Lecture 6 241018

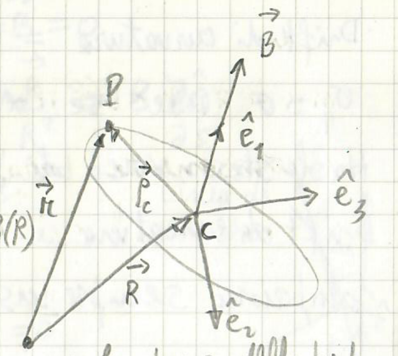
- Il pdf delle lezioni puo' essere scaricato da
- http://www.fisgeo.unipg.it/~fiandrin/didattica_fisica/cosmic_rays1819/

Motion in magnetic fields

ii) $\vec{\nabla}_{\parallel} B$

$\vec{r} = \vec{\rho}_c + \vec{R}$ *

Sviluppo il campo $\vec{B}$ in serie $\vec{B}(\vec{r}) = \vec{B}(\vec{R}) + \vec{\rho}_c \cdot \vec{\nabla} B(\vec{R})$



Sost. nell'equ.

$m\ddot{\vec{r}} = q\vec{v} \times \vec{B}$

* la posiz. della part. è espressa in termini della posizione di C e del raggio ist. di curvatura rispetto a C

$m(\ddot{\vec{R}} + \ddot{\vec{\rho}}_c) =$

$= q(\dot{\vec{R}} + \dot{\vec{\rho}}) \times [\vec{B}(\vec{R}) + \vec{\rho}_c \cdot \vec{\nabla}\vec{B}(\vec{R})]$

con la condizione che $|\rho \cdot \nabla B| \ll 1$

Prendo il val. medio in un periodo di Larmor

Si fa vedere che $\langle \rho \rangle = \langle \dot{\rho} \rangle = \langle \ddot{\rho} \rangle = 0$ (*)(*)

L'unico termine che contribuisce $\neq 0$ è   Vedi avanti

$\dot{\vec{\rho}} \times (\vec{\rho} \cdot \vec{\nabla}) B = \frac{1}{2}\Omega \rho^2 [\hat{e}_2 \times (\hat{e}_3 \cdot \vec{\nabla}) - \vec{e}_3 \times (\hat{e}_2 \cdot \vec{\nabla})] \hat{e}_1 B$

$= \frac{1}{2}\omega \rho^2 [(\hat{e}_3 \cdot \vec{\nabla} B)(\vec{e}_2 \times \vec{e}_1) - (\vec{e}_2 \cdot \vec{\nabla} B)(\hat{e}_3 \times \hat{e}_1)]$

$= \frac{1}{2}\omega \rho^2 [-(\hat{e}_3 \cdot \vec{\nabla} B)\hat{e}_3 - (\hat{e}_2 \cdot \vec{\nabla} B)\hat{e}_2]$

$= -\frac{1}{2}\omega \rho^2 q \vec{\nabla} B$

Quindi

$m\ddot{\vec{R}} = q[\dot{\vec{R}} \times \vec{B}(\vec{R})] - \frac{1}{2}\omega \rho_c^2 \vec{\nabla} B + \cdots$

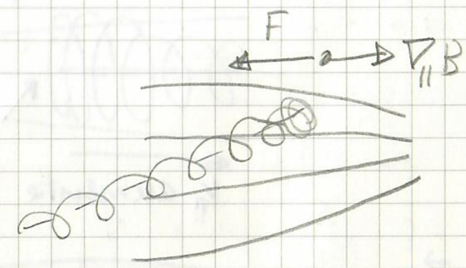
La componente parallela di $\ddot{\vec{R}}$

$m\ddot{\vec{R}} \cdot \hat{e}_1 = \underbrace{q[\dot{\vec{R}} \times \vec{B}(\vec{R})] \cdot \hat{e}_1}_{=0 \text{ perché } \perp} - \frac{1}{2}\omega^2 \rho_c \underbrace{(\vec{\nabla} B) \cdot \hat{e}_1}_{\vec{\nabla}_{\parallel} B}$

$\Rightarrow \boxed{m\frac{d\vec{v}_{\parallel}}{dt} = -\frac{1}{2}\omega^2 \rho_c \vec{\nabla}_{\parallel} B = -\frac{1}{2}\frac{v_\perp^2}{B}(\vec{\nabla}_{\parallel} B)}$

Il centro di guida è accelerato nel verso opposto al gradiente del campo.

Se si muove verso regioni con campo più forte, verrà respinta*, indipendentemente dal segno della carica o dalla direz. di B.



**più precisamente decelerata*

Motion in magnetic fields

Come conseguenze si ha inversione del moto e variazione di angolo di pitch α

Infatti $v^2 = v_\parallel^2 + v_\perp^2 = \text{cost.}$ perché l'unica forza che agisce è quella di Lorentz

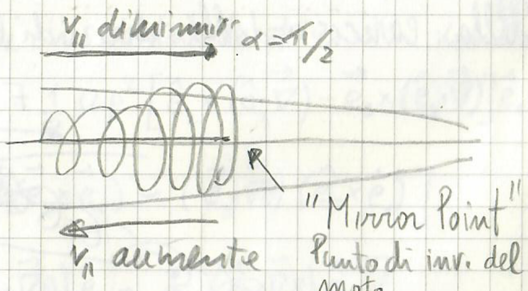
Quindi ~~sin~~ $\cos\alpha = \frac{v_\parallel}{v} \to 0$

quando $v_\parallel \to 0$

Man mano che la particella avanza nel grad. $v_\parallel$ diminuisce e α aumenta

Quando $v_\parallel = 0 \to v_\perp = v$ e $\alpha = \pi/2$ e la forza F è $F = -\frac{1}{2} v^2 \left(\frac{\nabla_\parallel B}{B}\right)$ max

La particella inverte il moto



Se

si può avere una trappola magnetica

⊛ Infatti nel piano ⊥ a B, ~~[illegible]~~

⊛ nel rif. del centro di guida con assi $\hat e_1, \hat e_2, \hat e_3$

$$\vec\rho_c = \rho(\hat e_2 \sin\Omega t + \hat e_3 \cos\Omega t)$$

e quindi

$$\dot{\vec\rho} = \Omega\rho(\hat e_2\cos\Omega t - \hat e_3\sin\Omega t) + \sin\Omega t\frac{d}{dt}(\rho\hat e_2) + \cos\Omega\frac{d}{dt}(\rho\hat e_3)$$

$$\ddot{\vec\rho} = \Omega^2\rho(-\hat e_2\sin\Omega t - \hat e_3\cos\Omega t) + \dot\Omega\rho(\hat e_2\cos\Omega t - \hat e_3\sin\Omega t) + 2\Omega\cos\Omega t\frac{d}{dt}(\rho\hat e_2) - 2\Omega\sin\Omega t\frac{d}{dt}(\rho\hat e_3) + \sin\Omega t\frac{d^2}{dt^2}(\rho\hat e_2) + \cos\Omega t\frac{d^2}{dt^2}(\rho\hat e_3)$$

Dato che $\langle\sin\Omega t\rangle = \langle\cos\Omega t\rangle = 0 \Rightarrow \langle\rho\rangle = \langle\dot\rho\rangle = \langle\ddot\rho\rangle = 0$

I termini con $\dot{\vec\rho} \times (\vec\rho\cdot\vec\nabla)$ contengono sì termini con $\sin^2\Omega t$ e $\cos^2\Omega t$ che danno $\langle\ \rangle \neq 0$

Motion in magnetic fields

Pitch angle scattering

Quando il campo presenta irregolarità, generati con una distribuzione casuale, può essere rappresentato come

$$B(\vec{r},t) = \vec{B}_0 + \delta\vec{B}(\vec{r},t)$$

Camp medio — Irregolarità random in $\vec{r}$ e t

La densità di energia è $u = \frac{B^2}{8\pi}$

Siamo interessati ai valori medi (p. es. nel tempo) o su tutti

$$\langle u \rangle = \frac{\langle B^2 \rangle}{8\pi}$$

$$B^2 = \vec{B}\cdot\vec{B} = B_0^2 + \delta B^2 + 2\vec{B}_0\cdot\delta\vec{B}$$

$$\langle B^2\rangle = \langle B_0\rangle + \langle \delta B^2\rangle + \underbrace{2\vec{B}_0\cdot\langle\delta\vec{B}\rangle}_{=0}$$

Scriviamo il campo $\delta\vec{B}$ come sovrapposiz. di onde piane

$$\delta\vec{B} = \frac{1}{(2\pi)^{3/2}}\int d^3k\, \delta\tilde{B}(k)\, e^{i\vec{k}\cdot\vec{r}}$$

Perciò

$$\delta B^2 = \frac{1}{(2\pi)^3}\int d^3k\, d^3k'\, \delta\tilde{B}(k)\,\delta\tilde{B}(k')\, e^{i\vec{r}\cdot(\vec{k}-\vec{k}')}$$

L'energia totale del campo è

$$E_T = \int d^3r\, u = \frac{1}{8\pi}\int d^3r\, \delta B^2 =$$

$$= \frac{1}{(2\pi)^3}\int d^3r\, d^3k\, d^3k'\, \frac{\delta B_k\, \delta B_{k'}}{8\pi}\, e^{i\vec{r}\cdot(\vec{k}-\vec{k}')}$$

L'integr. su d^3r dà $\delta^3(k-k')(2\pi)^3$

$$E_T = \frac{1}{8\pi}\int \delta B^2(\vec{k})\, d^3k = \frac{1}{8\pi}\int k^2\delta B^2(k)\, dk\, d\Omega$$

$$w(k) = \frac{1}{8\pi}\int k^2 \delta B^2(\vec{k})\, d\Omega \qquad [T^2/m^{-1}]$$

è la densità spettrale di potenza

Quindi $E_T = \int w(k)\, dK$

NB: al vett. K è associata $\omega = kV$ con V = vel. di propag. dell'onda e $K = 2\pi/\lambda$

Motion in magnetic fields

· $W(k)$ da l'energia associata a ciascuna comp. di Fourier

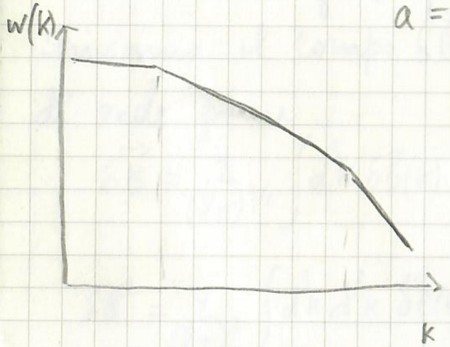
· A ciascuna componente k è associata una λ che definisca l'ampiezza della regione spaziale su cui l'onda agisca, i.e. la "scala" tipica dell'onda componente

Di solito $W(k) \propto k^{\alpha}$ con l'indice α che in generale dipende da k –

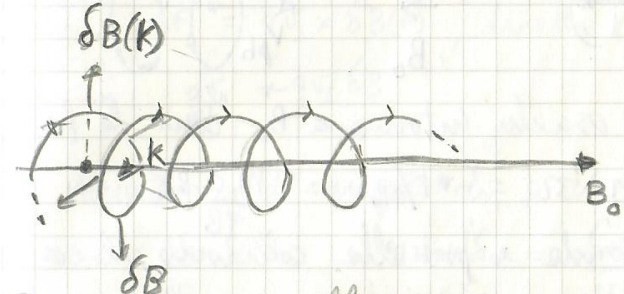
La distrib. $W(k)$ dipende dal tipo di turbolenza MHD che determina la distribuzione delle irregolarità

$\alpha = a - 2$

$$a = \begin{cases} 1/3 & \text{Kolmogorov} \\ 1/2 & \text{Kraichnan} \\ 0 & \text{Random disc.} \\ 1 & \text{white noise} \end{cases}$$

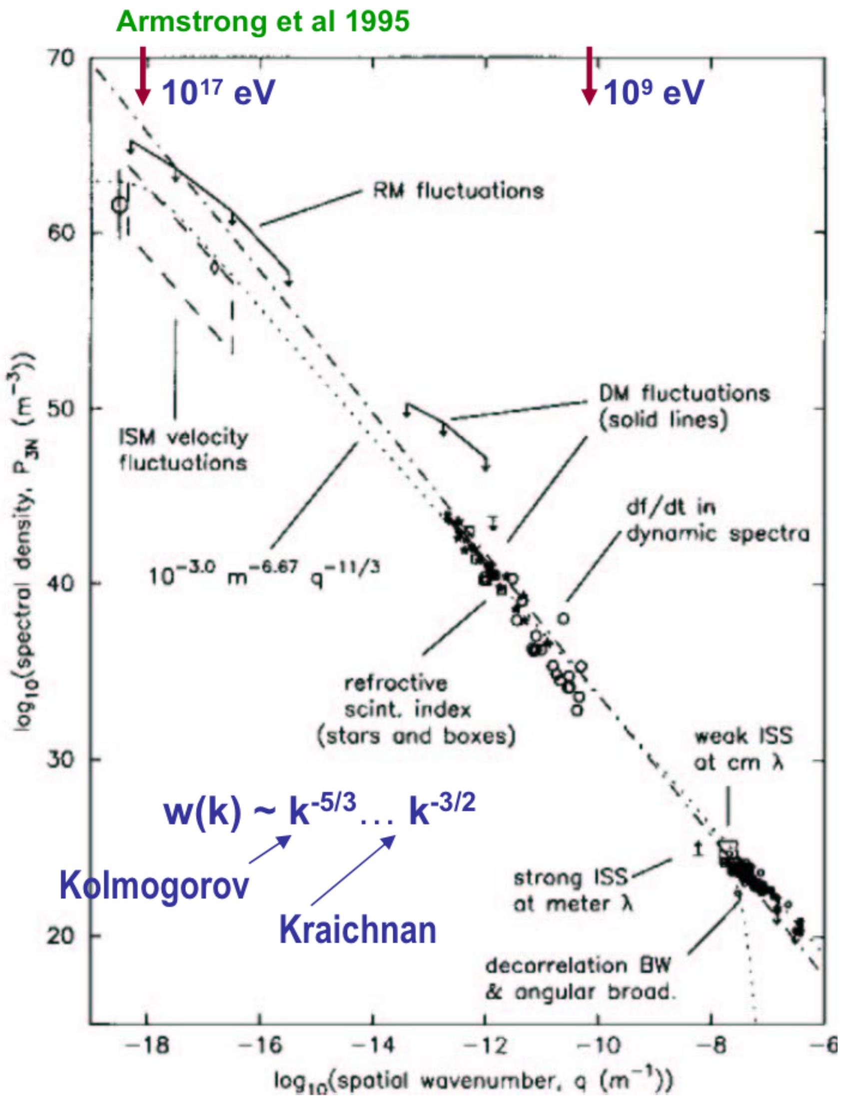


Il campo è quindi rappresentato come la sovrappos. di onde con numero d'onda k e freq. $\omega = \omega(k)$, ciascuna delle quali trasporta una energia $\sim W(k)\,dk$ –



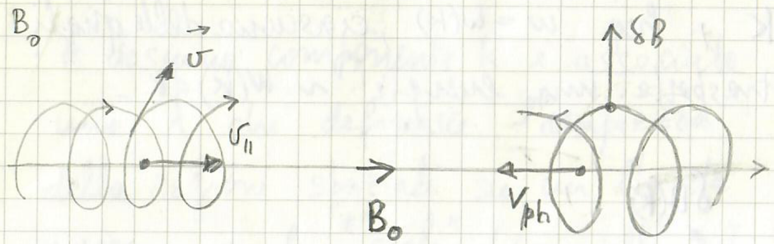
Possiamo, a scopo illustrativo, considerare una $\vec{\delta B} \perp B_0$ che si propaga lungo una linea del campo medio $\vec{B}_0$ –
In tal caso δB "ruota" intorno alla direzione di B_0 con frequenza ω

interstellar turbulence



Motion in magnetic fields

Supponiamo ora che una particella viaggi con velocità $\vec{v}$ t.c. abbia una comp. $v_\parallel$ lungo



La particella oscilla intorno a B_0 con la frequ. di Larmor $\Omega = p/qB = \frac{m_0 c}{qB}\beta\gamma$

Quando onda e particella collidono, il campo $\vec{\delta B}$ esercita una forza di Lorentz sulla particella

Dato che part. e onda si muovono l'una verso l'altra, la frequ. dell'onda "vista" dalla part. subisce uno spost. Doppler $\omega' = \gamma(\omega - \vec{k}\cdot\vec{v})$

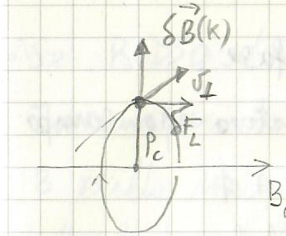
∴ Per semplicità supponiamo $v \ll c \Rightarrow \gamma \simeq 1$

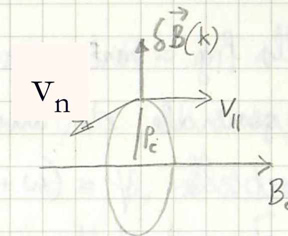
Dato che $\vec{k}$ e $\vec{v}$ hanno versi opposti

$\omega' \approx \omega + k v_\parallel$

Per facilitare la visualizz., separiamo in comp $V_\perp$ e $V_\parallel$ della vel. della particella

Nel caso di angoli di pitch arbitrari l'interazione è una combinazione delle due componenti





Prendiamo il caso di $v_\perp$.

Forza $\delta\vec{F}_L = q\vec{v}_\perp \times \vec{\delta B}(k)$, diretta come $\vec{B}_0$

Per cui $\frac{dv_\parallel}{dt} = \frac{q v_\perp \delta B}{m}$

La variazione di angolo di pitch

$$\cos\alpha = \frac{v_\parallel}{v} \Rightarrow -\sin\alpha\,\delta\alpha = \frac{\delta v_\parallel}{v} \quad (v = \text{cost.})$$

$$\Rightarrow \delta\alpha = -\frac{\delta v_\parallel}{v\sin\alpha} = -\frac{\delta v_\parallel}{v_\perp}$$

Perciò $\delta\alpha = -\frac{q \cancel{v_\perp} \delta B\,\delta t}{m \cancel{v_\perp}} = -\left(\frac{qB_0}{m}\right)\left(\frac{\delta B}{B_0}\right)\delta t$

$\Omega = \frac{qB_0}{m}$ freq. di Larmor

$$\delta\alpha = -\Omega\,\delta t\left(\frac{\delta B}{B_0}\right)$$

δt è l'intervallo di tempo in cui particella e campo δB sono in fase e $\Omega\delta t$ è la fraz. di tempo che sono in fase in un periodo di Larmor

Motion in magnetic fields

Nella fig. part. e onde sono in fase.

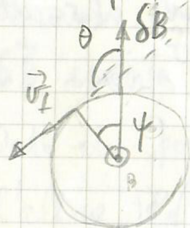
In generale c'è una fase relativa tra campo e particelle $\psi = (\omega + k v_\parallel - \Omega)t + \phi$

che dipende dalla ~~fase~~ frequ. relativa

$$R = \omega + k v_\parallel - \Omega$$

Se $R \neq 0$ $\psi = \psi(t)$, la diff. di fase varia nel tempo

Se $R = 0$ $\psi = \phi$ = cost., la diff. di fase non dipende dal tempo



La forza di Lorentz è

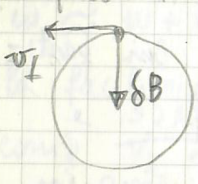
$$\delta \vec{F} = q \vec{v}_\perp \times \delta \vec{B} \quad \text{cioè}$$

$$\delta F = q v_\perp \delta B \sin\theta$$

Ma $\theta = \psi + \pi/2 \Rightarrow$

$$\delta F = q v_\perp \delta B \cos\psi$$

Se $\psi = \pi$ si ottiene una diffusione opposta a quella in cui $\psi = 0$ perché la forza di Lorentz

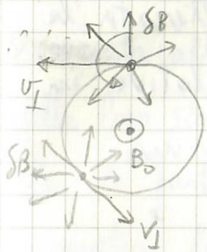


ha verso opposto al precedente

• Se $R \gg 0$ $\omega' = \omega + k v_\parallel \gg \Omega$. $R \approx \omega'$

In un periodo di Larmor $1/\Omega$, il campo B oscilla molte volte intorno a $B_0 \Rightarrow$ Assume tutte le possibili orientazioni rispetto a $\vec{v}_\perp$

per cui $\langle \delta F \rangle = 0$.

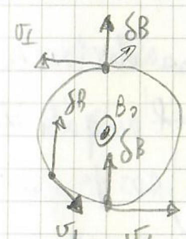


La particella vede un campo alla freq. ω'; dato che $\omega' \gg \Omega$ il valore medio di $\cos\psi(t)$ in un periodo di Larmor è

$$\langle \cos\psi \rangle_{Larmor} = \frac{1}{T} \int_0^T \cos\omega' t \, dt \approx 0$$

• Se $R \ll 0$ $\omega' \ll \Omega$ e $R \approx -\Omega$

In tal caso, in un periodo di Larmor, la fase dell'onda varia di poco, cioè la dir. di $\delta \vec{B}$ è $\approx$ fissa mentre $\vec{v}_\perp$ le ruota intorno



$\Rightarrow v_\perp$ assume tutte le possibili orientazioni rispetto a δB

In tal caso

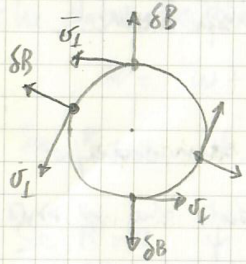
$$\langle \cos\psi \rangle_{Larmor} \approx \frac{1}{T} \int_0^T \cos \Omega t \, dt$$

$$= \Omega \int_0^T \cos \Omega t \, dt = 0$$

Motion in magnetic fields

Perciò se $|R| \gg 0$, in un periodo la forza netta è ≈ 0 e non c'è deflessione in α

$\therefore \quad R = \omega - k v_{\parallel} - \Omega = 0$



$\Rightarrow \psi(t) = \phi = \text{cost.}$

$\vec{\delta B}$ e particelle sono in fase, mantengono inalterata le loro orientaz. relative in un periodo di Larmor

Allora $\delta F = q v_\perp \delta B \cos\phi = \text{cost.}$ e

$$\delta\alpha = -\Omega\,\delta t \left(\frac{\delta B}{B_0}\right)\cos\phi \quad \text{in un tempo } \delta t$$

$$\delta\alpha_{Larmor} = \int \delta\alpha = -\int_0^{1/\Omega} \Omega\,\delta t \left(\frac{\delta B}{B_0}\right)\cos\phi$$

$$= \left(\frac{\delta B}{B_0}\right)\cos\Phi$$

In un periodo di Larmor o su una distanza $k = 2\pi/\lambda$, la particella cambia l'angolo di pitch di una quantità netta $\left(\frac{\delta B}{B_0}\right)\cos\phi$

La fase relativa ϕ determina l'intensità della deflessione. Se ϕ è random, anche $\delta\alpha$ lo è

- Dal dominio della frequ. si può passare a quello "spaziale" dei numeri d'onda $k = 2\pi/\lambda$ utilizzando $\omega = k V_{ph} \Rightarrow \lambda = \frac{2\pi V_{ph}}{\omega}$ che definisce la scala fisica su cui agisce la comp $\delta B(k)$.

Lo spettro di potenza $W(k)$ fornisce l'energia associata alle irregolarità magnetiche in funzione della loro scala fisica

- Per le particelle che si propagano nel campo magnetico, la scala associata alla frequ. di Larmor è il raggio di Larmor ρ_g.

La condizione di risonanza di ciclotrone si traduce nella condizione $\rho_g \sim \lambda$ per avere interazione significativa tra onda e particella.

$$\rho_g = \frac{p}{m\omega}$$

Se $\omega \ll \Omega \rightarrow \rho_g \gg \lambda$ la particella non interagisce in media con le irregolarità e segue il campo medio $\vec{B}_0$

Motion in magnetic fields

Se $\omega' \gg \Omega$ cioè $\lambda \gg \rho_g$, non si ha deflessione da risonanza. Tuttavia in questo limite invece si applica l'approx del centro di guida poiché $\frac{\rho_g}{\lambda} \sim \rho \frac{\nabla B}{B} \ll 1$

In tal caso si può dimostrare che la quantità
$$\frac{p_\perp^2}{B} \approx cost.$$

cioè $\quad p^2 \frac{\sin^2\alpha}{B} \approx cost. \quad$ ovvero $\quad \frac{\sin^2\alpha}{B} = cost.$

Quindi
$$\frac{2\sin\alpha\cos\alpha\, d\alpha}{B} - \frac{\sin^2\alpha\, dB}{B^2} = 0$$

$$\frac{d\alpha}{tg\alpha} = 2\frac{dB}{B} \qquad \delta\alpha = 2 tg\alpha \frac{\delta B}{B}$$

Nel caso $\rho_g \sim \lambda$, l'angolo di pitch può cambiare in modo significativo in un girorodio.
Se la particella interagisce con "parecchie" onde che hanno una fase random si può avere una deviazione in angolo significativa su una lunghezza caratteristica, quella di scattering λ_{sc}

Dato che $\rho_g = \frac{cp}{qB_0} = \frac{R}{B_0} \qquad R = \frac{cp}{q}$ = rigidità magn.

e $\rho_g \sim \lambda$, possiamo associare una rigidità magnetica con ciascuna scala fisica nello spettro di potenza: $\lambda \sim R/B_0$

Sono queste irregolarità che forniscono i centri di scattering più efficienti in angolo di pitch

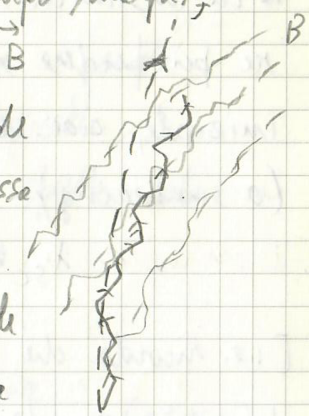
Motion in magnetic fields

Possiamo immaginare il campo nell'ISM come un campo medio $\vec{B}_0$ con sovrapposti treni d'onda monocromatici che si propagano in tutte le direzioni con fasi casuali.

Possiamo immaginare che ciascuna particella subisca l'azione di una particolare comp. del campo solo per una lunghezza d'onda λ prima di incontrare un'altra onda con la stessa λ ma con fase arbitraria rispetto all'onda precedente.

Così le particelle interagiscono successivamente con "molte" onde di lungh. d'onda λ con fasi relative casuali (e quindi casuale rispetto alla fase delle particelle) viaggiando rapidamente nel campo finché la deviazione cumulativa dell'angolo di pitch diventa grande e le particelle iniziano a interagire con un altro treno d'onde

La variazione $\delta\alpha$ in un giroradio implica uno spostamento casuale del centro di guida $\delta r_{cg} \approx \rho_g \delta\alpha$; cioè le particelle si spostano in modo casuale attraverso le linee di campo magn., cioè diffondono nel campo $\vec{B}$



Dopo aver interagito con N onde della stessa λ cioè con la stessa densità di energia ma fasi casuali, da ciascuna delle quali subisce una deviazione $\delta\alpha \approx \left(\frac{\delta B}{B_0}\right)$, la deviazione media complessiva è

$$\delta\phi = \sum_1^N \delta\alpha_i \qquad \text{con } \langle\delta\phi\rangle = 0$$

e

$$\langle\delta\phi^2\rangle = \sum_1^N (\delta\alpha_i)^2 = N\delta\alpha^2 = N\left(\frac{\delta B}{B_0}\right)^2$$

$$\Rightarrow \langle\delta\phi\rangle = \sqrt{\langle\delta\phi^2\rangle} = \left(\frac{\delta B}{B_0}\right)\sqrt{N}$$

Perciò per avere una deviazione di $\pi/2$ sono necessarie $N = \frac{\pi^2}{4}\left(\frac{B_0}{\delta B}\right)^2$ interazioni o "collisioni" con le onde

Motion in magnetic fields

$N = \frac{\pi^2}{4}\left(\frac{B_0}{\delta B}\right)^2$ è il # di collisioni necessarie affinché la particella "perda memoria" della sua direzione iniziale.

La distanza che la particella deve percorrere per perdere memoria della sua direzione iniziale, cioè la lunghezza di diffusione (o scattering) è

$$\lambda_{sc} = N\lambda \approx \frac{\pi^2}{4}\rho_g\left[\frac{B_0}{\delta B(k)}\right]^2$$

[i.e. ricorda che alla risonanza $\lambda \propto \rho_g$]

λ_{sc} dipende da $\delta B^2(k)$, la densità di E associata alla scala fisica $2\pi/k = \lambda$.

Lo spettro di potenza del campo è dato in T^2/m^{-1}, cioè $\delta B^2(k) = w(k)dk \sim w(k)k$

$$\Rightarrow \lambda_{sc} \approx \frac{\pi^2}{4}\rho_g \frac{B_0^2}{w(k)k}$$

$$k = \frac{2\pi}{\rho_g} \Rightarrow \lambda_{sc} \approx \frac{\pi}{8}\rho_g^2 \frac{B_0^2}{w(k)}$$

$$\rho_g = \frac{cp}{qB_0} = \frac{R}{B_0} \Rightarrow \lambda_{sc} \approx \frac{\pi}{8}\frac{R^2}{w(k)}$$

Dato che $w(k) \propto k^{-\alpha} \propto \rho_g^{+\alpha}$ $\boxed{a = \alpha - 2}$ Vedi p. dietro

$$\lambda_{sc} \propto \rho_g^{2+\alpha} = \rho^{+a}$$

cioè $\lambda_{sc} \propto R^{+a}$

La lunghezza di diffusione dipende dalla rigidità R delle particelle.

L'indice α dipende dallo spettro di potenza delle irregolarità δB

Il coeff. di diffusione D

$$D = \frac{\lambda v}{3} \propto cR^{+a} \quad (v \approx c)$$

dipende dalla rigidità magnetica

Di conseguenza anche il tempo di residenza dei RC dipende dalla rigidità, dato che

$$\tau \approx \frac{H^2}{D}$$ con H spessore del disco o dell'alone

$$\Rightarrow \tau \sim R^{-a}$$

Motion in B fields: classical approach

Guiding center decomposition:

Parallel and normal components to the field line: $\mathbf{V}=\mathbf{V}_p+\mathbf{V}_n$ and

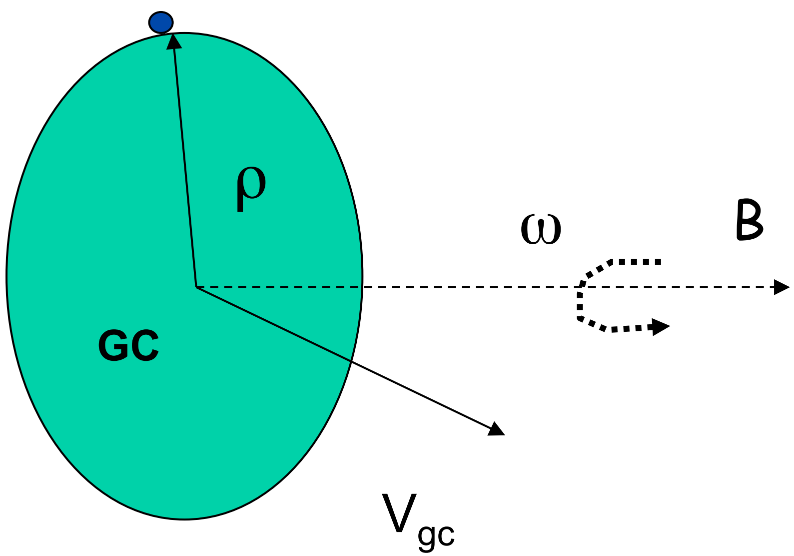
$\mathbf{V}_n$ is decomposed in a drift and a gyration with Larmor radius $\boldsymbol{\rho}=\mathbf{P}_n/\mathbf{Bq}$ and frequency

$$\boldsymbol{\omega}=\mathbf{qB/m} \Rightarrow \mathbf{V}=\mathbf{V}_p+\mathbf{V}_D+\boldsymbol{\omega}\mathbf{x}\boldsymbol{\rho}=\mathbf{V}_{gc}+\boldsymbol{\omega}\mathbf{x}\boldsymbol{\rho}$$

The motion is then described by a traslation of a point, the Guiding Center, plus a gyration around GC normal to B

<u>Parallel and normal components are decoupled</u>

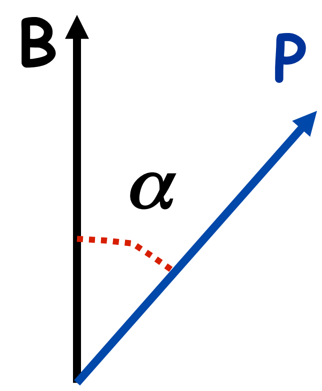
If $dB/Bdt \ll \omega/2\pi$



$\mathbf{V_n=V\sin\alpha}$

$\mathbf{V_p=V\cos\alpha}$

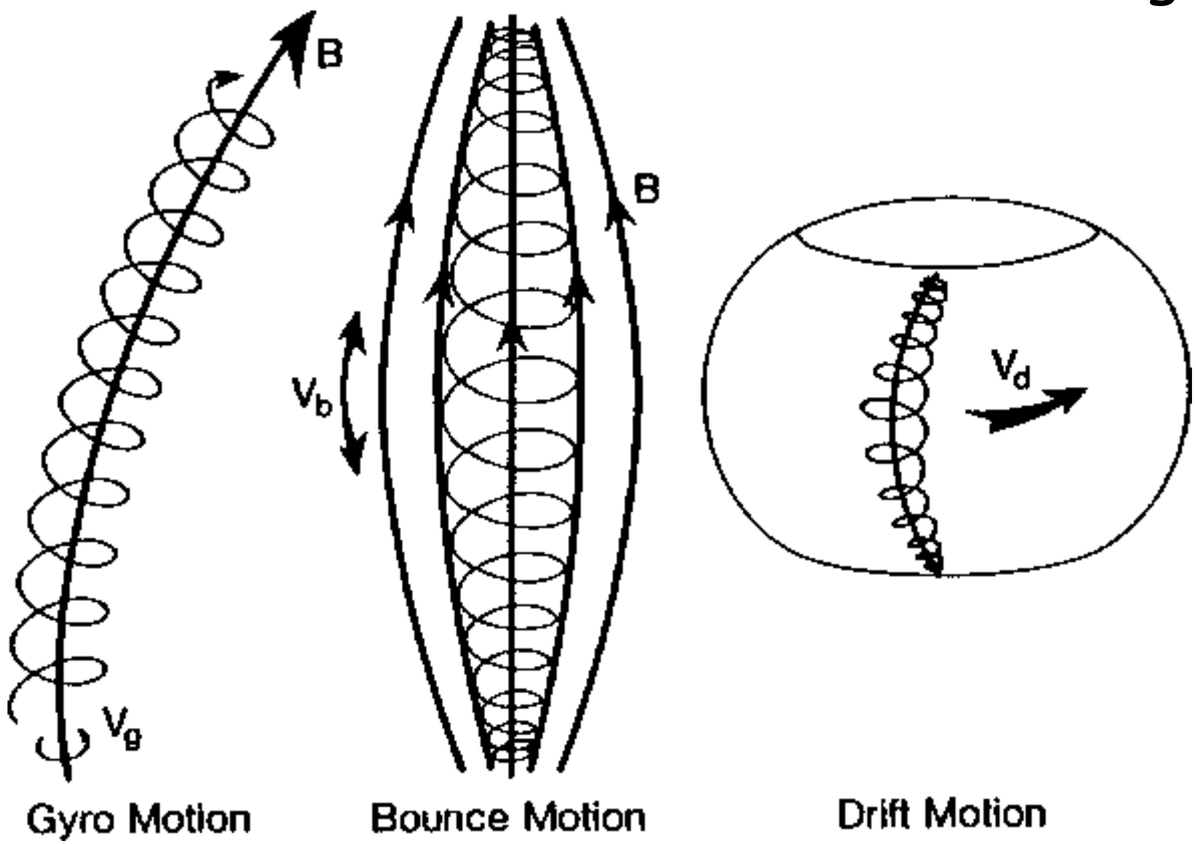
The "Pitch" Angle



Motion in B fields: classical approach

As a consequence of the decoupling, the motion can be decomposed in 3 quasi-periodic components:

- gyration around the field line
- bouncing between the mirror points along the field line
- drifting normal to the field line and to the field gradient



A more powerful approach: adiabatic invariants in B fields (1)

Guiding center equations are an enormous improvement wrt the Lorenz equation but drift and mirroring equations do not allow long-range predictions of particle location, if no axial simmetry is present

What is missing? The "constants of motion", analogous to the conservation of E, P, and angular momentum

Fortunately,in mechanical systems undergoing to periodic motion in which the force changes slightly over a period, approximate constants do exist➔ the adiabatic invariants

A more powerful approach: adiabatic invariants in B fields

The classical Hamilton-Jacobi theory defines adiabatic invariants for periodic motion: the action-angle variables

$$\mathbf{J_i = \oint p_i dq_i}$$

With p_i and q_i action angle variables canonically conjugated and the integral is taken over a full period of motion

dJ/dt~0 provided that changes in the variables occur slowly compared to the relevant periods of the system and the rate of change is constant

Because there are 3 periodic motions, 3 adiabatic invariants can be defined

For a charged particle in a magnetic field, the conjugate momentum is $\underline{\mathbf{P}}=\underline{\mathbf{p}} + \mathbf{q}\underline{\mathbf{A}}$, with **A** vector potential of magn field

Simple example→ Mechanical pendulus: if the lenght increases only weakly during one swing, then Energy x Period, E•T, is a quasi-constant of motion, i.e. an adiabatic invariant

1st invariant: gyromotion

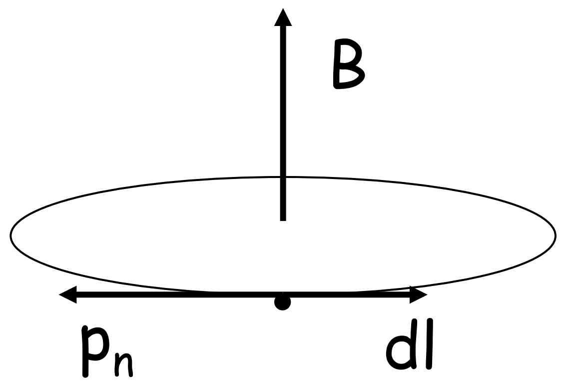
The so-called first adiabatic invariant is obtained by integrating $\mathbf{P}$ from equation (4.10) around the gyration orbit, where $d\mathbf{l}$ is an element of the particle path around the orbit.

$$
\begin{aligned}
J_1 &= \oint [\mathbf{p} + q\mathbf{A}] \cdot d\mathbf{l} \\
&= p_\perp \cdot 2\pi\rho + q\oint \mathbf{A} \cdot d\mathbf{l} \\
&= p_\perp \cdot 2\pi \frac{p_\perp}{Bq} + q\oint \nabla \times \mathbf{A} \cdot d\mathbf{S}
\end{aligned} \tag{4.11}
$$

where $d\mathbf{S}$ is an element of the area enclosed by the path. Therefore,

$$
\begin{aligned}
J_1 &= \frac{2\pi p_\perp^2}{Bq} + q\oint \mathbf{B} \cdot d\mathbf{S} \\
&= \frac{2\pi p_\perp^2}{Bq} - qB\pi\rho^2 \\
&= \frac{2\pi p_\perp^2}{Bq} - \frac{\pi p_\perp^2}{Bq} = \frac{\pi p_\perp^2}{qB}
\end{aligned} \tag{4.12}
$$

The second term in (4.12) is negative because $d\mathbf{S}$ as defined by the particle orbit points in the opposite direction to $\mathbf{B}$.



1st invariant: gyromotion

$$\frac{dJ_1}{dt} = \frac{\pi d}{q dt}\left(\frac{p_n^2}{B}\right) \qquad \frac{d}{dt}\left(\frac{p_n^2}{B}\right) = \left(\frac{dp_n^2}{Bdt} - \frac{p_n^2}{B^2}\frac{\partial B}{\partial t}\right)$$

Moltiplico per B/p^2 $$\frac{B}{p_n^2}\frac{d}{dt}\left(\frac{p_n^2}{B}\right) = \left(\frac{dp_n^2}{p_n^2 dt} - \frac{1}{B}\frac{\partial B}{\partial t}\right)$$

Let |B| vary in time uniformly. Because B varies in time, there an induction field such that

$$\oint \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint \mathbf{E} \cdot d\mathbf{l} = -\oint \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\pi \rho^2 \frac{\partial B}{\partial t} \qquad (4.14)$$

The energy change in one revolution or in one gyroperiod τ_g is therefore

$$\Delta W = -q \oint \mathbf{E} \cdot d\mathbf{l} = q \pi \rho^2 \frac{\partial B}{\partial t} \qquad (4.15)$$

<u>The fundamental assumption is that dB/dt does not vary over a gyroperiod, dB/dt~cost per $t < \tau_g$</u>

Hence

$$\frac{dW}{dt} = \frac{\Delta W}{\tau_g} = q \pi \rho^2 \frac{\partial B}{\partial t} \cdot \frac{Bq}{2\pi \, m_o \gamma}$$

$\tau_g = 2\pi/\omega = 2\pi m_o \gamma / qB$

$$= \left(\frac{p_n^2}{2 m_o \gamma B}\right) \frac{\partial B}{\partial t} \qquad (4.16)$$

1st invariant: gyromotion

Also,

$$\frac{dW}{dt} = \frac{d}{dt}(\gamma m_0 c^2) = m_0 c^2 \frac{d\gamma}{dt} \tag{4.17}$$

where

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left[1 + \frac{p_\perp^2}{m_0^2 c^2}\right]^{1/2}$$

$$= \frac{1}{2m_0^2 c^2 \gamma}\frac{dp_\perp^2}{dt} \tag{4.18}$$

$\gamma = E_n / m_o c^2$
$= [p_n{}^2 + (m_o c^2)^2]^{1/2} / m_o c^2 =$
$= [(p_n / m_o c^2)^2 + 1]^{1/2}$

Equate (4.16) and (4.17) using (4.18) for $d\gamma/dt$ to obtain

$$\frac{1}{B}\frac{\partial B}{\partial t} = \frac{1}{p_\perp^2}\frac{dp_\perp^2}{dt}.$$

Therefore $$\frac{B}{p_n^2}\frac{d}{dt}\left(\frac{p_n^2}{B}\right) = \left(\frac{dp_n^2}{p_n^2 dt} - \frac{1}{B}\frac{\partial B}{\partial t}\right) = 0$$

and $$\frac{p_\perp^2}{B} = \text{constant} \tag{4.19}$$

and it follows that $\mu = p_\perp^2/2m_0 B$ is also constant.

Adiabatic invariants in B fields: 1st invariant

If B field varies only weakly in 1 gyroradius, i.e. $dB/Bdt \ll w_L/2\pi$, or $Bd\rho/dB \ll \rho$, then

$$\mu = \frac{p_\perp^2}{2m_o B} \approx \mathbf{const.} \quad \text{or} \quad I_1 = J_1/p = \sin^2\alpha/B \approx const.$$

•When α=90°, mirroring occurs,
and $B_m = B/\sin^2\alpha$ defines the mirror field value which is the same at all the mirror points along the particle trajectory, i.e. particle reflection occurs always at B_m =constant.

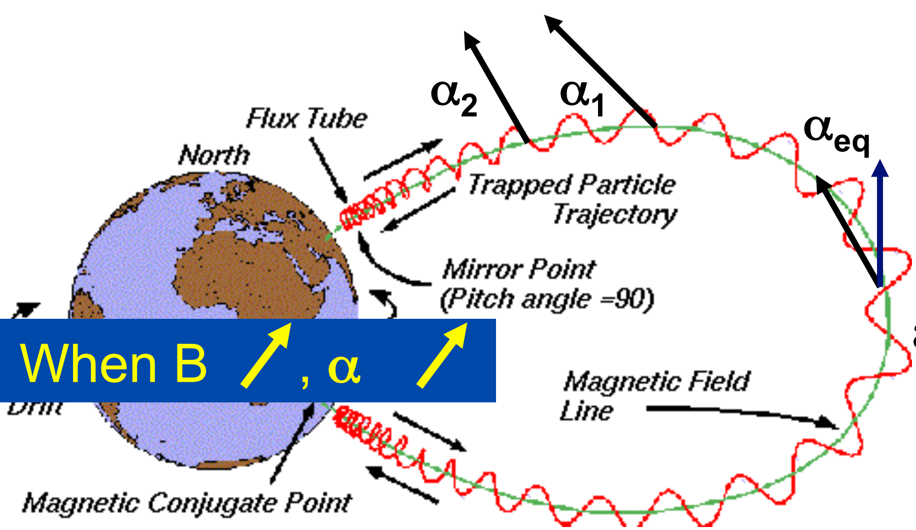
At magn equator, α is minimum and

$$\sin^2\alpha_{eq} = B_{eq}/B_m$$

$$\sin^2\alpha = B/B_m$$

any field value→<u>α depends only on the field</u>

B_m is an adiabatic invariant because identical to an adiabatic invariant and because B_{eq} is a constant α_{eq} is an adiabatic invariant too



Adiabatic invariants in B fields: 2nd invariant

Bouncing→ $\mathbf{J}_2 = \oint \left(\vec{p} + q\vec{A} \right) \cdot \mathbf{d}\vec{s}$ with ds element of path along the field line

If B field varies weakly on a scale comparable with the distance traveled along the field by the particle during one gyration → $\nabla_p B/B \ll \omega_L / 2\pi v_p$ then $J_2 \sim$const.

The 2nd term gives

$$\begin{aligned} \oint q\mathbf{A} \cdot d\mathbf{s} &= q \int \nabla \times \mathbf{A} \cdot d\mathbf{S} \\ &= q \int \mathbf{B} \cdot d\mathbf{S} \\ &= 0 \end{aligned} \tag{4.30}$$

since the integration path along the field line encloses a negligible area and no magnetic flux.

Therefore

$$J_2 = \oint \mathbf{p} \cdot d\mathbf{s} = \oint p \cos\alpha \, ds = \oint p_{\parallel} \, ds = \text{constant} \tag{4.31}$$

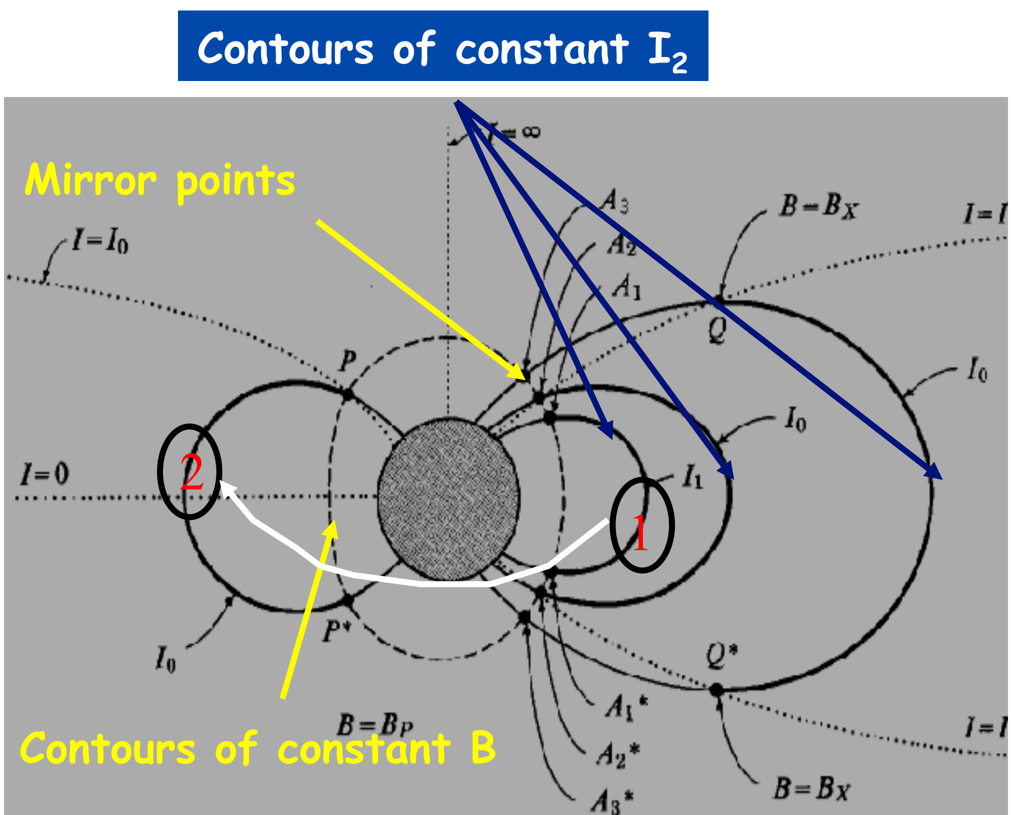
Da dimostrare..

Adiabatic invariants in B fields: 2nd invariant

J_2 does not depend on particle properties but only on field structure, because

$$\cos\alpha=(1-\sin^2\alpha)^{1/2}=[1-B(s)/B_m]^{1/2}$$

$$\mathbf{J_2 = p\int_{s'}^{s}\sqrt{1-B(s)/B_m}\,ds \rightarrow I_2 = \frac{J_2}{2p} \approx const}$$



The primary use of I_2 is to find surfaces mapped out during bouncing and drifting. A particle initially on curve 1, with a given I, will drift on curve 2 (with the same I) and return to 1, mirroring at B_m in both the hemispheres throughout the drifting.

At each longitude there is ONLY one curve –or field line segment- having the required value of I. <u>The particle will follow a trajectory made of field line segments such that I is constant.</u>

Adiabatic invariants in B fields: 3rd invariant

Drifting→ $\mathbf{J}_3 = \oint (\vec{p} + q\vec{A}) \cdot \mathbf{d}\vec{l}_D$ with dl_D element along the long drift path

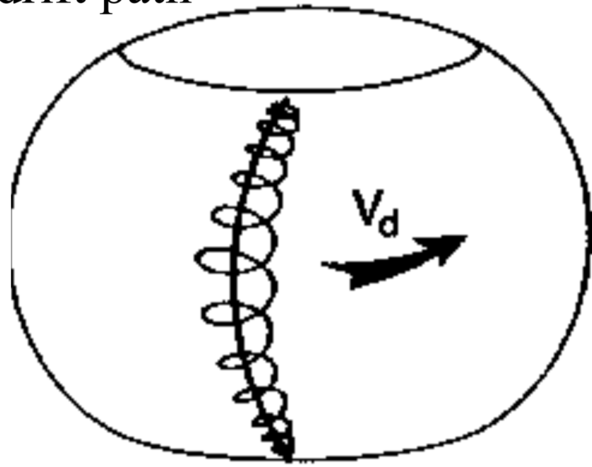
If B field varies only weakly in the area encircled by particle during the gyration or drift motion i.e. $\nabla \mathbf{B}_n / \mathbf{B} << \omega_L / 2\pi v_n$ or $\nabla \mathbf{B}_n / \mathbf{B} << \omega_L / 2\pi v_D$

then J_3~const.

$$\mathbf{J}_3 = \oint_{drift} (\mathbf{q}\vec{\mathbf{A}} + \vec{\mathbf{p}}) \cdot \mathbf{d}\bar{\mathbf{l}}_\mathbf{D} \approx \oint_{drift} \mathbf{q}\vec{\mathbf{A}} \cdot \mathbf{d}\bar{\mathbf{l}}_\mathbf{D} = \int (\nabla \times \vec{A}) \cdot dS = \int \vec{B} \cdot d\vec{S} = \mathbf{q}\boldsymbol{\Phi} \approx \mathbf{const}$$

The 3rd invariant is prop to magnetic flux Φ enclosed by drift path

Important to describe drifts paths during slow changes of B.
In slowly changing fields 1st and 2nd invariant are conserved
but E can change, e.g. due to slow compression/expansion of field
or secular variations of the field.
Conservation of Φ requires particles to move inward/outward
reversibly on the orbit during changes.
Rapid changes, i.e. dB/dt>>B$\mathbf{w_D}$, will cause permanent changes in
Φ and therefore in particle orbits, e.g. solar storms, CME,…



Motion periods: gyration

The periods of three components are characteristic with a precise hierarchy: $\tau_L << \tau_b << \tau_D$ at the approximation of guiding center and of adiabatic invariant approach.
Gyration motion: an istantaneous circular motion normal to the field line.

The frequency of the motion is given by the Larmor frequency $\tau_L = 2\pi/\omega = 2\pi\ m/qB$ with a Larmor radius $\rho = p_n/qB = p\sin\alpha/qB$.

For relativistic particles $\tau_L = 2\pi\ m_o\gamma\ /qB$ and $\gamma = E/m_o c^2 \rightarrow \tau_L = 2\pi E\ /qBc^2$
Typical ranges are $10^{-3} - 10^{-6}$ sec (i.e. kHz - Mhz freq. range)

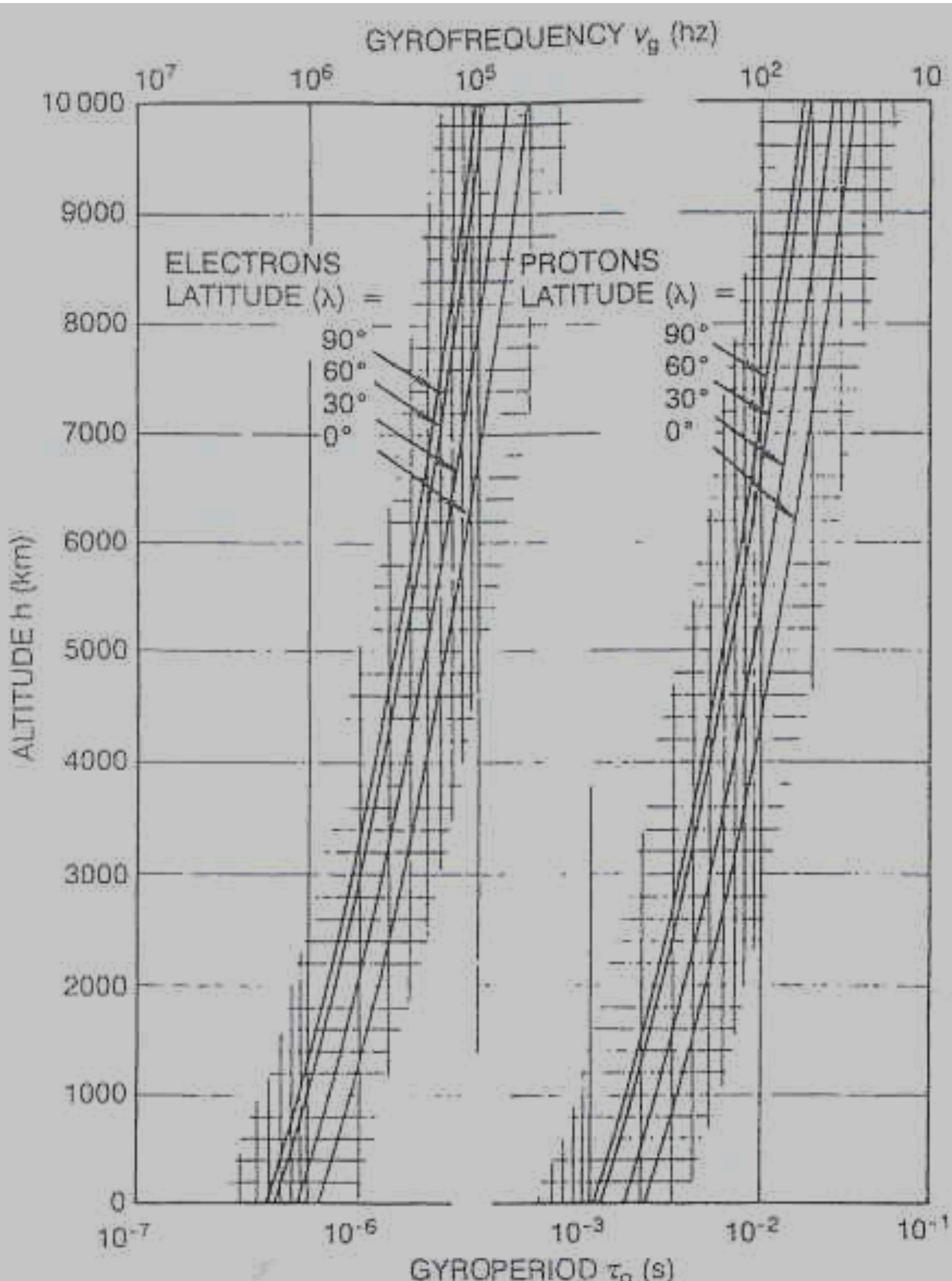


Figure B.1. Gyration frequencies and periods of trapped electrons and protons in a centered, dipole magnetic field.

Larmor period

Motion periods: bouncing

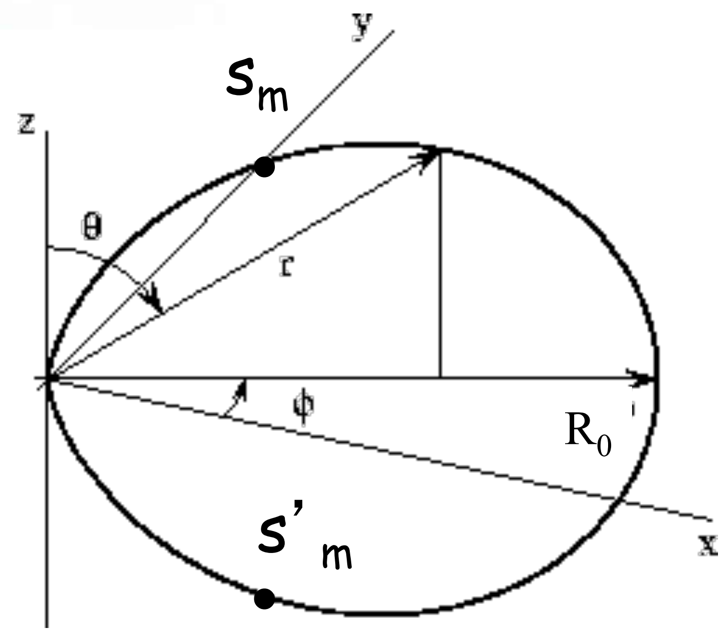
Bouncing motion: between the mirror points S, S' where $\alpha=90^o$. The period is given by

$$\tau_b = 2\int_{s_m}^{s'_m} \frac{ds}{v_\parallel(s)} = \frac{2}{v}\int_{s_m}^{s'_m} \frac{ds}{\cos\alpha(s)}$$

$$= \frac{2}{v}\int_{s_m}^{s'_m} \frac{ds}{\sqrt{\left[1-\frac{B(s)}{B_m}\right]}} = \frac{2}{v}\int_{s_m}^{s'_m} \frac{ds}{\sqrt{\left[1-\frac{B(s)}{B_{eq}}\sin^2\alpha_{eq}\right]}}$$

For a dipole field

$$\tau_b = 0.117\left(\frac{R_0}{R_E}\right)\frac{1}{\beta}[1-0.4635(\sin\alpha_{eq})^{3/4}]\ \mathrm{s}$$



It depends only on R_0, the equatorial distance of the field line from the dipole center and on the particle speed β. There is only a weak dependence on the pitch angle

Motion periods: drift

The drift period is given the average drift speed over a bounce period $\langle d\Phi/dt \rangle = \Delta\Phi/\tau_b$ as $\tau_D = 2\pi / \langle d\Phi/dt \rangle$.
In a dipole, after a numerical integration, the drift period is given by

$$\tau_d = \frac{2\pi q B_0 R_E^3}{mv^2} \frac{1}{R_0} [1 - 0.3333(\sin \alpha_{eq})^{0.62}] \tag{4.46}$$

This approximation can be simplified by collecting all constant factors to give

$$\tau_d = C_d \cdot \left(\frac{R_E}{R_0} \right) \frac{1}{\gamma \beta^2} [1 - 0.3333(\sin \alpha_{eq})^{0.62}] \tag{4.47}$$

where

$$C_d = 1.557 \times 10^4 \text{ s for electrons}$$

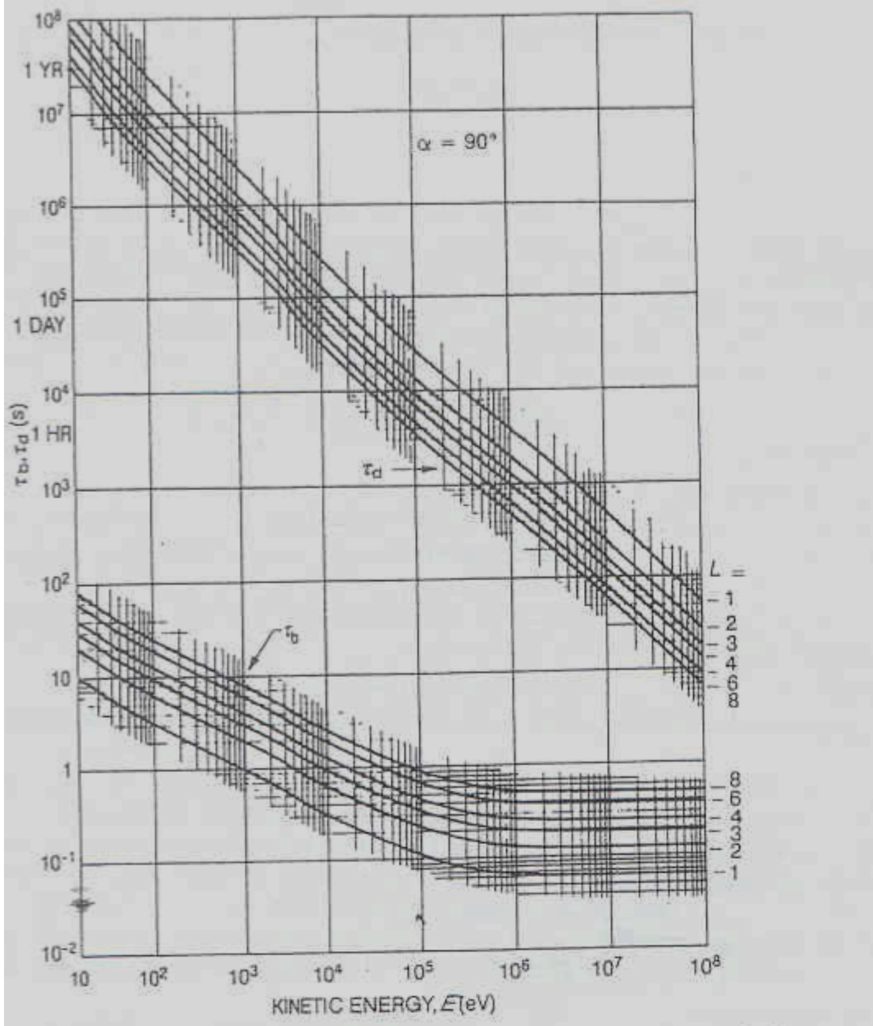
and

$$C_d = 8.481 \text{ s for protons}$$

$\gamma\beta^2 = (E/m_o c^2)[E^2 - (m_o c^2)^2]/E^2 = E/m_o c^2 - m_o c^2/E \rightarrow$ for relativistic particles the period scales as $1/E$

Time periods

Elettroni



.2. Bounce (τ_b) and drift (τ_d) periods of equatorially trapped electrons.

Protoni

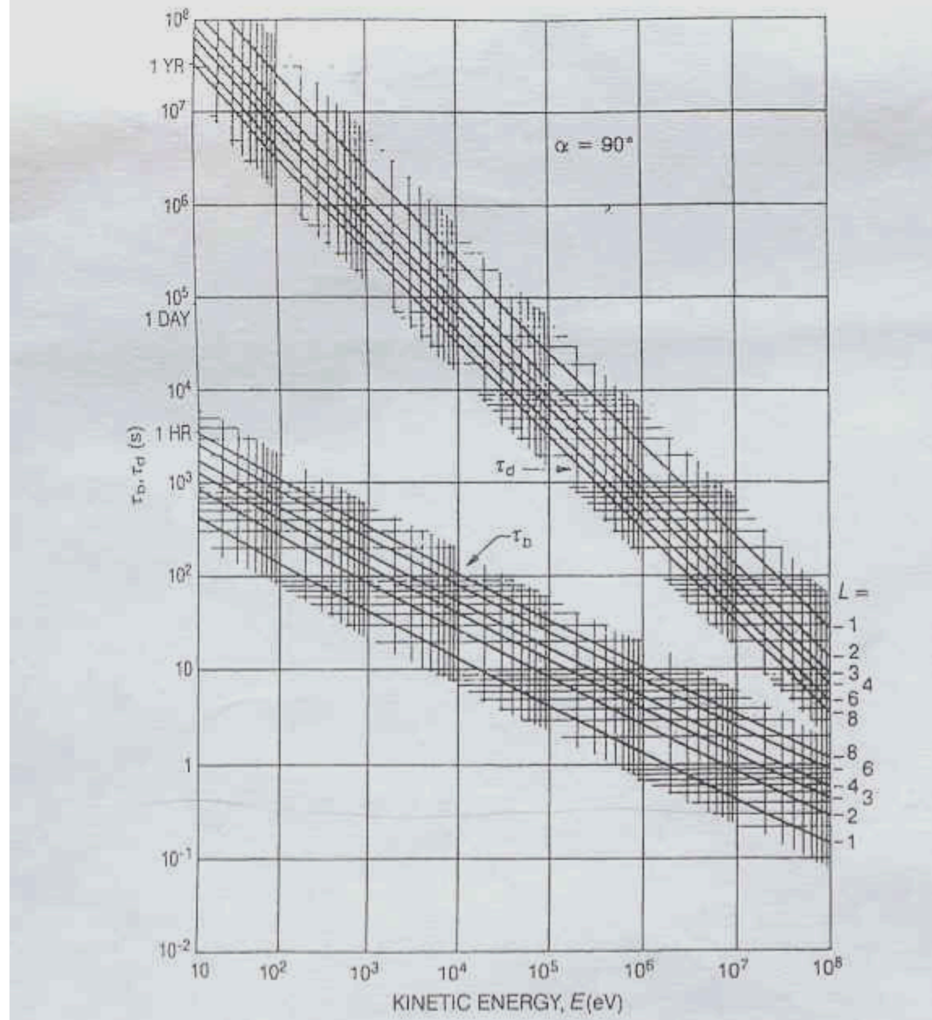
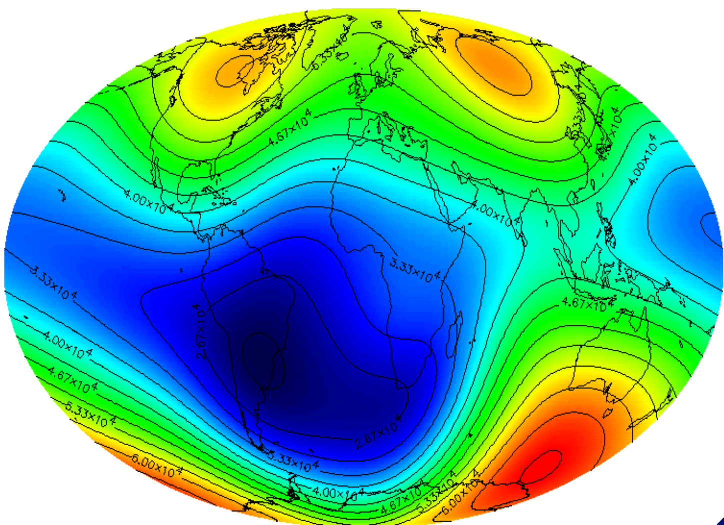


Figure B.3. Bounce (τ_b) and drift (τ_d) periods of equatorially trapped protons

Adiabatic invariants in B fields: coordinates

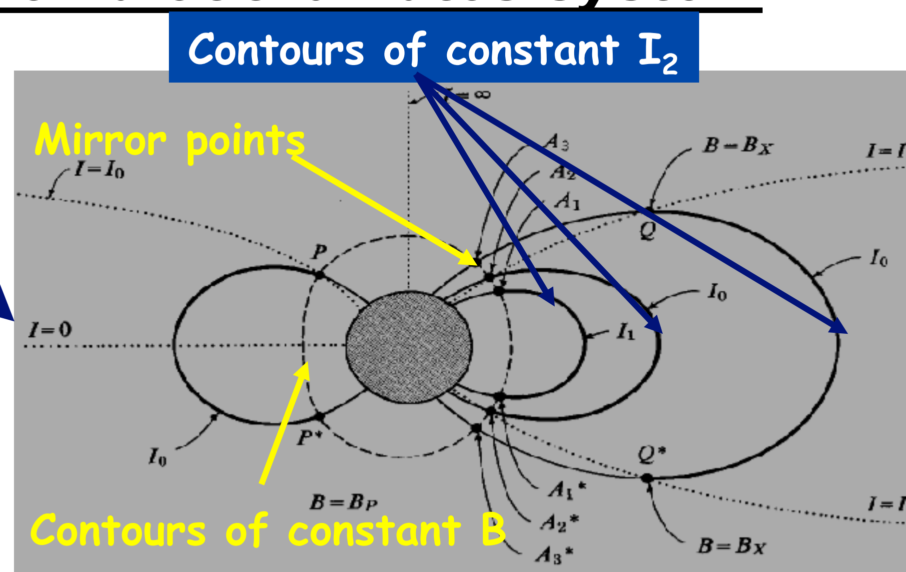
Any reference system based on geocentric coordinates does not allow insights into the relationships between the particle distributions at different locations due to lack of simmetry in the irregular geomagnetic field



What is needed is a coord system based on trapped particle motion which will have naturally identical values for equivalent magnetic positions

<u>Adiabatic invariants provide such a coordinates system</u>

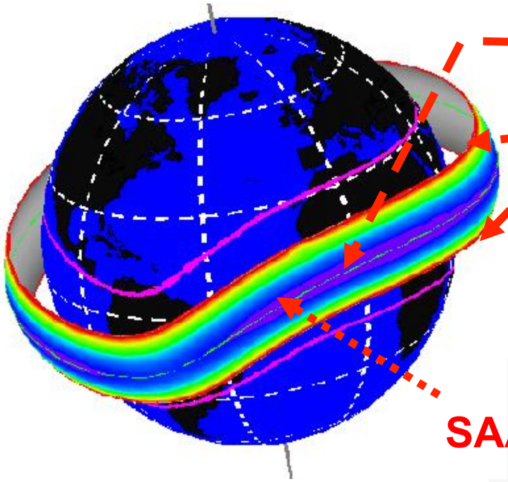
To conserve invariants particles will move following segments of field lines such that B_m (or α_o), L, Φ are conserved

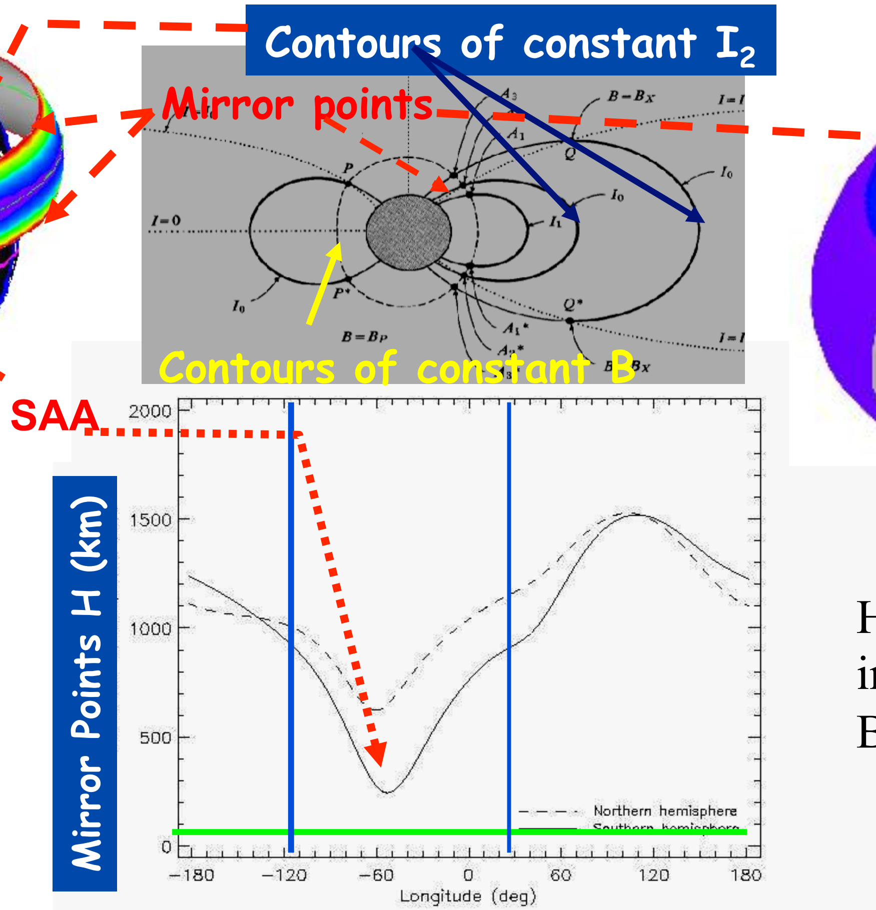


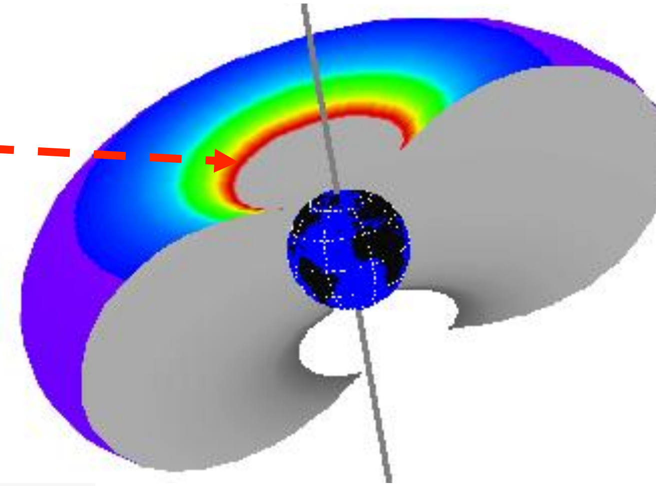
Adiabatic invariants in B fields:drift shells (1)

The ensemble of field lines segments of constant invariants forms the surface mapped out by the guiding center of a particle during its motion:

the drift shell







H_{mir} not const due to irregularities in earth's B field

<u>All the particles with the same invariants map out the same drift shell, i.e. are equivalent from magnetic point of view</u>

Adiabatic invariants in B fields:drift shells (2)

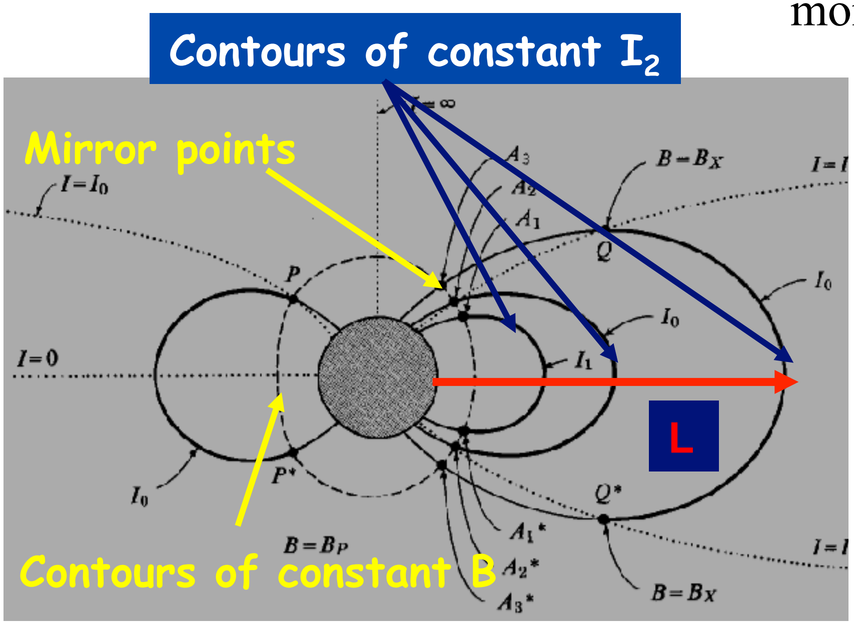
Adiabatic Invariants are difficult to visualize and interpret in a simple way, due to their complicate definition what is needed is to build more easily readable coords derived from AI:

μ ➔ B_m or $\boldsymbol{\alpha}_{eq}$, because are very easy to interpret and are still AI

→all the particles with same B_m and $\boldsymbol{\alpha}_{eq}$ will mirror at same location

For I_2 a dipole analogy: in a dipole field, all particles with same AI will cross magn. equator at same distance R_o from dipole axis, i.e. particles will remain on field lines having the same R_0

➔ $R_0=f_D(I_D,B_D,M_D)$ with f_D known function of dipole AI of the particle and magn moment of dipole

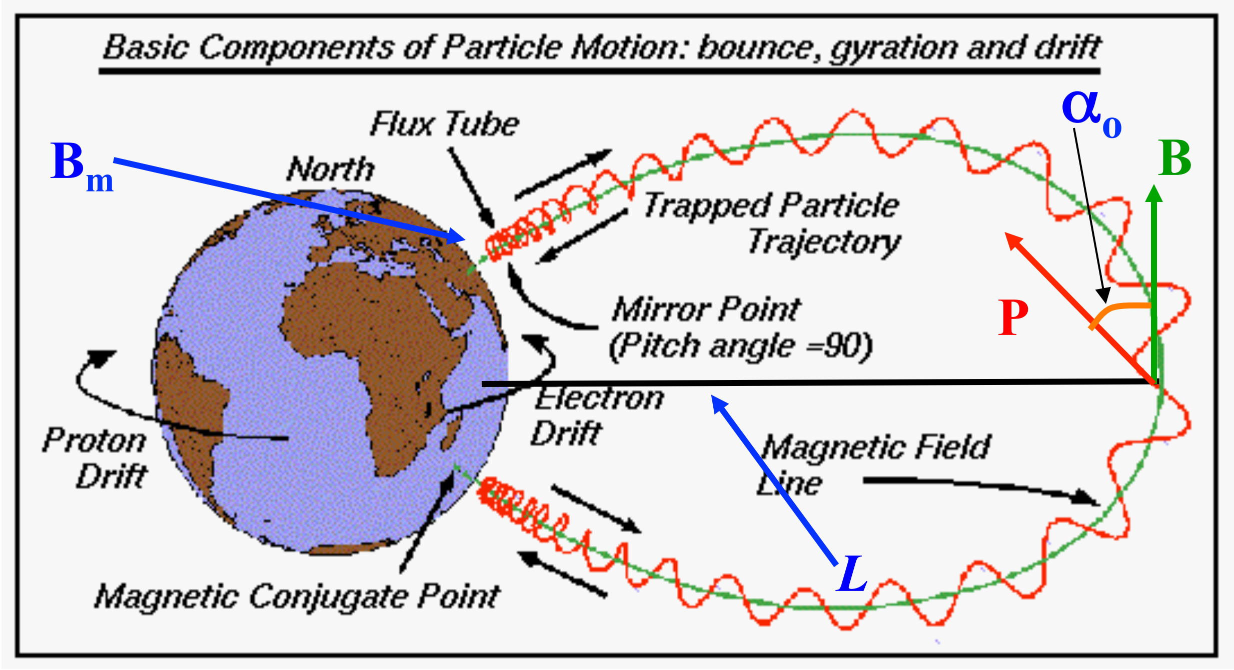


For real earth's field a new variable is defined based on dipole f_D: by definition the equivalent equatorial radius, L, called McIllwain parameter, is given by $LR_E=f_D(I,B,M_E)$

Particles will follow paths such that L=const.

NOTE: L=const. does not imply R const.!!!

Motion in Earth's Magnetic Field



3 *quasi-periodic* motion comp. — ***Adiab. Invariants***

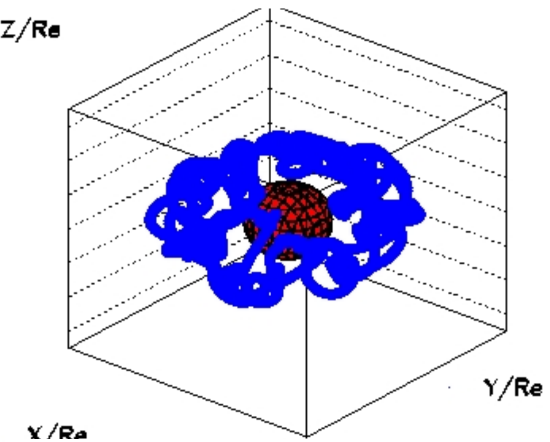
- ***Gyration*** **with Larmor freq.** ⟺ ***B_m*** **or α_o**
- ***Bouncing*** **betw. mirror points** ⟺ ***Shell Par. L***
- ***East-West*** **drift** ⟺ ***Mag. Flux Φ***

Particles with the same adiabatic invariants (L,a_o) or (L,B_m) have same motion in the Earth's field

Adiabatic invariants in B fields:validity

Adiabatic approach is an approximate description and validity requires small changes during relevant motion periods

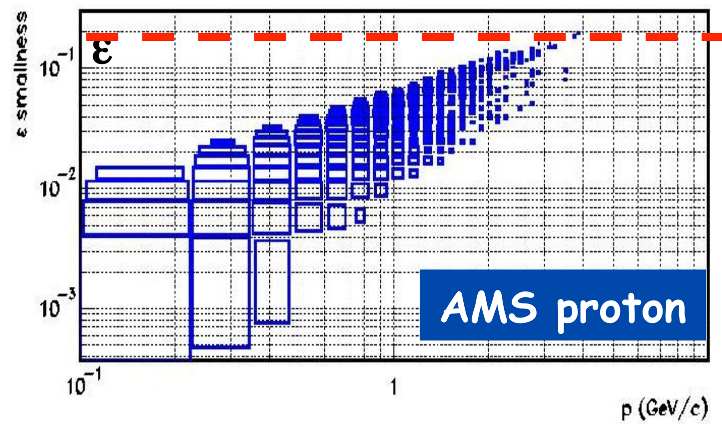
Validity of AI requires time scale for gyration,bouncing and drifting to be well separated by a smallness parameter $1/\boldsymbol{\varepsilon(p)}=1/(\rho_{eq}/R_{eq})\boldsymbol{>>1}$ with ρ and R Larmor and field line curvature at equ and momentum p.



This leads to a limit in P for AI description

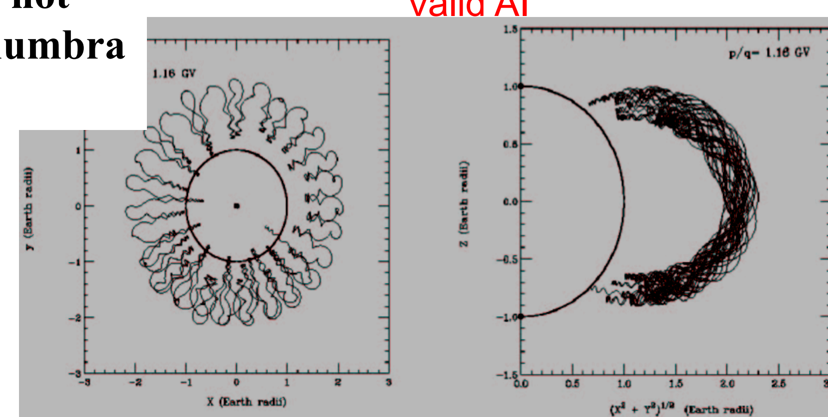
In a dipole field can be shown (Shultz,1974)

$$p_{max} = 10/L^2 \; (\varepsilon \le 0.1 - 0.2)$$



If $\varepsilon > \varepsilon_{max}$, motion becomes chaotic and AI are not conserved, like in penumbra regions

Valid AI



Motion frequencies

$$f_G \propto \frac{qB}{E} = 5K \div 1MHz$$

$$>>$$

$$f_B \propto \frac{\beta \cdot g(\alpha_0, \vartheta_m)}{L} = 0.03 \div 30H_z$$

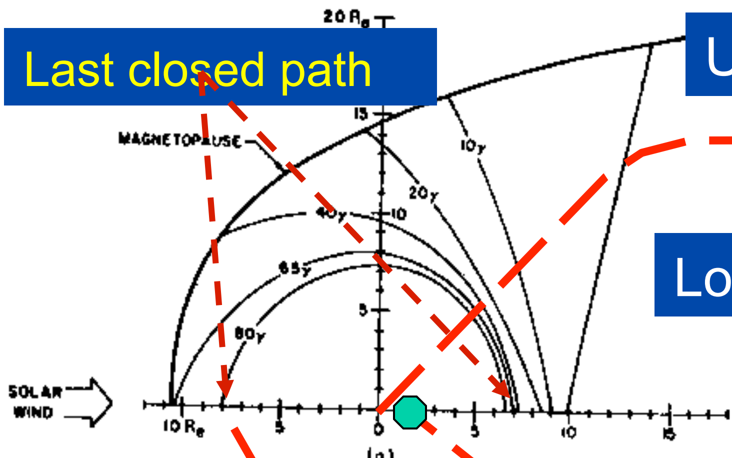
$$>>$$

$$f_D \propto \frac{E \cdot \beta^2 \cdot L}{h(\alpha_0, \vartheta_m)} \propto 2 \div 20mH_z$$

The Radiation Belts (1)

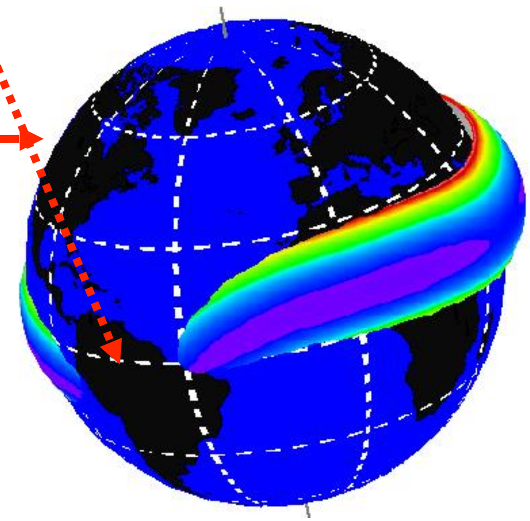
The radiation belts are formed by all the drift shells envelopped from last closed path down to atmosphere limit where shells intercept earth

Last closed path



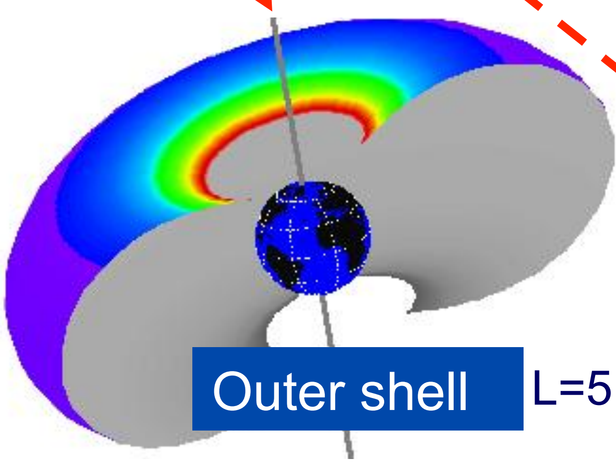
Upper Limit: 7-8 R_E

Lower Limit: ~1.01 R_E



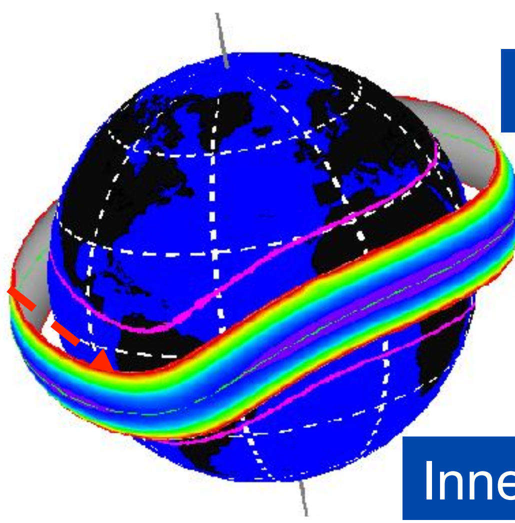
Atmospheric shell

L=1.05 R_E



Outer shell

L=5.5 R_E



Inner shell

L=1.2 R_E

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http://www.spenvis.oma.be/spenvis/help/background/traprad/traprad.html