)\prod_{i=1}^{k}\frac{i((i-1)\lambda+\nu)}{i(\mu-\lambda)+\gamma(1-p^i)+s} \\ &\quad +\sum_{j=1}^{k}\frac{m_{(j)}(0)}{j((j-1)\lambda+\nu)}\prod_{i=j}^{k}\frac{i((i-1)\lambda+\nu)}{i(\mu-\lambda)+\gamma(1-p^i)+s},\ k\geq 1, \end{aligned} \tag{2.11}$$

where $\tilde{m}_{(0)}(s) = 1/s$. In order to invert the Laplace transform we first write the finite product as a finite sum as follows

$$\prod_{i=j}^{k} \frac{i((i-1)\lambda+\nu)}{i(\mu-\lambda)+\gamma(1-p^i)+s} = \sum_{i=j}^{k} \frac{C_i}{i(\mu-\lambda)+\gamma(1-p^i)+s},$$

where the coefficients C_i, $i = j, j+1, \ldots, k$, are equal to

$$C_i = \frac{k! \prod_{r=j}^{k}((r-1)\lambda+\nu)}{\prod_{\substack{r=j\\ r\neq i}}^{k}((r-i)(\mu-\lambda)+\gamma(p^i-p^r))}.$$

Plugging these two expression into (2.11) yields

$$\begin{aligned}
\tilde{m}_{(1)}(s) &= \frac{\nu}{\mu-\lambda+\gamma(1-p)}\Big(\frac{1}{s}-\frac{1}{\mu-\lambda+\gamma(1-p)+s}\Big)+\frac{m_{(1)}(0)}{\mu-\lambda+\gamma(1-p)+s},\\
\tilde{m}_{(k)}(s) &= \frac{1}{s}\prod_{i=1}^{k}\frac{i((i-1)\lambda+\nu)}{i(\mu-\lambda)+\gamma(1-p^i)} - \sum_{i=1}^{k}\frac{1/(i(\mu-\lambda)+\gamma(1-p^i))}{i(\mu-\lambda)+\gamma(1-p^i)+s}\frac{k!\prod_{r=1}^{k}((r-1)\lambda+\nu)}{\prod_{\substack{r=1\\ r\neq i}}^{k}((r-i)(\mu-\lambda)+\gamma(p^i-p^r))}\\
&\quad+\sum_{j=1}^{k} m_{(j)}(0)\sum_{i=j}^{k}\frac{1/j((j-1)\lambda+\nu)}{i(\mu-\lambda)+\gamma(1-p^i)+s}\frac{k!\prod_{r=j}^{k}((r-1)\lambda+\nu)}{\prod_{\substack{r=j\\ r\neq i}}^{k}((r-i)(\mu-\lambda)+\gamma(p^i-p^r))},\ k>1.
\end{aligned}$$

Inverting the Laplace transform immediately yields the result of Lemma 2.1. ■

Remark 2.1 *We would like to note that the recursive scheme for the calculation of the Laplace transform of the transient factorial moments,* (2.10)*, involves no subtractions and is thus numerically stable, so it can be easily used for the numerical evaluation of the transient factorial moments.*

Step 4 Derive the transient distribution from the factorial moments

$$\pi_n(t) = \sum_{k=n}^{\infty}\binom{k}{n} m_{(k)}(t)\frac{(-1)^{k-n}}{k!} = \frac{1}{n!}\sum_{k=0}^{\infty} m_{(k+n)}(t)\frac{(-1)^k}{k!},\ n\geq 0, \tag{2.12}$$

We will numerically exploit this last step in Section 6. Moreover, in the following proposition we present the transient distribution in the special case that the system is initially empty, i.e. $\mathbb{P}[N(0)=0]=1$.

Proposition 2.1 *The transient distribution $\pi_n(t) = \mathbb{P}[N(t)=n]$ of the number of individuals for the general model described in Section 1, in the case $\mathbb{P}[N(0)=0]=1$, is given as follows:*

$$\begin{aligned}
\pi_n(t) &= \sum_{k=n}^{\infty}\frac{(-1)^{k-n}}{n!(k-n)!}\Bigg[k!\prod_{i=1}^{k}\frac{(i-1)\lambda+\nu}{i(\mu-\lambda)+\gamma(1-p^i)}\\
&\quad-\sum_{i=1}^{k}\frac{e^{-(i(\mu-\lambda)+\gamma(1-p^i))t}}{(i(\mu-\lambda)+\gamma(1-p^i))}\frac{k!\prod_{r=1}^{k}((r-1)\lambda+\nu)}{\prod_{\substack{r=1\\ r\neq i}}^{k}((r-i)(\mu-\lambda)+\gamma(p^i-p^r))}\Bigg],\ n\geq 0.
\end{aligned} \tag{2.13}$$

3 Equilibrium analysis

In this section we demonstrate that the previously applied methodology can be used for the derivation of the equilibrium distribution of $\{N(t) : t \geq 0\}$. Let $\pi_n = \lim_{t\to\infty}\mathbb{P}[N(t)=n]$ be the equilibrium distribution of the stochastic process $\{N(t) : t \geq 0\}$. Then, the balance equations of the model at hand read as follows:

$$(\nu+\gamma)\pi_0 = \mu\pi_1+\gamma\sum_{k=0}^{\infty}(1-p)^k\pi_k, \tag{3.1}$$

$$\begin{aligned}
(\nu+n\lambda+\gamma+n\mu)\pi_n &= (\nu+(n-1)\lambda)\pi_{n-1}+(n+1)\mu\pi_{n+1}\\
&\quad+\gamma\sum_{k=n}^{\infty}\binom{k}{n}p^n(1-p)^{k-n}\pi_k,\ n=1,2,\ldots.
\end{aligned} \tag{3.2}$$

Under the stability condition there exists a unique non-trivial solution of the balance equations (3.1) and (3.2) plus the normalizing condition $\sum_{n=0}^{\infty} \pi_n = 1$. For this reason, in the next paragraph we discuss the stability condition of the system.

3.1 Sufficient condition for stability

We first define the embedded discrete time Markov chain, say $\{N_n : n = 0, 1, 2, \ldots\}$, at the various transition epochs of the continuous time process $\{N(t) : t \geq 0)\}$. In particular the embedded Markov chain $\{N_n : n = 0, 1, 2, \ldots\}$ is defined on the discrete state space $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$ with transition probabilities

$$p_{ij} = \begin{cases} \dfrac{i\lambda + \nu}{i\lambda + \nu + i\mu + \gamma}, & \text{if } j = i+1,\ i \geq 0, \\ \dfrac{\gamma}{i\lambda + \nu + i\mu + \gamma}\dbinom{i}{j} p^j (1-p)^{i-j} + \dfrac{i\mu}{i\lambda + \nu + i\mu + \gamma}\,\delta_{j,i-1}, & \text{if } j = 0, 1, \ldots, i,\ i \geq 0. \end{cases}$$

For the derivation of a sufficient condition for stability (positive recurrence) we use Pakes' [60] theorem: an irreducible and aperiodic Markov chain $\{N_n : n = 0, 1, 2, \ldots\}$ is ergodic provided that for all but finitely many values of i, the drift at state i, $\mathbb{E}[N_{n+1} - N_n | N_n = i]$, is bounded above by a negative constant. For our model observe that $\{N_n : n = 0, 1, 2, \ldots\}$ is irreducible, aperiodic and the drift is written as follows:

$$\begin{aligned} \mathbb{E}[N_{n+1} - N_n | N_n = i,\, i > 0] &= \frac{i\lambda + \nu}{i\lambda + \nu + i\mu + \gamma}(i+1) + \frac{i\mu}{i\lambda + \nu + i\mu + \gamma}(i-1) \\ &\quad + \frac{\gamma}{i\lambda + \nu + i\mu + \gamma}\sum_{j=0}^{i} j\binom{i}{j} p^j (1-p)^{i-j} - i \\ &= \frac{(i\lambda + \nu)(i+1) + i\mu(i-1) + \gamma p i}{i\lambda + \nu + i\mu + \gamma} - i \\ &= \frac{i(\lambda - \mu - \gamma(1-p)) + \nu}{i\lambda + \nu + i\mu + \gamma}. \end{aligned}$$

Hence, if $\lambda < \mu + \gamma(1-p)$ the drift is bounded by a negative constant for $i \geq \lfloor \frac{\nu}{\mu + \gamma(1-p) - \lambda} \rfloor + 1$.

Note that the condition for stability, $\lambda < \mu + \gamma(1-p)$, is a sufficient condition for the continuous time Markov chain (CTMC) $\{N(t) : t \geq 0\}$ to be positive recurrent. In particular, it will become clear after the calculation of the transient factorial moments, that the above condition guarantees the finiteness of the limiting first moment, i.e., $\lim_{t\to\infty} \mathbb{E}[N(t)] < \infty$.

3.2 Equilibrium moments and distribution

We define the probability generating function (PGF) $\Pi(z)$ as

$$\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n,\ |z| \leq 1.$$

Multiplying both sides of the balance equations (3.1) and (3.2) with z^0 and z^n, respectively, and summing over all $n = 0, 1, \ldots$, yields after some straightforward calculations the following equation for the PGF

$$[\nu(1-z) + \gamma]\Pi(z) + (z-1)(\mu - \lambda z)\frac{d\,\Pi(z)}{dz} = \gamma\Pi(1 - p + pz). \tag{3.3}$$

An alternative would be to take the limit as $t \to \infty$ in (2.3). This type of functional equation can be referred to as a q-difference-differential equation. A first step towards its solution can be found in [9]. Difference-differential equations are quite frequent in various sources such as geometry, physics, mathematics and engineering, and in most situations have proven to be quite difficult to solve analytically. Significant progress has been noted in the case of linear difference-differential equations with constant coefficients, cf. [9, Equation (4)]. Unfortunately, in our case the difference-differential equation satisfied by the PGF, cf. (3.3), exhibits significant differences from the equations studied in [9] and to the best of our knowledge can not be solved using the traditional tools as the following remark demonstrates.

Remark 3.1 *As a first attempt to solve equation* (3.3) *we treat it as a usual differential equation, i.e. we first multiply both sides of equation* (3.3) *with* $(z-1)^{\gamma/(\mu-\lambda)}(1-\lambda z/\mu)^{\nu/\lambda-\gamma/(\mu-\lambda)}$. *Then, after some straightforward manipulations we obtain*

$$\frac{d}{dz}\Big[(z-1)^{\gamma/(\mu-\lambda)}(1-\lambda z/\mu)^{\nu/\lambda-\gamma/(\mu-\lambda)}\Pi(z)\Big] = \frac{\gamma}{\mu}(z-1)^{-1+\gamma/(\mu-\lambda)}(1-\lambda z/\mu)^{-1+\nu/\lambda-\gamma/(\mu-\lambda)}\Pi(1-p+pz).$$

Taking into account the normalization constant, $\Pi(1)=1$, *equation* (3.3) *yields*

$$\Pi(z) = \frac{\gamma}{\mu}(1-z)^{-\gamma/(\mu-\lambda)}(1-\lambda z/\mu)^{\nu/\lambda-\gamma/(\mu-\lambda)}\int_z^1 (1-t)^{-1+\gamma/(\mu-\lambda)}(1-\lambda t/\mu)^{-1+\nu/\lambda-\gamma/(\mu-\lambda)}\Pi(1-p+pt)dt.$$

After a change of variable the above integral equation assumes the form of a homogeneous second kind linear Volterra integral equation, see [6, 74],

$$\Pi(z) = \int_1^{1-p+pz} K(z,t)\Pi(t)dt.$$

The relationship between ordinary differential equations and Volterra integral equations has been established in [74, Section 1.8, page 18]. Of course it is notable that in our case we demonstrated a relation between a difference-differential equation and a Volterra integral equation. For an analytical solution of this type of Volterra integral equations little is known; Using the Picard method it can be easily proven that there exists a unique solution to the Volterra integral equation, see [57, Chapter 3, page 29]. Moreover, using the resolvent kernel approach a solution of the Volterra integral equation can be theoretically obtained, see [57, Chapter 3, Theorem 3.2]. There exist, also, several approaches to solve numerically this type of Volterra integral equations, see e.g. [6, 57, 74]. Some of the numerical methods are the degenerate kernel methods using Taylor series approximation or orthogonal expansions, the projection method, the interpolation method, the successive approximation method, etc. These methods are mainly used for asymptotic results, error estimates and for the control of numerical stability.

So one idea in order to solve the ordinary difference-differential equation is to use the classical tools for differential equations, reducing the original equation to a Volterra integral equation and then use any of the numerical approaches suggested above. However, our aim is to derive the exact solution. To this end, we perform the same analysis as in section 2:

Step 1 We first define the equilibrium factorial moments

$$m_{(k)} = \mathbb{E}[N(N-1)(N-2)\cdots(N-k+1)],\ k=1,2,\ldots.$$

We demonstrate in the next steps how we can obtain a recursion for the calculation of all the factorial moments. Then, we reconstruct the entire equilibrium distribution from its moments. As it is well known one cannot always reconstruct the distribution from its moments. In [11, Chapter 30], there is an interesting analysis on sufficient and necessary conditions under which such a construction is possible. Moreover, in [14, page 65] it is stated that if all moments exist and the moment generating function exists in some neighborhood of 0, then the distribution is uniquely determined from its moments. With these two references we have now the theoretical background validating our approach.

Step 2 Define the factorial moment generating function (FMGF)

$$M(z) := \sum_{k=0}^{\infty} m_{(k)}\frac{z^k}{k!} = \mathbb{E}[\sum_{k=0}^{\infty}\binom{N}{k}z^k] = \mathbb{E}[(1+z)^N] = \Pi(1+z).$$

A change of variable, $z := z+1$, in (3.3) reveals that

$$[\gamma-\nu z]M(z) + z(\mu-\lambda-\lambda z)\frac{d\,M(z)}{dz} = \gamma M(pz). \tag{3.4}$$

Alternatively, taking the limit as $t\to\infty$ in (2.4) yields (3.4). It is notable that, setting $z=p^x$ in equation (3.4) reveals a delay differential equation, cf. [9, Exercise 16, page 83]. Hence, one can also use the existing numerical approaches for delay differential equations [7].

Step 3 Determine the factorial moments.

Similarly to the transient case, we obtain the factorial moments.

Lemma 3.1 *The equilibrium factorial moments* $m_{(k)} = E[N(N-1)(N-2)\cdots(N-k+1)]$ *of the equilibrium number of individuals for the general model described in Section 1 are given as follows:*

$$m_{(k)} = k! \prod_{i=1}^{k} \frac{\lambda(i-1)+\nu}{i(\mu-\lambda)+\gamma(1-p^i)}, \ k \geq 1, \tag{3.5}$$

as long as $i(\mu-\lambda)+\gamma(1-p^i) > 0$ *for all* $i = 1, 2, \ldots, k$.

Proof. Equating the coefficients of z^k, $k = 0, 1, 2, \ldots$, in equation (3.4), we obtain

$$(k(\mu-\lambda)+\gamma(1-p^k))m_{(k)} = k(\lambda(k-1)+\nu)m_{(k-1)}, \ k = 1, 2, \ldots.$$

Note that this first order difference equation has the solution presented in (3.5). ∎

Remark 3.2 (Finiteness of factorial moments) *We would like to note that the factorial moments of the model at hand exist as long as the denominator of* (3.5) *is strictly positive. This behavior is also observed in the case of total catastrophes. In particular for the birth/immigration-death model with total catastrophes, see e.g. [16], the model at hand is always ergodic, however the k-th factorial moment exists as long as* $\gamma + i(\mu-\lambda) > 0$ *for all* $i = 1, 2, \ldots, k$, $k = 1, 2, 3, \ldots$, *see [64]. For our model, from the calculation of the transient factorial moments* (3.5), *it is immediately evident that the k-th equilibrium factorial moment is finite if and only if* $\gamma(1-p^i) + i(\mu-\lambda) > 0$ *for all* $i = 1, 2, \ldots, k$, $k = 1, 2, 3, \ldots$. *Hence, all moments will be finite if* $\lambda < \mu$.

Step 4 Derive the equilibrium distribution from the factorial moments

$$\pi_n = \sum_{k=n}^{\infty} \binom{k}{n} m_{(k)} \frac{(-1)^{k-n}}{k!} = \frac{1}{n!} \sum_{k=0}^{\infty} m_{(k+n)} \frac{(-1)^k}{k!}. \tag{3.6}$$

In Proposition 3.1 we present the equilibrium distribution of the model at hand.

Proposition 3.1 *For* $\lambda < \mu/2$, *the equilibrium distribution of the general model described in Section 1 is given by*

$$\pi_n = \sum_{k=n}^{\infty} (-1)^{k-n} \binom{k}{n} \prod_{i=1}^{k} \frac{\lambda(i-1)+\nu}{i(\mu-\lambda)+\gamma(1-p^i)}, \ n \geq 0. \tag{3.7}$$

Proof. Given the expression for the factorial moments, $m_{(k)}$, we immediately obtain expression (3.7). However, we need to be cautious with the subtle matters of the radius of convergence of the two generating functions. Note that, for $\lambda < \mu$, all factorial moments are finite and the FMGF converges inside the region $|z| < |\mu-\lambda|/\lambda$ (around zero). Indeed, this is immediately evident by considering the ratio test

$$\lim_{k\to\infty} \left| \frac{m_{(k+1)} z^{k+1}/(k+1)!}{m_{(k)} z^k/k!} \right| = \lim_{k\to\infty} \left| \frac{(\nu+\lambda k)z}{(\mu-\lambda)(k+1)+\gamma(1-p^{k+1})} \right| = \frac{\lambda|z|}{|\mu-\lambda|}.$$

However, note that the series expression for the equilibrium distribution (3.7) converges for $\lambda < \mu/2$. ∎

Remark 3.3 (Analytic continuation of the equilibrium distribution) *As briefly demonstrated in the proof of Proposition 3.1 the series expression for the equilibrium equation is valid for* $\lambda < \mu/2$. *This is of course a sufficient condition for the convergence of the equilibrium distribution. From a theoretical point of view the series expression for the calculation of the equilibrium distribution,* (3.7), *can be analytically continued in the entire region of the stability condition. Unfortunately, we were not able to find a general result for the analytic continuation of* *this particular series. In the special case that $p = 0$ we can easily demonstrate the analytical continuation of the equilibrium distribution. In particular, we have that equation* (3.7) *after some straightforward manipulations yields*

$$\begin{aligned}\pi_n &= \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}\prod_{j=1}^{k+n}\frac{j(\frac{\nu}{\lambda}+j-1)}{\frac{\gamma}{\mu-\lambda}+j}\left(\frac{\lambda}{\mu-\lambda}\right)^{k+n}\\ &= \left(\frac{\lambda}{\mu-\lambda}\right)^{n}\prod_{j=1}^{n}\frac{\frac{\nu}{\lambda}+j-1}{\frac{\gamma}{\mu-\lambda}+j}\,{}_2F_1\left(\frac{\nu}{\lambda}+n, n+1; \frac{\gamma}{\mu-\lambda}+n+1; -\frac{\lambda}{\mu-\lambda}\right),\end{aligned} \tag{3.8}$$

where we assume that $\lambda < \mu/2$ for the series to converge. Using Euler's integral representation formula, see [34, equation 12, page xiii],

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx,\ \mathbb{R}(c) > \mathbb{R}(b) > 0,\ |z| < 1,$$

we obtain that equation (3.8) *assumes the form*

$$\begin{aligned}\pi_n &= \left(\frac{\lambda}{\mu-\lambda}\right)^{n}\prod_{j=1}^{n}\frac{\frac{\nu}{\lambda}+j-1}{\frac{\gamma}{\mu-\lambda}+j}\frac{\Gamma(\frac{\gamma}{\mu-\lambda}+n+1)}{\Gamma(n+1)\Gamma(\frac{\gamma}{\mu-\lambda})}\int_0^1 x^n(1-x)^{\gamma/(\mu-\lambda)-1}(1+\frac{\lambda}{\mu-\lambda}x)^{-\nu/\lambda-n}dx\\ &= \frac{1}{n!}\frac{\lambda^n}{(\mu-\lambda)^{n+1}}\prod_{j=1}^{n}(\frac{\nu}{\lambda}+j-1)\int_0^1 x^n(1-x)^{\gamma/(\mu-\lambda)-1}(1+\frac{\lambda}{\mu-\lambda}x)^{-\nu/\lambda-n}dx\\ &= \frac{1}{n!}\frac{\lambda^n(\mu-\lambda)^{\nu/\lambda}}{\gamma}\prod_{j=1}^{n}(\frac{\nu}{\lambda}+j-1)\int_0^1(1-y^{(\mu-\lambda)/\gamma})^n(\mu-\lambda y^{(\mu-\lambda)/\gamma})^{-\nu/\lambda-n}dy,\end{aligned} \tag{3.9}$$

where in the last equation we used the transform $1 - x = y^{(\mu-\lambda)/\gamma}$. Note that this last equation is identical to equation (51) in [16, Theorem 3.2, page 106] for $\gamma = 1$. Moreover, this last expression is valid for $\lambda < \mu$, hence we were able to analytically continue the expression for the equilibrium distribution from the region $\lambda < \mu/2$ to $\lambda < \mu$. Note, that when $p = 0$, the model reduces to the birth/immigration-death model with total catastrophes, hence the model is always stable. To this end, we can similarly demonstrate how to obtain the equilibrium distribution for the region $\lambda \geq \mu$, cf. (53) in [16, Theorem 3.2, page 106].

Note that in the case of binomial transitions the formula for the equilibrium distribution always involves the calculation of an alternating infinite series, cf. [23, 24, 25].

4 Relaxation time

The speed of convergence to stationarity, see [17], is characterized by the quantity

$$\mathscr{R} := \inf\{r > 0 : m_{(1)} - \mathbb{E}[N(t)] = O(e^{-t/r}) \text{ as } t \to \infty\},$$

called the relaxation time of the stochastic process $\{N(t) : t \geq 0\}$. In (2.5) we obtained an expression for the transient factorial moments and in (3.5) we calculated the equilibrium factorial moments. Using these two expressions we immediately obtain

$$m_{(1)} - m_{(1)}(t) = \left(\frac{\nu}{\mu-\lambda+\gamma(1-p)} - m_{(1)}(0)\right)e^{-(\mu-\lambda+\gamma(1-p))t}, \tag{4.1}$$

hence, conclude that

$$\mathscr{R} = \frac{1}{\mu-\lambda+\gamma(1-p)}. \tag{4.2}$$

5 Ordering and limiting regimes

To emphasize the dependence of a single parameter of the model, while keeping the rest of the parameters fixed, in the rest of this section we will denote $N(t)$, $m_{(k)}(t) = \mathbb{E}[N(t)(N(t)-1)(N(t)-2)\cdots(N(t)-k+1)]$ and $m_{(k)} = \mathbb{E}[N(N-1)(N-2)\cdots(N-k+1)]$ by $N(t;x)$, $m_{(k)}(t;x)$ and $m_{(k)}(x)$, respectively, for the parameter $x \in \{\lambda, \nu, p, \mu, \gamma\}$ under consideration.

5.1 Strong stochastic order

In [10] the author proves that, if we consider two *similar* processes, in which the birth parameters in the first process exceed the corresponding birth parameters in the second and the reverse inequality holds for all other backward parameters, then the second stochastic process is stochastically smaller than the first one. Recall that a random variable Y is called stochastically smaller than a random variable X in the strong stochastic sense, symbolically $X \geq_{st} Y$, if $\mathbb{P}[X > x] \geq \mathbb{P}[Y > x]$, for all real x. This definition can be extended to stochastic processes. For further details on stochastic orders we refer the reader to [66].

For our model we can easily establish the following lemma.

Lemma 5.1 *We have, for all* $t \geq 0$

(a) *if* $\lambda_1 \geq \lambda_2$ *and* $N(0;\lambda_1) = N(0;\lambda_2)$, *then* $N(t;\lambda_1) \overset{\text{st}}{\geq} N(t;\lambda_2)$,

(b) *if* $\nu_1 \geq \nu_2$ *and* $N(0;\nu_1) = N(0;\nu_2)$, *then* $N(t;\nu_1) \overset{\text{st}}{\geq} N(t;\nu_2)$,

(c) *if* $p_1 \geq p_2$ *and* $N(0;p_1) = N(0;p_2)$, *then* $N(t;p_1) \overset{\text{st}}{\geq} N(t;p_2)$,

(d) *if* $\mu_1 \leq \mu_2$ *and* $N(0;\mu_1) = N(0;\mu_2)$, *then* $N(t;\mu_1) \overset{\text{st}}{\geq} N(t;\mu_2)$,

(e) *if* $\gamma_1 \leq \gamma_2$ *and* $N(0;\gamma_1) = N(0;\gamma_2)$, *then* $N(t;\gamma_1) \overset{\text{st}}{\geq} N(t;\gamma_2)$.

Proof. Couple all the random variables in the two systems: Poisson immigration process (ν), birth (λ) and death (μ) processes of customers, occurrences of catastrophes (γ), thinning (binomial reduction) of the number of customers (p). Furthermore, divide the customers present in the first system in two parts: those that are also present in the second system and those that are not present in the second system. In this way, we have, pathwise, that at any time t all customers present in the second system are also present in the first system. As a consequence, the number of customers in the second system is, at any time t, stochastically smaller than the number of customers in the first system. This idea for a proof follows the same line of reasoning as in [58]. ∎

Remark 5.1 *An alternative idea for the proof of Lemma 5.1 is to use the Kirstein equations [48] that give sufficient conditions for the stochastic domination of continuous-time Markov chains in terms of the elements of their infinitesimal generators, see [5, Theorem 7.1, page 589].*

5.2 Factorial moment ordering

In Section 3.2 we obtained an explicit expression for the factorial moments $m_{(k)} = \mathbb{E}[N(N-1)(N-2)\cdots(N-k+1)]$ of the equilibrium number of individuals in the model, cf. (3.5). Note that the strong stochastic order implies the factorial moment order, cf. [66, Theorem 5.C.4, page 254]. In particular, a random variable Y is called stochastically smaller than a random variable X in the factorial moment order, symbolically $X \geq_{fm} Y$, if

$$\mathbb{E}\binom{X}{i} \geq \mathbb{E}\binom{Y}{i}, \quad \text{for all } i \in \mathbb{N}_0.$$

This definition, as well, can be extended to stochastic processes.

Corollary 5.1 *We have, for all* $t \geq 0$

(a) *if* $\lambda_1 \geq \lambda_2$ *and* $N(0;\lambda_1) = N(0;\lambda_2)$, *then* $N(t;\lambda_1) \geq_{fm} N(t;\lambda_2)$,

(b) *if* $\nu_1 \geq \nu_2$ *and* $N(0;\nu_1) = N(0;\nu_2)$*, then* $N(t;\nu_1) \geq_{fm} N(t;\nu_2)$*,*

(c) *if* $p_1 \geq p_2$ *and* $N(0;p_1) = N(0;p_2)$*, then* $N(t;p_1) \geq_{fm} N(t;p_2)$*,*

(d) *if* $\mu_1 \leq \mu_2$ *and* $N(0;\mu_1) = N(0;\mu_2)$*, then* $N(t;\mu_1) \geq_{fm} N(t;\mu_2)$

(e) *if* $\gamma_1 \leq \gamma_2$ *and* $N(0;\gamma_1) = N(0;\gamma_2)$*, then* $N(t;\gamma_1) \geq_{fm} N(t;\gamma_2)$*.*

Proof. This is an immediate consequence of the strong stochastic order established in Lemma 5.1 and the fact that the usual stochastic order implies the factorial moment order, see [66, Theorem 5.C.4, page 254]. ∎

5.3 Stationary limiting regimes

We now turn our attention to the behavior of the model under certain limiting regimes. Note that the overall catastrophe time of a customer is a geometric sum of exponentially distributed random variables with rate γ and so we can easily see that it is also exponentially distributed with parameter $\gamma(1-p)$. Hence, $\gamma(1-p)$ can be thought of as the total effective catastrophe rate per customer.

Under this perspective, if we have two models with the same parameters λ, ν and μ that differ only in γ and p, but with $\gamma(1-p) = \gamma^*$ fixed, we can think that the models have identical birth rates λ, immigration rates ν, death rates μ and effective catastrophe rates per customer γ^* and differ only in the 'level of synchronization' p.

Lemma 5.2 *Let* $\gamma_1 \leq \gamma_2$ *and* $p_1 \leq p_2$ *such that* $\gamma_1(1-p_1) = \gamma_2(1-p_2) = \gamma^*$ *then*

$$m_{(k)}(p_1,\gamma_1) \geq m_{(k)}(p_2,\gamma_2), \;\; k = 1, 2, \ldots,$$

which implies that $N(\infty;p_1,\gamma_1) \geq_{fm} N(\infty;p_2,\gamma_2)$*.*

Proof. The proof of the lemma becomes evident upon observing the form of the factorial moments stated in equation (3.5). ∎

In Section 5.4 we show that the above factorial moment stochastic order, can be extended to the transient factorial moments, revealing that if $N(0;p_1,\gamma_1) = N(0;p_2,\gamma_2)$, $\gamma_1 \leq \gamma_2$ and $p_1 \leq p_2$ such that $\gamma_1(1-p_1) = \gamma_2(1-p_2) = \gamma^*$ then $N(t;p_1,\gamma_1) \geq_{\text{fm}} N(t;p_2,\gamma_2)$, $t > 0$.

We are also interested in the two extreme cases that $p \to 1^-$ or $p \to 0^+$ under the condition that $\gamma(1-p) = \gamma^*$ is fixed.

The case $p \to 1^-$ corresponds to no synchronization, since then $\gamma = \gamma^*/(1-p) \to \infty$, hence in this case the customers depart almost singly at the catastrophe epochs. On the contrary, the case $p \to 0^+$ corresponds to full synchronization, since then $\gamma = \gamma^*/(1-p) \to \gamma^*$, hence in the latter case almost all present customers depart simultaneously from the system when a catastrophe occurs.

We are interested in studying the equilibrium behavior of the system for the case where λ, ν, μ and γ^* are kept fixed in the two limiting cases $p \to 1^-$ and $p \to 0^+$.

The case $p \to 1^-$ corresponds exactly to the birth(λ)/immigration(ν)-death($\mu + \gamma^*$) process. Note that taking the limit in (3.3) as $p \to 1^-$, under the condition that $\gamma = \gamma^*/(1-p)$, yields

$$\nu\Pi(z) - (\mu + \gamma^* - \lambda z)\frac{d\,\Pi(z)}{dz} = 0.$$

Of course, solving this last equation is trivial. In this case we obtain $\Pi(z) = \left(\frac{\mu+\gamma^*-\lambda}{\mu+\gamma^*-\lambda z}\right)^{\nu/\lambda}$. Note that the solution corresponds to the generating function of the Pascal (or otherwise known as negative binomial) distribution with parameters ν/λ and success probability $\lambda/(\mu+\gamma^*)$, see [76, Proposition 1, page 186]. In addition, the factorial moments of the equilibrium distribution are given in Lemma 5.3.

Lemma 5.3 *For a system with birth rate λ, immigration rate ν, death rate μ and effective catastrophe rate per customer γ^* the factorial moments $m_{(k)}^{(I)} = E[N^{(I)}(N^{(I)}-1)(N^{(I)}-2)\cdots(N^{(I)}-k+1)]$ of the equilibrium number of customers in the system, when $p \to 1^-$, are given by*

$$m_{(k)}^{(I)} = \frac{\prod_{i=1}^{k}(\nu+\lambda(i-1))}{(\gamma^*+\mu-\lambda)^k}, \; k \geq 1. \tag{5.1}$$

The case $p \to 0^+$ corresponds exactly to the birth(λ)/immigration(ν)-death(μ) process with total catastrophes(γ^*). Note that taking the limit in (3.3) as $p \to 0^+$ under the condition that $\gamma = \gamma^*/(1-p)$ yields

$$(\nu(1-z)+\gamma^*)\Pi(z) + (z-1)(\mu-\lambda z)\frac{d\,\Pi(z)}{dz} = \gamma^*.$$

For a solution to the above differential equation and the derivation of the equilibrium distribution the interested reader is referred to [16, Theorem 3.1, page 99]. For the limiting case $p \to 0^+$, we obtain the factorial moments of the equilibrium distribution in Lemma 5.4.

Lemma 5.4 *For a system with birth rate λ, immigration rate ν, death rate μ and effective catastrophe rate per customer γ^* the factorial moments $m_{(k)}^{(C)} = E[N^{(C)}(N^{(C)}-1)(N^{(C)}-2)\cdots(N^{(C)}-k+1)]$ of the equilibrium number of customers in the system, when $p \to 0^+$ are given by*

$$m_{(k)}^{(C)} = k!\prod_{i=1}^{k}\frac{\nu+\lambda(i-1)}{\gamma^*+i(\mu-\lambda)}, \; k \geq 1. \tag{5.2}$$

Remark 5.2 *One can easily verify that*

$$\lim_{\substack{p_2\to 1^-\\ \gamma_2(1-p_2)=\gamma^*}} m_{(k)}(p_2,\gamma_2) := m_{(k)}^{(I)} \;\leq\; m_{(k)} \;\leq\; m_{(k)}^{(C)} =: \lim_{\substack{p_1\to 0^+\\ \gamma_1(1-p_1)=\gamma^*}} m_{(k)}(p_1,\gamma_1), \;\forall\, k \geq 0.$$

Hence, according to the definition of the factorial moments order, the queue length distribution is bigger (in the factorial moments order) than the queue length distribution of the birth(λ)/immigration(ν)-death($\mu+\gamma^*$) process and is smaller (in the factorial moments order) than the queue length distribution of the birth(λ)/immigration(ν)-death(μ) with total catastrophes($\gamma*$) process:

$$N^{(I)} \leq_{fm} N \leq_{fm} N^{(C)}.$$

Furthermore, the factorial moment ordering can be extended to a probability generating function order (see, e.g. [66, Section 5.C, page 255])

$$\mathbb{E}[(z+1)^{N^{(I)}}] \leq \mathbb{E}[(z+1)^{N}] \leq \mathbb{E}[(z+1)^{N^{(C)}}], \;\forall\; z \in (0,1).$$

5.4 Transient limiting regimes

In the previous section, 5.3, we investigated the behavior of the equilibrium factorial moments, $m_{(k)} = E[N(N-1)\cdots(N-k+1)]$, under the scaling effect that the total effective catastrophe rate per customer $\gamma(1-p)$ remains fixed. In this section we extend the above result to the transient factorial moments, $m_{(k)}(t) = \mathbb{E}[N(t)(N(t)-1)(N(t)-2)\cdots(N(t)-k+1)]$, proving a factorial moment order under the scaling $\gamma(1-p) = \gamma^*$ for the stochastic process $\{N(t) : \; t \geq 0\}$.

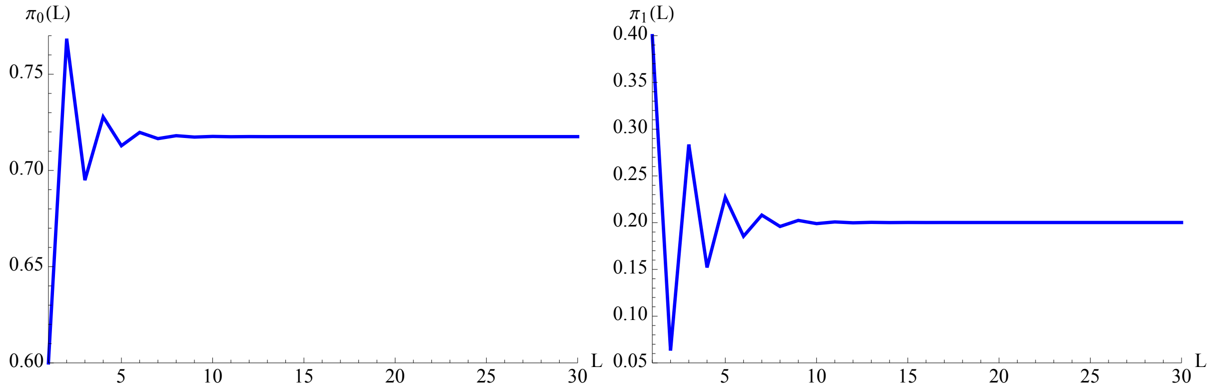
Consider two models with the same parameters λ, ν and μ that differ only in γ and p, but with $\gamma(1-p) = \gamma^*$ fixed, i.e. two models that have identical birth rates λ, immigration rates ν, death rates μ and effective catastrophe rates per customer γ^* and differ only in the 'level of synchronization' p.

Lemma 5.5 *Assume that $m_{(k)}(0;p_1,\gamma_1) = m_{(k)}(0;p_2,\gamma_2)$ and let $\gamma_1 \leq \gamma_2$ and $p_1 \leq p_2$ such that $\gamma_1(1-p_1) = \gamma_2(1-p_2) = \gamma^*$ then*

$$m_{(k)}(t;p_1,\gamma_1) \geq m_{(k)}(t;p_2,\gamma_2), \; t \geq 0,$$

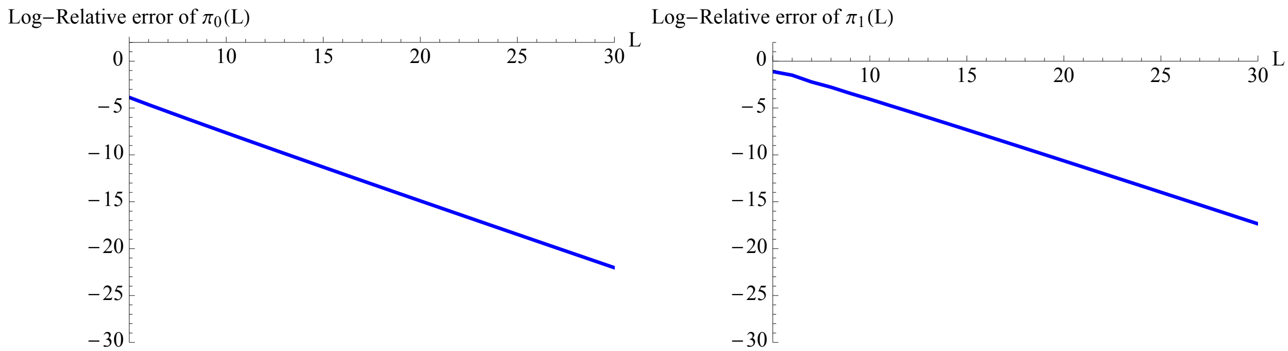
which implies that $N(t;p_1,\gamma_1) \geq_{fm} N(t;p_2,\gamma_2)$ for all $t \geq 0$.

Proof. The proof of the lemma becomes evident upon observing the form of the Laplace transform stated in equation (2.11). ∎



(a) $\pi_0(L)$ vs the truncation level. (b) $\pi_1(L)$ vs the truncation level.

Figure 2: Equilibrium distribution versus the truncation level, for $\lambda = 1$, $\nu = 1$, $\mu = 3$, $p = 0.5$ and $\gamma = 1$.



(a) Log-abs relative error of $\pi_0(L)$ vs the truncation level. (b) Log-abs relative error of $\pi_1(L)$ vs the truncation level.

Figure 3: Logarithmic scale for the absolute relative error versus the truncation level, for $\lambda = 1$, $\nu = 1$, $\mu = 3$, $p = 0.5$ and $\gamma = 1$.

6 Numerical results

In this section we present some numerical results that demonstrate the numerical efficiency of the formula (3.7) for the derivation of the equilibrium distribution and equation (2.12) for the calculation of the transient distribution. For numerical purposes we consider the corresponding truncated version of the equilibrium distribution and the transient distribution:

$$\pi_n(L) = \sum_{k=n}^{L} (-1)^{k-n} \binom{n}{k} \prod_{j=1}^{k} \frac{\nu + \lambda(j-1)}{(\mu - \lambda)j + \gamma(1 - p^j)}, \ n \geq 0, \tag{6.1}$$

$$\pi_n(t; L(t)) = \frac{1}{n!} \sum_{k=0}^{L(t)} m_{(k+n)}(t) \frac{(-1)^k}{k!}, \ n \geq 0, \tag{6.2}$$

where $m_{(1)}(t)$ and $m_{(k)}(t)$, $k > 1$, are given in (2.5) and (2.6), respectively.

In all numerical experiments we perform in this section we keep all but one parameters fixed and study the effect of the selected parameter in the performance of the model. To what follows, we assume that the birth rate is $\lambda = 1$, the immigration rate is $\nu = 1$, the death rate is $\mu = 3$, the survival probability is $p = 0.5$ and the catastrophe occurrence rate is $\gamma = 1$. We selected the parameters in such a way that the condition stated in remark 3.3 is satisfied, namely that $\lambda < \mu/2$.

For the selection of the appropriate truncation level we first define the absolute relative error as

$$\frac{|\pi_n(L) - \pi_n(L+1)|}{\pi_n(L+1)} \text{ and } \frac{|\pi_n(t;L(t)) - \pi_n(t;L(t)+1)|}{\pi_n(t;L(t)+1)}$$

and select the truncation level as

$$\min\{L : \frac{|\pi_n(L) - \pi_n(L+1)|}{\pi_n(L+1)} < 0.0005\} \text{ and } \min\{L(t) : \frac{|\pi_n(t;L(t)) - \pi_n(t;L(t)+1)|}{\pi_n(t;L(t)+1)} < 0.0005\}.$$

Using equation (6.1) we plot in Figure 2a the equilibrium probability for an empty system, $\pi_0(L)$, versus the truncation level, L, and in Figure 2b we plot $\pi_1(L)$ versus the truncation level, L. Moreover, in Figure 3a we plot the absolute relative error for the calculation of $\pi_0(L)$ versus the truncation level, L, and in Figure 3b we plot the absolute relative error for the calculation of $\pi_1(L)$ versus the truncation level, L. In order to demonstrate the rate of improvement in the approximation as a function of the truncation level, in Figures 3a and 3b, we have used a logarithmic scale.

In the following table we numerically calculate the equilibrium probabilities for various values of the survival probability p. As expected, based on Lemma 5.1 assertion c, as the survival probability increases the cumulative equilibrium distribution decreases.

p	π_0	π_1	π_2	π_3	π_4
0.1	0.752271	0.179233	0.0485400	0.0139684	0.00415731
0.3	0.735468	0.190005	0.0526917	0.0152718	0.00455538
0.5	0.717554	0.200171	0.0577526	0.0170853	0.00515113
0.7	0.698307	0.209695	0.0636868	0.0195302	0.00603887
0.9	0.677591	0.218343	0.0704395	0.0227497	0.00735521

We next numerically calculate the expectation and the variance of the number of individuals in steady state for various values of the survival probability p.

p	$\mathbb{E}[N]$	$Var[N]$
0.1	0.344828	0.364129
0.3	0.370370	0.384060
0.5	0.400000	0.408421
0.7	0.434783	0.438555
0.9	0.476190	0.476732

In Figure 4 we present the transient distribution as time evolves, in particular we plot the distribution for the states 0 and 1, under the assumption that originally the system was empty.

In Section 4 we obtained the relaxation time, in particular we showed that $\mathscr{R} = \frac{1}{\mu-\lambda+\gamma(1-p)}$. For the set of values used in this section, $\lambda = 1$, $\nu = 1$, $\mu = 3$, $p = 0.5$ and $\gamma = 1$, we obtain that $\mathscr{R} = 2/5$. The relaxation time reveals that the transient distribution converges rather quickly to its stationary counterpart. More specifically, in [1] the authors present a simple calculation of the time required for $\mathbb{E}[N(t)|N(0) = 0]$ to first be (and remain) within a q percentage of the equilibrium limit $m_{(1)}$, say t_q. Namely, expression (4.1) yields

$$\begin{aligned} m_{(1)} - m_{(1)}(t_q) &= \left(\frac{\nu}{\mu - \lambda + \gamma(1-p)} - m_{(1)}(0) \right) e^{-(\mu-\lambda+\gamma(1-p))t_q} \Leftrightarrow \\ q\, m_{(1)} &= \frac{\nu}{\mu - \lambda + \gamma(1-p)} e^{-(\mu-\lambda+\gamma(1-p))t_q}, \end{aligned}$$

which upon substitution of $m_{(1)} = \nu/\mu - \lambda + \gamma(1-p)$ reveals after some straight forward manipulations that

$$t_q = -\frac{\ln q}{\mu - \lambda + \gamma(1-p)}.$$

Hence, the time required to reach the equilibrium limit within a $q = 1\%$ percentage is approximately $t_q = 1.84207$, which is in absolute accordance with the behavior seen in figures 4a and 4b.

The effect of the birth rate λ, the immigration rate ν, the death rate μ, the survival probability p and the catastrophe occurrence rate γ on the equilibrium distribution of the number of individuals in the model, as well as on the equilibrium factorial moments appears to be exactly as stated in Lemma 5.1 and Corollary 5.1. For this reason, we select to present in Figure 5 the equilibrium distribution for states 0 and 1 versus the birth rate λ. As expected, as the birth rate increases, π_0 and π_1 decreases.

7 Variants of the general model

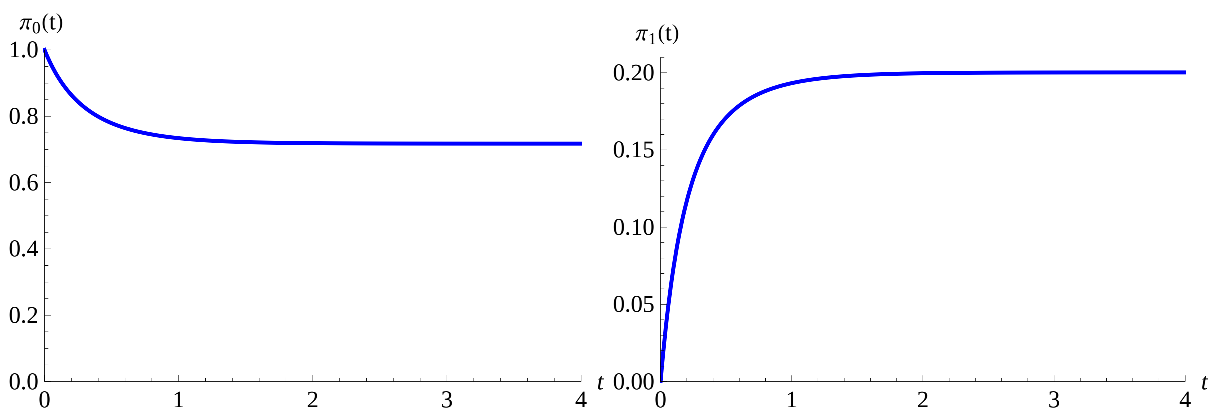
In this section we present several models, that can be considered as variants of the general model described in Section 1. We have tested the methodology presented in Section 3 for several models. In particular we considered separately the case of multiple binomial catastrophe mechanisms, the case of batch arrivals, the case of an underlying random environment and finally the case of emigration. In all but the last scenario we were able to obtain a stable recursive formula for the factorial moments and in these cases we were also able to reconstruct the equilibrium distribution from its factorial moments. The only exception we observed is the case of the birth/immigration-death/emigration process with binomial catastrophes, in this case the PGF depends on the unknown probability π_0, hence we can repeat the procedure we discussed in Section 3, however all factorial moments in this case will depend on π_0. In this scenario, we were able to numerically evaluate the unknown probability π_0, within a predefined relative error, and obtain all factorial moments.

7.1 Two or more binomial catastrophes

The procedure described in Section 3 is also successful in the case of multiple types of synchronized catastrophes. To this end, consider the continuous time Markov chain introduced in Section 1 with the additional characteristic of multiple types of binomial catastrophes, each with different catastrophe rate, γ_ξ, and different catastrophe probabilities $1 - p_\xi$, $\xi = 1, 2, \ldots, J$. The generator of the new CTMC $Q = \left(q_{ij}\right)_{i,j\in\mathbb{N}_0}$ is given as follows:

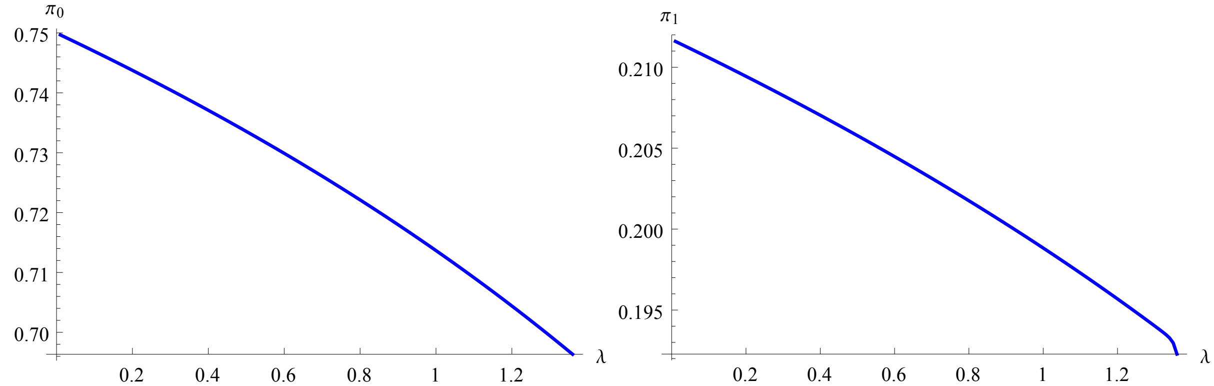
$$q_{ij} = \begin{cases} i\lambda + \nu, & \text{if } j = i+1,\ i \geq 0 \\ \binom{i}{j} p_1^j (1-p_1)^{i-j}\gamma_1 + \cdots + \binom{i}{j} p_J^j (1-p_J)^{i-j}\gamma_J + i\mu\,\delta_{j,i-1}, & \text{if } j = 0, 1, \ldots, i,\ i \geq 0. \end{cases}$$

Taking into account that all aforementioned rates are positive, we immediately deduce that the stochastic process $\{N(t) : t \geq 0\}$ is irreducible and regular. Regarding the stability condition, we can deduce, cf. Section 3.1, that $\lambda < \mu + \sum_{\xi=1}^{J} \gamma_\xi (1 - p_\xi)$ is a sufficient condition for the stability of the stochastic process.



(a) $\pi_0(t)$ (b) $\pi_1(t)$

Figure 4: Transient distribution versus time, for $\lambda = 1$, $\nu = 1$, $\mu = 3$, $p = 0.5$ and $\gamma = 1$.



(a) π_0 vs λ.

(b) π_1 vs λ.

Figure 5: Equilibrium distribution versus birth rate, for $\nu = 1$, $\mu = 3$, $p = 0.5$ and $\gamma = 1$.

Let π_n, $n \in \mathbb{N}_0$, be the equilibrium distribution of the CTMC, then, the balance equations of the model at hand read as follows:

$$(\nu + \sum_{j=1}^{J} \gamma_j)\pi_0 = \mu\pi_1 + \sum_{j=1}^{J} \gamma_j \sum_{k=0}^{\infty} (1-p_j)^k \pi_k, \tag{7.1}$$

$$(\nu + n\lambda + \sum_{j=1}^{J} \gamma_j + n\mu)\pi_n = (\nu + (n-1)\lambda)\pi_{n-1} + (n+1)\mu\pi_{n+1} + \sum_{j=1}^{J} \gamma_j \sum_{k=n}^{\infty} \binom{k}{n} p_j^n (1-p_j)^{k-n} \pi_k, \; n \geq 1. \tag{7.2}$$

There exists a unique non-trivial solution of the balance equations (7.1) and (7.2) plus the normalizing condition $\sum_{n=0}^{\infty} \pi_n = 1$.

Multiplying, now, both sides of the balance equations (7.1) and (7.2) with z^0 and z^n, respectively, and summing over all $n = 0, 1, \ldots$, yields after some straightforward calculations the following equation for the PGF, $\Pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$, $|z| \leq 1$,

$$(\nu(1-z) + \sum_{\xi=1}^{J} \gamma_\xi)\Pi(z) + (z-1)(\mu - \lambda z)\frac{d\,\Pi(z)}{dz} = \sum_{\xi=1}^{J} \gamma_\xi \Pi(1 - p_\xi + p_\xi z). \tag{7.3}$$

The factorial moments can then be found along the same lines as in Section 3, by considering the FMGF and equating the corresponding coefficients. Then, we obtain

$$m_{(k)} = k! \prod_{i=1}^{k} \frac{\nu + \lambda(i-1)}{i(\mu - \lambda) + \sum_{\xi=1}^{J} \gamma_\xi (1 - p_\xi^i)}, \; k \geq 1. \tag{7.4}$$

Moreover, the equilibrium distribution of the model is given by

$$\pi_n = \sum_{k=n}^{\infty} (-1)^{k-n} \binom{n}{k} \prod_{i=1}^{k} \frac{\nu + \lambda(i-1)}{i(\mu - \lambda) + \sum_{\xi=1}^{J} \gamma_\xi (1 - p_\xi^i)}, \; n \geq 0. \tag{7.5}$$

Note that the above formula is numerically stable and very efficient, this becomes apparent if we compare formula (7.5) with equation (3.7).

7.2 Batch arrivals

Consider a birth/immigration-death process with binomial catastrophes, in which immigration occurs in batches according to a Poisson process at rate ν. A batch includes i individuals with probability α_i, $i \geq 1$. Moreover, let $A(z) = \sum_{i=1}^{\infty} \alpha_i z^i$, $|z| \leq 1$, be the batch size PGF and $a_{(i)}$, $i \geq 1$, be the corresponding factorial moments. For the new CTMC we define the generator $Q = \left(q_{ij}\right)_{i,j\in\mathbb{N}_0}$, with

$$q_{ij} = \begin{cases} \nu\alpha_{j-i} + i\lambda\,\delta_{j,i+1}, & \text{if } j = i+1, i+2, \ldots, \ i \geq 0 \\ \binom{i}{j} p^j (1-p)^{i-j}\gamma + i\mu\,\delta_{j,i-1}, & \text{if } j = 0, 1, \ldots, i, \ i \geq 0. \end{cases}$$

Taking into account that all aforementioned rates are positive, we can immediately deduce that the stochastic process is irreducible and regular. Regarding the ergodicity of the CTMC we can deduce, cf. Section 3.1, that $\lambda < \mu + \gamma(1-p)$ is a sufficient condition for the stability of the stochastic process.

Let π_n, $n \in \mathbb{N}_0$, be the equilibrium distribution of the CTMC, then, the balance equations of the model at hand read as follows:

$$(\nu+\gamma)\pi_0 = \mu\pi_1 + \gamma\sum_{k=0}^{\infty}(1-p)^k\pi_k, \tag{7.6}$$

$$\begin{aligned}(\nu + n\lambda + \gamma + n\mu)\pi_n &= \nu\sum_{i=1}^{n}\alpha_i\pi_{n-i} + (n-1)\lambda\pi_{n-1} + (n+1)\mu\pi_{n+1} \\ &\quad + \gamma\sum_{k=n}^{\infty}\binom{k}{n}p^n(1-p)^{k-n}\pi_k, \ n \geq 1. \end{aligned}\tag{7.7}$$

There exists a unique non-trivial solution of the balance equations (7.6) and (7.7) plus the normalizing condition $\sum_{n=0}^{\infty}\pi_n = 1$.

Multiplying both sides of the balance equations (7.6) and (7.7) with z^0 and z^n, respectively, and summing over all $n = 0, 1, \ldots$, yields after some straight forward calculations the following equation for the PGF, $\Pi(z) = \sum_{n=0}^{\infty}\pi_n z^n$, $|z| \leq 1$,

$$[\nu(1-A(z)) + \gamma]\Pi(z) + (z-1)(\mu - \lambda z)\frac{d\,\Pi(z)}{dz} = \gamma\Pi(1-p+pz).$$

Proceeding as in Section 3 we can immediately obtain that the factorial moments satisfy the following recursion

$$m_{(k)} = \frac{1}{k(\mu-\lambda) + \gamma(1-p^k)}\left(\lambda k(k-1)m_{(k-1)} + \nu\sum_{i=0}^{k-1}\binom{k}{i}m_{(i)}a_{(k-i)}\right), \ k \geq 1.$$

The aforementioned recursive scheme for the factorial moments can be efficiently solved using numerical methods.

7.3 Random environment

In this section, we assume that our population model is regulated by a Markovian environment $\{E(t); t \geq 0\}$ on the state space $\{1, 2, \ldots, I\}$ whose infinitesimal generator $\boldsymbol{R}$ is given by

$$\boldsymbol{R} = \begin{pmatrix} -\widehat{r}_1 & r_{1,2} & \cdots & r_{1,I} \\ r_{2,1} & -\widehat{r}_2 & \cdots & r_{2,I} \\ \vdots & \vdots & \ldots & \vdots \\ r_{I,1} & r_{I,2} & \cdots & -\widehat{r}_I \end{pmatrix},$$

where $\widehat{r}_i = \sum_{j=1,\, j\neq i}^{I} r_{i,j}$. Moreover, let $\boldsymbol{p} = (p_1, p_2, \ldots, p_I)$ be the equilibrium distribution of the environment process, i.e., the unique solution of

$$\boldsymbol{p}\boldsymbol{R} = \boldsymbol{0} \ \text{ and } \ \boldsymbol{p}\boldsymbol{e} = 1,$$

where $\mathbf{0}$ is a row vector with all elements zero and $\boldsymbol{e}$ is a column vector with all elements one.

When the random environment is in state i, the immigration rate, the birth rate, the death rate, the catastrophe rate and the catastrophe probability are given by ν_i, λ_i, μ_i, γ_i and p_i, respectively, for $i = 1, 2, \ldots, I$.

Taking into account that all aforementioned rates are positive, we can immediately deduce that the stochastic process is irreducible and regular. Regarding the ergodicity of the CTMC we can deduce, cf. Section 3.1 and [28, 29], that $\sum_{i=1}^{I} p_i\lambda_i < \sum_{i=1}^{I} p_i\mu_i + \sum_{i=1}^{I} p_i\gamma_i(1-p_i)$ is a sufficient condition for the stability of the stochastic process, where $(p_1, p_2, \cdots, p_I)$ is the equilibrium distribution of the random environment.

Let $\pi_{i,n}$ denote the joint equilibrium probability that the environment is in phase i and that there are n individuals. It is easy to see that the number of individuals and the environment form a Markov chain on the state space $S = \{(i,n) : i \in \{1, 2, \ldots, I\},\ n \in \{0, 1, 2, \ldots\}\}$.

Then, the balance equations for the model read as follows

$$\begin{aligned}(\nu_i + \gamma_i + \widehat{r}_i)\pi_{i,0} &= \mu_i\pi_{i,1} + \gamma_i\sum_{k=0}^{\infty}(1-p_i)^k\pi_{i,k} \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{I} r_{j,i}\pi_{j,0},\ i = 1, 2, \ldots, I, \qquad (7.8)\\ (\nu_i + n(\lambda_i + \mu_i) + \gamma_i + \widehat{r}_i)\pi_{i,n} &= (\nu_i + \lambda_i(n-1))\pi_{i,n-1} + \mu_i(n+1)\pi_{i,n+1} \\ &\quad + \gamma_i\sum_{k=0}^{\infty}\binom{n+k}{k}(1-p_i)^k p_i^n\pi_{i,n+k} \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{I} r_{j,i}\pi_{j,n},\ n \geq 1,\ i = 1, 2, \ldots, I. \qquad (7.9)\end{aligned}$$

We define the partial PGFs as

$$\Pi_i(z) = \sum_{n=0}^{\infty}\pi_{i,n}z^n,\ |z| \leq 1,\ i = 1, 2, \ldots, I.$$

Multiplying the balance equations (7.8) and (7.9) with z^0 and z^n, respectively, and summing over all $n = 0, 1, \ldots$ yields

$$\begin{aligned}(\mu_i - \lambda_i z)(z-1)\Pi_i'(z) &= \nu_i(z-1)\Pi_i(z) + \gamma_i(\Pi_i(1-p_i+p_iz) - \Pi_i(z)) \\ &\quad + \sum_{j\neq i} r_{j,i}\Pi_j(z) - \widehat{r}_i\Pi_i(z),\ i = 1, 2, \ldots, I. \qquad (7.10)\end{aligned}$$

We adopt a matrix notation and we rewrite equation (7.10) as follows

$$\frac{d\,\mathbf{\Pi}(z)}{dz}(z-1)(\boldsymbol{\mu} - z\boldsymbol{\lambda}) - \mathbf{\Pi}(z)[(z-1)\boldsymbol{\nu} - \boldsymbol{\gamma} + \boldsymbol{R}] = \mathbf{\Pi}_{\boldsymbol{p}}(z)\boldsymbol{\gamma}, \qquad (7.11)$$

where

$$\begin{aligned}\boldsymbol{x} &= diag(x_1, x_2, \ldots, x_I),\\ \mathbf{\Pi}(z) &= (\Pi_1(z), \Pi_2(z), \ldots, \Pi_I(z)),\\ \mathbf{\Pi}_{\boldsymbol{p}}(z) &= (\Pi_1(1-p_1+p_1z), \Pi_2(1-p_2+p_2z), \ldots, \Pi_I(1-p_I+p_Iz)).\end{aligned}$$

We define

$$\boldsymbol{M}(z) = \sum_{k=0}^{\infty}\boldsymbol{m}_{(k)}\frac{z^k}{k!} = \mathbf{\Pi}(z+1).$$

Setting $z := z + 1$ in (7.11) and equating the coefficients of the difference-differential equation satisfied by the partial factorial moment generating function immediately yields

$$\boldsymbol{m}_{(k)}\boldsymbol{A}_k = \boldsymbol{m}_{(k-1)}\boldsymbol{B}_k, \qquad k = 1, 2, \ldots, \tag{7.12}$$

where

$$\begin{aligned} \boldsymbol{A}_k &= k(\boldsymbol{\mu} - \boldsymbol{\lambda}) + diag(1 - p_1^k, 1 - p_2^k, \ldots, 1 - p_K^k)\boldsymbol{\gamma} - \boldsymbol{R}, \\ \boldsymbol{B}_k &= k(\boldsymbol{\nu} + (k-1)\boldsymbol{\lambda}). \end{aligned}$$

It should be remarked that $\boldsymbol{m}_{(0)} = \boldsymbol{p}$, where $\boldsymbol{p}$ is the equilibrium distribution of the environment process.

Furthermore, in order to solve the system stated in equation (7.12) it is necessary that $\det(\boldsymbol{A}_k) \neq 0$ for all $k \geq 0$. This will provide us with the necessary condition for the calculation of the factorial moments. We can use then all moments to construct the equilibrium distribution in a similar manner as in Section 3.2. A different approach can be found in [38], in which an algorithmic approach is suggested for the study of Markovian trees subject to catastrophes using a G/M/1 type of modeling with level dependent structure. Moreover, the numerical approaches developed in [26, 45, 46] can be used to derive the equilibrium distribution.

7.4 Emigration

Consider a birth/immigration-death/emigration process with binomial catastrophes. For the new CTMC we define the generator $Q = \left(q_{ij}\right)_{i,j\in\mathbb{N}_0}$, with

$$q_{ij} = \begin{cases} i\lambda + \nu, & \text{if } j = i+1,\ i \geq 0 \\ \binom{i}{j} p^j (1-p)^{i-j}\gamma + (i\mu + \theta)\,\delta_{j,i-1}, & \text{if } j = 0, 1, \ldots, i,\ i \geq 0. \end{cases}$$

Note that, in this analysis we assume that the emigration rate is independent of the population size, however the analysis that follows can be easily extended to the case that the emigration rate is a linear function of the population size.

The transition diagram of the process is depicted in Figure 6. Taking into account that all aforementioned rates are positive, we can immediately deduce that the CTMC is irreducible and regular. Regarding the ergodicity of the CTMC we can deduce, cf. Section 3.1, that $\lambda < \mu + \gamma(1-p)$ is a sufficient condition for the stability of the stochastic process.

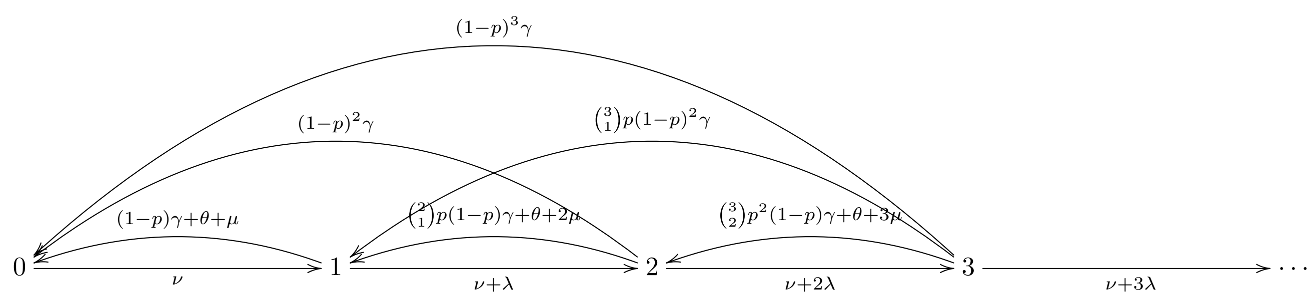


Figure 6: Transition rate diagram of $\{N(t) : t \geq 0\}$.

Let π_n, $n \geq 0$, be the equilibrium distribution of the CTMC, then, the balance equations of the model at hand read as follows:

$$(\nu+\gamma)\pi_0 = (\theta+\mu)\pi_1+\gamma\sum_{k=0}^{\infty}(1-p)^k\pi_k, \tag{7.13}$$

$$(\nu+n\lambda+\theta+\gamma+n\mu)\pi_n = (\nu+(n-1)\lambda)\pi_{n-1}+(\theta+(n+1)\mu)\pi_{n+1} +\gamma\sum_{k=n}^{\infty}\binom{k}{n}p^n(1-p)^{k-n}\pi_k. \tag{7.14}$$

Under the stability condition, there exists a unique non-trivial solution of the balance equations (7.13) and (7.14) plus the normalizing condition $\sum_{n=0}^{\infty}\pi_n=1$.

Multiplying the balance equations (7.13) and (7.14) with z^0 and z^n, respectively, and summing over all $n=0,1,\dots$ yields after some straightforward calculations the following equation for the PGF, $\Pi(z)=\sum_{n=0}^{\infty}\pi_n z^n,\ |z|\le 1$,

$$[(\nu+\theta+\gamma)z-\nu z^2-\theta]\Pi(z)+z(z-1)(\mu-\lambda z)\frac{d\,\Pi(z)}{dz}=\theta(z-1)\pi_0+\gamma z\Pi(1-p+pz). \tag{7.15}$$

Proceeding as in Section 3 we can immediately obtain that the factorial moments satisfy the following recursion for $k=1$

$$(\gamma(1-p)+\mu-\lambda)m_{(1)}+\theta m_{(0)}=\nu m_{(0)}+\theta\pi_0, \tag{7.16}$$

and, for $k\ge 2$,

$$(\gamma(1-p^k)+k(\mu-\lambda))m_{(k)}-(\nu-\gamma(1-p^{k-1})-(k-1)(\mu-2\lambda)-\theta)(k-1)m_{(k-1)} -(\nu+(k-2)\lambda)k(k-1)m_{(k-2)}=0. \tag{7.17}$$

For convenience we define $m_{(k)}=k!\,\xi_{(k)}$, then the above system of equations is rewritten as follows

$$(\gamma(1-p)+\mu-\lambda)\xi_{(1)}+\theta\xi_{(0)}=\nu\xi_{(0)}+\theta\pi_0,$$

and, for $k\ge 2$,

$$(\gamma(1-p^k)+k(\mu-\lambda))\xi_{(k)}-(\nu+(k-2)\lambda)\xi_{(k-2)}-(\nu-\gamma(1-p^{k-1})-(k-1)(\mu-2\lambda)-\theta)\xi_{(k-1)}=0.$$

Thus, we have

$$\xi_{(k)}=c_k^{(0)}\xi_{(k-2)}+c_k^{(1)}\xi_{(k-1)},\ k\ge 2, \tag{7.18}$$

where

$$c_k^{(0)}=\frac{\nu+(k-2)\lambda}{\gamma(1-p^k)+k(\mu-\lambda)},\qquad c_k^{(1)}=\frac{\nu-\gamma(1-p^{k-1})-(k-1)(\mu-2\lambda)-\theta}{\gamma(1-p^k)+k(\mu-\lambda)},$$

for $k\ge 2$.

We observe that equation (7.18) forms a second order difference equation. Furthermore, from the normalization equation we have that $\xi_{(0)}=1$, hence we are missing one more equation to solve the system. To this end, we observe that iterating formula (7.18) yields

$$\xi_{(k)}=\xi_{(k)}^{(0)}+\xi_{(k)}^{(1)}\pi_0,\ k\ge 0, \tag{7.19}$$

where $\xi_k^{(0)}$ and $\xi_k^{(1)}$ are two sequences defined by the following second order recurrence relations

$$\xi_{(k)}^{(0)}=c_k^{(0)}\xi_{(k-2)}^{(0)}+c_k^{(1)}\xi_{(k-1)}^{(0)},\ k\ge 2, \tag{7.20}$$

$$\xi_{(k)}^{(1)}=c_k^{(0)}\xi_{(k-2)}^{(1)}+c_k^{(1)}\xi_{(k-1)}^{(1)},\ k\ge 2, \tag{7.21}$$

where

$$\xi_{(0)}^{(0)}=1,\qquad \xi_{(1)}^{(0)}=\frac{\nu-\theta}{\gamma(1-p)+\mu-\lambda},\qquad \xi_{(0)}^{(1)}=0,\qquad \xi_{(1)}^{(1)}=\frac{\theta}{\gamma(1-p)+\mu-\lambda}.$$

In order now to fully determine $\xi_{(k)}$, $k \geq 1$, we still need to calculate the unknown probability π_0.

Keeping in mind that

$$\Pi(z) = \sum_{k=0}^{\infty} \xi_{(k)}(z-1)^k = \sum_{k=0}^{\infty} \xi_{(k)}^{(0)}(z-1)^k + \pi_0 \sum_{k=0}^{\infty} \xi_{(k)}^{(1)}(z-1)^k,$$

we set $z = 0$ and obtain

$$\pi_0 = \Pi(0) = \sum_{k=0}^{\infty} \xi_{(k)}(-1)^k = \sum_{k=0}^{\infty} \xi_{(k)}^{(0)}(-1)^k + \pi_0 \sum_{k=0}^{\infty} \xi_{(k)}^{(1)}(-1)^k,$$

leading to

$$\pi_0 = \frac{\sum_{k=0}^{\infty} \xi_{(k)}^{(0)}(-1)^k}{1 - \sum_{k=0}^{\infty} \xi_{(k)}^{(1)}(-1)^k}. \tag{7.22}$$

Once π_0 is determined all the factorial moments are also obtained by (7.19).

7.4.1 Numerical results

In this subsection we briefly demonstrate the numerical efficiency of calculating the factorial moments satisfying the recursion presented in equations (7.16) and (7.17).

To what follows, we assume that the birth rate is $\lambda = 1$, the immigration rate is $\nu = 1$, the death rate is $\mu = 3$, the emigration rate is $\theta = 1$, the survival probability is $p = 0.5$ and the catastrophe occurrence rate is $\gamma = 1$.

Note that in order to calculate the factorial moments, which will permit the numerical evaluation of the equilibrium distribution, it is imperative to first numerically evaluate the probability of empty system, π_0, as given by equation (7.22). In Figure 7 we demonstrate that we can truncate the two series involved in the calculation of π_0 at very small values and still obtain extremely accurate results. In particular we define

$$\pi_0(L) = \frac{\sum_{k=0}^{L} \xi_{(k)}^{(0)}(-1)^k}{1 - \sum_{k=0}^{L} \xi_{(k)}^{(1)}(-1)^k},$$

and plot $\pi_0(L)$ versus the truncation level, L. Moreover we define the absolute relative error as

$$\frac{|\pi_0(L) - \pi_0(L+1)|}{\pi_0(L+1)}$$

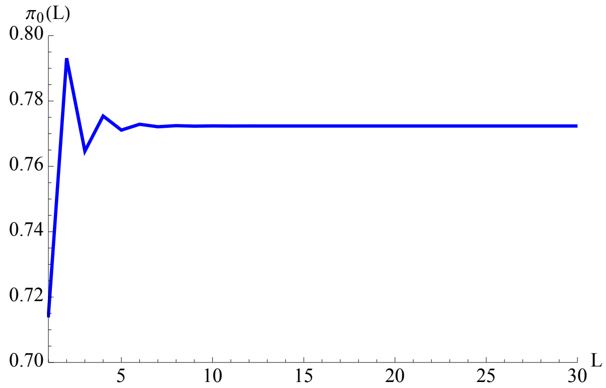
and select the truncation level as

$$\min\{L : \frac{|\pi_0(L) - \pi_0(L+1)|}{\pi_0(L+1)} < 0.0005\}.$$

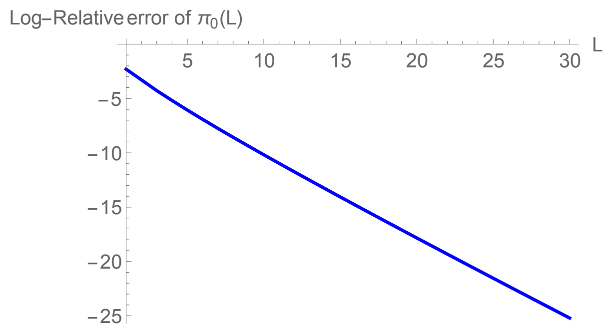
Note that, for the values of the parameters we have selected, it turns out that if $L = 7$ then the absolute value of the relative error performed in the calculation of $\pi_0(L)$ is 0.00043.

In addition, we present the values of the first ten factorial moments. We select $\lambda = 1$, $\nu = 1$, $\mu = 3$, $\theta = 1$, $p = 0.5$ and $\gamma = 1$.

k	$m_{(k)}/k!$	k	$m_{(k)}/k!$
1	0.308939853	6	0.003487042
2	0.112966362	7	0.001556596
3	0.044686867	8	0.000704202
4	0.018544053	9	0.000322018
5	0.007948599	10	0.000148496



(a) $\pi_0(L)$.



(b) Log-abs relative error of $\pi_0(L)$

Figure 7: Probability of empty system, $\pi_0(L)$, and logarithmically scaled absolute relative error versus the truncation level L, for $\lambda = 1$, $\nu = 1$, $\mu = 3$, $\theta = 1$, $p = 0.5$ and $\gamma = 1$.

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