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The Fibonacci Quarterly

Founded in 1963 by Verner E. Hoggatt, Jr. (1921-1980)
and Br. Alfred Brousseau (1907-1988)

THE OFFICIAL JOURNAL OF THE FIBONACCI ASSOCIATION
DEVOTED TO THE STUDY
OF INTEGERS WITH SPECIAL PROPERTIES

ON A CLASS OF NON-CONGRUENT AND NON-PYTHAGOREAN NUMBERS

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(*Submitted August 1993*)

In one of his famous results, Fermat showed that there exists no Pythagorean triangle with integer sides whose area is an integer square. His elegant method of proof is one of the first known examples in the history of the theory of numbers where the method of infinite descent is employed. Mohanty [3] has defined a Pythagorean number as the area of a Pythagorean triangle and studied properties of such numbers. Fermat has thus shown that no Pythagorean number can be an integer square.

To extend Fermat's result, one may ask if there exists a Pythagorean triangle whose area is p times a perfect square, p a given prime. It turns out that, for certain primes $p \equiv 1, 5, 7 \pmod 8$, this is the case; for example, the primes $p = 5, 7, 41$ have this property. For $p = 5$, the triangle $(3^4 - 1, 8, 3^4 + 1)$ has area $A = 5(3 \cdot 4)^2$. For $p = 7$, the triangle $(4^4 - 3^4, 2 \cdot 4^2 \cdot 3^2, 4^4 + 3^4)$ has area $A = 7 \cdot (3 \cdot 4 \cdot 5)^2$. For $p = 41$, the triangle $(5^4 - 4^4, 2 \cdot 5^2 \cdot 4^2, 5^4 + 4^4)$ has area $A = 41 \cdot (5 \cdot 4 \cdot 3)^2$. However, as shown below, no Pythagorean number can equal p times an integer square if p is a prime congruent to 3 (mod 8).

A natural question to ask is whether there exists a number $k \equiv 3 \pmod 8$ and a Pythagorean number which equals k times a square. There is no reason to believe that such a number of k does not exist. Furthermore, one may attempt to find infinitely many such numbers k.

In this paper the following result is proven. Let k be an odd squarefree positive integer with $k \equiv 3 \pmod 8$. Assume that k belongs to one of the following families:

Family (a): $k = p_1$, where p_1 is a prime with $p_1 \equiv 3 \pmod 8$.

Family (b): $k = p_1 p_2$, where p_1 and p_2 are primes such that $p_1 \equiv 5 \pmod 8$ and $p_2 \equiv 7 \pmod 8$, with p_1 being a quadratic nonresidue of p_2 (so, by quadratic reciprocity, p_2 is also a nonresidue of p_1).

Family (c): $k = p_1 p_2 \dots p_n, n \geq 2$, where $p_1 p_2 \dots p_n$ are distinct primes such that $p_1 \equiv 3 \pmod 8$, $p_2 \equiv \cdots \equiv p_n \equiv 1 \pmod 8$; the primes $p_2, \dots, p_n$ are all quadratic residues of each other, and they are all quadratic nonresidues of p_1 (so, by quadratic reciprocity, p_1 is a quadratic nonresidue of $p_2, \dots, p_n$ as well).

Family (d): $k = p_1 p_2 p_3 \cdots p_n$, $n \geq 3$, where $p_1, p_2, p_3, \dots, p_n$ are distinct primes such that $p_1 \equiv 5 \pmod 8$, $p_2 \equiv 7 \pmod 8$, and $p_3 \equiv \cdots \equiv p_n \equiv 1 \pmod 8$, with p_1 being a quadratic nonresidue of p_2 (so, by quadratic reciprocity, p_2 is a nonresidue of p_1 as well) and $p_3, \dots, p_n$ being quadratic residues of each other; and **either** with $p_3, \dots, p_n$ being quadratic residues of p_1 (so, by quadratic reciprocity, p_1 is a quadratic residue of $p_3, \dots, p_n$) and with $p_3, \dots, p_n$ being quadratic nonresidues of p_2 (so, by reciprocity, p_2 is a quadratic nonresidue of $p_3, \dots, p_n$) **or** vice-versa.

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Theorem: Let k be an odd squarefree positive integer, $k \equiv 3 \pmod 8$ and suppose that k belongs to one of the families (a)-(d) listed above. Then there is no Pythagorean triangle whose area equals k times an integer square.

Proof: Let (A, B, C) be a Pythagorean triple whose area is k times a square, $\frac{1}{2}AB = kD^2$. One easily sees that we may assume $(A, B) = 1$, for if it were otherwise, the problem would reduce to the case of a Pythagorean triple (A_1, B_1, C_1) with $(A_1, B_1) = 1$ and $\frac{1}{2}A_1B_1 = kD_1^2$. By assuming that (A, B, C) is a primitive Pythagorean triple, we may set $A = M^2 - N^2$, $B = 2MN$, $C = M^2 + N^2$, for positive integers M, N with $(M, N) = 1$ and $M + N \equiv 1 \pmod 2$. Thus, from $\frac{1}{2}AB = kD^2$, one obtains

$$(M - N)(M + N)MN = kD^2. \tag{1}$$

Since $(M, N) = 1$ and $M + N \equiv 1 \pmod 2$, we have

$$\begin{aligned}(M, N) &= (M, M + N) = (M, M - N) = (N, M - N)\\ &= (N, M + N) = (M - N, M + N) = 1.\end{aligned} \tag{2}$$

Thus, all the factors $M - N, M + N, M,$ and N on the left-hand side of (1) are pairwise relatively prime. Therefore, since k is squarefree, there are precisely four cases or possibilities and their ramifications.

The first possibility is that precisely one of the factors on the left-hand side of (1) is equal to k times a square, while the rest of them are perfect squares.

The second possibility is that one of $M + N, M - N, M,$ or N equals a times a square, another of the factors equals b times a square, and the other two factors are integer squares with $ab = k$ and $1 < a, b < k$.

The third possibility is that one of the factors equals a times a square, another equals b times a square, a third equals c times a square, and the fourth is just an integer square with $abc = k$ and $1 < a, b, c < k$.

The fourth possibility is that $M = aM_1^2$, $N = bN_1^2$, $M + N = cU^2$, $M - N = dV^2$, with $abcd = k$ and $1 < a, b, c, d < k$.

Case 1. Exactly one of $M + N, M - N, M,$ or N equals k times an integer square, while the remaining three are integer squares.

First, suppose $M = kM_1^2, N = N_1^2, M - N = U^2, M + N = V^2$. Consequently, we obtain

$$kM_1^2 - N_1^2 = U^2, \tag{3}$$
$$kM_1^2 + N_1^2 = V^2. \tag{4}$$

Thus, $2kM_1^2 = U^2 + V^2$ and $(U, V) = 1$ by (2). However, the last equation constitutes a contradiction, since $k \equiv 3 \pmod 4$, and it is well known that no prime congruent to 3 (mod 4) divides the sum of two relatively prime integer squares.

Next, suppose that $N = kN_1^2, M = M_1^2, M - N = U^2, M + N = V^2$. Thus,

$$M_1^2 - kN_1^2 = U^2, \tag{5}$$
$$M_1^2 + kN_1^2 = V^2. \tag{6}$$

Since $M+N \equiv 1 \pmod 2$, we also have $M_1+N_1 \equiv 1 \pmod 2$. But then equation (6) implies, by virtue of $k \equiv 3 \pmod 4$, that $M_1 \equiv 1 \pmod 2$ and $N_1 \equiv 0 \pmod 2$. Moreover, $(M_1, N_1)=1$, so $(N_1, U)=1$ as well. By adding (5) and (6), we obtain

$$2M_1^2 = U^2 + V^2. \tag{7}$$

Clearly, we may assume M_1, U, and V to be positive (recall $M, N \neq 0$), and since (2) implies that $(U, V)=1$, it follows (see [2], p. 427, lines 4 and 5) that

$$M_1 = m^2+n^2,\ U = m^2+2mn-n^2,\ V = n^2+2mn-m^2 \tag{8}$$

for positive integers m, n with $m+n \equiv 1 \pmod 2$ and $(m,n)=1$. Consequently, combining (6) and (8), we have

$$\begin{aligned} kN_1^2 &= V^2 - M_1^2 = (V-M_1)(V+M_1) \\ &= (2mn-2m^2)(2n^2+2mn) = 4mn(n-m)(n+m); \end{aligned}$$

thus,

$$kN_2^2 = (n-m)(n+m)\cdot m \cdot n, \tag{9}$$

where $N_1 = 2N_2$. Therefore, $(n^2-m^2, 2mn, m^2+n^2)$ is a primitive Pythagorean triple whose area equals kN_2^2. But $kN_1^2 = V^2 - M_1^2 \le V^2 = M+N$. Hence, $0 < n+m < M+N$; thus, an infinite descent with respect to the initial equation (1) is established.

Now suppose that $M = M_1^2, N = N_1^2, M-N = kU^2$, and $M+N = V^2$. Then

$$M_1^2 - N_1^2 = kU^2, \tag{10}$$
$$M_1^2 + N_1^2 = V^2. \tag{11}$$

Adding (10) and (11), we obtain

$$2M_1^2 = kU_1^2 + V^2. \tag{12}$$

Now, since $U \equiv V \equiv 1 \pmod 2$, (12) implies $2M_1^2 \equiv k+1 \pmod 8$; hence, $k \equiv 2M_1^2 - 1 \equiv \pm 1$ (mod 8). But $k \equiv 3 \pmod 8$, so this is a contradiction.

Finally, suppose that $M = M_1^2, N = N_1^2, M-N = U^2$, and $M+N = kV_1^2$. This leads to a contradiction, since $M+N = M_1^2 + N_1^2 = kV_1^2$, $k \equiv 3 \pmod 4$ and $(M_1, N_1) = 1$. This concludes the proof of Case 1.

Case 2. One of $M+N, M-N, M$, or N is a times a square, one is b times a square, and the other two are squares, with $ab = k \equiv 3 \pmod 8$ and $1 < a, b < k$. Note that $ab \equiv 3 \pmod 8$ implies that either $a \equiv 3, b \equiv 1 \pmod 8$ or vice versa, or $a \equiv 5, b \equiv 7 \pmod 8$ or vice versa. First, suppose that $a \equiv 1, b \equiv 3 \pmod 8$. Since $ab = k$ with $1 < a, b < k$, it follows that k belongs to Family (c) or Family (d) of the Theorem.

If k belongs to Family (c), then $k = p_1 \cdot p_2 \cdot \dots \cdot p_n$ with $p_1 \equiv 3 \pmod 8$ and $p_2 \equiv p_3 \equiv \dots \equiv p_n \equiv 1 \pmod 8$. Also, $a = q_1 \cdot q_2 \cdot \dots \cdot q_k$ and $b = p_1$ or $b = p_1 q_{k+1} q_{k+2} \dots q_{n-1}$, where the two sets of q's are disjoint and their union is $\{p_2, p_3, \dots, p_n\}$. All the various subcases of Case 2 lead to a congruence of the form $b \cdot R^2 \equiv e \cdot L^2 \pmod{q_1}$, with $(bR, q_1) = 1$ and where $e = 1, -1, 2$, or -2; thus, since $q_1 \equiv 1 \pmod 8$, b is a quadratic residue of q_1. On the other hand, according to the hypothesis, p_1 is a quadratic nonresidue and $q_{k+1}, q_{k+2}, \dots, q_{n-1}$ are all quadratic residues of q_1. Thus, b is a quadratic nonresidue of q_1, a contradiction.

If k belongs to Family (d), then $k = p_1 \cdot p_2 \cdot \cdots \cdot p_n$ with $p_1 \equiv 5$, $p_2 \equiv 7$, and $p_3 \equiv p_4 \equiv \cdots \equiv p_n \equiv 1 \pmod 8$. Thus, as above, $a = q_1 \cdot q_2 \cdot \cdots \cdot q_k$ and $b = p_1 p_2$ or $b = p_1 p_2 q_{k+1} \cdots q_{n-2}$, where the two sets of q's are disjoint and their union is $\{p_3, p_4, \ldots, p_n\}$. Again, as above, b is a quadratic residue of q_1. Also, according to the hypothesis, each of $q_{k+1}, q_{k+2}, \ldots, q_{n-2}$ are quadratic residues of q_1, and **either** p_1 is a quadratic residue of q_1 and p_2 is a quadratic nonresidue of q_1 **or** p_1 is a quadratic nonresidue of q_1 and p_2 is a quadratic residue of q_1. In any event, we see that b must be a quadratic nonresidue of q_1. This contradiction completes the proof of this subcase.

Since the proofs for the remaining subcases and cases are similar to those above, we omit the details, except to note that Legendre's theorem (see [2], p. 422) is used in these proofs.

Recall that a natural number k is a congruent number if there exist natural numbers a, b, and c with $a^2 + b^2 = c^2$ and $2ab = k$. We now have the following corollary.

Corollary: If k is an integer satisfying the hypothesis of the Theorem, then kd^2, for any positive integer d, is a non-congruent number.

Proof: Since an integer kd^2 is congruent if and only if there exist nonzero integers a, b, and c such that $a^2 + b^2 = c^2$ and $2ab = kd^2$, if kd^2 were a congruent number, then we would have $(2a)^2 + (2b)^2 = (2c)^2$ and $\frac{1}{2}(2a)(2b) = k \cdot d^2$, which implies that $(2a, 2b, 2c)$ is a Pythagorean triangle whose area equals k times an integer square, contradicting the Theorem.

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ON WEIGHTED r-GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

The standard Fibonacci numbers have several well-known and familiar properties, among which are the fact that the ratio of successive terms approaches a fixed limit ϕ, and that the n^{th} Fibonacci number is asymptotic to ϕ^n. In this paper we extend these properties to a generalized class of Fibonacci sequences, giving necessary and sufficient conditions for such a sequence to be asymptotic to one of the form $n^{\nu-1}\lambda^n$. In that case, we show how to compute the limiting ratio between the solution and $n^{\nu-1}\lambda^n$, as well as proving that the ratio of successive terms of a solution must have λ as a limit.

The necessary and sufficient conditions mentioned above are stated in terms of the roots of a polynomial. Indeed, this polynomial is the characteristic polynomial associated with the difference equation defining a generalized Fibonacci sequence. We also discuss conditions that depend directly on the coefficients of the characteristic polynomial. As a special case, we derive results when the polynomial has negative real coefficients (except for the leading coefficient 1). More generally, we give a sufficient condition on the coefficients for the roots to satisfy the necessary and sufficient conditions discussed above.

2. PRELIMINARIES

Let $a_1, a_2, \ldots, a_r$ be arbitrary $r \geq 2$ complex numbers with $a_r \neq 0$, and let $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$ be any given sequence of complex numbers. The *weighted r-generalized Fibonacci sequence* $\{y_A(n)\}_{n=-r+1}^{+\infty}$ is the sequence generated by the difference equation with initial values:

$$\begin{cases} y_A(n) = \alpha_n, & n = -r+1, -r+2, \ldots, -1, 0; \\ y_A(n) = \sum_{i=1}^{r} a_i y_A(n-i), & n = 1, 2, 3, \ldots. \end{cases} \tag{1}$$

As a special case, when $a_i = 1$ for all i (the unweighted case), $\alpha_0 = 1$, and $\alpha_i = 0$ for $i = -r+1, \ldots, -1$, (1) generates the *r-generalized Fibonacci numbers* introduced by Miles [8]. Explicit representations for these numbers can be found in [3], [5], and [6].

The polynomial $p(x) = x^r - a_1 x^{r-1} - \cdots - a_{r-1}x - a_r$ is called the *characteristic polynomial* associated to (1), and any solution λ of the *characteristic equation* $p(x) = 0$ is called a *characteristic root* for (1).

The first result, whose proof can be found in Kelley and Peterson [7], for example (or also in Jeske [4] or Ostrowski [9, §12]), relates the general solution of (1) to its characteristic roots.

Theorem 1: Suppose (1) has characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$, respectively $(m_1 + m_2 + \cdots + m_k = r)$. Then (1) has r independent solutions $n^j \lambda_\ell^n$ $(j = 0, \ldots, m_{\ell-1};$ $\ell = 1, \ldots, k)$. Moreover, any solution of (1) is of the form

$$y_A(n) = \sum_{\ell=1}^{k} \sum_{j=0}^{m_\ell - 1} \beta_{\ell,j} n^j \lambda_\ell^n, \tag{2}$$

where the $\beta_{\ell,j}$ are determined by the initial condition $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$. □

Remark 2: Any independent solution $n^j \lambda_\ell^n$ $(j = 0, \ldots, m_{\ell-1};$ $\ell = 1, \ldots, k)$ can be generated from the initial conditions $\alpha_i = i^j \lambda_\ell^i$ for $i = -r+1, \ldots, -1$ and $\alpha_0 = 1$. □

3. THE MAIN RESULTS

The necessary and sufficient conditions we consider are given in terms of the roots of the characteristic polynomial associated to (1). To simplify, we introduce the following terminology.

The polynomial $p(x)$ is called *asymptotically simple* if, among its roots of maximal modulus, there is a unique root λ of maximal multiplicity ν. Then λ is called the *dominant root* of $p(x)$ and ν is called the *dominant multiplicity*. Also, the system (1) is *asymptotically simple with dominant root* λ *and dominant multiplicity* ν if its characteristic polynomial is.

Theorem 3: System (1) is asymptotically simple with dominant root λ and dominant multiplicity ν if and only if, for any initial condition A, the sequence

$$\left\{ \frac{y_A(n)}{n^{\nu-1}\lambda^n} \right\}_{n=1}^{+\infty}$$

converges to a limit L_A, with L_A not equal to 0 for at least one A.

Proof: To prove the *if* part, we observe from (2) and Remark 2 that the convergence for all A implies the convergence of the sequence

$$\left\{ \frac{n^j}{n^{\nu-1}} \left(\frac{\lambda_\ell}{\lambda} \right)^n \right\}_{n=1}^{+\infty} \tag{3}$$

for any $\ell = 1, \ldots, k$ and $j = 0, \ldots, m_\ell - 1$. But, for each ℓ, the convergence of the sequence (3) for $j = 0, \ldots, m_\ell - 1$ implies that $|\lambda_\ell| < |\lambda|$ or $|\lambda_\ell| = |\lambda|$ and $v > m_\ell$ for $\lambda_\ell \neq \lambda$ or $v \geq m_\ell$ for $\lambda_\ell = \lambda$. Moreover, all the limits are zero except for $j = m_\ell - 1$ when $\lambda_\ell = \lambda$ and $v = m_\ell$. Also, the convergence to a nonzero limit L_A for some A implies that at least one sequence (3) has a nonzero limit. Hence, λ must be the dominant root and v the dominant multiplicity. The *only if* part follows directly from Theorem 1. □

The next step is to relate $y_A(n)$ for arbitrary A to $y_0(n)$, the solution of (1) obtained for the initial conditions $\alpha_i = 0$, for $i = -r+1, \ldots, -1$, and $\alpha_0 = 1$. The matrix approach allows us to obtain the desired relation.

Let T be the (r, r)-matrix defined by

$$T = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_r \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

and $Y_A(n)$ be the $(r, 1)$-matrix defined by

$$Y_A(n) = \begin{bmatrix} y_A(n) \\ y_A(n-1) \\ \vdots \\ y_A(n-r+1) \end{bmatrix}, \quad n = 0, 1, 2, \ldots.$$

Hence, $Y_A(0) = A$ [if we consider A as an $(r, 1)$-matrix]. Therefore, we have $Y_A(n+1) = TY_A(n)$ and $Y_A(n) = T^n A$.

Let us also define the $(r, 1)$-matrices Y_i $(i = 0, 1, \ldots)$ by

$$Y_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow (i+1)^{\text{th}} \text{ entry}, \\ \\ \\ \\ \end{matrix} \quad i = 0, 1, \ldots, r-1,$$

and $Y_i = 0$ for $i = r, r+1, \ldots$. Let $Y_i(n) = T^n Y_i$ for $n = 0, 1, 2, \ldots$. Hence $TY_i(n) = Y_i(n+1)$, and also

$$Y_i(n) = \begin{bmatrix} y_i(n) \\ y_i(n-1) \\ \vdots \\ y_i(n-r+1) \end{bmatrix},$$

where $\{y_i(n)\}_{n=-r+1}^{+\infty}$ is the solution of (1) with the initial condition $A = Y_i$ ($\alpha_{-j} = 1$ if $j = i$ and $\alpha_{-j} = 0$ if $j \neq i$ for $j = 0, 1, \ldots, r-1$).

Since $A = \sum_{i=0}^{r-1} \alpha_{-i} Y_i$, it follows that

$$Y_A(n)=\sum_{i=0}^{r-1}\alpha_{-i}Y_i(n). \tag{4}$$

From these definitions and notation, we have the following direct result.

Lemma 4: Let $a_i=0$ for $i=r+1, r+2, \ldots$, then we have:

(a) $TY_i=a_{i+1}Y_0+Y_{i+1}$;

(b) for $i\geq 0$,

$$Y_i(n)=\sum_{j=1}^{n}a_{i+j}Y_0(n-j)+Y_{i+n}, \quad n=0,1,2,\ldots;$$

(c) for any A,

$$Y_A(n)=\alpha_0Y_0(n)+\begin{cases}\displaystyle\sum_{i=1}^{r-1}\alpha_{-i}\sum_{j=1}^{n}a_{i+j}Y_0(n-j)+\sum_{i=1}^{r-1-n}\alpha_{-i}Y_{i+n}, & 0\leq n<r-1,\\ \displaystyle\sum_{i=1}^{r-1}\alpha_{-i}\sum_{j=1}^{r-i}a_{i+j}Y_0(n-j), & n\geq r-1;\end{cases} \tag{5}$$

(d) for $n\geq 0$, we have

$$y_A(n)=\alpha_0y_0(n)+\sum_{i=1}^{r-1}\alpha_{-i}\sum_{j=1}^{r-i}a_{i+j}y_0(n-j). \tag{6}$$

Proof: ***(a)*** is a direct consequence of the definitions and notation. ***(b)*** is easily obtained by induction. ***(c)*** follows by substitution of ***(b)*** into (4). We obtain ***(d)*** by considering the first entry in (5). □

The identity (6) leads to the next result.

Theorem 5: Let λ be any nonzero complex number and ν be a positive integer.

(a) The sequence

$$\left\{\frac{y_A(n)}{n^{\nu-1}\lambda^n}\right\}_{n=1}^{+\infty}$$

converges for any A (with limit L_A) if and only if the sequence

$$\left\{\frac{y_0(n)}{n^{\nu-1}\lambda^n}\right\}_{n=1}^{+\infty}$$

converges (with limit L_0). Moreover, the limits are related by the formula

$$L_A=\left[\alpha_0+\sum_{i=1}^{r-1}\alpha_{-i}\lambda^i\sum_{j=i}^{r-1}\frac{a_{1+j}}{\lambda^{1+j}}\right]L_0. \tag{7}$$

(b) Moreover, in (a), $L_A\neq 0$ for some A if and only if $L_0\neq 0$. In that case,

$$L_0=\left[\delta_{\nu 1}+(-1)^{\nu-1}\sum_{j=1}^{r-1}\frac{a_{1+j}}{\lambda^{1+j}}\sum_{i=1}^{j}i^{\nu-1}\right]^{-1}. \tag{8}$$

Proof: ***(a)*** From (6), we have

$$\frac{y_A(n)}{n^{\nu-1}\lambda^n} = \alpha_0 \frac{y_0(n)}{n^{\nu-1}\lambda^n} + \sum_{i=1}^{r-1} \alpha_{-i} \sum_{j=1}^{r-i} \frac{a_{i+j}}{\lambda^j} \left(\frac{n-j}{n}\right)^{\nu-1} \frac{y_0(n-j)}{(n-j)^{\nu-1}\lambda^{n-j}},$$

and (7) follows. For ***(b)***, (1) must be asymptotically simple with dominant root λ and dominant multiplicity ν. Then, for $\alpha_0 = \delta_{\nu 1}, \alpha_{-i} = (-i)^{\nu-1}\lambda^{-i}$ $(i = 1, \ldots, r-1)$, we have $y_A(n) = n^{\nu-1}\lambda^n$ and $L_A = 1$ in (a). it follows that

$$L_0 = \left[\delta_{\nu 1} + (-1)^{\nu-1} \sum_{i=1}^{r-1} i^{\nu-1} \sum_{j=i}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}\right]^{-1},$$

and we get (8). □

Immediate consequences of these results are the next two theorems.

Theorem 6 (Ratio of weighted r-generalized Fibonacci sequences): Assume (1) asymptotically simple with dominant root λ and dominant multiplicity ν. If $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$ and $B = (\beta_{-r+1}, \beta_{-r+2}, \ldots, \beta_{-1}, \beta_0)$ are sequences of r complex numbers such that $L_B \neq 0$, then

$$\lim_{n\to+\infty} \frac{y_A(n)}{y_B(n)} = \frac{L_A}{L_B} = \frac{\alpha_0 + \sum_{i=1}^{r-1} \alpha_{-i}\lambda^i \sum_{j=1}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}}{\beta_0 + \sum_{i=1}^{r-1} \beta_{-i}\lambda^i \sum_{j=1}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}}. \quad \square$$

Theorem 7 (Ratio of consecutive terms): Assume (1) asymptotically simple with dominant root λ and dominant multiplicity ν. If $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_{-1}, \alpha_0)$ is such that $L_A \neq 0$, then

$$\lim_{n\to+\infty} \frac{y_A(n+1)}{y_A(n)} = \lambda. \quad \square$$

This last result has already been obtained for the unweighted r-generalized Fibonacci numbers (see, e.g., [2] and [3]).

4. THE CASE OF NONNEGATIVE a_i's

In this section we assume the a_i's are nonnegative real numbers and $a_r > 0$. The following lemma is well known for the unweighted case (see, e.g., [9, §12] or [2, Lemma 2]) and is given without proof.

Lemma 8: There exists a unique real strictly positive characteristic root λ for (1). Moreover, λ is a simple characteristic root and all other characteristic roots of (1) have moduli $\leq \lambda$. □

Theorem 9: Let $a_1, \ldots, a_r$ be nonnegative real numbers with $a_r > 0$, and let λ be the unique positive real number of Lemma 8. Then the following are equivalent:

(a) (1) is asymptotically simple with dominant root λ and dominant multiplicity 1;

(b) the greatest common divisor $\mathrm{GCD}\{i \mid a_i > 0\} = 1$.

Proof: (a)⇒(b). Suppose $\mathrm{GCD}\{i \mid a_i > 0\} = d > 1$, then $p(x)$ is a polynomial in $y = x^d$, which has a unique greatest root $\lambda_d > 0$ from Lemma 8. Hence, the d^{th} roots of λ_d are all roots of

$p(x)$ with the same modulus as $\lambda_d^{1/d}$, which contradicts (a). For *(b)⇒(a)*, see Ostrowski [9, Th. 12.2]. □

Example 10: For the unweighted case ($a_i = 1$ for all i), Lemma 8 implies (1) is asymptotically simple with dominant root λ and multiplicity 1, and we get:

(a) $$L_0 = \left[1 + \sum_{j=1}^{r-1} \frac{j}{\lambda^{1+j}}\right]^{-1};$$

(b) $$\frac{L_A}{L_B} = \frac{\alpha_0 + \sum_{i=1}^{r-1} \alpha_{-i} \lambda^i \sum_{j=i}^{r-1} \frac{1}{\lambda^{1+j}}}{\beta_0 + \sum_{i=1}^{r-1} \beta_{-i} \lambda^i \sum_{j=i}^{r-1} \frac{1}{\lambda^{1+j}}}.$$

Dence [1] obtained similar results in terms of all roots of $p(x)$. Here we obtain the result in terms of only the largest root λ. □

5. THE GENERAL CASE

For arbitrary complex number a_i's, we do not have a result similar to Theorem 9. In Theorem 9, the nonnegativity of the a_i's is important. The next result and example illustrate this fact.

Theorem 11: Assume (1) asymptotically simple with dominant root λ and dominant multiplicity ν, then $\mathrm{GCD}\{i \mid a_i \neq 0\} = 1$.

Proof: Similar to *(a)⇒(b)* of Theorem 9. □

Unfortunately, the condition $\mathrm{GCD}\{i \mid a_i \neq 0\} = 1$ is not a sufficient condition for the general case.

Example 12: Let $p(x) = (x-1)(x-i) = x^2 - (1+i)x + i$, where $i = \sqrt{-1}$. Then $a_i = 1+i$, $a_2 = -i$, and $\mathrm{GCD}\{i \mid a_i \neq 0\} = 1$. But the characteristic roots 1 and i have the same modulus and multiplicity, hence (1) is not asymptotically simple. □

The general problem is to find criteria which are equivalent to or imply that (1) is asymptotically simple. In the sequel, we consider one such criterion which could certainly be weakened.

Let λ be *a priori* any characteristic root of $p(x)$, and set

$$b_i = \sum_{j=i}^{r-1} \frac{a_{1+j}}{\lambda^{1+j}}, \quad i = 0, 1, 2, \ldots.$$

Then $b_0 = 1$ and $b_i = 0$ for $i \geq r$. Consider $\alpha_{-i} = \lambda^{-i}$ for $i = 0, \ldots, r-1$. Then, from (6),

$$\lambda^n = y_0(n) + \sum_{i=1}^{r-1} \lambda^{-i} \sum_{j=1}^{r-i} a_{i+j} y_0(n-j), \quad n \geq 0,$$

and

$$1 = \frac{y_0(n)}{\lambda^n} + \sum_{j=1}^{r-1} b_j \frac{y_0(n-j)}{\lambda^{n-j}}. \tag{9}$$

Let $z(n)=\frac{y_0(n)}{\lambda^n}-\frac{y_0(n-1)}{\lambda^{n-1}}$. We have $z(0)=1$, and from (9),

$$z(n)=-\sum_{j=1}^{r-1}b_j z(n-j),\quad n=1,2,3,\ldots. \tag{10}$$

It follows that

$$\sum_{j=0}^{n}b_j z(n-j)=\begin{cases}1, & \text{for } n=0,\\ 0, & \text{for } n=1,2,3,\ldots.\end{cases}$$

We can now prove the following convergence result for the sequence $\{z(n)\}_{n=0}^{+\infty}$.

Lemma 13: If $\sum_{j=1}^{r-1}|b_j|<1$, then $\{z(n)\}_{n=0}^{+\infty}$ converges to 0.

Proof: From (10), it follows that

$$|z(n)|\le\left[\sum_{j=1}^{r-1}|b_j|\right]\max\{|z(n-j)|: j=1,\ldots,r-1\},\ \text{for } n\ge 1.$$

From this inequality, the sequence $\{M_n=\max\{|z(j)|: j\ge n\}\}_{n=0}^{+\infty}$ is a decreasing sequence, and $M_n\le[\sum_{j=1}^{r-1}|b_j|]M_{n-r+1}$ for any $n\ge r-1$. Hence $\{M_n\}_{n=0}^{+\infty}$ converges to 0 and the result follows since $|z(n)|\le M_n$. □

From the definition, we have

$$\frac{y_0(n)}{\lambda^n}=\sum_{j=0}^{n}z(j)\quad\text{and}\quad\sum_{j=0}^{n}b_j=\sum_{j=0}^{r-1}b_j,\ \text{for } n\ge r-1.$$

Let us consider the following product for $n\ge r-1$:

$$\begin{aligned}\frac{y_0(n)}{\lambda^n}\sum_{j=0}^{r-1}b_j&=\left(\sum_{j=0}^{n}z(j)\right)\left(\sum_{j=0}^{n}b_j\right)\\ &=\sum_{j=0}^{n}\sum_{k=0}^{j}b_k z(j-k)+\sum_{j=1}^{r-1}b_k\sum_{k=n-j+1}^{n}z(k)=1+\sum_{j=1}^{r-1}b_j\sum_{k=0}^{j-1}z(n-k).\end{aligned} \tag{11}$$

Taking the limit in (11), we get

$$\lim_{n\to+\infty}\frac{y_0(n)}{\lambda^n}=\left[\sum_{j=0}^{r-1}b_j\right]^{-1},$$

and we have proved the following result.

Theorem 14: Let λ be a characteristic root of $p(x)$ and set

$$b_i=\sum_{j=i}^{r-1}\frac{a_{1+j}}{\lambda^{1+j}},\ \text{for } i=0,1,\ldots,r-1.$$

If

$$\sum_{j=1}^{r-1}|b_j|<1, \tag{12}$$

then (1) is asymptotically simple with dominant root λ and dominant multiplicity 1. □

The next result illustrates that the condition (12) is satisfied in many cases.

Theorem 15: For any fixed sequence $\{a_i\}_{i=2}^r$ $(r \geq 2, a_r \neq 0)$ of complex numbers, there exists a positive real number R such that, for any a_1 with $|a_1| > R$, the sequence of a_i's satisfies the condition of Theorem 14 for a root λ of the characteristic polynomial $p(x)$.

Proof: Using Lemma 8, let R_1 be the unique positive root of the equation

$$x^r = |a_2|x^{r-2} + 2|a_3|x^{r-3} + \cdots + (r-1)|a_r|.$$

Let us note that we have

$$|a_2| + 2|a_3|R_1^{-1} + 3|a_4|R_1^{-2} + \cdots + (r-1)|a_r|R_1^{-r+2} = R_1^2.$$

Set $R = rR_1$. Suppose $|a_1| > R$ and let λ be any root of maximum modulus of $p(x)$. Since the sum of all the roots of $p(x)$ (with multiplicity) is equal to a_1, we see that λ must have a modulus greater than or equal to $|a_1|/r$. Thus, we have $|\lambda| > R_1$. Then

$$\begin{aligned}\sum_{i=1}^{r-1}|b_i| &\leq \sum_{j=2}^{r}(j-1)|a_j||\lambda|^{-j}\\ &< |\lambda|^{-2}\sum_{j=2}^{r}(j-1)|a_j||\lambda|^{-j+2}\\ &= |\lambda|^{-2}R_1^2 < 1.\end{aligned}$$

Thus, the condition of Theorem 14 is satisfied. □

Remark 16: The last result is intuitively clear. It indicates that, if we increase $|a_1|$, eventually there will be only one root of maximum modulus. But increasing $|a_1|$ means that $p(x)$ behaves like $q(x) = x^r - a_1 r^{r-1} = x^{r-1}(x - a_1)$. Moreover, increasing $|a_1|$ implies the modulus of the largest characteristic root increases also, and since the expression $\sum_{j=1}^{r-1}|b_j|$ does not contain a_1, it will eventually be less than 1 because each term contains negative powers of $|\lambda|$. □

Example 17: As an explicit example, consider the case $r = 3, a_1 = a, a_2 = i$, and $a_3 = -ai$, where $i = \sqrt{-1}$ and a is a complex number with $|a| > 1$. In this case, $|a| > 1$ is one of the characteristic roots of the polynomial $p(x) = (x-a)(x^2 - i)$, and we can check that $\sum_{j=1}^{r-1}|b_j| = 1/|a|^2$, which implies that the condition in Theorem 14 is satisfied. Hence, $\lim_{n\to+\infty}[y_0(n)]/a^n$ exists and is equal to $a^2(a^2+i)/(a^4+1)$. Note that when $|a| \leq 1$ the condition in Theorem 14 is not satisfied and the limit does not exist. □

Remark 18: For the nonnegative a_i's, condition (12) is useless because it implies $a_1 > 0$. Indeed, we have, for λ as given by Lemma 8,

$$\sum_{j=1}^{r-1} b_j = \sum_{i=1}^{r-1}\frac{ia_{i+1}}{\lambda^{i+1}} = \sum_{i=1}^{r-1}\frac{(i-1)a_{i+1}}{\lambda^{i+1}} + 1 - \frac{a_1}{\lambda}$$

and if $a_1 = 0$ then $\sum_{j=1}^{r-1} b_j \geq 1$. □

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ON SOME BASIC LINEAR PROPERTIES OF THE SECOND-ORDER INHOMOGENEOUS LINE-SEQUENCE

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1. INTRODUCTION

A second-order line-sequence is inhomogeneous if its recurrence relation includes a nonzero constant k, such as the following:

$$u_n = u_{n-2} + u_{n-1} + 1, \tag{1.1}$$

where $k = 1$ is the inhomogeneous term.

A line-sequence generated by (1.1), according to the convention adopted in (2.1) of [3], is represented by

$$I_{u_0,u_1} : \ldots u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, \ldots, \tag{1.2}$$

where u_n is the n^{th} element counting from u_0 in both directions, and the pair u_0, u_1 is referred to as a generating pair. The algebraic properties of these sequences have been investigated by Bicknell and Bergum [1], and the general solution of an arbitrary order inhomogeneous sequence has been obtained by Liu [6]. In this article we investigate some basic linear properties of these line-sequences. We shall first treat the simple case of line-sequences generated by (1.1) in some detail. Later on we shall extend the treatment to more general cases.

Some samples of the inhomogeneous line-sequences given by (1.1) are:

$$I_{0,-1} : \ldots 4, -4, 1, -2, [0, -1], 0, 0, 1, 2, \ldots; \tag{1.3}$$

$$I_{-1,0} : \ldots -4, 1, -2, 0, [-1, 0], 0, 1, 2, 4, \ldots. \tag{1.4}$$

For reasons to be explained later, we say that these constitute an inhomogeneous Fibonacci pair. Also, for convenience, the terms of the line-sequences will be represented by

$$I_n : \ldots I_{-3}, I_{-2}, I_{-1}, I_0, I_1, I_2, I_3, \ldots, \tag{1.5}$$

where $I_0 = -1$ is the origin, $I_1 = 0$, and so on.

We shall call the change of a sequential relation from the homogeneous case to the corresponding inhomogeneous case an *inhomogeneous transformation*, and those relations that remain unchanged in form (*inhomogeneously*) *covariant*. As it turns out, many well-known sequential relations are found to be inhomogeneous covariant.

2. THE INHOMOGENEOUS HARMONIC CASE

We define the following inhomogeneous operations in relation to (1.1) and (1.2).

Definition 1: Addition is defined to be addition of corresponding numbers in the line-sequences, together with the inhomogeneous constant 1. Thus,

$$I_{i,j} = I_{i',j'} + I_{i'',j''}, \tag{2.1}$$

where $i = i' + i'' + 1$ and $j = j' + j'' + 1$. We refer to this operation as *inhomogeneous addition*.

Definition 2: Multiplication by a scalar h is defined in the sense of repeated addition. Thus,

$$I_{i,j} = hI_{i',j'}, \tag{2.2}$$

where h is a scalar, $i = hi' + h - 1$ and $j = hj' + h - 1$. We refer to this operation as *inhomogeneous multiplication*.

Definition 3: The inner product of two line-sequences is defined as follows:

$$(I_{i,j}, I_{i',j'}) = (i+1)(i'+1) + (j+1)(j'+1). \tag{2.3}$$

Two line-sequences are said to be orthogonal if and only if their inner product is zero, normal if and only if one's self inner product is one. The length of a line-sequence is defined as the (positive) square root of its inner product with itself.

Definition 4: Two line-sequences are said to be congruent if and only if they constitute the same set of numbers; equal if and only if they are congruent and have the same set of generating numbers. We refer to this as the *uniqueness of generating numbers*.

It is clear that the set I of line-sequences spans a vector space [2] referred to as an *inhomogeneous-harmonic (IH-)space*, where the first predicate signifies the type of operations and the second the recurrence relation. Furthermore, it can be verified easily that the line-sequences (1.3) and (1.4) form an orthonormal pair that serves as the basis set for this space. An arbitrary line-sequence in this space can then be resolved into its inhomogeneous basis components as follows:

$$I_{i,j} = (i+1)I_{0,-1} + (j+1)I_{-1,0}. \tag{2.4}$$

Applying (1.5), this equation can also be expressed in terms of I_n's:

$$I_{i,j} = (i+1)I_{I_{-1},I_0} + (j+1)I_{I_0,I_1}. \tag{2.5}$$

Following are some examples illustrating the inhomogeneous operations.

Example 1: Let $I_{a,b}$ be the identity element of addition. Then, for an arbitrary line-sequence $I_{i,j}$, we must have $I_{a,b} + I_{i,j} = I_{i,j}$. By (2.1), we have $I_{a,b} + I_{i,j} = I_{a+i+1,b+j+1}$. Comparing the right-hand sides of these equations, we obtain $a = b = -1$. Hence, the additive identity is a sequence of -1's:

$$I_{-1,-1}: \ldots -1, -1, [-1, -1], -1, -1, \ldots. \tag{2.6}$$

Example 2: By (2.1), we have

$$I_{i,j} + I_{-i-2,-j-2} = I_{-1,-1}; \tag{2.7}$$

hence, $I_{-i-2,-j-2}$ is the inverse element of $I_{i,j}$.

Example 3: Letting $h = -1$ in (2.2), we find that

$$-I_{i,j} = I_{-i-2,-j-2}, \tag{2.8}$$

which is the negative element equation. Together with (2.7), we see that the inverse element is just the negative element. In particular,

$$-I_{-1,-1} = I_{-1,-1}, \tag{2.9}$$

which confirms once more that $I_{-1,-1}$ is indeed the identity element of addition. Combining (2.7) and (2.8), we have

$$I_{i,j} - I_{i,j} = I_{-1,-1}, \tag{2.10}$$

which is the equation of elimination.

Applying (2.8) to (2.1), we obtain

$$I_{i,j} = I_{i',j'} - I_{i'',j''}, \tag{2.11}$$

where $i = i' - i'' - 1$ and $j = j' - j'' - 1$. This is the subtraction formula.

Example 4: Letting $h = 0$ in (2.2), we obtain

$$0I_{i,j} = I_{-1,-1}. \tag{2.12}$$

This is the equation of zero (scalar) multiplication.

Applying (2.2) and (2.1) successively, we have $iI_{0,-1} + jI_{-1,0} = I_{i-1,-1} + I_{-1,j-1} = I_{i-1,j-1}$. Thus, $iI_{0,-1} + jI_{-1,0} = I_{-1,-1}$ if and only if $i = j = 0$. This confirms the linear independence of the two basis vectors.

Example 5: Applying (2.2), we have

$$(a+b)I_{i,j} = I_{(a+b)i+(a+b)-1,\,(a+b)j+(a+b)-1}.$$

Applying (2.2) and (2.1) successively, we have

$$\begin{aligned} aI_{i,j} + bI_{i,j} &= I_{ai+a-1,\,aj+a-1} + I_{bi+b-1,\,bj+b-1} \\ &= I_{(a+b)i+(a+b)-1,\,(a+b)j+(a+b)-1}. \end{aligned}$$

Comparing these results, we have

$$(a+b)I_{i,j} = aI_{i,j} + bI_{i,j}. \tag{2.13}$$

This is the *right distributive property* of scalar multiplication.

Again applying (2.1) and (2.2) successively, on the one hand, we have

$$\begin{aligned} h(I_{i',j'} + I_{i'',j''}) &= hI_{i'+i''+1,\,j'+j''+1} \\ &= I_{h(i'+i''+1)+h-1,\,h(j'+j''+1)+h-1}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} hI_{i',j'} + hI_{i'',j''} &= I_{hi'+h-1,\,hj'+h-1} + I_{hi''+h-1,\,hj''+h-1} \\ &= I_{h(i'+i''+1)+h-1,\,h(j'+j''+1)+h-1}. \end{aligned}$$

Comparing these results, we find that

$$h(I_{i',j'} + I_{i'',j''}) = hI_{i',j'} + hI_{i'',j''}. \tag{2.14}$$

This is the *left distributive property* of scalar multiplication.

Example 6: Let A and B denote the pair of Golden ratios, so $A+B=1$ and $AB=-1$.

Parallel to the homogeneous case (see [4], (2.1) and (2.2)), applying (2.2), we obtain

$$I_{0,-1} + AI_{-1,0} = I_{0,-1} + I_{-1,A-1} = I_{0,A-1} \tag{2.15}$$

and

$$I_{0,-1}+BI_{-1,0}=I_{0,-1}+I_{-1,B-1}=I_{0,B-1}. \tag{2.16}$$

Subtracting (2.16) from (2.15), then applying (2.4) and (2.12), we find

$$I_{-1,0}=(I_{0,A-1}-I_{0,B-1})/(A-B), \tag{2.17}$$

which is the inhomogeneous version of Binet's formula.

This result indicates that the right-hand sides of (2.15) and (2.16) constitute the inhomogeneous Golden pair.

In terms of matrix representation, let

$$F'=\begin{bmatrix} I_{0,-1} \\ I_{-1,0} \end{bmatrix}, \quad G'=\begin{bmatrix} I_{0,B-1} \\ I_{0,A-1} \end{bmatrix}, \quad M=\begin{bmatrix} 1 & B \\ 1 & A \end{bmatrix};$$

then it can be shown that

$$MF'=G' \quad \text{and} \quad M^{-1}G'=F'. \tag{2.18}$$

Thus, these matrix relations are inhomogeneously covariant to their homogeneous counterparts (see [4], (4.8) and (4.9)).

The sum of the inhomogeneous Golden pair then gives the inhomogeneous Lucas line-sequence

$$I_{1,0}=I_{0,A-1}+I_{0,B-1}, \tag{2.19}$$

which generates the inhomogeneous Lucas line-sequence

$$L'_n: \ldots -5, 2, -2, 1, 0, 2, 3, 6, \ldots, \tag{2.20}$$

where we adopt L' to represent inhomogeneous Lucas numbers, and where $L'_0=1$, $L'_1=0$ is the pair of generating numbers.

Note that (2.19) is another example of inhomogeneous covariance to its homogeneous counterpart (see [4], (3.1)). Note also that the line-sequence (2.20) is congruent to the one generated by the first Fibonacci basis vector $F_{1,0}$ with an inhomogeneous term $k=1$. The second Fibonacci basis vector $F_{0,1}$ with an inhomogeneous term $k=1$ generates the inhomogeneous line-sequence congruent to the two inhomogeneous basis vectors (1.3) and (1.4). For this reason, we are justified to refer to (1.3) and (1.4) as the inhomogeneous Fibonacci pair, as we have done above.

Furthermore, applying (2.4) to (2.19), we obtain the expression of the inhomogeneous Lucas line-sequence in terms of the inhomogeneous basis components,

$$I_{1,0}=2I_{0,-1}+I_{-1,0}. \tag{2.21}$$

This is another example of inhomogeneous covariance relating to the homogeneous relation between Lucas and Fibonacci line-sequences: $F_{2,1}=2F_{1,0}+F_{0,1}$.

3. THE TRANSLATIONAL PROPERTIES

The *translation operation* on the inhomogeneous line-sequence is defined in the same way as that on the homogeneous line-sequence (see [3], (3.2)) with the following appropriate modifications.

Definition 5: The translation operator T_i, $i =$integer, acting on a line-sequence shifts all of its elements i places to the right if $i > 0$, forming a new but congruent line-sequence

$$T_i I_{u_0, u_1} = I_{u_i, u_{i+1}}. \tag{3.1}$$

We say that translation is a congruent operation, because it preserves congruency of the line-sequence. In particular, for the additive identity, we have

$$T_i I_{-1,-1} = I_{-1,-1}. \tag{3.2}$$

Namely, the additive identity is translationally invariant.

Since translation preserves congruence, translation must be distributive over addition of line-sequences:

$$T_i(I_{u_0,u_1} + I_{u_0',u_1'}) = T_i I_{u_0,u_1} + T_i I_{u_0',u_1'}. \tag{3.3}$$

This is the *left distributive property* of translation. Using (3.1) and (2.1), we have

$$T_i(I_{u_0,u_1} + I_{u_0',u_1'}) = I_{u_i+u_i'+1, u_{i+1}+u_{i+1}'+1}. \tag{3.4}$$

Since translation preserves congruency, translation after repeated addition is the same as repeated addition after translation. Hence, multiplication and translation commute:

$$h(T_i I_{u_0,u_1}) = T_i(h I_{u_0, u_1}). \tag{3.5}$$

Definition 6: Obviously, T_i is uniquely defined; thus, two translations are said to be equal if and only if both effect the same shift of the elements in a line-sequence.

Definition 7: Addition of two translations on a line-sequence is defined to be the sum of the two translated line-sequences,

$$(T_i + T_j)I_{u_0,u_1} = T_i I_{u_0,u_1} + T_j I_{u_0,u_1}. \tag{3.6}$$

Namely, addition of translations is distributive over line-sequences. This is the *right distributive property* of translation. Obviously, addition of the translation operations is commutative,

$$T_i + T_j = T_j + T_i. \tag{3.7}$$

Applying (3.1) and (2.1) to (3.6), we obtain

$$(T_i + T_j)I_{u_0,u_1} = I_{u_i+u_j+1,\, u_{i+1}u_{j+1}+1}. \tag{3.8}$$

Therefore, the sum of two translations does not preserve the congruence of the line-sequence it operates on.

Definition 8: By the product notation, $T_i \circ T_j$, we mean successive applications of the respective translation on a line-sequence, the result of which is such that all the elements shift $i + j$ places. Hence, this is equivalent to the application of a single operation on that line-sequence. That is,

$$T_i \circ T_j = T_{i+j}. \tag{3.9}$$

Obviously, the translations commute with respect to the order of application,

$$T_i \circ T_j = T_j \circ T_i. \tag{3.10}$$

Applying (3.9) and (3.1), we have

$$(T_i \circ T_j)I_{u_0,u_1} = I_{u_{i+j},u_{i+j+1}}. \tag{3.11}$$

Letting $i = j$ and adopting the exponential convention for repeated translation, it follows from (3.9) that $T_i^2 = T_{2i}$. In general, we have

$$T_i^n = T_{ni}. \tag{3.12}$$

We illustrate the foregoing results with the following examples.

Example 7: Putting $i = j$ in (3.8), we obtain $(T_i + T_i)I_{u_0,u_1} = I_{2u_i+1,2u_{i+1}+1}$. Letting $h = 2$ in (3.5), we find that $2T_iI_{u_0,u_1} = I_{2u_i+1,2u_{i+1}+1}$. Comparing these results, we have

$$T_i + T_i = 2T_i. \tag{3.13}$$

Hence, we conclude that the scalar multiplication of a translation is equivalent to the repeated addition of that translation.

Example 8: Putting $j = i+1$ in (3.8) and applying (1.1), we get $(T_i + T_{i+1})I_{u_0,u_1} = I_{u_{i+2},u_{i+3}}$. This induces the recurrence formula of translation: $T_i + T_{i+1} = T_{i+2}$.

Let $i = 0$ and $I = T_0$, the identity of translation, then we have

$$I + T - T^2 = 0. \tag{3.14}$$

So the pleasant equation of translation (see [5], (2.16)) is inhomogeneously covariant.

Example 9: From (2.5), we have $I_{u_0,u_1} = (u_0+1)I_{I_{-1},I_0} + (u_1+1)I_{I_0,I_1}$. Applying translation on both sides and using (3.1), (3.5), and (2.1), we obtain

$$T_iI_{u_0,u_1} = I_{(u_0+1)I_{i-1}+(u_1+1)I_i+u_0+u_1+1,\,(u_0+1)I_i+(u_1+1)I_{i+1}+u_0+u_1+1}. \tag{3.15}$$

Thus, by Definition 4, the uniqueness of generating numbers, we arrive at the following formula relating u_i to the corresponding pair of I_i's:

$$u_i = (u_0+1)I_{i-1} + (u_1+1)I_i + u_o + u_1 + 1. \tag{3.16}$$

Putting $u_0 = L_0' = 1$ and $u_i = L_i' = 0$ in (3.16), we obtain the expression of the inhomogeneous Lucas numbers L_i' in terms of the inhomogeneous Fibonacci numbers:

$$L_i' = 2I_{i-1} + I_i + 2. \tag{3.17}$$

Applying (1.1), this becomes

$$L_i' = I_{i-1} + I_{i+1} + 1, \tag{3.18}$$

which is the inhomogeneous version of the relation between the Lucas numbers and the Fibonacci numbers: $L_i = F_{i-1} + F_{i+1}$.

From (3.15), we find that

$$T_iI_{1,0} = I_{2I_{i-1}+I_i+2,\,2I_i+I_{i+1}+2}. \tag{3.19}$$

Substitute (3.17) into (3.19) to obtain

$$T_iI_{L_0',L_1'} = I_{L_i',L_{i+1}'}, \tag{3.20}$$

which is none other than the translation formula for the inhomogeneous Lucas line-sequence.

Example 10: Applying (1.1) and (1.2) and using (3.1), we obtain $(T_{-1}+T_1)I_{i,0} = I_{-2,1}+I_{0,2}$.

Applying (2.1) and (2.4) to the right-hand side and using (2.12), we obtain $(T_{-1}+T_1)I_{1,0} = 5I_{-1,0}$, which translates into $(T_{i-1}+T_{i+1})I_{1,0} = 5T_iI_{-1,0}$.

Applying (2.20) and (1.5) to both sides, respectively, we find that

$$L'_{i-1}+L'_{i+1} = 5I_i, \tag{3.21}$$

which is another relation covariant to its homogeneous counterpart $L_{i-1}+L_{i+1} = 5F_i$,

Example 11: From (3.1), and putting $u_j = u'_0$ and $u_{j+1} = u'_i$, we have $T_i(T_jI_{u_o,u_1}) = T_iI_{u_j,u_{j+1}} = I_{u'_i,u'_{i+1}}$, where, by (3.16), we have

$$u'_i = (u'_0+1)I_{i-1}+(u'_1+1)I_i+u'_0+u'_1+1 \quad \text{and} \quad u'_0 = u_j = (u_0+1)I_{j-1}+(u_1+1)I_j+u_0+u_1+1.$$

On the other hand, applying (3.9) and (3.1), we obtain $T_i(T_jI_{u_0,u_1}) = T_{i+j}I_{u_0,u_1} = I_{u_{i+j},u_{i+j+1}}$, where, by (3.16), we have

$$u_{i+j} = (u_0+1)I_{i+j-1}+(u_1+1)I_{i+j}+u_0+u_1+1.$$

By Definition 4, $u'_i = u_{i+j}$. Since u_0 is an independent parameter, the coefficients of u_0 must be equal in the two expressions. This leads to the following relation: $I_{j-1}I_{i-1}+I_jI_1+I_{j-1}+I_j+I_{i-1}+I_i+2 = I_{i+j-1}+1$. Putting $i=j$, we obtain the relation

$$(I_{i-1}+1)^2+(I_i+1)^2 = I_{2i-1}+1, \tag{3.22}$$

which is the inhomogeneous version of the relation $F_{i-1}^2+F_i^2 = F_{2i-1}$. Likewise, we obtain the relation

$$(L'_i+1)(I_i+1) = I_{2i}+1, \tag{3.23}$$

which is the inhomogeneous version of the relation $L_iF_i = F_{2i}$.

Example 12: Starting from $I_{I_i,I_{i+1}} = (A-B)I_{I_i,I_{i+1}}/(A-B)$ and applying (2.13) and (2.2), we obtain the translational form of the inhomogeneous version of Binet's formula:

$$I_{I_i,I_{i+1}} = \frac{1}{A-B}\left(I_{AI_i+A-1,\,AI_{i+1}+A-1} - I_{BI_i+B-1,\,BI_{i+1}+B-1}\right). \tag{3.24}$$

Similarly, applying (2.5) to (2.19), we obtain $I_{1,0} = AI_{I_0,I_1}+BI_{I_0,I_1}+2I_{I_{-1},I_0}$. Applying translation on both sides and using (3.20) and (2.2), we get

$$I_{L'_i,L'_{i+1}} = I_{AI_i+A-1,\,AI_{i+1}+A-1}+I_{BI_i+B-1,\,BI_{i+1}+B-1}+2I_{I_{i-1},I_i}. \tag{3.25}$$

This is the translational form of Binet's formula for the inhomogeneous Lucas numbers.

4. THE INHOMOGENEOUS ANHARMONIC CASE

An anharmonic recurrence relation with an inhomogeneous constant term can be expressed in general as follows:

$$u_n = cu_{n-2}+bu_{n-1}+k, \tag{4.1}$$

where b and c, called the anharmonic parameters, are nonzero constants not both equal to one, and k is the inhomogeneous constant term. An anharmonic line-sequence is represented by

$$J_{u_0,u_1}: \ldots u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, \ldots. \tag{4.2}$$

The corresponding terms in the line-sequence are represented by

$$J_n: \ldots J_{-3}, J_{-2}, J_{-1}, J_0, J_1, J_2, J_3, \ldots, \tag{4.3}$$

where J_0 is the origin.

It is easy to see from (4.1) that anharmonic addition of anharmonic line-sequences is incompatible with the translational invariance of the additive identity. Therefore, we shall try harmonic operations as defined below.

Definition 9: Addition is inhomogeneous, that is, addition of corresponding terms in the line-sequences, together with the inhomogeneous constant k. Thus,

$$J_{i,j} = J_{i',j'} + J_{i'',j''}, \tag{4.4}$$

where $i = i' + i'' + k$ and $j = j' + j'' + k$.

Definition 10: Multiplication by a scalar h is defined in the sense of repeated addition. That is,

$$J_{i,j} = hJ_{i',j'}, \tag{4.5}$$

where h is a scalar, $i = hi' + (h-1)k$, and $j = hj' + (h-1)k$.

Definition 11: The inner product of two line-sequences is defined as follows:

$$(J_{i,j}, J_{i',j'}) = (i+k)(i'+k) + (j+k)(j'+k). \tag{4.6}$$

Two line-sequences are said to be orthogonal if and only if their inner product is zero, normal if and only if one's self inner product is one. The length of a line-sequence is defined as the (positive) square root of its inner product with itself.

Furthermore, let J_{u_0,u_1} denote the additive identity, then, for an arbitrary line-sequence $J_{i,j}$, we have $J_{u_0,u_1} + J_{i,j} = J_{i,j}$. However, by (4.4), we have $J_{u_0,u_1} + J_{i,j} = J_{u_0+i+k,\, u_1+j+k}$. Therefore, $u_0 = u_1 = -k$. So we find the additive identity

$$J_{-k,-k}: \ldots -k, -k, [-k, -k], -k, -k, \ldots. \tag{4.7}$$

On the other hand, the line-sequence of the additive identity must be translationally invariant, namely, $u_0 = u_1 = u_2$. By (4.4), $u_2 = cu_0 + bu_1 + k$, so we must have $u_0 = -k/(c+b-1)$.

Comparing these results, we arrive at the condition between the anharmonic parameters:

$$c + b = 2. \tag{4.8}$$

Then it is obvious that the set J of anharmonic line-sequences together with the inhomogeneous operations defined above constitute a vector space referred to as an *inhomogeneous-anharmonic (IA-)space*.

Let J_{u_{-1},u_0} and J_{u_0,u_1} denote the pair of basis vectors. The orthogonality requirement leads to the following combinations of basis pair choices, differing in parity: $J_{1-k,-k}$ or $J_{-1-k,-k}$ and $J_{-k,1-k}$ or $J_{-k,-1-k}$.

We choose the following combination consistent with previous works, with both the anharmonic parameters and the inhomogeneous constant specified:

$$J_{1-k,-k}(c,b;k):\ldots,\frac{c+b^2}{c^2}-k,-\frac{b}{c}-k,[1-k,-k],c-k,cb-k; \tag{4.9}$$

$$J_{-k,1-k}(c,b;k):\ldots,\frac{-b}{c^2}-k,\frac{1}{c}-k,[-k,1-k],b-k,c+b^2-k. \tag{4.10}$$

If $c=b=k=1$, this pair reduces to the harmonic basis pair (1.3) and (1.4) above. If $k=0$, it reduces to the homogeneous basis pair (see [3], (4.2) and (4.3)). An arbitrary line-sequence in this space can be resolved into its basis components according to the formula:

$$J_{i,j}=(i+k)J_{1-k,-k}+(j+k)J_{-k,1-k}. \tag{4.11}$$

Note that putting $h=-1$ in (4.5) gives

$$-J_{i,j}=J_{-i-2k,-j-2k}. \tag{4.12}$$

This is the negative element equation, which reduces to (2.8) if $k=1$.

Putting $h=0$ in (4.5) gives

$$0J_{i,j}=J_{-k,-k}. \tag{4.13}$$

This is the zero multiplication equation, which reduces to (2.12) if $k=1$.

Note that

$$-J_{-k,-k}=J_{-k,-k}, \tag{4.14}$$

which confirms that $J_{-k,-k}$ is indeed the additive identity.

5. THE HOMOGENEOUS ANHARMONIC CASE

It is also possible to combine line-sequences generated by (4.1), but with different inhomogeneous constant terms. To avoid confusion, we represent the set of line-sequences under these types of operations by $H_{i,j}(c,b;k)$. We define the following operations.

Definition 12: Addition of two line-sequences is defined as addition of corresponding terms in the line-sequences:

$$H_{i,j}(c,b;k)=H_{i',j'}(c,b;k')+H_{i'',j''}(c,b;k''), \tag{5.1}$$

where $i=i'+i''$, $j=j'+j''$, and $k=k'+k''$. We refer to this as *homogeneous addition*.

The additive identity is, of course, $H_{0,0}(c,b;0)$, namely, a sequence of zeros, and the inverse element of $H_{i,j}(c,b;k)$ is $H_{-i,-j}(c,b;-k)$.

Definition 13: Multiplication by a scalar h is defined as

$$H_{i,j}(c,b;k)=hH_{i',j'}(c,b;k'), \tag{5.2}$$

where h is scalar, $i=hi'$, $j=hj'$, and $k=hk'$. We refer to this as *homogeneous multiplication*.

Definition 14: The inner product of two line-sequences is defined as follows:

$$(H_{i,j},H_{i',j'})=(i+k)(i'+k')+(j+k)(j'+k'). \tag{5.3}$$

Two line-sequences are said to be orthogonal if and only if their inner product is zero, normal if and only if one's self inner product is one. The length of a line-sequence is defined as the (positive) square root of its inner product with itself.

Thus, it is clear that the set H of line-sequences spans a vector space, referred to as a *homogeneous-anharmonic* (*HA-*)*space*. Obviously, the set of basis of this three-dimensional space is given by:

$$H_{1,0}(c,b;\,0)\colon \ldots, \frac{c+b^2}{c^2}, -\frac{b}{c}, [1,0], c, cb, \ldots; \tag{5.4}$$

$$H_{0,1}(c,b;\,0)\colon \ldots, \frac{-b}{c^2}, \frac{1}{c}, [0,1], b, c+b^2, \ldots; \tag{5.5}$$

$$H_{0,0}(c,b;\,1)\colon \ldots, \frac{b-c}{c^2}, \frac{-1}{c}, [0,0], 1, b+1, \ldots. \tag{5.6}$$

An arbitrary H line-sequence can then be decomposed into its basis components as follows:

$$H_{i,j}(c,b;\,k) = iH_{1,0}(c,b;\,0) + jH_{0,1}(c,b;\,0) + kH_{0,0}(c,b;\,1). \tag{5.7}$$

Since the operations employed in [1] are basically compatible with the homogeneous operations, many results arrived therein can be derived directly in terms of H line-sequences, but not directly in terms of I line-sequences, which undergo inhomogeneous operations. Following are some examples.

Example 13: From (5.1), we have

$$H_{0,0}(1,1;\,1) = H_{1,1}(1,1;\,0) + H_{-1,-1}(1,1;\,1), \tag{5.8}$$

which corresponds to (1.4) in [1];

$$H_{1,2}(1,1;\,1) = H_{1,2}(1,1;\,0) + H_{0,0}(1,1;\,1), \tag{5.9}$$

which corresponds to (1.14) in [1];

$$H_{2,1}(1,1;\,1) = H_{2,1}(1,1;\,0) + H_{0,0}(1,1;\,1), \tag{5.10}$$

which corresponds to (1.22) in [1]; and so forth.

Example 14: Substituting $c = b = 1$ into (5.7), we obtain

$$H_{i,j}(1,1;\,k) = iH_{1,0}(1,1;\,0) + jH_{0,1}(1,1;\,0) + kH_{0,0}(1,1;\,1), \tag{5.11}$$

which corresponds to (2.2) in [1], or to (1.13) in [1] if $k = 1$.

Substituting (5.8) into (5.11) and using (5.2) and the distributive property of multiplication, we obtain

$$H_{i,j}(1,1;\,k) = iH_{1,0}(1,1;\,0) + jH_{0,1}(1,1;\,0) + kH_{1,1}(1,1;\,0) + H_{-k,-k}(1,1;\,k), \tag{5.12}$$

which corresponds to (2.3) in [1], or to (1.6) in [1] if $k = 1$.

Using (5.1) and (5.2), we have

$$H_{0,0}(1,1;\,k) = H_{1,2}(1,1;\,k) - H_{1,0}(1,1;\,0) - 2H_{0,1}(1,1;\,0).$$

Substituting into (5.11), we obtain

$$H_{i,j}(1,1;k) = (i-k)H_{1,0}(1,1;0) + (j-2k)H_{0,1}(1,1;0) + kH_{1,2}(1,1;1), \tag{5.13}$$

which corresponds to (2.6) in [1]. It reduces to (1.33) in [1] if $k=1$. And so forth.

Following is a table of some equivalence and correspondence ($\rightarrow$) relations.

TABLE 1. Some Equivalence and Correspondence Relations

No.	Relations	References
1	$J_{i,j}(1,1;0) = F_{i,j}$	[3], (1.3)
2	$J_{i,j}(c,b;0) = G_{i,j}$	[3], (4.1)
3	$J_{i,j}(1,1;1) = I_{i,j}$	(1.1)
4	$H_{1,1}(1,1;0) \rightarrow F_n$	[1], p. 193
5	$H_{1,1}(1,1;1) \rightarrow c_n$	[1], (1.2)
6	$H_{1,2}(1,1;1) \rightarrow c_n^*$	[1], (1.3)
7	$H_{0,0}(1,1;1) \rightarrow c_n'$	[1], (1.4)
8	$H_{a,b}(1,1;1) \rightarrow c_n(a,b)$	[1], (1.5)

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CONSTRUCTION OF 2*n CONSECUTIVE n-NIVEN NUMBERS

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1. INTRODUCTION

Fix a natural number, $n \geq 2$, as our base. For a a natural number, define $s(a)$ to be the sum of the digits of a written in base n. Define $\nu(a)$ to be the number of digits of a written in base n, i.e., $n^{\nu(a)-1} \leq a < n^{\nu(a)}$. For a and b natural numbers, denote the product of a and b by $a*b$. For a and b natural numbers written in base n, let ab denote the concatenation of a and b, i.e., $ab = a*n^{\nu(b)}+b$. Denote concatenation of k copies of a by a_k, i.e.,

$$a_k = a + a*n^{\nu(a)} + a*n^{2*\nu(a)} + \cdots + a*n^{(k-1)*\nu(a)} = a*\frac{n^{k*\nu(a)}-1}{n^{\nu(a)}-1}.$$

Definition: We say a is an *n-Niven number* if a is divisible by its base n digital sum, i.e., $s(a)|a$.

Example: For $n=11$, we have $15 = 1*11+4*1$, so $s(15) = 1+4 = 5$. Since $5|15$, 15 is an 11-Niven number.

It is known that there can exist at most $2*n$ consecutive n-Niven numbers [3]. It is also known that, for $n=10$, there exist sequences of twenty consecutive 10-Niven numbers (often just called Niven numbers) [2]. In [1], sequences of six consecutive 3-Niven numbers and four consecutive 2-Niven numbers were constructed. Mimicking a construction of twenty consecutive Niven numbers in [4], we can prove Grundman's conjecture.

Conjecture: For each $n \geq 2$, there exists a sequence of $2*n$ consecutive n-Niven numbers.

Before giving a constructive proof of this conjecture, we give some notation and results that will give us necessary congruence conditions for a number, α, to be the base n digital sum of the first of $2*n$ consecutive n-Niven numbers, β.

For any prime p, let $a(p)$ be such that $p^{a(p)} \leq n$ but $p^{a(p)+1} > n$. For any prime p, let $b(p)$ be such that $p^{b(p)}|(n-1)$ but $p^{b(p)+1}\nmid(n-1)$. Let $\mu = \prod_p p^{a(p)-b(p)}$.

Theorem 1: A sequence of $2*n$ consecutive n-Niven numbers must begin with a number congruent to $n^{\mu*m}-n$ modulo $n^{\mu*m}$ (but not congruent to $n^{\mu*m+1}-n$ modulo $n^{\mu*m+1}$) for some positive integer m.

Proof: It is shown in [3] that the first of $2*n$ consecutive n-Niven numbers, β, must be congruent to 0 modulo n. Suppose $\beta \equiv n^{m'}-n \bmod n^{m'}$ but $\beta \not\equiv n^{m'+1}-n \bmod n^{m'+1}$. We will show that $\mu|m'$. It is enough to show $p^{a(p)-b(p)}|m'$ for all p. Among the n consecutive numbers $s(\beta)$, $s(\beta+1), \ldots, s(\beta+n-1)$, there is a multiple of $p^{a(p)}$. Similarly for $s(\beta+n)$, $s(\beta+n+1), \ldots, s(\beta+2*n-1)$. By the definition of an n-Niven number, this means $p^{a(p)}|s(\beta+i)$, $s(\beta+i)|(\beta+i)$, $p^{a(p)}|s(\beta+n+j)$, and $s(\beta+n+j)|(\beta+n+j)$ for some i,j in $0,1,\ldots,n-1$. But $s(\beta+i) = S(\beta)+i$

and $s(\beta+n+j)=s(\beta)+n+j-m'*(n-1)$. So, $p^{a(p)}|(n+j-i)$ and $p^{a(p)}|(n+j-i-m'*(n-1))$, and therefore, $p^{a(p)}|m'*(n-1)$. Since $p^{b(p)}$ is the highest power of p dividing $n-1$, we obtain $p^{a(p)-b(p)}|m'$. □

Corollary 1: A sequence of $2*n$ consecutive n-Niven numbers must consist of numbers having at least μ digits written in base n.

Another result of this theorem is to get restrictions on the digital sum, α, of the first of $2*n$ consecutive n-Niven numbers.

Corollary 2: If $\alpha=s(\beta)$ for β the first of $2*n$ consecutive n-Niven numbers, then for m as in Theorem 1 and for

$$\gamma=\mathrm{lcm}(\alpha,\alpha+1,\ldots,\alpha+n-1) \tag{1}$$

and

$$\gamma'=\mathrm{lcm}(\alpha+n-\mu*m*(n-1),\alpha+n+1-\mu*m*(n-1),\ldots,\alpha+2*n-1-\mu*m*(n-1)), \tag{2}$$

we have $\gcd(\gamma,\gamma')\,|\,\mu*m*(n-1)$.

Proof: For β the first of $2*n$ consecutive n-Niven numbers and for α the base n digital sum of β, since $\beta\equiv 0 \bmod n$, we get

$$(\alpha+i)|(\beta+i) \text{ for } i=0,1,\ldots,n-1$$

and, by Theorem 1, we get

$$(\alpha+n+j-\mu*m*(n-1))\,|\,(\beta+n+j) \text{ for } j=0,1,\ldots,n-1.$$

These imply $\beta\equiv\alpha \bmod \gamma$ and $\beta\equiv\alpha-\mu*m*(n-1) \bmod \gamma'$. These two congruences are compatible if and only if $\gcd(\gamma,\gamma')\,|\,\mu*m*(n-1)$. □

Finally, we will need the following three lemmas in our construction.

Lemma 1: For $\delta=\mathrm{lcm}(\gamma,\gamma')$ there exist positive integer multiples of δ, say $k*\delta$ and $k'*\delta$ so that $\gcd(s(k*\delta),s(k'*\delta))=n-1$. Further, this is the smallest the greatest common divisor of the digital sums of any two integral multiples of δ can be.

Proof: Since $(n-1)|\delta$, we see that $(n-1)|k*\delta$ for any $k\in\mathbb{Z}$. Since $n-1$ is one less than our base, $(n-1)|s(k*\delta)$, so the smallest the greatest common divisor can be is $n-1$.

Now let $a\mathbf{a}b0_\ell$ be the base n expansion of δ with a and b nonzero digits and $\mathbf{a}$ a block of digits of length ℓ'. We can suppose without loss of generality that $\mathbf{a}$ ends in a digit other than $n-1$, for if it does end in $n-1$ we can consider $(n+1)*\delta$ in place of δ. Since $\delta<n^{\ell+\ell'+2}$, there is a multiple of δ between any two multiples of $n^{\ell+\ell'+2}$, so there is some multiple of δ between $(n-1)*n^{\ell+\ell'+2}$ and $n^{\ell+\ell'+3}$, i.e., some κ so that the base n representation of $\kappa*\delta$ is $(n-1)\mathbf{a}'$ with $\nu(\mathbf{a}')=\ell+\ell'+2$. Then, for $k=\kappa*n^{\ell+\ell'+2}+1$ and $k'=(n^{\ell+2*\ell'+4}+1)*k$, we get $(n-1)\mathbf{a}'a\mathbf{a}b0_\ell$ as the base n representation of $k*\delta$ and $(n-1)\mathbf{a}'a(\mathbf{a}+1)(b-1)\mathbf{a}'a\mathbf{a}b0_\ell$ as the base representation of $k'*\delta$. Then we see $s(k*\delta)=n-1+s(\mathbf{a}')+s(a)+s(\mathbf{a})+s(b)$, while $s((k'*\delta))=n-1+2*s(\mathbf{a}')+2*s(a)+2*s(\mathbf{a})+1+2*s(b)-1$; thus, $s(k'*\delta)=2*s(k*\delta)-(n-1)$. This means $\gcd(s(k*\delta), s(k'*\delta))=n-1$. □

Remark 1: It follows from the proof that we can choose k, k' in the lemma with $\nu(k*\delta)\le 5+2*\ell+2*\ell'$ and $\nu(k'*\delta)\le 9+3*\ell+4*\ell'$ when δ [or $(n+1)*\delta$ if $\mathbf{a}$ ends in $n-1$] has

$\ell+\ell'+2$ digits in base n. Since ℓ is the number of terminal zeros in δ and ℓ' is the number of digits strictly between the first and last nonzero digit of δ, we have

$$v(k*\delta)\le 5+2*(\ell+\ell')\le 5+2*(v(\delta)-1).$$

Since $\delta<(\alpha+n-1)^{2*n}$, we have $v(\delta)\le 2*n*(\log_n(\alpha+n-1)+1)$.. This inequality leads to

$$v(k*\delta)\le 5+2*(2*n*(\log_n(\alpha+n-1)+1)-1)\le 5+4*n*(\log_n(\alpha+n-1)+1).$$

Similarly,

$$v(k*\delta)\le 2*(5+4*(n*(\log_n(\alpha+n-1)+1))).$$

This comes into play in constructing a "growth condition" in the next section.

Lemma 2: For any positive integer z, if $\alpha\equiv z \bmod \gamma$, then $(n-1)|(\alpha-s(z))$.

Proof: This is equivalent to showing $\alpha\equiv s(z) \bmod (n-1)$. We know $z\equiv s(z) \bmod (n-1)$ as $n-1$ is one less than our base. Since $(n-1)|\gamma$, we get $z\equiv\alpha \bmod (n-1)$ which, taken with the previous congruence, gives the result. □

Lemma 3: For positive integers x, y, z, if $\gcd(x,y)|z$ and $z\ge x*y$, then we can express z as a nonnegative linear combination of x and y.

Proof: That we can write z as a linear combination of x and y follows from the extended Euclidean algorithm. To see that we can obtain a nonnegative linear combination, suppose $z=r*x+t*y$. Since $x,y,z>0$, at least one of r and t is positive. If they are both nonnegative, we are done, so suppose without loss of generality that $r<0$. Then $z=z+(y*x-x*y)=(r+y)*x+(t-x)*y$. We can repeat this until we have a nonnegative coefficient on x, so assume without loss of generality that $r+y\ge 0$. If $t-x\ge 0$, then we have a nonnegative linear combination and so are done. This means we are left to consider $r<0, t>0, r+y\ge 0$, and $t-x<0$. However, if $z=r*x+t*y$ with $r<0, x>0$, then $t*y>z$ so that $(t-x)*y>z-x*y\ge 0$ by hypothesis. But $y>0$ and $(t-x)*y\ge 0$ means $t-x\ge 0$, a contradiction. □

2. CONSTRUCTION

In this section we shall construct an α that can serve as the digital sum of the first of $2*n$ consecutive n-Niven numbers. We then use this α to actually construct the first of $2*n$ consecutive n-Niven numbers, β, with $\alpha=s(\beta)$. We present the construction using the results of the previous section. In that section, we derived congruence restrictions on the digital sum of the first of $2*n$ consecutive n-Niven numbers (if such a sequence exists). We now use these restrictions to construct such a sequence.

Let $a(p), b(p)$, and μ be as in the previous section. For our construction, we specifically fix $m=\prod_{p|n} p$. For p a prime, define $c(p)$ by

$$p^{c(p)}|(\mu*m*(n-1)-i) \text{ for some } i=1,2,\dots,2*n-1$$

and

$$p^{c(p)+1}\nmid(\mu*m*(n-1)-i) \text{ for any } i=1,2,\dots,2*n-1.$$

To produce an α satisfying $\gcd(\gamma,\gamma')|\mu*m*(n-1)$, we impose the following condition.

Congruence Condition I: For all $p \nmid n$ with $c(p) > a(p)$, we require

$$\alpha \equiv 1, 2, \ldots, p^{a(p)+1} - n \bmod p^{a(p)+1}$$

or

$$\alpha + n - \mu * m * (n-1) \equiv 1, 2, \ldots, p^{a(p)+1} - n \bmod p^{a(p)+1}. \tag{3}$$

This assures that the "prime to n" part of $\gcd(\gamma, \gamma')$ will divide $\mu * (n-1)$. But, for $p|n$, we require stronger conditions in order to have an α for which $\gcd(\gamma, \gamma') | \mu * m * (n-1)$.

Congruence Condition II: For all $p|n$, we require both of the following:

$$\alpha + n - \mu * m * (n-1) \equiv 1, 2, \ldots, p^{a(p)+2} - n \bmod p^{a(p)+2}; \tag{4}$$

$$\alpha \equiv p^{a(p)+1} - n \bmod p^{a(p)+1}. \tag{5}$$

Remark 2: There exist α simultaneously satisfying these conditions. It is clear we can find an α satisfying Condition I for every p. For Condition II, (5) is equivalent to

$$\alpha \equiv p^{a(p)+1} - n, 2 * p^{a(p)+1} - n, \ldots, p * p^{a(p)+1} - n \bmod p^{a(p)+2}. \tag{6}$$

Then (4) restricts α to one of $p^{a(p)+2} - n$ consecutive residue classes modulo $p^{a(p)+2}$, but at least one of these must also be a solution to (6) since those solutions are spaced every $p^{a(p)+1}$. and $p^{a(p)+1} > n$ implies $p^{a(p)+2} - n > p^{a(p)+1}$.

Finally, as there are infinitely many α satisfying Congruence Conditions I and II, we are free to choose one as large as we like. We choose α large enough to satisfy the following

Growth Condition:

$$\begin{aligned} \alpha \geq\ & (n-1) * (\mu * m + 2 * n * (\log_n(\alpha + n - 1) + 1)) \\ & + (n-1)^2 * 2 * (5 + 4 * n * (\log_n(\alpha + n - 1) + 1))^2. \end{aligned} \tag{7}$$

Again, it is possible to find such an α because the left-hand side grows linearly while the right-hand side grows logarithmically in α.

Theorem 2: Any α satisfying Congruence Conditions I and II and the Growth Condition is the digital sum of the first of $2*n$ consecutive n-Niven numbers. In particular, for each $n \geq 2$, there exists a sequence of $2*n$ consecutive n-Niven numbers.

Proof: We start with an α satisfying Congruence Conditions I and II and the Growth Condition. For $\gamma = \operatorname{lcm}(\alpha, \alpha+1, \ldots, \alpha+n-1)$ and $\gamma' = \operatorname{lcm}(\alpha + n - \mu * m * (n-1), \ldots, \alpha + 2 * n - 1 - \mu * m * (n-1))$, we can solve

$$b \equiv \alpha \bmod \gamma \text{ and } b \equiv \alpha - \mu * m * (n-1) \bmod \gamma'. \tag{8}$$

To see this, note that, for $p \nmid n$, we have $v_p(\mu * m * (n-1)) = a(p)$ and Congruence Condition I assures that $v_p(\gcd(\gamma, \gamma')) \leq a(p)$. For $p|n$, we have $v_p(\mu * m * (n-1)) = a(p) + 1$ and, by (5), $v_p(\gcd(\gamma, \gamma')) \leq a(p)$.

Let b be the least positive solution to (8). Any other solution to (8) differs from the minimal positive one by a multiple of $\delta = \operatorname{lcm}(\gamma, \gamma')$. We can modify b by adding multiples of δ to create a number, b', so that

but
$$b' \equiv n^{\mu*m} - n \mod n^{\mu*m}$$
$$b' \not\equiv n^{\mu*m+1} - n \mod n^{\mu*m+1}. \tag{9}$$

This is possible by Congruence Condition II: For $p|n$, Condition II assures that $\alpha \equiv p^{a(p)+1} - n \mod p^{a(p)+1}$. Since $\mu*m*(n-1) \equiv 0 \mod p^{a(p)+1}$, we have $\alpha + n - \mu*m*(n-1) \equiv 0 \mod p^{a(p)+1}$.Now (8) assures $b + n \equiv 0 \mod p^{a(p)+1}$. By Condition II, $v_p(\delta) \le a(p) + 1 = v_p(\mu*m)$, so

$$b \equiv n^{\mu*m} - n \mod \prod_{p|n} p^{v_p(\delta)}.$$

This means we can add multiples of δ to b to get b' as above.

Our next task is to modify b' by concatenating copies of multiples of δ so that we obtain a number, β, with $s(\beta) = \alpha$. Since δ is less than the product of the $2*n$ numbers $\alpha, \alpha+1, \ldots, \alpha+2*n-1-\mu*m*(n-1)$, the largest of which has $v(\alpha+n-1) \le \log_n(\alpha+n-1)+1$, we get

$$v(\delta) \le 2*n*(\log_n(\alpha+n-1)+1).$$

Since b was the minimal solution to (8), we have $v(b) \le v(\delta)$. We created b' by adding multiples of δ to b. Keeping track of the digits, we see that

$$v(b') \le \mu*m + v(\delta) + 1$$

as we modify b to get a terminal 0 with $\mu*m-1$ penultimate $(n-1)$'s. To do this by adding multiples of δ, we will be left with not more than $v(\delta)+1$ digits in front of the penultimate $(n-1)$'s, since we can first choose a multiple of δ less than $n*\delta$ to change the second base n digit (from right) of b to $n-1$ and then choose a multiple of $n*\delta$ less than $n^2*\delta$ to change the third base n digit (from right) to $n-1$, and so on. We continue until we add a multiple of $n^{\mu*m-2}*\delta$ less than $n^{\mu*m-1}*\delta$ to change the $\mu*m$ base n digit to $n-1$. A final multiple of $n^{\mu*m-1}*\delta$ may need to be added to assure that the $\mu*m+1$ digit is not $n-1$.

Since each digit can contribute at most $n-1$ to the digital sum, we get

$$s(\delta) \le 2*n*(\log_n(\alpha+n-1)+1)*(n-1)$$

and

$$s(b') \le (\mu*m + 2*n*(\log_n(\alpha+n-1)+1))*(n-1).$$

Since $b' \equiv b \equiv \alpha \mod \gamma$, Lemma 2 gives $(n-1)|(\alpha - s(b'))$. By Lemma 1, there exist k and k' so that $\gcd(s(k*\delta), s(k'*\delta)) = n-1$; thus,

$$\gcd(s(k*\delta), s(k'*\delta))|(\alpha - s(b')).$$

Remark 1 says that our k and k' may be chosen so that

$$s(k*\delta) \le (n-1)*(5+2*(2*n*(\log_{(n)}(\alpha+n-1)+1)))$$

and

$$s(k'*\delta) \le (n-1)*2*(5+2*(2*n*(\log_{(n)}(\alpha+n-1)+1))).$$

These two inequalities and the Growth Condition assure $\alpha - s(b') \ge s(k*\delta)*s(k'*\delta)$, so we can use Lemma 3 with $z = \alpha - s(b')$, $x = s(k*\delta)$, and $y = s(k'*\delta)$. We conclude that there are nonnegative integers r and t such that $\alpha - s(b') = r*s(k*\delta) + t*s(k'*\delta)$. But then

$$\alpha = r*s(k*\delta)+t*s(k'*\delta)+s(b'), \text{ so}$$
$$\alpha = s((k*\delta)_r(k'*\delta)_t b').$$

Using $(k*\delta)_r(k'*\delta)_t b' \equiv n^{\mu*m}-n \mod n^{\mu*m}$, we get

$$\alpha+i = s((k*\delta)_r(k'*\delta)_t b'+i)$$

and $\qquad$ (10)

$$\alpha+n+i-\mu*m*(n-1) = s((k*\delta)_r(k'*\delta)_t b'+n+i)$$

for $i=0,1,\dots,n-1$. Since $(k*\delta)_r(k'*\delta)_t b' \equiv b \mod \delta$, (8) assures

$$(\alpha+i) \mid ((k*\delta)_r(k'*\delta)_t b'+i)$$

and $\qquad$ (11)

$$(\alpha+n+i) \mid ((k*\delta)_r(k'*\delta)_t b'+n+i)$$

for all $i=0,1,\dots,n-1$. By (10), (11), and the definition of an n-Niven number, $(k*\delta)_r(k'*\delta)_t b'$ is the first of $2*n$ consecutive n-Niven numbers. □

Remark 3: We note that we have proved something stronger than the theorem, namely, that there exist infinitely many sequences of $2*n$ consecutive n-Niven numbers, since there exist infinitely many α satisfying Condition I, Condition II, and the Growth Condition.

3. EXAMPLES

Example 1: For $n=2$ we get $\mu=2, m=2$, and the conditions (3)-(5), (7),

$$\alpha \equiv 0,1 \mod 3,$$
$$\alpha \equiv 6 \mod 8,$$
$$\alpha \geq 36033.$$

Taking, for example, $\alpha = 36046$, we get the base 2 representations:

$$b = 1_5001_40101_30010010101_70_31011010_51_40_41010_{(2)};$$

$$\delta = 101_301101101011010110110_410011010101010_31_601001_400_{(2)}.$$

Then, letting $b' = b+7*\delta$, we get the right number of penultimate 1's:

$$b' = 1011001_3001_300101010_31101010_31101101_50_31_501_30101_30_{(2)}.$$

We easily see that $s(b')=37$. Now we want to follow Lemma 1 to get multiples of δ with relatively prime base 2 digital sums. First, we want $\delta' = (n+1)*\delta$ as **a** has a terminal $n-1$. Using δ' in place of δ, we get $k = 2*2^{62}+1$ and $k' = (2^{122}+1)k = 2^{185}+2^{122}+2^{63}+1$. Then we see that

$$k*\delta' = 10_3110010010_410_410_310_31_3001_90101_401_4011010_310_3$$
$$110010010_410_410_310_31_3001_90101_401_40110100_{(2)}$$

with $s(k*\delta') = 64$ and

$$k'*\delta' = 10_3110010010_410_410_310_31_3001_90101_401_4011010_3$$
$$10_3110010010_410_410_310_31_3001_90101_401_401_30_4$$
$$110010010_410_410_310_31_3001_90101_401_4011010_310_3$$
$$110010010_410_410_310_31_3001_90101_401_40110100_{(2)}$$

with $s(k'*\delta')=127$. It is easy to see that

$$\alpha - s(b') = 36009 = 517*64+23*127 = 517*s(k*\delta')+23*s(k'*\delta'),$$

so

$$s((k*\delta')_{517}(k'*\delta')_{23}b') = 36046 = \alpha.$$

Thus, $(k*\delta')_{517}(k'*\delta')_{23}b'$ is the base 2 representation of the first number in a sequence of 4 consecutive 2-Niven numbers.

We note that the Growth Condition, while assuring we can get α as a digital sum, results in large numbers. In practice, much smaller α satisfying Congruence Conditions I and II can be digital sums of $2*n$ consecutive n-Niven numbers.

Example 2: For $n=2$, we get $\mu=2, m=2$ and the congruence conditions $\alpha \equiv 0, 1 \bmod 3$ and $\alpha \equiv 6 \bmod 8$. $\alpha=6$ is such an α (although it clearly does not satisfy the Growth Condition). This leads to $b=342=101010110_{(2)}$ and $\delta=420=110100100_{(2)}$. It is easy to see that $\beta=b+14*\delta=6222$ has base 2 expansion $1100001001110_{(2)}$, so $s(\beta)=\alpha$. This means β is the first of a sequence of four consecutive 2-Niven numbers.

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A GENERALIZATION OF THE "ALL OR NONE" DIVISIBILITY PROPERTY

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1. INTRODUCTION

In [6], Juan Pla proved the following interesting theorem.

Theorem 1.1: Let h_n be the general term of a given sequence of integers such that $h_{n+2} = h_{n+1} + h_n$, where h_0 and h_1 are arbitrary integers. Let c be an arbitrary integer other than -2, -1, 0, and 1. Let D be any divisor of $c^2 + c - 1$ other than 1. Then, the sequence $\{w_n\}$, where $w_n = ch_{n+1} - h_n$, for $n \geq 0$, is such that either (a) D divides every w_n or (b) D divides no w_n.

We would like to point out a more interesting fact that, essentially, the above theorem is the corollary of the following.

Theorem 1.2: Let $\{f_n\}$ be the Fibonacci sequence, that is, $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. Let $f(x) = x^2 - x - 1$. Then, for $n \in \mathbf{Z}$, we have

$$x^n \equiv f_n x + f_{n-1} \pmod{f(x)}. \tag{1.1}$$

Proof: Equation (1.1) holds for $n = 0$ since $x^0 = f_0 x + f_{-1}$. Assume that (1.1) holds for $n = k$, $k \geq 0$, that is, $x^k \equiv f_k x + f_{k-1} \pmod{f(x)}$. Then $x^{k+1} \equiv f_k x^2 + f_{k-1} x \equiv f_k(x+1) + f_{k-1}x = f_{k+1}x + f_k \pmod{f(x)}$. This means that (1.1) holds for all $n \geq 0$. Now assume that (1.1) holds for $n = -k$, $k \geq 0$, that is, $x^{-k} \equiv f_{-k}x + f_{-k-1} \pmod{f(x)}$. Then $x^{-k-1} \equiv f_{-k} + f_{-k-1}x^{-1} \pmod{f(x)}$. Since $x(x-1) \equiv 1 \pmod{f(x)}$, we have that $x^{-1} \equiv x - 1 \pmod{f(x)}$, and so $x^{-k-1} \equiv f_{-k} + f_{-k-1}(x-1) = f_{-k-1}x + f_{-k-2} \pmod{f(x)}$. This means that (1.1) holds also for all $n < 0$. □

Now we apply Theorem 1.2 to prove Theorem 1.1. We have $h_n = h_1 f_n + h_0 f_{n-1}$ and $h_{n+1} = h_1 f_{n+1} + h_0 f_n$ for $n \geq 0$ (see [2]), whence $w_n = -h_1(-f_{n+1}c + f_n) - h_0(-f_n c + f_{n-1})$. In (1.1), taking $x = -c$, we get $w_n \equiv -h_1(-c)^{n+1} - h_0(-c)^n = (-c)^n(ch_1 - h_0) = (-c)^n w_0 \pmod{c^2 + c - 1}$. Since D divides $c^2 + c - 1$ and $D > 1$, we have $\gcd(c, D) = 1$. If D divides w_n for some $n \geq 0$, then D divides w_0. This leads to the fact that D divides w_n for all $n \geq 0$. □

In this paper we generalize the result of Theorem 1.2 to the case of k^{th}-order homogeneous recursion sequence with constant coefficients in Section 3. In Section 4 we generalize the interesting result of Theorem 1.1, correspondingly, i.e., we give and prove the main result of this paper. Some necessary preliminaries are given in Section 2.

2. PRELIMINARIES

Let the sequence $\{h_n\} = \{h_n\}_{n \geq 0}$ be defined by the recurrence relation

$$h_{n+k} = a_1 h_{n+k-1} + \cdots + a_{k-1}h_{n+1} + a_k h_n, \tag{2.1}$$

and the initial condition

$$h_0 = c_0,\ h_1 = c_1, \ldots, h_{k-1} = c_{k-1}, \tag{2.2}$$

where $a_1, \ldots, a_k$ and $c_0, \ldots, c_{k-1}$ are constants. Then we call $\{h_n\}$ a k^{th}-order **Fibonacci-Lucas sequence** or simply an **F-L sequence**, and we call h_n the n^{th} **F-L number**. The polynomial

$$f(x) = x^k - a_1 x^{k-1} - \cdots - a_{k-1}x - a_k \tag{2.3}$$

is called the **characteristic polynomial** of $\{h_n\}$. If $f(\theta) = 0$, then we call θ a **characteristic root** of $\{h_n\}$. The set of F-L sequences satisfying (2.1) is denoted by $\Omega(a_1, \ldots, a_k)$ and also by $\Omega(f(x))$.

If $a_k \neq 0$, then (2.1) can be rewritten as

$$h_n = (h_{n+k} - a_1 h_{n+k-1} - \cdots - a_{k-1}h_{n+1}) / a_k, \tag{2.4}$$

whence, from the given values of $h_0, h_1, \ldots, h_{k-1}$, we can calculate the values of $h_{-1}, h_{-2}, \ldots$. Therefore, in the case $a_k \neq 0$, we may consider $\{h_n\}$ as $\{h_n\}_{-\infty}^{+\infty}$. For convenience, we always assume that $a_k \neq 0$ whenever we refer to $\Omega(a_1, \ldots, a_k)$.

Obviously, $\Omega(a_1, \ldots, a_k)$ is a linear space [3] under the operations $\{h_n\} + \{w_n\} = \{h_n + w_n\}$ and $\lambda\{h_n\} = \{\lambda h_n\}$. Let $\{u_n^{(i)}\}$,. $0 \leq i \leq k-1$, be a sequence in $\Omega = \Omega(a_1, \ldots, a_k)$ with the initial condition $u_n^{(i)} = \delta_{ni}$ for $0 \leq n \leq k-1$, where δ is the Kronecker function. Then we call $\{u_n^{(i)}\}$ the i^{th} **basic sequence** in Ω. Construct a map, $\Omega \to \mathbf{R}^k$ such that each sequence $\{h_n\} \in \Omega$, with initial condition (2.2), corresponds to $(c_0, c_1, \ldots, c_{k-1})$. Clearly, this map is an isomorphism, and the basic sequences $\{u_n^{(0)}\}, \{u_n^{(1)}\}, \ldots, \{u_n^{(k-1)}\}$ form a base in Ω. Thus, we have the following lemmas.

Lemma 2.1: Let $\Omega = \Omega(a_1, \ldots, a_k)$. Let $\{u_n^{(i)}\}, 0 \leq i \leq k-1$, be the i^{th} basic sequence in Ω and let $\{h_n\}$ be an arbitrary sequence in Ω. Then $\{h_n\}$ can be represented uniquely by $\{u_n^{(0)}\}, \{u_n^{(1)}\}, \ldots, \{u_n^{(k-1)}\}$, as

$$h_n = \sum_{i=0}^{k-1} h_i u_n^{(i)} \text{ for } n \in \mathbf{Z}. \tag{2.5}$$

Lemma 2.2: Under the condition of Lemma 2.1, we have

$$h_{n+1} = (a_1 h_{k-1} + a_2 h_{k-2} + \cdots + a_k h_0) u_n^{(k-1)} + \sum_{i=0}^{k-2} h_{i+1} u_n^{(i)} \text{ for } n \in \mathbf{Z}. \tag{2.6}$$

Proof: Let $\{w_n\} = \{h_{n+1}\}$. Then $w_0 = h_1, \ldots, w_{k-2} = h_{k-1}$ and (2.1) implies $w_{k-1} = h_k = a_1 h_{k-1} + a_2 h_{k-2} + \cdots + a_k h_0$. Thus, the lemma is proved by Lemma 2.1. □

In (2.6), replacing $\{h_n\}$ by $\{u_n^{(0)}\}, \ldots, \{u_n^{(k-1)}\}$, respectively, we obtain

Lemma 2.3: Let $\{u_n^{(i)}\}, 0 \leq i \leq k-1$, be the i^{th} basic sequence in $\Omega(a_1, \ldots, a_k)$. Then, for $n \in \mathbf{Z}$, we have

$$u_{n+1}^{(0)} = a_k u_n^{(k-1)} \quad \text{and} \quad u_{n+1}^{(i)} = a_{k-i} u_n^{(k-1)} + u_n^{(i-1)} \text{ for } 1 \leq i \leq k-1. \tag{2.7}$$

Lemma 2.4: Under the condition of Lemma 2.3, we have

$$u_n^{(i)} = \sum_{j=0}^{i} a_{k-i+j} u_{n-1-j}^{(k-1)},\ i = 0, \ldots, k-1,\ n \in \mathbf{Z}. \tag{2.8}$$

Proof: From (2.7), (2.8) holds for $i = 0$. Assume (2.8) holds for i, $0 \le i < k-1$. Then (2.7) and the induction hypothesis imply that

$$u_n^{(i+1)} = a_{k-i-1}u_{n-1}^{(k-1)} + u_{n-1}^{(i)} = a_{k-i-1}u_{n-1}^{(k-1)} + \sum_{j=0}^{i} a_{k-i+j}u_{n-2-j}^{(k-1)}$$
$$= \sum_{j=0}^{i+1} a_{k-(i+1)+j}u_{n-1-j}^{(k-1)},$$

and we are done. □

From (2.7) and (2.8), we observe that the $(k+1)^{\text{th}}$ basic sequence in $\Omega(a_1, \dots, a_k)$ plays an important role, so that we call it the **principal sequence** in Ω and denote it by $\{u_n^{(k-1)}\} = \{u_n\}$.

Now, substituting (2.8) into (2.5), we get

Lemma 2.5: Let $\{u_n\}$ be the principal sequence in $\Omega = \Omega(a_1, \dots, a_k)$. Let $\{h_n\}$ be an arbitrary sequence in Ω. Then

$$h_n = \sum_{i=0}^{k-1} b_{k-1-i}u_{n-i} \quad \text{for } n \in \mathbb{Z}. \tag{2.9}$$

where

$$b_{k-1} = h_{k-1} \quad \text{and} \quad b_{k-1-i} = \sum_{j=0}^{k-1-i} a_{i+1+j}h_{k-2-j} \quad \text{for } 1 \le i \le k-1. \tag{2.10}$$

3. A PROPERTY OF THE CHARACTERISTIC POLYNOMIAL OF A k^{th}-ORDER F-L SEQUENCE

Theorem 3.1: Let $\{u_n^{(i)}\}$, $0 \le i \le k-1$, be the i^{th} basic sequence in $\Omega(f(x))$, where $f(x)$ is denoted by (2.3). Then

(a) $x^n \equiv \sum_{i=0}^{k-1} u_n^{(i)}x^i \pmod{f(x)}$ for $n \in \mathbb{Z}$. (3.1)

(b) If, besides (3.1), we have $x^n \equiv \sum_{i=0}^{k-1} v_n^{(i)}x^i \pmod{f(x)}$, where each of the $v_n^{(i)}$'s $(i = 0, \dots, k-1)$ is independent of x, then $u_n^{(i)} = v_n^{(i)}$, $i = 0, \dots, k-1$.

Proof: Part (b) is proved by the uniqueness of the remainder of x^n over $f(x)$. Now we must prove part (a). By the definition of $\{u_n^{(i)}\}$, $i = 0, \dots, k-1$, (3.1) holds for $n = 0$. Assume that (3.1) holds for $n = m$, $m \ge 0$. Then, from the induction hypothesis and (2.7), we have

$$x^{m+1} \equiv x\sum_{i=0}^{k-1} u_m^{(i)}x^i = u_m^{(k-1)}x^k + \sum_{i=0}^{k-2} u_m^{(i)}x^{i+1}$$
$$\equiv u_m^{(k-1)}(a_1x^{k-1} + \cdots + a_{k-1}x + a_k) + \sum_{i=0}^{k-2} u_m^{(i)}x^{i+1}$$
$$= a_ku_m^{(k-1)} + \sum_{i=1}^{k-1} (a_{k-i}u_m^{(k-1)} + u_m^{(i-1)})x^i = \sum_{i=0}^{k-1} u_{m+1}^{(i)}x^i \pmod{f(x)}.$$

This implies that (3.1) holds for all $n \ge 0$.

Now assume that (3.1) holds for $n = -m$, $m \ge 0$. Then

$$x^{-m-1} \equiv x^{-1}\left(\sum_{i=0}^{k-1} u_{-m}^{(i)} x^i\right) = \sum_{i=1}^{k-1} u_{-m}^{(i)} x^{i-1} + u_{-m}^{(0)} x^{-1} \pmod{f(x)}. \tag{3.2}$$

From $x(x^{k-1} - a_1 x^{k-2} - \cdots - a_{k-1}) \equiv a_k \pmod{f(x)}$ and $a_k \neq 0$, we obtain

$$x^{-1} \equiv (x^{k-1} - a_1 x^{k-2} - \cdots - a_{k-1}) / a_k \pmod{f(x)}. \tag{3.3}$$

Substituting (3.3) into (3.2) and noting that $u_{-m}^{(0)} / a_k = u_{-m-1}^{(k-1)}$ we get, by (2.7)

$$\begin{aligned} x^{-m-1} &\equiv \sum_{i=1}^{k-1} u_{-m}^{(i)} x^{i-1} + u_{-m-1}^{(k-1)}(x^{k-1} - a_1 x^{k-2} - \cdots - a_{k-1}) \\ &= u_{-m-1}^{(k-1)} x^{k-1} + \sum_{i=1}^{k-1} (u_{-m}^{(i)} - a_{k-i} u_{-m-1}^{(k-1)}) x^{i-1} \\ &= u_{-m-1}^{(k-1)} x^{k-1} + \sum_{i=1}^{k-1} u_{-m-1}^{(i-1)} x^{i-1} = \sum_{i=0}^{k-1} u_{-m-1}^{(i)} x^i \pmod{f(x)}. \end{aligned}$$

This implies that (3.1) holds also for $n < 0$. □

Corollary: Under the condition of Theorem 3.1, if $f(\theta) = 0$, then

$$\theta^n = \sum_{i=0}^{k-1} u_n^{(i)} \theta^i \text{ for } n \in \mathbf{Z}. \tag{3.4}$$

It can be observed that the results in [1], [4], and [5] may be obtained easily by using (3.4).

4. A GENERALIZATION OF THE "ALL OR NONE" DIVISIBILITY PROPERTY

Theorem 4.1: Let $\{h_n\}$ be an arbitrary sequence in $\Omega(a_1, \ldots, a_k) = \Omega(f(x))$, where $a_1, \ldots, a_k$ are integers and $f(x)$ is denoted by (2.3). Let $c \in \mathbf{Z}$, $f(c) \neq \pm 1$. Let D be a divisor of $f(c)$ other than 1, and $\gcd(c, D) = 1$. Suppose that

$$w_n = \sum_{i=0}^{k-1} g_{k-1-i}(c) h_{n+k-1-i}, \tag{4.1}$$

where

$$g_{k-1}(x) = x^{k-1} \quad \text{and} \quad g_{k-1-i}(x) = \sum_{j=0}^{k-1-i} a_{i+1+j} x^{k-2-j} \text{ for } 1 \leq i \leq k-1. \tag{4.2}$$

Then either D divides w_n for all $n \geq 0$ or D divides no w_n.

To prove the theorem, we need the following lemmas.

Lemma 4.1:

$$\sum_{i=0}^{k-1} g_{k-1-i}(x) u_{n-i} \equiv x^n \pmod{f(x)}, \tag{4.3}$$

where $\{u_n\}$ is the principal sequence in $\Omega(f(x))$.

Proof: Let $\{u_n^{(i)}\}$, $0 \leq i \leq k-1$, be the i^{th} basic sequence. From Theorem 3.1 and Lemma 2.4, we have

$$x^n \equiv \sum_{t=0}^{k-1} u_n^{(t)} x^t = x^{k-1}u_n + \sum_{t=0}^{k-2} x^t \sum_{i=0}^{t} a_{k-t+i}u_{n-1-i}$$
$$= x^{k-1}u_n + \sum_{i=0}^{k-2} u_{n-1-i} \sum_{t=i}^{k-2} a_{k-t+i}x^t = \sum_{i=0}^{k-2} u_{n-1-i} \sum_{j=0}^{k-2-i} a_{i+2+j}x^{k-2-j} + u_n x^{k-1}$$
$$= \sum_{i=0}^{k-2} u_{n-1-i}g_{k-2-i}(x) + g_{k-1}(x)u_n = \sum_{i=0}^{k-1} g_{k-1-i}(x)u_{n-i} \pmod{f(x)}. \quad \square$$

Lemma 4.2:

$$\sum_{i=0}^{k-1} b_i c^i = w_0, \tag{4.4}$$

where b_i $(0 \le i \le k-1)$ is denoted by (2.10).

Proof:

$$\sum_{i=0}^{k-1} b_i c^i = h_{k-1}c^{k-1} + \sum_{i=0}^{k-2} b_{k-1-(k-1-i)}c^t$$
$$= h_{k-1}c^{k-1} + \sum_{i=0}^{k-2} c^i \sum_{j=0}^{i} a_{k-i+j}h_{k-2-j} = h_{k-1}c^{k-1} + \sum_{j=0}^{k-2} h_{k-2-j} \sum_{i=j}^{k-2} a_{k-i+j}c^i$$
$$= h_{k-1}c^{k-1} + \sum_{j=0}^{k-2} h_{k-2-j} \sum_{i=0}^{k-2-j} a_{2+j+i}c^{k-2-i} = h_{k-1}g_{k-1}(c) + \sum_{j=0}^{k-2} h_{k-2-j}g_{k-2-j}(c)$$
$$= \sum_{j=0}^{k-1} g_{k-1-j}(c)h_{k-1-j} = w_0,$$

by (4.2) and (4.1). $\square$

Proof of Theorem 4.1: From (4.1) and Lemma 2.5, we have

$$w_n = \sum_{j=0}^{k-1} g_{k-1-j}(c)h_{n+k-1-j} = \sum_{j=0}^{k-1} g_{k-1-j}(c) \sum_{i=0}^{k-1} b_{k-1-i}u_{n+k-1-j-i}$$
$$= \sum_{i=0}^{k-1} b_{k-1-i} \sum_{j=0}^{k-1} g_{k-1-j}(c)u_{n+k-1-j-i}.$$

In Lemma 4.1, taking $x = c$, we get

$$\sum_{j=0}^{k-1} g_{k-1-j}(c)u_{n+k-1-j-i} \equiv c^{n+k-1-i} \pmod{f(c)},$$

whence, from Lemma 4.2,

$$w_n \equiv \sum_{i=0}^{k-1} b_{k-1-i}c^{n+k-1-i} = c^n \sum_{i=0}^{k-1} b_{k-1-i}c^{k-1-i} = c^n w_0 \pmod{f(c)}.$$

Because $\gcd(c, D) = 1$, if D divides w_n for some $n \ge 0$, then D must divide w_0, so D divides w_n for all $n \ge 0$. $\square$

Example 1: Let $f(x)=x^3-x^2-x-1$, then $k=3$, $a_1=a_2=a_3=1$. Let $c=-2$, then $f(c)=-11$. Take $D=11$, then $\gcd(c,D)=1$. Assume that $\{h_n\}\in\Omega(f(x))$ and $h_0=0, h_1=h_2=1$. From (4.2) and (4.1), we have $g_2(c)=(-2)^2=4$, $g_1(c)=1\times(-2)+1=-1$, $g_0(c)=1\times(-2)=-2$, and $w_n=4h_{n+2}-h_{n+1}-2h_n$, respectively. Since $w_0=4h_2-h_1-2h_0=3$ and 11 does not divide 3, thus 11 divides no w_n.

Example 2: Let $f(x)=x^3-x^2+2x-3$, then $k=3, a_1=1, a_2=-2, a_3=3$. Let $c=3$, then $f(c)=21$. Take $D=7$, then $\gcd(c,D)=1$. Assume that $\{h_n\}\in\Omega(f(x))$ and that $h_0=h_2=1$, $h_1=-1$. From (4.2) and (4.1), we have $g_2(c)=3^2=9$, $g_1(c)=(-2)\times 3+3=-3$, $g_0(c)=3\times 3=9$, and $w_n=9h_{n+2}-3h_{n+1}+9h_n$, respectively. Since $w_0=9h_2-3h_1+9h_0=21$ and 7 divides 21, thus 7 divides w_n for all $n\geq 0$.

Concluding Remark: Theorem 3.1 can be seen in [7], which was published in Chinese in 1993. Some other applications of Theorem 3.1 and its corollary to the identities involving F-L numbers, congruence relations, modular periodicities, divisibilities, etc., are also stated in [7].

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ON A QUESTION OF COOPER AND KENNEDY

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In [2], Cooper and Kennedy note the following: If

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3}, \tag{1}$$

then

$$x_n^2 = Ax_{n-1}^2 + Bx_{n-2}^2 + Cx_{n-3}^2 + Dx_{n-4}^2 + Ex_{n-5}^2 + Fx_{n-6}^2, \tag{2}$$

where the coefficients A, B, C, D, E, F may be expressed in terms of a, b, c. They ask: **Is there a similar formula for third powers**? The answer is: **YES**. The reason is the following: Sequences which are solutions of linear recurrences with constant coefficients have ordinary generating functions which are *rational*. Conversely, if a rational function has no pole in $z = 0$, its Taylor coefficients fulfill a linear recurrence with constant coefficients. If

$$f := \sum_{n\geq 0} a_n z^n \quad \text{and} \quad g := \sum_{n\geq 0} b_n z^n \tag{3}$$

are two (formal) series, their HADAMARD product is defined to be

$$f \odot g := \sum_{n\geq 0} a_n b_n z^n. \tag{4}$$

And rational functions are closed under the Hadamard product! (See [1], p. 85.)

The larger (any maybe even more important) class of *holonomic* functions (solutions of linear differential equations with polynomial coefficients) is also closed under the Hadamard product. Their Taylor coefficients fulfill linear recursions with polynomial coefficients. There is a MAPLE package, GFUN, which computes (among many other things) the Hadamard product (see [4]).

There is another very useful program, EKHAD, written by Doron Zeilberger [3], which should be mentioned. With it, we find, for example, recursions for the d^{th} powers of the Fibonacci numbers F_n in almost no time. In the following, F_n^d will be a solution of the given recursion.

$d = 1$ $\quad x_{n+2} - x_{n+1} - x_n = 0,$

$d = 2$ $\quad x_{n+3} - 2x_{n+2} - 2x_{n+1} + x_n = 0,$

$d = 3$ $\quad x_{n+4} - 3x_{n+3} - 6x_{n+2} + 3x_{n+1} + x_n = 0,$

$d = 4$ $\quad x_{n+5} - 5x_{n+4} - 15x_{n+3} + 15x_{n+2} + 5x_{n+1} - x_n = 0,$

$d = 5$ $\quad x_{n+6} - 8x_{n+5} - 40x_{n+4} + 60x_{n+3} + 40x_{n+2} - 8x_{n+1} - x_n = 0,$

$d = 6$ $\quad x_{n+7} - 13x_{n+6} - 104x_{n+5} + 260x_{n+4} + 260x_{n+3} - 104x_{n+2} - 13x_{n+1} + x_n = 0.$

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Announcement

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JACOBSTHAL REPRESENTATION POLYNOMIALS

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1. PRELIMINARIES

Consider two sequences of polynomials $\{J_n(x)\}$, the *Jacobsthal polynomials*, and $\{j_n(x)\}$, the *Jacobsthal-Lucas polynomials*, defined recursively [3] by

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x), \quad J_0(x) = 0, \ J_1(x) = 1, \tag{1.1}$$

and

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x), \quad j_0(x) = 2, \ j_1(x) = 1, \tag{1.2}$$

respectively.

Observe that $J_n(1/2) = F_n$ and $j_n(1/2) = L_n$, the n^{th} Fibonacci and Lucas numbers, respectively. When $x = 1$, we obtain the Jacobsthal and Jacobsthal-Lucas numbers [8], respectively. (Other number sequences derived from (1.1) and (1.2) which are of some interest are generated by $x = 1/4$.)

For $\{J_n(x)\}$ and $\{j_n(x)\}$, the characteristic equation is

$$\lambda^2 - \lambda - 2x = 0 \tag{1.3}$$

with roots

$$\left.\begin{aligned} \alpha(x) &= \frac{1+\sqrt{8x+1}}{2}, \\ \beta(x) &= \frac{1-\sqrt{8x+1}}{2}, \end{aligned}\right\} \tag{1.4}$$

so that

$$\left.\begin{aligned} \alpha(x)+\beta(x) &= 1, \\ \alpha(x)\beta(x) &= -2x, \\ \alpha(x)-\beta(x) &= \sqrt{8x+1} = \Delta(x), \end{aligned}\right\} \tag{1.5}$$

whence

$$\Delta(1) = 3. \tag{1.5a}$$

Moreover,

$$\left.\begin{aligned} \alpha^2(x)+2x &= \Delta(x)\alpha(x), \\ \beta^2(x)+2x &= -\Delta(x)\beta(x). \end{aligned}\right\} \tag{1.6}$$

Comparison might be made between our definition (1.1) and that in [2] for Jacobsthal polynomials. The correspondence is simple: x in [2] $\leftrightarrow$ $2x$ in (1.1). While the nomenclature in [2] serves a very valuable purpose leading to elegant results and extensions, we prefer to retain the factor $2x$ for consistency with our notation for Pell polynomials [10].

To the best of my knowledge, properties of the Jacobsthal-Lucas polynomials defined fully in (1.2), and the corresponding numbers [8] generated when $x = 1$, are generally due to the present author, as an appropriate companion to those of the Jacobsthal polynomials (1.1). (Our (3.10), (3.11), and (3.12) do occur in [12], though in a heavily camouflaged form.) When it is convenient (e.g., for brevity), the polynomials given by (1.1) and (1.2) will simply be referred to collectively as *Jacobsthal-type polynomials*, or, as in the title of the paper, more simply still as *Jacobsthal polynomials.*

Aspects of Jacobsthal polynomials (1.1), which are documented in other sources (e.g., [1], [2], [12]) will not in general be duplicated in this presentation, though the basic features must recur.

Goals of This Paper

Aims of this presentation are:

(i) to exhibit some basic properties of the polynomials (Tables 1 and 2) which generalize the properties of the corresponding numbers in [8];

(ii) to reveal some of the salient features of the diagonal functions generated by (1.1) and (1.2);

(iii) to examine the properties of the "augmented" polynomials developed from (1.1) and (1.2) by the addition of an appropriate constant.

2. THE JACOBSTHAL-TYPE POLYNOMIALS

Tables 1 and 2 list the first few polynomials of (1.1) and (1.2) of these Jacobsthal-type sequences.

TABLE 1. Jacobsthal Polynomials $\{J_n(x)\}: 0 \le n \le 10$

$J_0(x)=0$	$J_6(x)=1+8x+12x^2$
$J_1(x)=1$	$J_7(x)=1+10x+24x^2+8x^3$
$J_2(x)=1$	$J_8(x)=1+12x+40x^2+32x^3$
$J_3(x)=1+2x$	$J_9(x)=1+14x+60x^2+80x^3+16x^4$
$J_4(x)=1+4x$	$J_{10}(x)=1+16x+84x^2+160x^3+80x^4$
$J_5(x)=1+6x+4x^2$	

TABLE 2. Jacobsthal-Lucas Polynomials $\{j_n(x)\}: 0 \le n \le 10$

$j_0(x)=2$	$j_6(x)=1+12x+36x^2+16x^3$
$j_1(x)=1$	$j_7(x)=1+14x+56x^2+56x^3$
$j_2(x)=1+4x$	$j_8(x)=1+16x+80x^2+128x^3+32x^4$
$j_3(x)=1+6x$	$j_9(x)=1+18x+108x^2+240x^3+144x^4$
$j_4(x)=1+8x+8x^2$	$j_{10}(x)=1+20x+140x^2+400x^3+400x^4+64x^5$
$j_5(x)=1+10x+20x^2$	

Equivalent expressions for $\{J_n(x)\}$ in Table 1 are given in [2] with $x \leftrightarrow 2x$, as mentioned in Section 1.

3. BASIC PROPERTIES OF THE JACOBSTHAL-TYPE POLYNOMIALS

Generating Functions

$$\sum_{i=1}^{\infty} J_i(x)y^{i-1} = (1-y-2xy^2)^{-1}, \tag{3.1}$$

$$\sum_{i=1}^{\infty} j_i(x)y^{i-1} = (1+4xy)(1-y-2xy^2)^{-1}. \tag{3.2}$$

Binet Forms

$$J_n(x) = \frac{\alpha^n(x)-\beta^n(x)}{\Delta(x)}, \tag{3.3}$$

$$j_n(x) = \alpha^n(x)+\beta^n(x). \tag{3.4}$$

Simson Formulas

$$J_{n+1}(x)J_{n-1}(x) - J_n^2(x) = (-1)^n(2x)^{n-1}, \tag{3.5}$$

$$\left.\begin{aligned} j_{n+1}(x)j_{n-1}(x) - j_n^2(x) &= -\Delta^2(x)(-1)^n(2x)^{n-1} \\ &= -\Delta^2(x)(J_{n+1}(x)J_{n-1}(x) - J_n^2(x)) \end{aligned}\right\}. \tag{3.6}$$

Summation Formulas

$$\sum_{i=1}^{n} J_i(x) = \frac{J_{n+2}(x)-1}{2x}, \tag{3.7}$$

$$\sum_{i=0}^{n} j_i(x) = \frac{j_{n+2}(x)-1}{2x}. \tag{3.8}$$

Explicit Closed Forms

$$J_n(x) = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-r}{r}(2x)^r, \tag{3.9}$$

$$j_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-r}\binom{n-r}{r}(2x)^r. \tag{3.10}$$

Important Interrelationships

$j_n(x)J_n(x) = J_{2n}(x)$	[by (3.3), (3.4)],	(3.11)
$j_n(x) = J_{n+1}(x) + 2xJ_{n-1}(x)$	[from (3.1), (3.2)],	(3.12)
$\Delta^2(x)J_n(x) = j_{n+1}(x) + 2xj_{n-1}(x)$	[by (1.6), (3.3), (3.4)],	(3.13)
$J_n(x) + j_n(x) = 2J_{n+1}(x)$	[a form of (3.12)],	(3.14)
$\Delta^2(x)J_n(x) + j_n(x) = 2j_{n+1}(x)$	[a form of (3.13)],	(3.15)
$\Delta(x)J_n(x) + j_n(x) = 2\alpha^n(x)$	[by (3.3), (3.4)],	(3.16)
$\Delta(x)J_n(x) - j_n(x) = -2\beta^n(x)$	[by (3.3), (3.4)],	(3.17)

$$J_m(x)j_n(x)+J_n(x)j_m(x)=2J_{m+n}(x) \qquad \text{[by (3.3), (3.4)]}, \tag{3.18}$$

$$j_m(x)j_n(x)+\Delta^2(x)J_m(x)J_n(x)=2j_{m+n}(x) \qquad \text{[by (3.3), (3.4)]}, \tag{3.19}$$

whence $(m=n)$

$$j_n^2(x)+\Delta^2(x)J_n^2(x)=2j_{2n}(x), \tag{3.20}$$

the left-hand side being a sum of squares. Putting $m=n$ in (3.18) reduces the formula to (3.11). Readers are invited to discover formulas corresponding to (3.18) and (3.19) when the + sign on the left-hand side is replaced by a – sign (leading in the second instance to a difference of squares).

A neat differentiation worth recording is

$$\frac{dj_n(x)}{dx}=2nJ_{n-1}(x), \tag{3.21}$$

which differs appreciably from analogous derivatives for other "Lucas-type" polynomials, namely, those for which the initial term (i.e., when $n=0$) has the value 2 (see [7]). Less exciting is the companion result

$$\Delta^2(x)\frac{dJ_n(x)}{dx}=2nj_{n-1}(x)-4J_n(x). \tag{3.22}$$

Column Generators of $\{J_n(x)\}$ and $\{j_n(x)\}$

Formulas (3.1) and (3.2) disclose the methods for producing the polynomials $\{J_n(x)\}$ and $\{j_n(x)\}$, i.e., the rows in Tables 1 and 2. Columns in Table 1 are readily seen to be generated by $(2xy^2)^0(1-y)^{-1}$, $(2xy^2)(1-y)^{-2}$, $(2xy^2)^2(1-y)^{-3}$, $(2xy^2)^3(1-y)^{-4}$, $(2xy^2)^4(1-y)^{-5}$, ..., i.e., the r^{th} column is born from

$$(2xy^2)^{r-1}(1-y)^{-r} \quad (r\geq 1). \tag{3.23}$$

The column generator for the r^{th} column in Table 2 is conceived to be

$$(2xy^2)^{r-1}\left[\frac{1}{(1-y)^{r-1}}+\frac{1}{(1-y)^r}\right] \quad (r\geq 1)$$

$$=(2xy^2)^{r-1}\frac{2-y}{(1-y)^r}. \tag{3.24}$$

Associated Sequences

Suppose we define the k^{th} *associated sequences* $\{J_n^{(k)}(x)\}$ and $\{j_n^{(k)}(x)\}$ of $\{J_n(x)\}$ and $\{j_n(x)\}$ to be, respectively $(k\geq 1)$,

$$J_n^{(k)}(x)=J_{n+1}^{(k-1)}(x)+2xJ_{n-1}^{(k-1)}(x) \tag{3.25}$$

and

$$j_n^{(k)}(x)=j_{n+1}^{(k-1)}(x)+2xj_{n-1}^{(k-1)}(x), \tag{3.26}$$

where $J_n^{(0)}(x)=J_n(x)$ and $j_n^{(0)}(x)=j_n(x)$. Accordingly,

$$J_n^{(1)}(x)=j_n(x) \quad \text{[by (3.12)]} \tag{3.27}$$

and

$$j_n^{(1)}(x) = \Delta^2(x)J_n(x) \quad \text{[by (3.13)]} \tag{3.28}$$

are the generic members of the *first associated sequences* $\{J_n^{(1)}(x)\}$ and $\{j_n^{(1)}(x)\}$.

Repeated manipulation of the above formulas eventually reveals that

$$\begin{cases} J_n^{(2m)}(x) = j_n^{(2m-1)}(x) = \Delta^{2m}(x)J_n(x), \\ J_n^{(2m+1)}(x) = j_n^{(2m)}(x) = \Delta^{2m}(x)j_n(x). \end{cases} \tag{3.29}$$

Thus, for $m = 1, n = 5$,

$$\begin{cases} J_5^{(2)}(x) = j_5^{(1)}(x) = (8x+1)J_5(x) = 1+14x+52x^2+32x^3, \\ J_5^{(3)}(x) = j_5^{(2)}(x) = (8x+1)j_5(x) = 1+18x+100x^2+160x^3. \end{cases}$$

Another approach [7] may be employed to discover the formulas (3.29).

4. DIAGONAL FUNCTIONS

Inherent in the structure of $\{J_n(x)\}$ and $\{j_n(x)\}$ are the rising and descending diagonals which are fashioned in a manner analogous to those for Chebyshev and Fermat polynomials [4], [5].

Rising Diagonals

Imagine parallel upward-slanting lines in Tables 1 and 2 in which there exist the *rising diagonal functions* $\{R_1(x)\}$ and $\{r_1(x)\}$, respectively. Some of these are, say,

$$R_0(x) = 0,\ R_1(x) = R_2(x) = R_3(x) = 1,\ R_4(x) = 1+2x, \ldots, R_{10}(x) = 1+14x+40x^2+8x^3 \tag{4.1}$$

and

$$r_0(x) = 2,\ r_1(x) = r_2(x) = 1,\ r_3(x) = 1+4x, \ldots, r_{10}(x) = 1+18x+80x^2+56x^3. \tag{4.2}$$

Generating functions unfold by the usual technique. We have

$$\sum_{i=1}^{\infty} R_i(x)t^{i-1} = (1-t-2xt^3)^{-1} \tag{4.3}$$

and

$$\sum_{i=0}^{\infty} r_i(x)t^i = (2-t)(1-t-2xt^3)^{-1}. \tag{4.4}$$

Alternatively, see (4.10),

$$\sum_{i=1}^{\infty} r_i(x)t^{i-1} = (1+4xt^2)(1-t-2xt^3)^{-1}. \tag{4.4$'$}$$

Comparing (4.3) with (4.4), and taking into account the different initial values of i therein, we arrive at

$$r_n(x) = 2R_{n+1}(x) - R_n(x), \tag{4.5}$$

i.e.,

$$r_n(x) + R_n(x) = 2R_{n+1}(x), \tag{4.5$'$}$$

which bears a formal correspondence with (3.14).

Inherent in (4.3) and (4.4) are the recurrence relations ($n \geq 3$)

$$R_n(x) = R_{n-1}(x) + 2xR_{n-3}(x) \tag{4.6}$$

and

$$r_n(x) = r_{n-1}(x) + 2xr_{n-3}(x). \tag{4.7}$$

Explicit closed forms are

$$R_n(x) = \sum_{r=0}^{\left[\frac{n-1}{3}\right]} \binom{n-1-2r}{r} (2x)^r \tag{4.8}$$

and

$$r_n(x) = 1 + \sum_{r=1}^{\left[\frac{n}{3}\right]} \frac{n-r}{r} \binom{n-1-2r}{r-1} (2x)^r. \tag{4.9}$$

Recall from (1.5) that $2x = -\alpha(x)\beta(x)$, so that (4.8) and (4.9) allow us to express $R_n(x)$ and $r_n(x)$ in terms of $\alpha(x)$ and $\beta(x)$.

Combinatorial calculations (including Pascal's formula) may be employed to establish (4.8) and (4.9) from the recurrence formulas. Proofs by inductive methods may also be applied, but these are somewhat tortuous and are omitted as a leisure activity for the dedicated reader who can convert a tedious activity into a pleasurable challenge.

From (4.5) and (4.6), it follows immediately ($n \to n+1$) that

$$r_n(x) = R_n(x) + 4xR_{n-2}(x) \quad (n \geq 2). \tag{4.10}$$

This result also follows directly from (4.4)$'$. Combining (4.5)$'$ and (4.10), we deduce that

$$r_n^2(x) - R_n^2(x) = 8xR_{n+1}(x)R_{n-2}(x). \tag{4.11}$$

Oddly, there is no result like (4.10) in which $R_n(x)$ (possibly with a factor) and $r_n(x)$ are interchanged, as in (4.6) and (4.7), for descending diagonals. (Why is this so?) A similar situation exists for Pell-type polynomials (cf. [13]).

Differential equations (partial) of the first order are readily determined from (4.3) and (4.4) on writing

$$R \equiv R(x, t) = \sum_{i=1}^{\infty} R_i(x) t^{i-1} \tag{4.12}$$

and

$$r \equiv r(x, t) = \sum_{i=0}^{\infty} r_i(x) t^i. \tag{4.13}$$

These are

$$2t^3 \frac{\partial R}{\partial t} - (1 + 6xt^2) \frac{\partial R}{\partial x} = 0 \tag{4.14}$$

and

$$2t^3 \left(\frac{\partial r}{\partial t} + R \right) - (1 + 6xt^2) \frac{\partial r}{\partial x} = 0. \tag{4.15}$$

Theoretically, there exists a pair of ordinary differential equations derivable from $R_n(x)$ and $r_n(x)$ (see [4], [5]), but so far their nature has not been vouchsafed to the writer.

Coming now to descending diagonal polynomials, we encounter a surprisingly felicitous ease with the mathematics (as also occurs, e.g., in [4], [5]).

Descending Diagonals

Formed in a similar way to the rising diagonal polynomials, except that we now imagine systems of parallel downward-slanting lines (cf. [4] for Chebyshev and Fermat polynomials), we behold the *descending diagonal functions* $\{D_i(x)\}$ and $\{d_i(x)\}$, respectively.

Some of these are, say,

$$\begin{aligned} &D_0(x)=0,\ D_1(x)=1,\ D_2(x)=1+2x, \ldots, \\ &D_5(x)=1+8x+24x^2+32x^3+16x^4 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &d_0(x)=2,\ d_1(x)=1+4x,\ d_2(x)=1+6x+8x^2, \ldots, \\ &d_5(x)=1+12x+56x^2+128x^3+144x^4+64x^5. \end{aligned} \tag{4.17}$$

Patterns of behavior are readily discernible from the formation of the generating functions

$$\sum_{n=1}^{\infty} D_n(x)t^{n-1}=[1-(1+2x)t]^{-1}, \tag{4.18}$$

and

$$\sum_{n=1}^{\infty} d_n(x)t^{n-1}=(1+4x)[1-(1+2x)t]^{-1}, \tag{4.19}$$

whence $(n \geq 1)$

$$D_n(x)=(1+2x)^{n-1} \tag{4.20}$$

and

$$d_n(x)=(1+4x)(1+2x)^{n-1}, \tag{4.21}$$

leading to

$$d_n(x)=(1+4x)D_n(x), \tag{4.22}$$

$$\frac{D_n(x)}{D_{n-1}(x)}=\frac{d_n(x)}{d_{n-1}(x)}=1+2x \quad (n \geq 2), \tag{4.23}$$

i.e., $d_n(x)D_{n-1}(x)=d_{n-1}(x)D_n(x)$,

$$\frac{d_n(x)}{D_n(x)}=1+4x \quad (n \geq 1), \tag{4.24}$$

$$d_n(x)=D_{n+1}(x)+2xD_n(x), \tag{4.25}$$

$$(1+4x)^2D_n(x)=d_{n+1}(x)+2xd_n(x), \tag{4.26}$$

$$J_5(x)D_{n-1}(x)=D_{n+1}(x)+2xD_{n-1}(x), \tag{4.27}$$

and

$$J_5(x)d_{n-1}(x)=d_{n+1}(x)+2xd_{n-1}(x). \tag{4.28}$$

See Table 1 for $J_5(x)$. *En passant*, notice that $1+2x$ and $1+4x$, occurring in (4.18)-(4.24) and (4.26), may be expressed variously in terms of polynomials in Table 1, Table 2, (4.1), (4.2), (4.16), and (4.17).

Observe that the summed forms in (4.27)-(4.28) preclude the possibility of any associated sequence properties of $\{D_n(x)\}$ and $\{d_n(x)\}$ analogous to those for $\{J_n(x)\}$ and $\{j_n(x)\}$. (Put $k=1$ in (3.25) and (3.26) for the comparison and contrast.)

Quartet: Differential Equations

Write

$$D \equiv D(x,t) = \sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1-(1+2x)t]^{-1} \tag{4.29}$$

and

$$d \equiv d(x,t) = \sum_{n=1}^{\infty} d_n(x)t^{n-1} = (1+4x)[1-(1+2x)t]^{-1} \tag{4.30}$$

using (4.5) and (4.6).

Without difficulty one derives, from (4.20), (4.21), (4.29), and (4.30),

$$2t\frac{\partial D}{\partial t} - (1+2x)\frac{\partial D}{\partial x} = 0, \tag{4.31}$$

$$2t\frac{\partial d}{\partial t} - (1+2x)\left\{\frac{\partial d}{\partial x} - 4D\right\} = 0, \tag{4.32}$$

$$(1+2x)\frac{dD_n}{dx}(x) = 2(n-1)D_n(x) \tag{4.33}$$

$$\frac{dd_n(x)}{dx} = 2\{2D_n(x) + (n-1)d_{n-1}(x)\}. \tag{4.34}$$

More generally,

$$(1+2x)^k \frac{\partial^k D}{\partial x^k} = (2t)^k \frac{\partial^k D}{\partial t^k} = \frac{k!\{(1+2x)2t\}^k}{[1-(1+2x)t]^{k+1}} \tag{4.35}$$

and

$$\frac{d^{n-1}D_n(x)}{dx^{n-1}} = (n-1)!2^{n-1}. \tag{4.36}$$

Roots

Clearly, from (4.20) and (4.21), the polynomial equations $D_n(x)=0$ (of degree $n-1$) and $d_n(x)=0$ (of degree n) have multiple roots, namely, an $(n-1)$-fold root $-1/2$ in the first case and an $(n-1)$-fold root $-1/2$ together with a root $x=-1/4$ in the second case.

Diagonal Numbers

Substitute $x=1$ in (4.1), (4.2), (4.16), and (4.17). Then the skeletal profiles of the bodies fleshed out by the polynomials reduce to

$$\begin{array}{rcllllllllllll} n &=& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \{R_n\} &=& 0 & 1 & 1 & 1 & 3 & 5 & 7 & 13 & 23 & 37 & 63 & 109 \\ \{r_n\} &=& 2 & 1 & 1 & 5 & 7 & 9 & 19 & 33 & 51 & 89 & 155 & 267 \\ \{D_n\} &=& 0 & 1 & 3 & 9 & 27 & 81 & 243 & 729 & 2187 & \cdots & \cdots & \cdots \\ \{d_n\} &=& 2 & 5 & 15 & 45 & 135 & 405 & 1215 & 3645 & \cdots & \cdots & \cdots & \cdots \end{array} \tag{4.37}$$

So, e.g., by (4.6), (4.7), (4.10), (4.22), and (4.23),

$$\left.\begin{array}{l} R_n = R_{n-1} + 2R_{n-3}, \\ r_n = r_{n-1} + 2r_{n-3}, \\ r_n = R_n + 4R_{n-2}, \\ d_n = 5D_n = 15D_{n-1} \text{ since } D_n = 3D_{n-1}. \end{array}\right\} \tag{4.38}$$

Diagonal numbers for (say) Fibonacci, Pell, Fermat, and Chebyshev polynomials *inter alia* could be tabulated, along with the numbers for their cognate "Lucas" polynomials. See, e.g., [4], [5], [11], [13], and [14].

Reverting to (4.1), (4.2), (4.16), and (4.17), we may find mild interest in substituting $x = 1/2$ and $x = 1/4$.

Bizarre Afterthought

What of any interest might eventuate if we imagined rising rising diagonals, descending descending diagonals, rising descending diagonals, and other combinations of the two elementary dichotomous concepts of rising and (falling) descending?

Conjectures

$$R_{m+n}(x) = R_{m+1}(x)R_n(x) + 2xR_m(x)R_{n-2}(x) + 2xR_{m-1}(x)R_{n-1}(x) \tag{4.39}$$

and

$$r_{m+n}(x) = R_{m+1}(x)r_n(x) + 2xR_m(x)r_{n-2}(x) + 2xR_{m-1}(x)r_{n-1}(x). \tag{4.40}$$

5. AUGMENTED JACOBSTHAL-TYPE REPRESENTATION POLYNOMIALS

New symbolism and terminology are now required.

Following the situation for the number sequences $\{\mathcal{J}_n\}$ and $\{\hat{j}\}$ described in (3.4) and (3.5) of [8], we introduce the *augmented Jacobsthal representation polynomial sequence* $\{J_n(x)\}$ defined by

$$\mathcal{J}_{n+2}(x) = \mathcal{J}_{n+1}(x) + 2x\mathcal{J}_n(x) + 3, \quad \mathcal{J}_0(x) = 0, \ \mathcal{J}_1(x) = 1, \tag{5.1}$$

and the *augmented Jacobsthal-Lucas representation polynomial sequence* $\{\hat{j}_n(x)\}$ defined by

$$\hat{j}_{n+2}(x) = \hat{j}_{n+1}(x) + 2x\hat{j}_n(x) + 5, \quad \hat{j}_0(x) = 0, \ \hat{j}_1(x) = 1. \tag{5.2}$$

Some of these are, for example,

$$\mathcal{J}_0(x) = 0, \mathcal{J}_1(x) = 1, \mathcal{J}_2(x) = 4, \mathcal{J}_3(x) = 7 + 2x, \ldots, \mathcal{J}_8(x) = 22 + 102x + 160x^2 + 56x^3 \tag{5.3}$$

and

$$\hat{j}_0(x) = 0, \hat{j}_1(x) = 1, \hat{j}_2(x) = 6, \hat{j}_3(x) = 11 + 2x, \ldots, \hat{j}_8(x) = 36 + 162x + 240x^2 + 72x^3. \tag{5.4}$$

The choice and the *raison d'être* of the constants +3 and +5 in (5.1) and (5.2) are explained in [8]. Properties of these new polynomial sequences $\{\mathcal{J}_n(x)\}$ and $\{\hat{j}_n(x)\}$ are worthy of consideration *per se*.

Replacing +3 and +5 more generally by $+c$ has been done in a separate paper which thus covers the four special polynomial sequences $\{J_n(x)\}$, $\{j_n(x)\}$, $\{\mathcal{J}_n(x)\}$, and $\{\hat{j}_n(x)\}$.

6. BASIC PROPERTIES OF $\{\mathcal{J}_n(x)\}$ and $\{\hat{j}_n(x)\}$

Generating Functions

Standard techniques lead readily to

$$\sum_{i=1}^{\infty} \mathcal{J}_i(x)y^{i-1} = \frac{1+2y}{1-2y-(2x-1)y^2+2xy^3}, \tag{6.1}$$

$$\sum_{i=1}^{\infty} \hat{j}_i(x)y^{i-1} = \frac{1+4y}{1-2y-(2x-1)y^2+2xy^3}. \tag{6.2}$$

Binet Forms

Examination of Table 1 and (5.3) leads to the somewhat surprising observation that

$$\mathcal{J}_n(x) = \frac{J_{n+2}(x)+2J_{n+1}(x)-3}{2x}. \tag{6.3}$$

Proof of (6.3): Checking quickly validates the cases $n = 0, 1, 2, 3, 4$ (say). Assume (6.3) is true for $n = k$ (fixed integer), i.e., suppose

$$\mathcal{J}_k(x) = \frac{J_{k+2}(x)+2J_{k+1}(x)-3}{2x} \quad \cdots \ (H).$$

Then

$$\begin{aligned}
\mathcal{J}_{k+1}(x) &= \mathcal{J}_k(x) + 2x\mathcal{J}_{k-1}(x) + 3 \quad \text{by (5.1)} \\
&= \frac{J_{k+2}(x)+2J_{k+1}(x)-3+2x[J_{k+1}(x)+2J_k(x)-3]}{2x} + 3 \\
&= \frac{J_{k+2}(x)+2xJ_{k+1}(x)+2\{J_{k+1}(x)+2xJ_k(x)\}-3}{2x} \\
&= \frac{J_{k+3}(x)+2J_{k+2}(x)-3}{2x},
\end{aligned}$$

where the hypothesis (H) has been applied.

Hence, (6.3) is true for $n = k+1$, and so, for all n.

Consequently, (6.3) is true, by induction.

Induction is used in a similar fashion to establish

$$\hat{j}_n(x) = \frac{J_{n+2}(x)+4J_{n+1}(x)-5}{2x}. \tag{6.4}$$

For example,

$$\begin{aligned} n=7 \Rightarrow \text{R.H.S.} &= \frac{(1+14x+60x^2+80x^3+16x^4)+4(1+12x+40x^2+32x^3)-5}{2x} \\ &= 31+110x+104x^2+8x^3 \\ &= \hat{j}_7(x) \quad \text{from Table 1 and (5.4).} \end{aligned}$$

Binet forms for $\mathcal{T}_n(x)$ and $\hat{j}_n(x)$ are obtainable by substituting for $J_n(x)$ from (3.3) in (6.3) and (6.4).

Simson Formulas

$$\mathcal{T}_{n+1}(x)\,\mathcal{T}_{n-1}(x)-\mathcal{T}_n^2(x)=(-2x)^{n-2}(2x-6)-3(J_{n-1}(x)+2J_{n-2}(x)), \tag{6.5}$$

$$\hat{j}_{n+1}(x)\hat{j}_{n-1}(x)-\hat{j}_n^2(x)=(-2x)^{n-2}(2x-20)-5(J_{n-1}(x)+4J_{n-2}(x)). \tag{6.6}$$

Summation Formulas

$$\sum_{i=1}^{n}\mathcal{T}_i(x)=\frac{\mathcal{T}_{n+2}(x)-3n-4}{2x}, \tag{6.7}$$

$$\sum_{i=1}^{n}\hat{j}_i(x)=\frac{\hat{j}_{n+2}(x)-5n-6}{2x}. \tag{6.8}$$

Explicit Closed Forms

$$\mathcal{T}_n(x) = J_n(x)+3\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1-r}{r+1}(2x)^r, \tag{6.9}$$

$$\hat{j}_n(x) = J_n(x)+5\sum_{r=0}^{\left[\frac{n-1}{2}\right]}\binom{n-1-r}{r+1}(2x)^r. \tag{6.10}$$

Spotting the second portion of the expressions in (6.9) was not easy. Induction provides us with a proof.

Proof of (6.9): Verification of (6.9) for $n=1,2,3$ is straightforward. Assume (6.9) is true for $n=1,2,3,\ldots,k-1,k$. Then, by (1.1) and the hypothesis,

$$\begin{aligned} \mathcal{T}_k(x)+2x\mathcal{T}_{k-1}(2)+3 &= J_{k+1}(x)+3\left\{1+\sum_{r=0}^{\left[\frac{k-1}{2}\right]}\binom{k-1-r}{r+1}(2x)^r+\sum_{r=0}^{\left[\frac{k-2}{2}\right]}\binom{k-2-r}{r+1}(2x)^{r+1}\right\} \\ &= J_{k+1}(x)+3\sum_{r=0}^{\left[\frac{k}{2}\right]}\binom{k-r}{r+1}(2x)^r \\ &= \mathcal{T}_{k+1}(x) \quad \text{by (6.9).} \end{aligned}$$

Being valid for $n=k+1$, the theorem is true for all n.

Pascal's formula for binomial coefficients has been applied in the proof of (6.9) when combining corresponding powers $(2x)^r$, $r=1,2,3,\ldots$. Also, we have absorbed $1=\binom{k-1}{0}$ into the constant $\binom{k-1}{1}$ to produce $\binom{k}{1}$, by Pascal's formula.

Arguments of a similar nature are applicable for (6.10).

Observe the simple, but important, connection between (6.3) and (6.4):

$$\hat{j}_n(x) - \mathcal{T}_n(x) = \frac{J_{n+1}(x) - 1}{x}. \tag{6.11}$$

This seems to be an appropriate place at which to conclude our theory, though more could be told.

7. CONCLUDING REMARKS

Possibilities for other avenues of development that present themselves include, for example:

(i) the extension of the theory in this paper to negative subscripts [9];

(ii) convolutions for Jacobsthal-type polynomials (cf. [13]);

(iii) further work on diagonal functions, e.g., as in [14];

(iv) research into Jacobsthal-type polynomials along the lines of that for Pell-type polynomials in [13] and in a series of papers by Mahon and Horadam, e.g., [10].

Initial exploration of some of these opportunities has commenced.

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THE APPEARANCE OF FIBONACCI AND LUCAS NUMBERS IN THE SIMULATION OF ELECTRICAL POWER LINES SUPPLIED BY TWO SIDES

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INTRODUCTION

In the analysis of some physical structures, the possibility of modeling them with an electrical circuit is particularly important because it allows the determination of the characteristic behavior by means of a simple circuital analysis. Moreover, it is also interesting to have a different method of measurement evaluation, comparable with the "direct" one, which sometimes either is not simple or requires the use of computer programs which on some occasions do not go into convergence. Finally, it can make a contribution to the mathematical interest in testing of network software algorithms for solving linear equation systems.

In this article, a symmetrical ladder network is used as a model for the simulation of electrical power lines. Fibonacci and Lucas numbers come out from the analysis of the power distribution among the users. The electrical characteristics of the ladder network have also been determined in a closed form using a theory previously developed by the author [1].

1. MODELING OF A POWER ELECTRIC LINE

Let us consider a high voltage electric line, supplied by the two sides, which gives power to users distributed along the line, as in Figure 1.

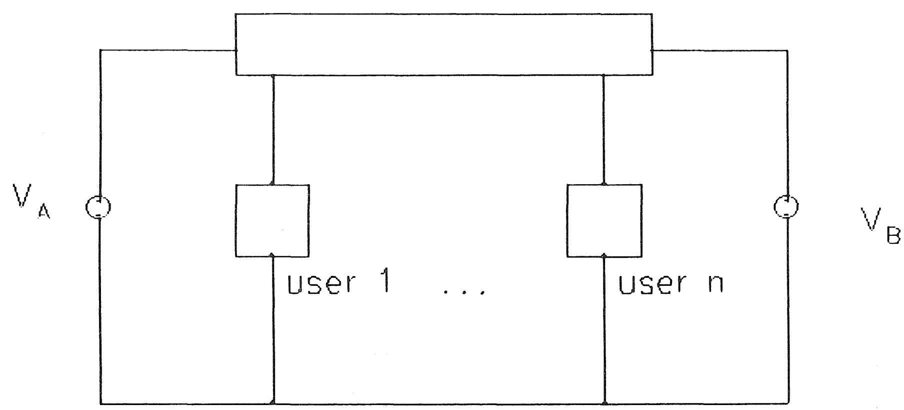

FIGURE 1. The Electrical Power Line Supplied by Two Sides

A ladder structure (Fig. 2) can be used as a discrete electrical model of the power line. For the sake of simplicity, we consider n users who have equal consumption, represented by n equal vertical impedances Z_2, placed at equidistant points characterized by equal horizontal impedances Z_1.

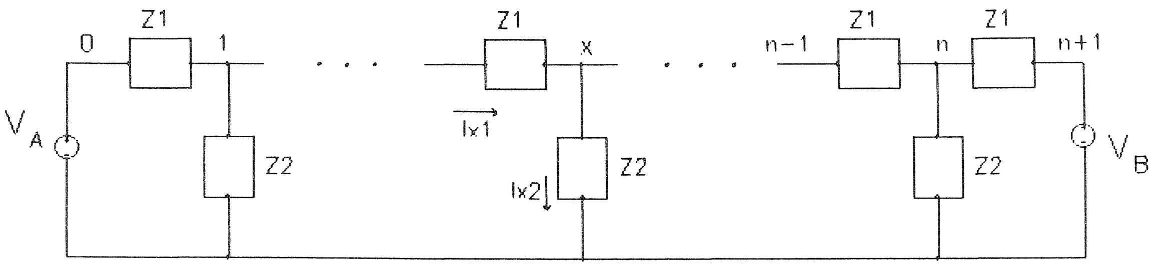

FIGURE 2. Ladder Network as a Model of the Power Line

2. ANALYSIS OF THE LADDER NETWORK

In order to analyze the network of Figure 2, we can use the superimposition of the effects in the networks of Figures 3 and 4. The analysis of these networks can be done starting from the study of the network of Figure 5, by adding a "load" impedance.

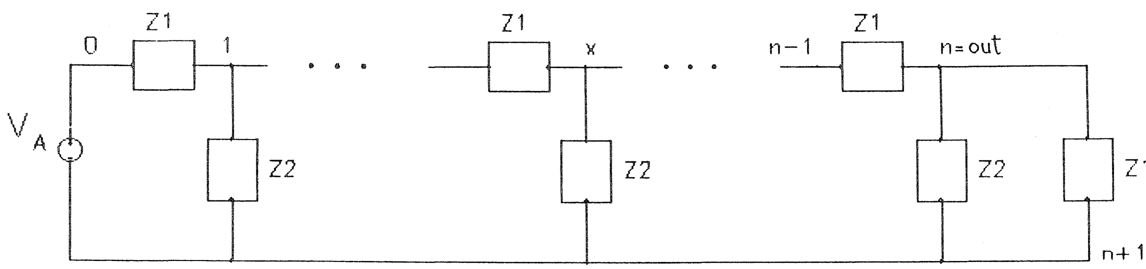

FIGURE 3. Ladder Network Supplied by V_A

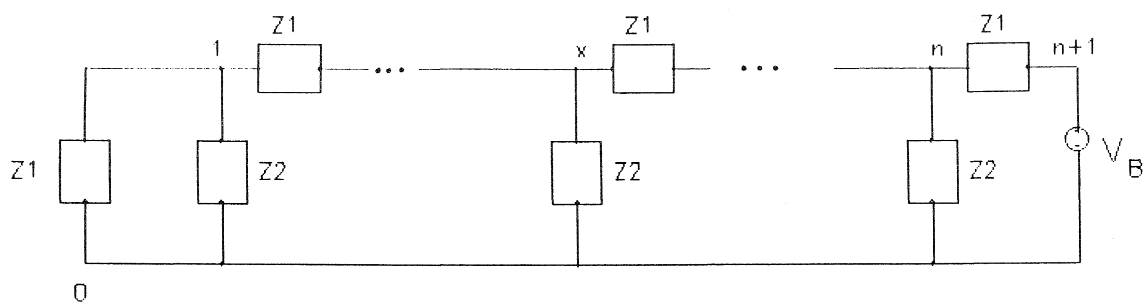

FIGURE 4. Ladder Network Supplied by V_B

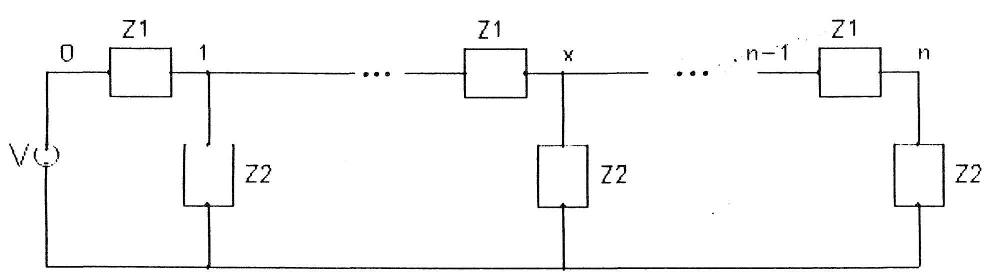

FIGURE 5. Ladder Network with *n* Identical Cells

In [1] a new fast method for the ladder network characterization in Figure 5 was presented; by using this method, all the electrical parameters of a ladder network formed by n identical cells can be written directly by means of both a parameter that characterizes the single cell [the "cell factor" $K(s) = \mathrm{Z}_1(s)/\mathrm{Z}_2(s)$] and the polynomials in K whose coefficients are the entries of two numerical triangles, named DFF [3] and DFFz [4], here reported:

n	K^0	K^1	K^2	K^3
0	1			
1	1	1		
2	1	3	1	
3	1	6	5	1
	...			

DFF Triangle

$$\text{Entry} = \binom{n+K}{n-K}$$

n	K^0	K^1	K^2	K^3
0	1			
1	2	1		
2	3	4	1	
3	4	10	6	1
	...			

DFFz Triangle

$$\text{Entry} = \binom{n+K+1}{n-K}$$

The mathematical properties of triangles and polynomials have been presented in [2]. Let us call b_n and B_n the polynomials whose coefficients are the entries of DFF and DFFz triangles, respectively. These polynomials coincide with the polynomials defined by Morgan-Voyce and then investigated by Swamy [7] and Lahr [5] and [6].

All the electrical characteristics of the network represented in Figure 5 can be expressed directly in a closed form by means of these polynomials if all the cells are equal.

The networks drawn in Figures 3 and 4 are very similar to that of Figure 5. The only difference is in the fact that the last cell of the Figure 5 network has a "load" impedance of infinite value. It is possible to write the electrical expressions for the Figure 3 and Figure 4 networks as simply as for the Figure 5 ones and also in closed form.

For the Figure 3 network, we have (see [5], p. 275) that the transfer function is given by

$$\mathrm{G}'_{\mathrm{n}}(K) = \frac{\mathrm{V}_{\mathrm{out}}}{\mathrm{V}_{\mathrm{A}}} = \frac{1}{B_n(K)}, \tag{1}$$

while the voltage at the generical x^{th} node is given by

$$\mathrm{V}'_{\mathrm{x}}(K) = \mathrm{V}_{\mathrm{A}} \frac{B_{n-x}(K)}{B_n(K)} \quad (0 \le \mathrm{x} \le n+1), \tag{2}$$

with $B_{-1}(K) = 0$.

The voltage behavior for the network of Figure 4 is symmetrical. For that reason, we can write

$$\mathrm{V}''_{\mathrm{x}}(K) = \mathrm{V}_{\mathrm{B}} \frac{B_{x-1}(K)}{B_n(K)} \quad (0 \le \mathrm{x} \le n+1). \tag{3}$$

By the application of the superimposition of the effects, we can write, for the network represented in Figure 2, the following expression for the node voltages:

$$\mathrm{V}_{\mathrm{x}}(K) = \mathrm{V}'_{\mathrm{x}}(K) + \mathrm{V}''_{\mathrm{x}}(K) = \mathrm{V}_{\mathrm{A}} \frac{B_{n-x}(K)}{B_n(K)} + \mathrm{V}_{\mathrm{B}} \frac{B_{x-1}(K)}{B_n(K)} \quad (0 \le \mathrm{x} \le n+1). \tag{4}$$

Denoting by I_{x1} and I_{x2} the currents flowing into the x^{th} cell horizontal and vertical impedances, respectively, we can write similar expressions, using the following property of Morgan-Voyce polynomials, $b_x = B_x - B_{x-1}$ (see [1], [5]-[7]):

$$I_{x1} = \frac{1}{Z_1}[V_x - V_{x-1}] = \frac{1}{Z_1}\left[-V_A \frac{b_{n-x+1}(K)}{B_n(K)} + V_B \frac{b_{x-1}(K)}{B_n(K)}\right] \quad (1 \le x \le n+1); \tag{5}$$

$$I_{x2} = \frac{V_x}{Z_2} = \frac{1}{Z_2}\left[V_A \frac{B_{n-x}(K)}{B_n(K)} + V_B \frac{B_{x-1}(K)}{B_n(K)}\right] \quad (1 \le x \le n). \tag{6}$$

Let us now consider the case of odd n, for which the middle point exists for the voltage and the vertical current and is defined for $x = m = (n+1)/2$. In this point, from (4), we can write

$$V_m = (V_A + V_B)\frac{B_{(n-1)/2}}{B_n}. \tag{7}$$

In the middle vertical impedance, we also have

$$I_{m2} = \frac{1}{Z_2}(V_A + V_B)\frac{B_{(n-1)/2}}{B_n}. \tag{8}$$

In the case of even n, we can reason analogously by considering the middle horizontal current, whose value is given by

$$I_{m1} = \frac{1}{Z_1}(-V_A + V_B)\frac{b_{n/2}}{B_n} \tag{9}$$

being $x = m = (n+2)/2$, while expressions (4)-(6) are always valid.

We are mainly interested in determining the power dissipated in the vertical impedances (because only these have a physical meaning), which is given by the voltage-current product:

$$P_{x2} = \frac{1}{Z_2}\left[V_A \frac{B_{n-x}(K)}{B_n(K)} + V_B \frac{B_{x-1}(K)}{B_n(K)}\right]^2 \quad (1 \le x \le n); \tag{10}$$

$$P_{m2} = \frac{1}{Z_2}(V_A + V_B)^2\left[\frac{B_{(n-1)/2}}{B_n}\right]^2 \quad (n \text{ odd}). \tag{11}$$

The Fibonacci and Lucas numbers appear in the case of $K = 1$, which corresponds to $Z_1 = Z_2 = R$. In this case, $B_x = F_{2x+2}$ and $b_x = F_{2x+1}$. Consequently, we have

$$V_x = V_A \frac{F_{2(n+1-x)}}{F_{2(n+1)}} + V_B \frac{F_{2x}}{F_{2(n+1)}} \quad (0 \le x \le n+1), \tag{12}$$

$$V_m = (V_A + V_B)\frac{F_{n+1}}{F_{2n+2}} = (V_A + V_B)\frac{1}{L_{n+1}} \quad (n \text{ odd}), \tag{13}$$

$$I_{x1} = \frac{1}{R}\left[-V_A \frac{F_{2(n+1-x)+1}}{F_{2(n+1)}} + V_B \frac{F_{2x-1}}{F_{2(n+1)}}\right] \quad (1 \le x \le n+1), \tag{14}$$

$$I_{x2} = \frac{1}{R}\left[V_A \frac{F_{2(n+1-x)}}{F_{2(n+1)}} + V_B \frac{F_{2x}}{F_{2(n+1)}}\right] \qquad (1 \le x \le n), \tag{15}$$

$$I_{m1} = \frac{1}{R}(-V_A + V_B)\frac{1}{L_{n+1}} \qquad (n \text{ even}), \tag{16}$$

$$I_{m2} = \frac{1}{R}(V_A + V_B)\frac{1}{L_{n+1}} \qquad (n \text{ odd}), \tag{17}$$

from which:

$$P_{x2} = \frac{1}{R}\left[V_A \frac{F_{2(n+1-x)}}{F_{2(n+1)}} + V_B \frac{F_{2x}}{F_{2(n+1)}}\right]^2 \qquad (1 \le x \le n); \tag{18}$$

$$P_{m2} = \frac{1}{R}(V_A + V_B)^2\left[\frac{1}{L_{n+1}}\right]^2 \qquad (n \text{ odd}). \tag{19}$$

The last two relations show that the power consumption of the users is also a function of the Fibonacci and Lucas numbers.

3. EXAMPLE

Let us consider the power dissipation in the vertical impedances in the case of $n = 3$, shown in Figure 6 below.

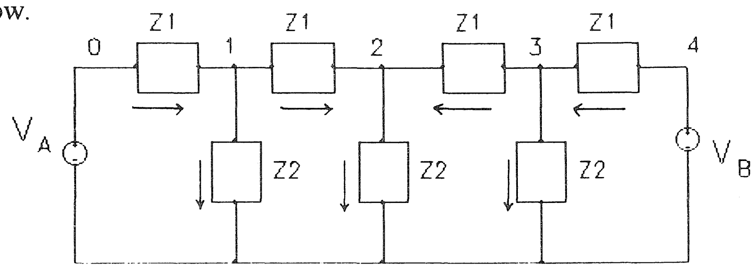

FIGURE 6. Example

In the generical case of different values between the horizontal and vertical impedances, we have, from (9):

$$P_{x2} = \frac{1}{Z_2}\left[V_A \frac{B_{3-x}(K)}{B_3(K)} + V_B \frac{B_{x-1}(K)}{B_3(K)}\right]^2 \qquad (1 \le x \le 3), \tag{20}$$

that is,

$$\begin{cases} P_{12} = \dfrac{1}{Z_2}\left[V_A \dfrac{B_2(K)}{B_3(K)} + V_B \dfrac{B_0(K)}{B_3(K)}\right]^2 = \dfrac{1}{Z_2}\left[\dfrac{V_A(K^2+4K+3)+V_B}{K^3+6K^2+10K+4}\right]^2, \\ P_{22} = \dfrac{1}{Z_2}[V_A + V_B]^2\left[\dfrac{B_1(K)}{B_3(K)}\right]^2 = \dfrac{1}{Z_2}[V_A + V_B]^2\left[\dfrac{K+2}{K^3+6K^2+10K+4}\right]^2, \\ P_{32} = \dfrac{1}{Z_2}\left[V_B \dfrac{B_2(K)}{B_3(K)} + V_A \dfrac{B_0(K)}{B_3(K)}\right]^2 = \dfrac{1}{Z_2}\left[\dfrac{V_B(K^2+4K+3)+V_A}{K^3+6K^2+10K+4}\right]^2. \end{cases} \tag{21}$$

In the particular case of $Z_1 = Z_2 = R$, we have

$$P_{x2} = \frac{1}{R}\left[V_A \frac{F_{8-2x}(K)}{F_8(K)} + V_B \frac{F_{2x}(K)}{F_8(K)} \right]^2 \quad (1 \le x \le 3), \tag{22}$$

from which:

$$\begin{cases} P_{12} = [8V_A + V_B]^2 / 441\,R, \\ P_{22} = [V_A + V_B]^2 / 49\,R, \\ P_{32} = [V_B + 8V_A]^2 / 441\,R. \end{cases} \tag{23}$$

4. PARTICULAR SUPPLY VALUES

In the analysis of the symmetrical ladder network, which models the power electrical line, we can consider some particular cases for the values of V_B and V_A.

1) If $V_B = V_A > 0$, the network is completely symmetrical and the current flows as in the direction, for example, indicated in Figure 6, if n is odd. When n is even, in the middle horizontal impedance, the current is zero.

2) If $V_A = -V_B$, and only from the mathematical point of view, only the case n odd is interesting. In this case, in the middle point, all the electrical characteristics (voltage, vertical current and power) are zero.

3) In the case $V_B = V_A + \Delta V$, where ΔV can be positive or negative and $\Delta V \ll V_A, V_B$, we have a slightly unbalanced situation and, as a consequence, there is a small difference in the electrical parameter values. This is a real case and the computation can be of practical importance: if one of the supplies does not have enough power (owing to a lack of power), the other one can provide it. We can write:

$$V_x = V_A \frac{B_{n-x} + B_{x-1}}{B_n} + \Delta V \frac{B_{x-1}}{B_n}; \; \Delta V_x = \Delta V \frac{B_{x-1}}{B_n} \quad (1 \le x \le n) \tag{24}$$

and

$$I_{x2} = \frac{1}{Z_2}\left[V_A \frac{B_{n-x} + B_{x-1}}{B_n} + \Delta V \frac{B_{x-1}}{B_n} \right]; \; \Delta I_{x2} = \frac{1}{Z_2} \Delta V \frac{B_{x-1}}{B_n} \quad (1 \le x \le n), \tag{25}$$

so that

$$\Delta P_{x2} = \Delta V_x \Delta I_{x2} = \frac{1}{Z_2} \Delta V^2 \left[\frac{B_{x-1}}{B_n} \right]^2 \quad (1 \le x \le n) \tag{26}$$

which, in the case of $Z_1 = Z_2 = R$, is equal to

$$\Delta P_{x2} = \frac{1}{R} \Delta V^2 \left[\frac{F_{2x}}{F_{2n+2}} \right]^2 \tag{27}$$

and, in the middle point, for n odd, is equal to

$$\Delta P_m = \frac{1}{R} \Delta V^2 \left[\frac{1}{L_{n+1}} \right]^2. \tag{28}$$

This means that the power variation is strongly dependent on the number of cells n (i.e., the number of the users) upon whom the line is modeled and is also a function of Fibonacci and Lucas numbers.

For example, if $n = 3$, for a variation of 1%, we have that

$$R \cdot \Delta P_m = 2.041 \mu W \cdot \Omega \tag{29}$$

while, for a variation of 10%, we have that

$$R \cdot \Delta P_m = 0.204 mW \cdot \Omega, \tag{30}$$

where, in the case of 10 cells, we have, for $\Delta V = 1\%$,

$$R \cdot \Delta P_m = 2.52 nW \cdot \Omega, \tag{31}$$

and, for $\Delta V = 10\%$,

$$R \cdot \Delta P_m = 0.25 \mu W \cdot \Omega. \tag{32}$$

CONCLUSION

A symmetrical ladder network with a high number of cells can be considered as a good model for the investigation of the behavior of an electrical power line. In the particular case of equal impedances, the electrical characteristics can be written as a function of Fibonacci and Lucas numbers.

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A NOTE ON THE BRACKET FUNCTION TRANSFORM

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Let (a_n) be a given sequence. The bracket function transform (s_n) is defined by

$$s_n = \sum_{k=1}^{n} \left[\frac{n}{k}\right] a_k. \tag{1}$$

Let $S(x)$ denote the formal power series of the sequence (s_n), that is,

$$S(x) = \sum_{n=1}^{\infty} s_n x^n.$$

H. W. Gould [2] pointed out that

$$S(x) = \frac{1}{1-x} \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}. \tag{2}$$

The aim of this paper is to study the effect of the terms $\frac{1}{1-x}$, $\frac{1}{1-x^n}$, and x^n in (2). We replace these terms with the powers $\frac{1}{(1-x)^r}$, $\frac{1}{(1-x^n)^s}$, and x^{tn} and find the coefficients of the modified series.

First, we study the effect of the term $\frac{1}{1-x}$. If the term $\frac{1}{1-x}$ is deleted from (2), that is, if

$$T(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \tag{3}$$

then $T(x) = (1-x)S(x)$ and, consequently,

$$t_n = s_n - s_{n-1} = \sum_{k=1}^{n} \left(\left[\frac{n}{k}\right] - \left[\frac{n-1}{k}\right] \right) a_k = \sum_{d|n} a_d \tag{4}$$

(see [2], Eq. (8)). More generally, let

$$T(x) = \frac{1}{(1-x)^r} \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \quad r \in \mathbf{R}. \tag{5}$$

What are the coefficients of $T(x)$?

Let $\binom{a}{n} = \frac{a(a-1)\cdots(a-n+1)}{n!}$, $a \in \mathbf{R}$. Then

$$\sum_{n=0}^{\infty} \binom{a}{n} x^n = (1+x)^a \tag{6}$$

(see [1], Eq. (1.1)). Thus,

$$\frac{1}{(1-x)^{r-1}} = \sum_{n=0}^{\infty} (-1)^n \binom{-r+1}{n} x^n. \tag{7}$$

It is known (see [2], Eq. (5)) that

$$\frac{1}{(1-x)(1-x^k)} = \sum_{n=0}^{\infty}\left[\frac{n+k}{k}\right]x^n = \sum_{n=0}^{\infty}([n/k]+1)x^n. \tag{8}$$

Combining (7) and (8) and applying the Cauchy convolution, we obtain

$$\frac{1}{(1-x)^r(1-x^k)} = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^i\binom{-r+1}{i}\left[\frac{n-i+k}{k}\right]\right)x^n. \tag{9}$$

For the sake of brevity, we write

$$C(n,k,r) = \sum_{i=0}^{n}(-1)^i\binom{-r+1}{i}\left[\frac{n-i+k}{k}\right]. \tag{10}$$

Now we use (9) in finding the coefficients of $T(x)$ in (5). In fact,

$$T(x) = \sum_{k=1}^{\infty}a_k x^k \frac{1}{(1-x)^r(1-x^k)} = \sum_{k=1}^{\infty}a_k x^k \sum_{n=0}^{\infty}C(n,k,r)x^n$$

$$= \sum_{k=1}^{\infty}a_k x^k \sum_{n=k}^{\infty}C(n-k,k,r)x^{n-k} = \sum_{n=1}^{\infty}x^n\sum_{k=1}^{n}C(n-k,k,r)a_k,$$

which shows that the coefficients of $T(x)$ in (5) are

$$t_n = \sum_{k=1}^{n}C(n-k,k,r)a_k, \tag{11}$$

where $C(n-k,k,r)$ is as defined in (10). Note that

(i) if $r=1$, then $C(n-k,k,r)=[n/k]$, and thus $t_n = s_n$, which is the bracket function transform (1),

(ii) if $r=0$, then $C(n-k,k,r)=[n/k]-[(n-1)/k]$, and thus (11) reduces to (4).

Second, we study the effect of the term $\frac{1}{1-x^n}$. If the term $\frac{1}{1-x^n}$ is deleted from (2), that is, if

$$T(x) = \frac{1}{1-x}\sum_{n=1}^{\infty}a_n x^n, \tag{12}$$

then

$$t_n = \sum_{k=1}^{n}a_k. \tag{13}$$

More generally, let

$$T(x) = \frac{1}{1-x}\sum_{n=1}^{\infty}a_n\frac{x^n}{(1-x^n)^s}, \quad s\in\mathbf{R}. \tag{14}$$

What are the coefficients of $T(x)$?

By (6) we obtain

$$\frac{1}{(1-x)(1-x^k)^s} = (1+x+x^2+\cdots)\left(1-\binom{-s}{1}x^k+\binom{-s}{2}x^{2k}-\cdots\right) =$$

$$= (1+x+\cdots+x^{k-1}) + \left(1-\binom{-s}{1}\right)(x^k + x^{k+1} + \cdots + x^{2k-1})$$

$$+\left(1-\binom{-s}{1}+\binom{-s}{2}\right)(x^{2k}+x^{2k+1}+\cdots+x^{3k-1})+\cdots$$

$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{[n/k]}(-1)^i\binom{-s}{i}\right)x^n.$$

Applying Equation (1.9) of [1], we obtain

$$\frac{1}{(1-x)(1-x^k)^s} = \sum_{n=0}^{\infty}\binom{[n/k]+s}{[n/k]}x^n. \tag{15}$$

We can use this formula in finding the coefficients of $T(x)$. In fact,

$$T(x) = \sum_{k=1}^{\infty} a_k x^k \frac{1}{(1-x)(1-x^k)^s} = \sum_{k=1}^{\infty} a_k x^k \sum_{n=0}^{\infty}\binom{[n/k]+s}{[n/k]}x^n$$

$$= \sum_{k=1}^{\infty} a_k x^k \sum_{n=k}^{\infty}\binom{[n/k]+s-1}{[n/k]-1}x^{n-k} = \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n}\binom{[n/k]+s-1}{[n/k]-1}a_k,$$

which shows that the coefficients of $T(x)$ in (14) are

$$t_n = \sum_{k=1}^{n}\binom{[n/k]+s-1}{[n/k]-1}a_k. \tag{16}$$

Note that

(i) if $s=1$, then $\binom{[n/k]+s-1}{[n/k]-1} = [n/k]$, and thus $t_n = s_n$, which is the bracket function transform (1),

(ii) if $s=0$, then $\binom{[n/k]+s-1}{[n/k]-1} = 1$, and thus (16) reduces to (13).

Third, we study the effect of the term x^n. Let

$$T(x) = \frac{1}{1-x}\sum_{n=1}^{\infty} a_n \frac{x^{tn}}{1-x^n}, \quad t \in \mathbf{Z}^+. \tag{17}$$

Then, by (8),

$$\begin{aligned} T(x) &= \sum_{k=1}^{\infty} a_k x^{tk} \frac{1}{(1-x)(1-x^k)} = \sum_{k=1}^{\infty} a_k x^{tk} \sum_{n=0}^{\infty}([n/k]+1)x^n \\ &= \sum_{k=1}^{\infty} a_k x^{tk} \sum_{n=tk}^{\infty}\big([(n-tk)/k]+1\big)x^{n-tk} = \sum_{n=t}^{\infty} x^n \sum_{k=1}^{[n/t]}\big([n/k]-t+1\big)a_k, \end{aligned} \tag{18}$$

which shows that the coefficients of $T(x)$ in (17) are

$$t_n = \sum_{k=1}^{[n/t]}\big([n/k]-t+1\big)a_k. \tag{19}$$

Note that if $t = 1$, then $t_n = s_n$, which is the bracket function transform (1).

What is the effect of deleting the term x^n in (2), that is, what are the coefficients of

$$T(x) = \frac{1}{1-x}\sum_{n=1}^{\infty} a_n \frac{1}{1-x^n}? \tag{20}$$

Proceeding in a way similar to that in (18), we obtain the coefficients of $T(x)$ in (20) as

$$t_n = \sum_{k=1}^{\infty}([n/k]+1)a_k = s_n + a, \tag{21}$$

provided that the series $\sum_{k=1}^{\infty} a_k$ is convergent and its sum is equal to a.

Finally, we note that the three cases (5), (14), and (17) could be treated simultaneously. In fact, let

$$T(x) = \frac{1}{(1-x)^r}\sum_{n=1}^{\infty} a_n \frac{x^{tn}}{(1-x^n)^s}, \quad r, s \in \mathbf{R}, t \in \mathbf{Z}^+. \tag{22}$$

Then

$$t_n = \sum_{k=1}^{[n/t]} C(n-tk, k, r, s)a_k, \tag{23}$$

where

$$C(n, k, r, s) = \sum_{i=0}^{n}(-1)^i \binom{-r+1}{i}\binom{[(n-i)/k]+s}{[(n-i)/k]}.$$

This can be proved in a similar way to the above three cases. For the sake of brevity, we omit the details here.

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DYNAMICS OF THE ZEROS OF FIBONACCI POLYNOMIALS

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1. INTRODUCTION

The Fibonacci polynomials are defined by the recursion relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \tag{1}$$

with the initial values $F_1(x) = 1$ and $F_2(x) = x$. When $x = 1$, $F_n(x)$ is equal to the n^{th} Fibonacci number, F_n. The Lucas polynomials, $L_n(x)$ obey the same recursion relation, but have initial values $L_1(x) = x$ and $L_2(x) = x^2 + 2$.

Explicit expressions for the zeros of the Fibonacci and Lucas polynomials have been known for some time ([1], [2]). The zeros of $F_{2n}(x)$ are at the points

$$\pm 2i \sin\frac{k\pi}{2n}, \quad k = 0, 1, \ldots, n-1. \tag{2}$$

The zeros of the odd polynomials $F_{2n+1}(x)$ are at

$$\pm 2i \sin\left[\left(\frac{2k+1}{2n+1}\right)\cdot\frac{\pi}{2}\right], \quad k = 0, 1, \ldots, n-1. \tag{3}$$

Similarly, for the Lucas polynomials, the zeros of $L_{2n}(x)$ are at

$$\pm 2i \sin\left[\left(\frac{2k+1}{2n}\right)\cdot\frac{\pi}{2}\right], \quad k = 0, 1, \ldots, n-1, \tag{4}$$

and the zeros of $L_{2n+1}(x)$ are at

$$\pm 2 \sin\frac{k\pi}{2n+1}, \quad k = 0, 1, \ldots, n-1. \tag{5}$$

With a view toward finding clues to obtaining similar analytic expressions for the zeros of the Tribonacci polynomials [3] and other generalizations of the $F_n(x)$, it is of interest to study the properties of the above expressions in more detail, looking for patterns that may generalize. In what follows, it will be shown that the zeros of each $F_n(x)$ and $L_n(x)$ satisfy a number of relations among themselves, many of which can be derived without any knowledge of the explicit formulas given above. The results presented here divide into two parts: in §2, expressions for the elementary symmetric polynomials of the zeros of each polynomial are derived. Then in §3, the zeros are described in terms of points on the trajectories of a dynamical system. In §4, some comments are made regarding the generalization of these results to the Tribonacci case.

2. SYMMETRIC POLYNOMIALS

Consider the elementary symmetric polynomials $\sigma_j(x_1, x_2, \ldots, x_m)$ over the set $x_1, x_2, \ldots, x_m$, where $0 \le j \le m$. These polynomials are defined by the relation

$$\prod_{k=1}^{m}(t+x_k) = \sum_{j=0}^{m}\sigma_j(x_1, \ldots, x_m)\cdot t^{m-j}. \tag{6}$$

Clearly, σ_j is a polynomial of order j in its m arguments. Note that by multiplying out the left-hand side and comparing powers of t on each side, we can write the σ_j as

$$\sigma_j(x_1, \ldots, x_m) = \sum_{l_1=0}^{m-1}\sum_{l_2=l_1}^{m-1}\cdots\sum_{l_j=l_{j-1}}^{m-1}\prod_{i=1}^{j}x_{l_i}. \tag{7}$$

The idea in the following theorems is to derive general formulas for the symmetric polynomials over the zeros, using the following algebraic representations of the Fibonacci and Lucas polynomials [2]:

$$F_n(x) = \sum_{k=0}^{\left[\frac{(n-1)}{2}\right]}\binom{n-k-1}{k}x^{n-2k-1}, \tag{8}$$

and

$$L_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n}{n-k}\binom{n-k}{k}x^{n-2k}, \tag{9}$$

where $[p]$ means the greatest integer less than or equal to p.

First, let us consider the $F_n(x)$. Since the zeros of $F_n(x)$ are pure imaginary and come in complex conjugate pairs, we will concentrate on their magnitudes. Thus, for the even polynomials $F_{2n}(x)$, denote the zeros by

$$x_0 = 0, \quad \pm ix_k, \quad k = 1, 2, \ldots, n-1, \tag{10}$$

with $x_k > 0$ for $k > 0$. As for the odd polynomials, $F_{2n+1}(x)$, denote the zeros by

$$\pm ix_k, \quad k = 0, 1, \ldots, n-1, \tag{11}$$

where

$$x_k = 2\sin\left(\frac{2k+1}{2n+1}\cdot\frac{\pi}{2}\right), \quad k = 0, 1, \ldots, n-1. \tag{12}$$

Theorem 1: The j^{th} symmetric polynomial over the squares of the zeros of $F_{2n}(x)$ is given by

$$\sigma_j(x_1^2, \ldots, x_{n-1}^2) = \binom{2n-j-1}{j}. \tag{13}$$

Proof: Clearly, since the zeros are of the form given in formula (10) above, the $F_{2n}(x)$ can be factored as follows:

$$F_{2n}(x) = x\prod_{k=1}^{n-1}(x-ix_k)(x+ix_k). \tag{14}$$

We can then regroup this expression in the following manner:

$$
\begin{aligned}
F_{2n}(x) &= x\prod_{k=1}^{n-1}(x^2+x_k^2) \\
&= x\left\{x^{2n-2}+x^{2n-4}\sum_{k=1}^{n-1}x_k^2+x^{2n-6}\sum_{j\neq k}x_j^2x_k^2+x^{2n-8}\sum_{j\neq k\neq l}x_j^2x_k^2x_l^2+\cdots+\prod_{k=1}^{n-1}x_k^2\right\} \\
&= x^{2n-1}+\sum_{j=1}^{n-1}x^{2n-2j-1}\left\{\sum_{1=l_1<l_2<\cdots<l_j}\prod_{i=1}^{j}x_{l_i}^2\right\} \\
&= x^{2n-1}+\sum_{j=1}^{n-1}\sigma_j(x_1^2,\ldots,x_{n-1}^2)x^{2n-2j-1}.
\end{aligned} \tag{15}
$$

But we also know that

$$
\begin{aligned}
F_{2n}(x) &= \sum_{j=0}^{n-1}\binom{2n-j-1}{j}x^{2n-2j-1} \\
&= x^{2n-1}+\sum_{j=1}^{n-1}\binom{2n-j-1}{j}x^{2n-2j-1}.
\end{aligned} \tag{16}
$$

Setting the right-hand sides of equations (15) and (16) equal and equating the coefficient of each power of x, we arrive at the desired result. □

Alternatively, this theorem and those that follow can be proved by applying standard trigonometric identities to the explicit formulas for the zeros that were given in equations (2) through (5).

Corollary 1: The zeros of the even polynomials $F_{2n}(x)$ satisfy the following relations for fixed n:

(i) $\prod_{k=1}^{n-1}x_k^2 = n$,

(ii) $\sum_{k=0}^{n-1}x_k^2 = 2(n-1)$.

Proof: These follow immediately by setting $j=1$ and $j=n-1$, respectively, in the previous theorem. □

Turning now to the odd Fibonacci polynomials, the following result can be quickly proved in the same manner.

Theorem 2: The j^{th} symmetric polynomial over the zeros of $F_{2n+1}(x)$ is given by the expression

$$\sigma_j(x_1^2,\ldots,x_{n-1}^2)=\binom{2n-j}{j}. \tag{17}$$

Corollary 2: For fixed n, the zeros of $F_{2n+1}(x)$ satisfy the following relations:

(i) $\prod_{k=0}^{n-1}x_k^2 = 1$,

(ii) $\sum_{k=0}^{n-1}x_k^2 = 2n-1$.

Proof: In the previous theorem, set $j=1$ to obtain (i) and $j=n$ to obtain (ii). □

Theorems 1 and 2 have been checked numerically for the polynomials $F_1(x)$ through $F_{13}(x)$. The corollaries have been checked numerically for all values from $n=1$ to $n=20$, as well as for selected values up to $n=1000$. The numerical results show perfect agreement with the results predicted here.

3. THE DYNAMICS OF THE ZEROS

The goal here is to obtain the zeros of $F_n(x)$ as iterates of some function (independent of n) which maps the zeros of $F_{n-1}(x)$ to the zeros of $F_n(x)$. This procedure is complicated by the fact that the number of zeros increases with increasing n, but that will be dealt with below by breaking up the zeros into one parameter families, with n as the parameter. A second parameter, m, will distinguish one family from the next. Although the recursion relations derived below contain no information that is not already implicitly contained in formulas (2) through (5), it provides a different perspective on this information. Also, this recursion relation method can provide an algorithm that may be more efficient than other methods for numerical calculations of zeros for other classes of polynomials when the zeros do not have such simple analytic formulas.

As in earlier sections, rather than dealing directly with the zeros, $\pm ix_j$, we will deal only with their magnitudes, x_j. However, for our purposes here, it is convenient to alter our notation slightly. For a fixed value of n, label the magnitudes of the zeros in decreasing order as follows: $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$. The superscript labels the polynomial of which it is a zero, and the subscript labels the relative size of the zero. Using this ordering, $x_{n/2}^{(n)}$ always vanishes for even n. For a generic zero $x_m^{(n)}$ of $F_n(x)$, we will call m the *row number* of the zero, for reasons that will become apparent later. The idea is to find a function $f_m : \Re \to \Re$, independent of n, such that $f_m(x_m^{(n)}) = x_m^{(n+1)}$. As we will see below, the zeros $x_m^{(n)}$ for all n will then be obtainable by applying the appropriate f_m to the initial value $x=0$, and then iterating a certain number of times. The main result is Theorem 3 below.

Theorem 3: For all $n \geq 2$, the zero in the m^{th} row of $F_{n+1}(x)$ is related to the zero in the m^{th} row of $F_n(x)$ by the following mapping:

$$x_m^{(n+1)} = \frac{2}{\sqrt{1+\alpha_m(x_m^{(n)})}}, \tag{18}$$

where

$$\alpha_m(x) = \tan^2\left\{ m\pi \left[\frac{\tan^{-1}\sqrt{\frac{4-x^2}{x^2}} + 2\pi K_1}{\tan^{-1}\sqrt{\frac{4-x^2}{x^2}} + 2\pi K_1 + m\pi} \right] - 2\pi K_2 \right\}, \tag{19}$$

where K_1 and K_2 are a pair of integer constants.

Proof: Assume for the sake of definiteness that n is even. [If n is odd, the proof proceeds in an identical manner, except that the roles of equations (2) and (3) are reversed.] Referring to equations (2) and (3), the integer k is related to the row number m by $k = n-m$, so that these equations tell us that the zeros of $F_n(x)$ and $F_{n+1}(x)$ are at

$$x_m^{(n)} = 2\sin\left(\frac{n-m}{2n}\right)\pi = 2\cos\left(\frac{m\pi}{2n}\right), \tag{20}$$

$$x_m^{(n+1)} = 2\sin\left[\left(\frac{2n+1-2m}{2n+1}\right)\frac{\pi}{2}\right] = 2\cos\left(\frac{m\pi}{2n+1}\right), \tag{21}$$

where we have used the fact that $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$.

Now, note that

$$\exp\left(\frac{im\pi}{2n}\right) = \cos\frac{m\pi}{2n} + i\sin\frac{m\pi}{2n} = \frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}. \tag{22}$$

Taking the natural logarithm of the last equation gives

$$\frac{im\pi}{2n} = \ln\left[\frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}\right] + 2i\pi K_1, \tag{23}$$

where K_1 is an integer that specifies which branch of the Riemann surface is used to evaluate the logarithm. Now, solve for $2n$:

$$2n = \frac{im\pi}{\ln\left[\frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}\right] + 2\pi i K_1}. \tag{24}$$

Repeating the procedure of the previous paragraph, but this time applying it to $\exp\left(\frac{im\pi}{2n+1}\right)$, we find

$$2n+1 = \frac{im\pi}{\ln\left[\frac{x_m^{(n+1)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n+1)}}{2}\right)^2}\right] + 2\pi i K_2}. \tag{25}$$

where, again, K_2 is an integer constant.

Substituting equation (24) into equation (25) yields

$$\frac{im\pi}{\ln\left[\frac{x_m^{(n+1)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n+1)}}{2}\right)^2}\right] 2\pi i K_2} = \frac{im\pi}{\ln\left[\frac{x_m^{(n)}}{2} + i\sqrt{1-\left(\frac{x_m^{(n)}}{2}\right)^2}\right] 2\pi i K_1} + 1. \tag{26}$$

This result can be simplified. Note that, for any variable y such that $-2 \le y \le 2$, we can define a pair of polar coordinates (r, θ) via

$$\frac{y}{2} + i\sqrt{1-\left(\frac{y}{2}\right)^2} = r\exp i\theta. \tag{27}$$

Clearly,

$$r = 1, \quad \theta = \tan^{-1}\sqrt{\frac{4-y^2}{y^2}}. \tag{28}$$

Taking the natural logarithm of equation (27),

$$\ln\left[\frac{y}{2}+i\sqrt{1-\left(\frac{y}{2}\right)^2}\,\right] = \ln(r\exp i\theta) \\ = \ln r + i\theta \\ = i\tan^{-1}\sqrt{\frac{4-y^2}{y^2}}. \tag{29}$$

Finally, applying equation (29) to both sides of formula (26), and then solving for $x_m^{(n+1)}$ gives the desired result. □

Note that two integer constants, K_1 and K_2, appear in this result. From examining equation (19), it is clear that K_1 is completely arbitrary; changing its value will simply change the argument of the tangent by a multiple of 2π. Because of the periodicity of the tangent, the value of K_1 has no effect on the results and will henceforth be set to zero.

The second constant, K_2, enters into the proof in the same way but, curiously, its value *does* affect the positions of the zeros. Theorem 3 has been checked numerically by using it to predict the first 40 zeros for all cases from $m=1$ to $m=10$. In each case, the theorem gives the correct results, provided that K_2 is set equal to zero. Allowing K_2 to have nonzero values seems to lead to interesting effects; these are currently under investigation. But in the remainder of this paper, we will set $K_2 = 0$ (or, in other words, we will restrict ourselves to the principal branches of all logarithms), since this is the case that gives the correct zeros for the Fibonacci polynomials.

Theorem 3 tells us that families of $x_m^{(n)}$ with fixed m form trajectories of a dynamical system, with n playing the role of a discrete time variable. Points are moved along each trajectory by repeated iteration of the function $f_m(x) = 2/\sqrt{1+\alpha_m(x)}$. This situation is illustrated in Figure 1.

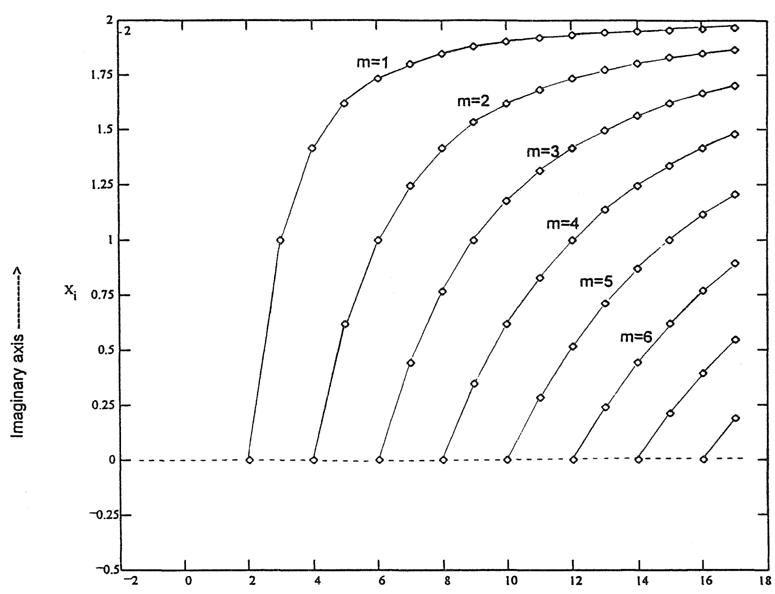

FIGURE 1: The Zeros of the Fibonacci Polynomials
(Only zeros with nonnegative imaginary part are shown)

It can be seen that the integer m labels how many rows the trajectory is from the outside of the diagram. It is also clear that each trajectory begins at a root of the form $x_m^{(n)} = x_m^{(2m)} = 0$. Since each iteration of f_m increases n by one, and since each trajectory starts at an initial value of $n_0 = 2m$, it takes $n-2m$ iterations to reach a fixed final value of n. As a consequence, we have the following corollary.

Corollary 3: The zero of $F_n(x)$ with row number m can be written as

$$x_m^{(n)} = f_m^{(n-2m)}(0), \tag{30}$$

where $f_m^{(j)}$ means the j^{th} iterate of f_m.

Some observations can be made about this result. First, it is clear from the form of $f_m(x)$ that each trajectory approaches an attracting fixed point situated at $x^{(\infty)} = 2$. This implies that as $n \to \infty$, $x_m^{(n)} \to 2$, for all m.

Second, a similar result is easily proved for the zeros of the Lucas polynomials by using the same method. In the Lucas case, we still have $f_m(x) = 2/\sqrt{1+\alpha_m(x)}$, but now the form of α_m changes:

$$\alpha_m(x) = \tan^2\left[(2m-1)\pi \cdot \frac{\tan^{-1}\sqrt{\frac{4-x^2}{x^2}}}{(2m-1)\pi + 2\tan^{-1}\sqrt{\frac{4-x^2}{x^2}}}\right]. \tag{31}$$

Here, we have again set $K_1 = K_2 = 0$. There is one complication arising here that did not occur in the Fibonacci case: iteration of the above function does not simply carry us along the m^{th} row. Instead, the trajectory jumps back and forth between two adjacent rows. More specifically, repeated use of α_m will give us the zeros in the m^{th} row for $L_{2n}(x)$ and those in row $m+1$ for $L_{2n+1}(x)$. This occurs because m enters the expressions for the even and odd zeros in the same manner for the Fibonacci case [compare the numerators of the last expressions in equations (20) and (21)], while in the corresponding expres-sions for the Lucas zeros, it enters through a factor of $(2m+1)$ in one case and $(2m-1)$ in the other. Although the trajectory now alternates rows, we still recover all of the zeros as we run over differing values of m, as has been verified numerically.

Some observations can also be made about the properties of the $\alpha_m(x)$. For the Fibonacci case, define

$$\beta_m^{(n)} = \sqrt{\frac{4-(x_m^{(n)})^2}{(x_m^{(n)})^2}} \quad \text{and} \quad \alpha_m^{(n)} = \alpha_m(x_m^{(n)}) = \tan^2\left\{\frac{\tan^{-1}\beta_m^{(n)}}{\tan^{-1}\beta_m^{(n)} + \pi} \cdot m\pi\right\}.$$

Then we have the following propositions.

Proposition 1: For all m and n,

$$\beta_m^{(n)} = \sqrt{\alpha_m^{(n-1)}}. \tag{32}$$

Proof: We know that

$$\beta_m^{(n)} = \sqrt{\frac{4-(x_m^{(n)})^2}{(x_m^{(n)})^2}} = \sqrt{\frac{4-f_m^2(x_m^{(n-1)})}{f_m^2(x_m^{(n-1)})}}. \tag{33}$$

Substituting $f_m(x) = 2 / \sqrt{a + \alpha_m(x)}$ into this expression and simplifying the fraction quickly leads to equation (32). □

Proposition 2: The argument of the tangent in $\alpha_m^{(n)}$ is always a rational multiple of π. In other words, the quantity

$$\frac{\tan^{-1}\beta_m^{(n)}}{\tan^{-1}\beta_m^{(n)} + \pi} \tag{34}$$

is rational for all n and m.

Proof: We know [by equations (20) and (21) or, alternately, by equations (2) and (3)] that all of the zeros can be written in the form $x_m^{(n)} = 2\cos\left(\frac{p}{q}\pi\right)$ for some pair of integers p and q (depending on m and n). Substituting this expression into the definition of $\beta_m^{(n)}$, we find that $B_m^{(n)} = \tan\frac{p}{q}\pi$, or $\tan^{-1}\beta_m^{(n)} = \frac{p}{q}\pi$. Substituting this into the quantity in formula (34), we find that it equals $\frac{p}{p+1}$, which is clearly rational. □

Note that $\beta_m^{(n)}$ describes the tangent of an angle inscribed in a right triangle of hypotenuse equal to 2, and adjacent side of length $x_m^{(n)}$. The hypotenuse remains constant, while the adjacent side increases in length with increasing n or decreasing m. A deeper understanding of the geometric meanings of $\alpha_m^{(n)}$ and $\beta_m^{(n)}$ may help provide some insight into the properties of the zeros of the Tribonacci polynomials and other generalizations of the $F_n(x)$.

4. TRIBONACCI POLYNOMIALS

The Fibonacci and Lucas polynomials have been generalized in various ways. The simplest generalization is that of the Tribonacci polynomials, $T_n(x)$ (see [3]), which obey the relation

$$T_{n+3}(x) = x^2 T_{n+2}(x) + x T_{n+1}(x) + T_n(x), \tag{35}$$

with $T_0(x) = 0$, $T_1(x) = 1$, $T_2(x) = x^2$. The $T_n(x)$ are often written in terms of the trinomial coefficients $\binom{m}{j}_3$, which are defined implicitly by the following equation [3]:

$$T_n(x) = \sum_{j=0}^{\left[\frac{2}{3}(n-1)\right]} \binom{n-j-1}{j}_3 x^{2n-3j-2}. \tag{36}$$

While numerical work has been done concerning the zeros of the Tribonacci polynomials, explicit expressions for them are not known, so deriving formulas of the sort presented in §2 of this paper would be of interest, as they could provide valuable clues to the possible forms the zeros could have. Below is a theorem giving expressions for the symmetric polynomials of the Tribonacci zeros. Again, these results are easily verified numerically. The proofs are omitted, as they are identical to those of §2, except that equation (36) replaces equation (8).

The zeros of the Tribonacci polynomials form a set that is invariant under rotations in the complex plane by multiples of $2\pi/3$, so the zeros can be divided into three subsets: $\{x_i\}$, $\{x_i e^{2\pi/3}\}$, and $\{x_i e^{-2\pi/3}\}$, for an appropriate set of x_i.

Theorem 4:

(i) The zeros of $T_{3n+1}(x)$ have elementary symmetric polynomials of the form

$$\sigma_j(x_1^3, \ldots, x_{2n}^3) = (-1)^j \binom{3n-j}{j}_3. \tag{37}$$

(ii) The zeros of $T_{3n+2}(x)$ satisfy the following relation:

$$\sigma_j(x_1^3, \ldots, x_{2n}^3) = (-1)^j \binom{3n-j+1}{j}_3. \tag{38}$$

(iii) The zeros of $T_{3n}(x)$ satisfy the following relation:

$$\sigma_j(x_1^3, \ldots, x_{2n-1}^3) = (-1)^j \binom{3n-j-1}{j}_3. \tag{39}$$

By setting $j = 1$ in the above theorem, we have the following corollary.

Corollary 4:

(i) The zeros of $T_{3n+1}(x)$ satisfy $\sum_{k=1}^{2n} x_k^3 = -(3n-1)$.

(ii) The zeros of $T_{3n+2}(x)$ satisfy $\sum_{k=1}^{2n} x_k^3 = -3n$.

(iii) The zeros of $T_{3n}(x)$ satisfy $\sum_{k=1}^{2n-1} x_k^3 = -(3n-2)$.

As for the results presented in §3 of this paper, their derivation depended on prior knowledge of the explicit formulas for the zeros of the $F_n(x)$. However, the logic could be reversed: *if* formulas analogous to the f_m could be found for the $T_n(x)$ by fitting functions to a few of the numerically known zeros, then explicit formulas for the positions of all the zeros could immediately be generated. Finding the f_m functions and finding the zeros are thus equivalent problems, but it could turn out that one form of the problem is easier than the other. Finding the f_m functions could be aided by further analysis of the geometrical content of the results of §3. and of how the geometry changes in the Tribonacci case.

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❖❖❖

SOME PROPERTIES OF THE GENERALIZED FIBONACCI SEQUENCES $C_n = C_{n-1} + C_{n-2} + r$

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The generalized Fibonacci sequences $\{C_n(a,b,r)\}$ defined by $C_n(a,b,r) = C_{n-1}(a,b,r) + C_{n-2}(a,b,r) + r$ with $C_1(a,b,r) = a$, $C_2(a,b,r) = b$, where r is a constant, have been studied in [2] and [3]. Again we take the initial value $C_0(a,b,r) = b-a-r$. The Fibonacci sequence arises as a special case, $F_n = C_n(1,1,0)$, while the Lucas sequence is $L_n = C_n(1,3,0)$.

The purpose of this note is to establish some properties of $C_n(a,b,r)$ by using the method of L. C. Hsu [1].

For the convenience of the reader, we introduce the following symbols:

I will be the identity operator;

E represents the shift operator;

E_i is the "i^{th} coordinate" shift operator ($i = 1, 2$);

$\nabla = I + E_2 - E_1$.

We also let $\binom{n}{i,j} = \dfrac{n!}{i!\,j!(n-i-j)!}$.

In [1], Hsu and Maosen gave the following proposition.

Proposition 1: Let $f(n,k)$ and $g(n,k)$ be any two sequences. Then the following reciprocal formulas hold:

$$g(n,k) = \nabla^n f(0,k) = \sum_{i+j+s=n} \binom{n}{i,j}(-1)^i f(i, k+j), \tag{1}$$

$$f(n,k) = \nabla^n g(0,k) = \sum_{i+j+s=n} \binom{n}{i,j}(-1)^i g(i, k+j). \tag{2}$$

From this point on, we briefly write C_n for $C_n(a,b,r)$.

Lemma 1: $C_k + C_{k+1} + C_{k+6} = 3C_{k+4}$. (3)

Proof:
$$\begin{aligned} C_k + C_{k+1} + C_{k+6} &= C_k + C_{k+1} + C_{k+5} + C_{k+4} + r \\ &= C_{k+2} - r + C_{k+4} + C_{k+3} + r + C_{k+4} + r \\ &= C_{k+4} - r + C_{k+4} + r + C_{k+4} = 3C_{k+4}. \end{aligned}$$

Theorem 1:
$$\left.\begin{aligned} C_{4n+6k} &= \sum_{i+j+s=n} \binom{n}{i,j} 3^{-n} C_{i+6(j+k)}, &(4)\\ C_{n+6k} &= \sum_{i+j+s=n} \binom{n}{i,j} (-1)^{n+i} 3^i C_{4i+6(j+k)}. &(5) \end{aligned}\right\}$$

Proof: We take $f(i,j) = (-1)^i C_{i+6j}$. Using Lemma 1,

$$\begin{aligned}\nabla f(i, j) &= (I + E_2 - E_1) f(i, j) = f(i, j) + f(i, j+1) - f(i+1, j)\\ &= (-1)^i C_{i+6j} + (-1)^i C_{i+6(j+1)} - (-1)^{i+1} C_{i+1+6j}\\ &= (-1)^i (C_{i+6j} + C_{i+6j+1} + C_{i+6j+6}) = (-1)^i 3C_{i+6j+4}.\end{aligned}$$

Hence, $\nabla \equiv 3E_1^4$. Thus, we obtain $g(n, k) = \nabla^n f(0, k) = 3^n E_i^{4n} f(0, k) = 3^n C_{4n+6k}$. By (1), we have

$$3^n C_{4n+6k} = \sum_{i+j+s=n} \binom{n}{i, j} C_{i+6(j+k)},$$

and, by (2), we get

$$(-1)^n C_{n+6k} = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^i 3^i C_{4i+6(j+k)},$$

completing the proof of Theorem 1.

We take $k = 0$ in Theorem 1 to Write Corollary 1.1, and $i = 0$ in Corollary 1.1 to derive Corollary 1.2.

Corollary 1.1: $$C_{4n} = \sum_{i+j+s=n} \binom{n}{i, j} 3^{-n} C_{i+6j}, \tag{6}$$

$$C_n = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^{n+i} 3^i C_{4i+6j}. \tag{7}$$

Corollary 1.2: $$C_n - (-1)^n \sum_{j=0}^{n} \binom{n}{j} C_{6j} \equiv 0 \pmod 3. \tag{8}$$

We can obtain Theorem 2, in a manner similar to that used to prove Theorem 1, by taking $f(i, j) = (-1)^i C_{6i+j}$ and expanding $\nabla f(i, j)$. Again, set $k = 0$ in Theorem 2 to write Corollary 2.1, and let $i = 0$ in (12) below to obtain Corollary 2.2.

Theorem 2: $$C_{4n+k} = \sum_{i+j+s=n} \binom{n}{i, j} 3^{-n} C_{6i+j+k}, \tag{9}$$

$$C_{6n+k} = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^{n+i} 3^i C_{4i+j+k}. \tag{10}$$

Corollary 2.1: $$C_{4n} = \sum_{i+j+s=n} \binom{n}{i, j} 3^{-n} C_{6i+j} \tag{11}$$

$$C_{6n} = \sum_{i+j+s=n} \binom{n}{i, j} (-1)^{n+i} 3^i C_{4i+j}. \tag{12}$$

Corollary 2.2: $$C_{6n} - (-1)^n \sum_{j=0}^{n} \binom{n}{j} C_j \equiv 0 \pmod 3. \tag{13}$$

Proposition 2: If a sequence $\{X_n\}$ satisfies

$$I = 2E^{-1} - E^{-3} \tag{14}$$

then

$$I = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i E^{-3n+2i}; \tag{15}$$

hence,

$$X_{3n} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i X_{2i}, \tag{16}$$

and

$$X_{3n+k} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i X_{2i+k}. \tag{17}$$

Proof: Use binomial expansions.

Lemma 2: $C_n = 2C_{n-1} - C_{n-3}$. (18)

Proof:

$$\begin{aligned} C_n &= C_{n-1} + C_{n-2} + r \\ &= C_{n-1} + C_{n-1} - C_{n-3} - r + r \\ &= 2C_{n-1} - C_{n-3}. \end{aligned}$$

Theorem 3:

$$\left.\begin{aligned} C_{3n} &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i C_{2i}, && (19) \\ C_{3n+k} &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} 2^i C_{2i+k}. && (20) \end{aligned}\right\}$$

Proof: Since C_n satisfies (14), Theorem 3 is proved by Proposition 2.

Our final corollary follows by setting $i = 0$ in (20).

Corollary 3.1: $C_{3n+k} - (-1)^n C_k \equiv 0 \pmod 2$. (21)

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ON THE PROPORTION OF DIGITS IN REDUNDANT NUMERATION SYSTEMS*

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1. INTRODUCTION

In the standard binary numeration system, an n-bit integer N is uniquely represented as the sum of powers of 2. Specifically,

$$N = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \cdots + a_2 2^2 + a_1 2^1 + a_0 2^0,$$

where a_i is either 0 or 1. As is common, N can be represented as an n-tuple of 0's and 1's, where the position of the bit determines the power of 2 involved. For example, in a 4-bit standard binary numeration system, $N = 0101 = 5$, since 5 is equivalent to $2^2 + 2^0$. Newman ([13], p. 2422) suggests that the Chinese used the binary numeration system around 3000 B.C.

Instead of powers of 2's, if Fibonacci numbers are used, then an alternate numeration system (viz. Zeckendorf [14]) occurs in which an integer N may have more than one representative. That is, let

$$N = a_{n-1}F_{n+1} + a_{n-2}F_n + \cdots + a_2F_4 + a_1F_3 + a_0F_2, \tag{1}$$

where F_i is the i^{th} Fibonacci number. For example, $1000 = 0110 = 5$ in the Fibonacci numeration system, where 5 is equivalent to both F_5 and $F_4 + F_3$. It is known (e.g., Brown [1]) that an n-tuple of 0's and 1's is a unique representative of N if every pair of 1's is separated by at least one 0. Under this restriction, we view 1000 as the representative of 5 and 0110 as the redundant representative. Brown [2] showed that, if one represents an integer by the n-tuple with the *most* 1's, then this representative is unique. In this case, we view 0110 as the representative of 5 and 1000 as the redundant representative.

Representations of this type have important advantages. For example, in a CD-ROM, three or more consecutive 1's cannot be read reliably (Davies [4]). Motivated by this, Klein [11] investigated Fibonacci-like representations of the form (1), where $F_i = F_{i-1} + F_{i-m}$ for $i > m+1$, and $F_i = i-1$ for $1 < i \leq m+1$. The case $m = 2$ corresponds to the Zeckendorf representation using Fibonacci numbers.

Kautz [9] uses such representations in a data transmission system where the receiver clock is synchronized to the transmitter clock using only the data. Toward this end, he uses code words in which there are neither strings of 1's of length greater than m nor strings of 0's of length greater than m.

Dimitrov and Donevsky [5] show that the number of steps required to multiply two n-bit numbers represented in the Zeckendorf numeration system using Quadranacci numbers is less than

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that required by two numbers represented as standard binary numbers. That is, even though the Zeckendorf representation requires more bits, its efficiency in the multiplication process more than compensates for extra operations because of larger word size. Indeed, the Zeckendorf representation outperforms both standard binary multiplication and multiplication using the more efficient multiplication algorithm for $n \to \infty$ when the number of bits in the standard binary representation is 131 through 1200.

The question posed and answered in this paper is: To what extent does redundancy occur in certain redundant numeration systems? The question has important consequences for both the efficiency of number representations and the transmission of data. We analyze redundancy in two ways: 1) the number of distinct representative n-tuples for some given n and 2) the proportion of digits used in nonredundant representatives. Table 1 shows the numeration systems considered in this paper and the corresponding recurrences, basis elements, and references.

TABLE 1. Selected Numeration Systems, Recurrences, and Basis Elements

Name	Recurrence	Basis elements	Reference
Standard binary	$F_i = 2F_{i-1}$	... $2^6\ 2^5\ 2^4\ 2^3\ 2^2\ 2^1\ 2^0$	[7,8,12,13]
Zeckendorf - - Fibonacci	$F_i = F_{i-1} + F_{i-2}$	... 21 13 8 5 3 2 1	[1,2,3,14]
Gen. Fibonacci - - Tribonacci - - Quadranacci	$F_i = F_{i-1} + F_{i-2} + \ldots + F_{i-m}$ $F_i = F_{i-1} + F_{i-2} + F_{i-3}$ $F_i = F_{i-1} + F_{i-2} + F_{i-3} + F_{i-4}$	 ... 44 24 13 7 4 2 1 ... 56 29 15 8 4 2 1	[3,9]
Generalization of Fibonacci Numbers	$F_i = F_{i-1} + F_{i-m}$ $F_i = F_{i-1} + F_{i-3}$ $F_i = F_{i-1} + F_{i-4}$	 ... 13 9 6 4 3 2 1 ... 10 7 5 4 3 2 1	[11]
m - ary Numbers	$F_i = mF_{i-1} - F_{i-2}$ $F_i = 3F_{i-1} - F_{i-2}$ $F_i = 4F_{i-1} - F_{i-2}$	 ... 144 55 21 8 3 1 ... 780 209 56 15 4 1	[11]

2. BINARY NUMERATION SYSTEMS

Consider a numeration system in which the basis elements are $(\ldots, F_4, F_3, F_2)$, where $F_i = F_{i-1} + F_{i-2} + \cdots + F_{i-m}$ for $i > m+1$, and $F_i = 2^{i-2}$ for $1 < i \le m+1$, where $m \ge 2$. Consider a representative n-tuple $T = (a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, where $a_i \in \{0, 1\}$. From [3] and [6], if no more than $m-1$ consecutive a_i's are 1, then T is a unique representative of $N = \sum_{i=0}^{n-1} a_i F_{i+2}$. We can write the regular expression (see [10], pp. 617-23) for the allowed representatives as

$$\mathbf{R} = (\lambda + \mathbf{1} + \mathbf{1}^2 + \mathbf{1}^3 + \cdots + \mathbf{1}^{m-1})(\mathbf{0}(\lambda + \mathbf{1} + \mathbf{1}^2 + \mathbf{1}^3 + \cdots + \mathbf{1}^{m-1}))^*. \tag{2}$$

Here, $\mathbf{a}^* = \{\lambda, a, aa, aaa, \ldots\}$, where λ is the *empty* string, and $\mathbf{1}^i$ denotes i consecutive 1's. Thus, this expression represents the set of strings consisting of substrings beginning with i 1's, for $0 \le i \le m-1$, followed by a sequence of substrings each of the form $0, 01, 011, \ldots,$ or 01^{m-1}. From (2), we can derive a generating function $N(x, y, z)$ for the number of representatives and the number of 0's and 1's in these representatives. Let x track the number of bits, y track the number of 0's, and z track the number of 1's. Then, a typical term in the power series expansion of $N(x, y, z)$ is $\xi_{nij}\, x^n y^i z^j$ for $n = i + j$, where ξ_{nij} is the number of representative n-tuples with i 0's and j 1's. We can write

$$N(x,y,z) = (1+xz+x^2z^2+\cdots+x^{m-1}z^{m-1})\left(\frac{1}{1-xy(1+xz+x^2z^2+\cdots+x^{m-1}z^{m-1}}\right), \tag{3}$$

where the first term represents the leftmost substring, which can be nothing, 1, 1^2, ..., or 1^{m-1}, while the second term represents the ways to choose 0, 01, 01^2, ..., and 01^{m-1}. We can rewrite (3) as follows:

$$N(x,y,z) = \left(\frac{1-x^mz^m}{1-xz}\right)\left(\frac{1}{1-xy\left(\frac{1-x^mz^m}{1-xz}\right)}\right).$$

From this we can generate, for example, the distribution of 16-tuples with i 1's for $0 \le i \le 15$, as shown in Figure 1. It is interesting that the number of representative n-tuples increases markedly from $m=2$ to $m=3$; for $m=7$, the distribution is almost binomial. The fact that it is not exactly binomial can be seen by its asymmetry. Capocelli, Cerbone, Cull, and Hollaway [3] derive an expression for the average proportion, $P_{1's}$, of bits that are 1, when the number n of bits is large. Table 2 shows this. In the Zeckendorf numeration system using Fibonacci numbers ($m=2$), the average proportion of 1's is near 25%. However, as m increases from 2, this value approaches 50%.

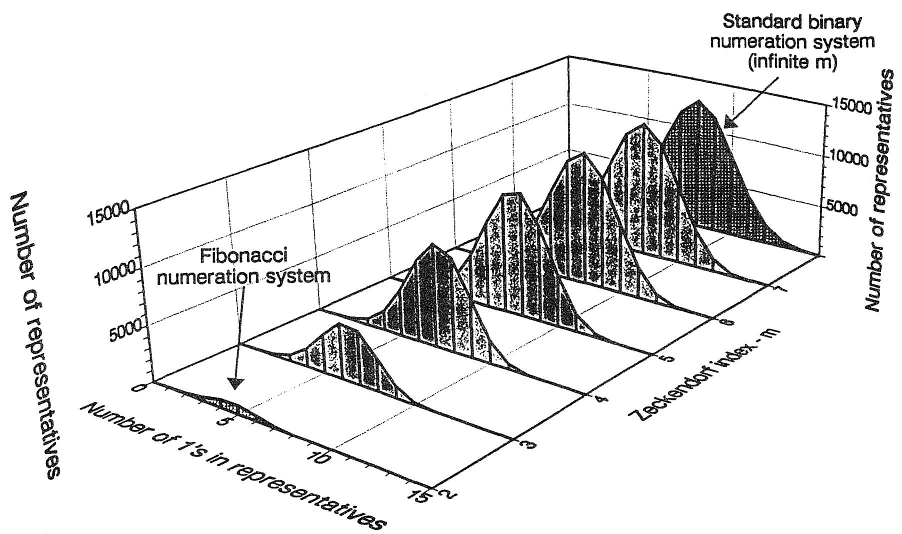

FIGURE 1. Distributions of 1's in 16-Tuple Zeckendorf Numeration Systems

TABLE 2 ([3], [11]). Average Proportion of 1's in Numeration Systems with Basis Elements $F_i = F_{i-1} + F_{i-2} + \cdots + F_{i-m}$ When the Number of Bits Is Large

m	2	3	4	5	6	7	8	∞
$P_{1's}$	**0.2764**	**0.3816**	**0.4337**	**0.4621**	**0.4782**	**0.4875**	**0.4929**	**0.5000**

Klein [11] considers numeration systems based on the recurrence $F_i = F_{i-1} + F_{i-m}$ for $i > m+1$, and $F_i = i-1$ for $1 < i \le m+1$, where $m \ge 2$. Consider a representative n-tuple $\boldsymbol{T}$ = $(a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, where $a_i \in \{0, 1\}$. From Theorem 1 in [6] it follows that, if every pair of 1's is separated by at least $m-1$ 0's, then T is a unique representation of $N=\sum_{i=0}^{n-1}a_iF_{i+2}$. For $m=2$, this is the Fibonacci numeration system in which no two 1's are adjacent. A regular expression for the allowed representatives is

$$\mathbf{R}=\mathbf{0}^*+(\mathbf{0}+\mathbf{10}^{m-1})^*\,\mathbf{10}^*. \tag{4}$$

The $\mathbf{1}$ in $\mathbf{10}^*$ represents the rightmost 1 in a string containing at least one 1. In this case, any number of 0's, as described by $\mathbf{0}^*$ occurs to its right. $(\mathbf{0}+\mathbf{10}^{m-1})^*$ represents a string consisting of a sequence of substrings of the form 0 and 10^{m-1}. It follows from this construction that each pair of 1's is separated by at least $m-1$ 0's.

Consider a generating function $N(x,y,z)$ to count the representative n-tuples and the 0's and 1's in these representatives. From (4), we can write

$$\begin{aligned}N(x,y,z)&=(1+xy+x^2y^2+\cdots)\\&\quad+[(1+(xy+x^my^{m-1}z)+(xy+x^my^{m-1}z)^2+\cdots)\,xz\,(1+xy+x^2y^2+\cdots)].\end{aligned} \tag{5}$$

Here, $(1+xy+x^2y^2+\cdots)$ counts the ways to choose no 0's, one 0, two 0's, and so forth, while $(xy+x^my^{m-1}z)$ counts the ways to choose either a single 0 or 10^{m-1}, and xz counts the choice of a single 1. Equivalent to (5) is the following:

$$N(x,y,z)=\frac{1}{1-xy}\left[1+\frac{xz}{1-xy-x^my^{m-1}z}\right]. \tag{6}$$

From this, we obtain the distribution of 16-tuples according to the number of 1's, as shown in Figure 2. It is interesting that, even for small m, the number of representative n-tuples is small compared to the standard binary numeration system, shown here truncated to 1000 in order to display detail.

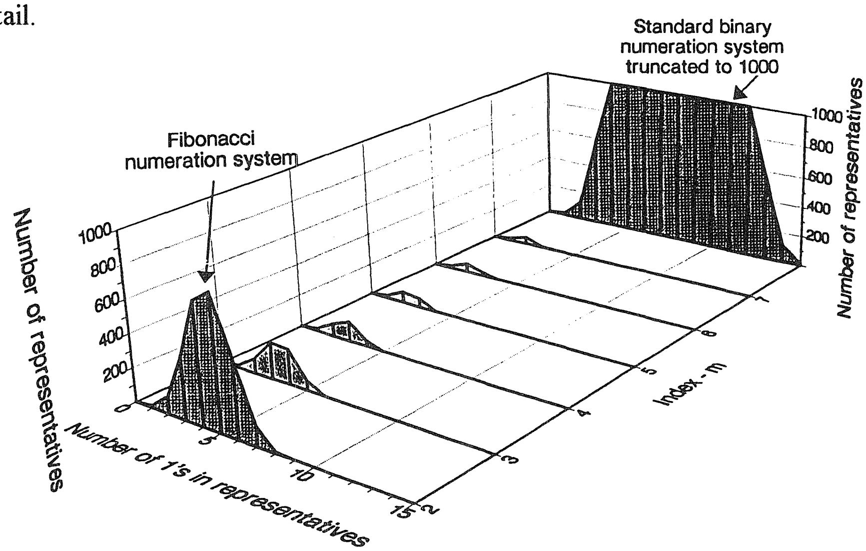

FIGURE 2. Distributions of 1's in 16-Tuple Numeration Systems Whose Basis Elements Are Generated by the Recurrence $F_i=F_{i-1}+F_{i-m}$

If we substitute 1 for y and z in (6), we achieve a generating function for the number of representative n-tuples, as follows:

$$N(x,1,1)=\frac{1}{1-x}\left[1+\frac{x}{1-x-x^m}\right]=\frac{1-x^m}{(1-x)(1-x-x^m)}. \tag{7}$$

Specifically, $\xi_{n\square\square}$, the coefficient of x^n in the power series representation of (7), is the number of representative n-tuples. We can write (7) as

$$N(x,1,1)=\frac{g_m}{1-\frac{x}{\alpha_m}}+\cdots, \tag{8}$$

where ... represents terms whose contribution to $\xi_{n\square\square}$ is negligible, for large n, compared to the term shown, and

$$g_m=\frac{1}{(1-\alpha_m)(a+m\alpha_m^{m-1})}.$$

Here, α_m is the dominant root, i.e., the singularity on the circle of convergence of $N(x,1,1)$. We are interested in the value of $\xi_{n\square\square}$ when n is large; thus, we write

$$\xi_{n\square\square}\sim\frac{1}{(1-\alpha_m)(a+m\alpha_m^{m-1})}\left(\frac{1}{\alpha_m}\right)^n, \tag{9}$$

where $f_n\sim g_n$ means $\lim_{n\to\infty}\frac{f_n}{g_n}=1$.

Consider now the proportion of bits that are 0 and 1 in the representatives counted by $\xi_{n\square\square}$. Substituting 1 for z in (6) yields $N(x,y,1)$, a generating function in which a typical term is $(\xi_{n0\square}+\xi_{n1\square}y^1+\xi_{n2\square}y^2+\cdots+\xi_{nn\square}y^n)x^n$, where $\xi_{ni\square}$ is the number of representative n-tuples with i 0's. Differentiating $N(x,y,1)$ with respect to y and setting $y=1$ yields a generating function in x in which a typical term is $(\xi_{n1\square}+2\xi_{n2\square}+n\xi_{nn\square})x^n=\Xi_n x^n$. Dividing Ξ_n by $\xi_{n\square\square}$ yields the average number of 0's in representative n-tuples. Dividing this by n gives the average proportion $P_{0\text{'s}}$ of bits that are 0. That is,

$$N_{0\text{'s}}(x)=\sum_{n\geq 0}\Xi_n x^n=\frac{d}{dy}N(x,y,1)\bigg|_{y=1}=\frac{(1-x^m)(x+(m-1)x^m)}{(1-x)(1-x-x^m)^2}+\cdots, \tag{10}$$

where $\cdots$ represents terms whose contribution to Ξ_n is negligible for large n, compared to the term shown. But $N_{0\text{'s}}(x)$ can be expressed as

$$N_{0\text{'s}}(x)=\frac{g_m^2(1-\alpha_m)(1+(m-1)\alpha_m^{m-1})}{\left(1-\frac{x}{\alpha_m}\right)^2}+\cdots, \tag{11}$$

where ... represents negligible terms. Therefore, from (11), we have

$$\Xi_n\sim g_m^2(1-\alpha_m(1+(m-1)\alpha_m^{m-1})\left(\frac{1}{\alpha_m}\right)^n n.$$

Thus, the proportion of digits that are 0 when the number n of digits is large is

$$P_{0\text{'s}}=\frac{1+(m-1)\alpha_m^{m-1}}{1+m\alpha_m^{m-1}}.$$

Table 3 summarizes these results. It includes $P_{1\text{'s}}$, the proportion of bits that are 1, which can be obtained from $P_{1\text{'s}} = 1 - P_{0\text{'s}}$. It also shows values for the proportion of 0's and 1's for large n. As m grows, the proportion of bits that are 1 approaches 0, as shown in the last row of the table. All entries in this row are approximations that apply when m is large. For example, the dominant root α_m of $1-x-x^m$ can be calculated as follows. Let $\alpha_m = e^{-\Delta}$. For small Δ, this can be approximated be the truncated series $1-\Delta$. Thus, $1-\alpha_m-\alpha_m^m = 0 \approx \Delta - e^{-m\Delta}$. For $t = m\Delta$, we find that $t \cong me^{-t}$, from which we obtain $\ln m \cong \ln t + t \cong t$. Thus, $1-\alpha_m-\alpha_m^m \approx 1-\alpha_m-\Delta \approx 1-\alpha_m - t/m$ $\approx 1-\alpha_m - \ln m/m$ or $\alpha_m \cong 1-(\ln m)/m$. By a similar calculation, the approximation for the number of representative n-tuples shown in the last row, second column of Table 3 can be derived.

TABLE 3. Asymptotic Approximations to the Number of Representative n-Tuples and the Proportion of 0's and 1's in Numeration Systems with Basis Elements $F_i = F_{i-1} + F_{i-m}$

m	Number of representative n-tuples	Proportion of bits that are 0	Proportion of bits that are 1	α_m
General m	$\frac{1}{(1-\alpha_m)(1+m\alpha_m^{m-1})}\left(\frac{1}{\alpha_m}\right)^n$	$\frac{(1+(m-1)\alpha_m^{m-1})}{(1+m\alpha_m^{m-1})}$	$\frac{(1-\alpha_m)}{\alpha_m(1+m\alpha_m^{m-1})}$	Dominant root of $1-x-x^m$
2	1.1708×1.6180^n	0.7236	0.2764	0.6180
3	1.3134×1.4656^n	0.8057	0.1943	0.6823
4	1.4397×1.3803^n	0.8492	0.1508	0.7245
5	1.5550×1.3247^n	0.8762	0.1238	0.7549
6	1.6621×1.2852^n	0.8948	0.1052	0.7781
7	1.7630×1.2554^n	0.9084	0.0916	0.7965
8	1.8587×1.2320^n	0.9188	0.0812	0.8117
$\to\infty$	$m/\ln^2 m$	$1 - 1/m$	$1/m$	$1-(\ln m)/m$

3. MULTIPLE-VALUED NUMERATION SYSTEMS

There has been less work on numeration systems with nonbinary digits. Klein [11] considers numeration systems based on the recurrence $F_i = mF_{i-1} - F_{i-2}$ for $i > 3$, $F_3 = m$, and $F_2 = 1$, where $m \geq 3$. Consider a representative n-tuple $T = (a_{n-1}, a_{n-2}, \ldots, a_1, a_0)$, where $a_i \in \{0, 1, \ldots, m-1\}$. From [11], if every pair of $m-1$'s is separated by at least one i, such that $i \in \{0, 1, \ldots, m-3\}$, then T is a unique representative of $N = \sum_{i=0}^{n-1} a_i F_{i+2}$. For this numeration system, we seek the proportion of digits that are $0, 1, \ldots, m-2$ and $m-1$. We use a generating function $N(x, y, z, w)$ in which x tracks the number of digits, y tracks the number of $m-1$'s, z tracks the number of $m-2$'s, and w tracks the number of 0's. By symmetry, the proportion of digits that are i, where i is restricted by $1 \leq i \leq m-3$, is the same as the proportion of 0's. Indeed, w can be viewed as tracking any i in the range $0 \leq i \leq m-3$.

We enumerate a representative according to whether it has 1) no $m-1$'s or 2) at least one $m-1$. For 1), there is no restriction on the digits, and the representatives are described by the regular expression, $\mathbf{P} = (\mathbf{0}+\mathbf{1}+\mathbf{2}+\cdots+\boldsymbol{m}-\mathbf{2})^*$. The power series expression for the number of representatives, in this case, is

$$1+(wx+(m-3)x+zx)+(wx+(m-3)x+zx)^2+(wx+(m-3)x+zx)^3+\cdots. \tag{12}$$

That is, the term wx represents a choice of a 0 that contributes 1 to the count of 0's, as tracked by w, and 1 to the count of digits, as tracked by x. Similarly, the term zx tracks the number of

$m-2$'s. The term $(m-3)x$ tracks the number of digits in $\{1, 2, \ldots, m-3\}$. Expression (12) can be written as

$$\frac{1}{1-wx-(m-3)x-zx}. \tag{13}$$

For 2), the regular expression that describes the allowed representatives is

$$[\mathbf{P}+(\boldsymbol{m}-\mathbf{1})(\boldsymbol{m}-\mathbf{2})^*(\mathbf{0}+\mathbf{1}+\mathbf{2}+\cdots+(\boldsymbol{m}-\mathbf{3}))]^*(\boldsymbol{m}-\mathbf{1})\mathbf{P}.$$

Here, the rightmost $\boldsymbol{m}-\mathbf{1}$ is the rightmost $m-1$ in the string. To its right is any substring consisting of the digits 0, 1, ..., and $m-2$, as described by $\mathbf{P}$ and enumerated by (13). The digits to the left of the rightmost $m-1$ can be chosen from 0, 1, 2, ..., $m-2$ (i.e., from $\mathbf{P}$) and from strings beginning in $m-1$, ending in a digit whose value is $m-3$ or less with no, one, two, etc. $m-2$'s in between. The choices for the digits to the left of the rightmost $m-1$ are enumerated by

$$1+\left[wx+(m-3)x+zx+\frac{yx[wx+(m-3)x]}{1-zx}\right]+\left[wx+(m-3)x+zx+\frac{yx[wx+(m-3)x]}{1-zx}\right]^2+\cdots.$$

Here, the choices of a substring beginning in $m-1$ are enumerated by $yx[wz+(m-3)x]/(1-zx)$, where yz represents the choice of the first digit $m-1$, $[wz+(m-3)x]$ represents choice of the last digit, 0, 1, ..., $m-2$, and $1/(1-zx)$ represents the choice of the $m-2$'s in between. Thus, the generating function for the choices of representatives is

$$N(x,y,z,w)=\frac{1}{1-wx-(m-3)x-zx}\left[1+\frac{yx}{1-wx-(m-3)x-zx-\dfrac{yx[wx+(m-3)x]}{1-zx}}\right]. \tag{14}$$

Substituting 1 for y, z, and w into (14) yields $N(x,1,1,1)$, where

$$N(x,1,1,1)=\frac{1}{1-mx+x^2} \tag{15}$$

is the generating function for the number of representative n-tuples in this numeration system. Specifically, $\xi_{n\square\square\square}$, the coefficient of x_n in the power series representation of (15), is the number of representative n-tuples. We prefer to write (15) as

$$N(x)=\frac{g_m}{1-\frac{x}{\alpha_m}}+\frac{h_m}{1-\frac{x}{\beta_m}}, \tag{16}$$

where

$$\alpha_m=\frac{m-\sqrt{m^2-4}}{2},\ \beta_m=\frac{m+\sqrt{m^2-4}}{2}\left(=\frac{1}{\alpha_m}\right),\ g_m=\frac{1}{1-\alpha_m^2},\text{ and } h_m=\frac{1}{1-\beta_m^2}.$$

That is, from (16), we can write $\xi_{n\square\square\square}=g_m(1/\alpha_m)^n+h_m(1/\beta_m)^n$. We are interested in the value of $\xi_{n\square\square\square}$ when n is large, so we use only the left term of the right side of (16). Thus,

$$\xi_{n\square\square\square}\sim\frac{1}{1-\alpha_m^2}\left(\frac{1}{\alpha_m}\right)^n. \tag{17}$$

Table 4 shows the values of g_m and $1/\alpha_m$ for various m.

TABLE 4. Asymptotic Approximations to the Number of Representative n-Tuples and Proportion of Digits in Numeration Systems with Basis Elements $F_i = mF_{i-1} - F_{i-2}$

m	Number of representative n-tuples	Proportion of digits that are i for $0 \le i \le m-3$	Proportion of digits that are $m-2$	Proportion of digits that are $m-1$	α_m
General m	$\frac{1}{1-\alpha_m^2}\left(\frac{1}{\alpha_m}\right)^n$	$\frac{\alpha_m}{1-\alpha_m^2}$	α_m	$\frac{\alpha_m(1-\alpha_m)}{(1+\alpha_m)}$	$\frac{m-\sqrt{m^2-4}}{2}$
3	1.1708×2.6180^n	0.4472	0.3820	0.1708	0.3820
4	1.0774×3.7321^n	0.2887	0.2679	0.1547	0.2679
5	1.0455×4.7913^n	0.2182	0.2087	0.1366	0.2087
6	1.0303×5.8284^n	0.1768	0.1716	0.1213	0.1716
7	1.0217×6.8541^n	0.1491	0.1459	0.1087	0.1459
8	1.0164×7.8730^n	0.1291	0.1270	0.0984	0.1270
$\to \infty$	$1.0000 \times m^n$	$1/m$	$1/m$	$1/m$	$1/m$

Substituting 1 for y and z in (14) yields $N(x,1,1,w)$. A typical term in the power series representation of this generating function is $(\xi_{n\square\square 0} + \xi_{n\square\square 1}w^1 + \xi_{n\square\square 2}w^2 + \cdots + \xi_{n\square\square n}w^n)x^n$, where $\xi_{n\square\square k}$ is the number of representative n-tuples with k 0's. Differentiating this with respect to w and setting $w = 1$ yields a generating function in x in which a typical term is $(\xi_{n\square\square 1} + 2\xi_{n\square\square 2} + \cdots + n\xi_{n\square\square n})x^n = \Xi_n x^n$. Dividing Ξ_n by $\xi_{n\square\square\square}$ yields the average number of 0's in representative n-tuples. Dividing this by n gives the average proportion of digits that are 0.

$$N_{0's}(x) = \sum_{n\ge 0} \Xi_n x^n = \frac{d}{dw} N(x,1,1,w)\bigg|_{w=1} = \frac{x}{(1-mx+x^2)^2}. \tag{18}$$

But $N_{0's}(x)$ can be expressed as

$$N_{0's}(x) = \frac{\dfrac{\alpha_m}{(1-\alpha_m^2)^2}}{\left(1-\dfrac{x}{\alpha_m}\right)^2} + \cdots, \tag{19}$$

where $\cdots$ represents terms whose contribution to Ξ_n is negligible, for large n, compared to the contributions from the term shown. Therefore, from (19), we have

$$\Xi_n \sim \frac{\alpha_m}{(1-\alpha_m^2)^2}\left(\frac{1}{\alpha_m}\right)^n n.$$

Thus, the proportion of digits that are 0 when the number of digits is large is $P_{0's} = \alpha_m/(1-\alpha_m^2)$. By an earlier observation, we can write $P_{m-3's} = \cdots = P_{1's} = P_{0's}$. Similarly, for the $m-2$'s, we have

$$N_{m-2's}(x) = \sum_{n\ge 0} \Xi_n x^n = \frac{d}{dz} N(x,1,z,1)\bigg|_{z=1} = \frac{(1-x^2)x}{(x^2-mx+1)^2} = \frac{\dfrac{\alpha_m}{1-\alpha_m^2}}{\left(1-\dfrac{x}{\alpha_m}\right)^2} + \cdots, \tag{20}$$

where $\cdots$ represents terms that can be neglected, when n is large. Therefore, from (20), we have

$$\Xi_n \sim \frac{\alpha_m}{1-\alpha_m^2}\left(\frac{1}{\alpha_m}\right)^n n \quad \text{and} \quad P_{m-2\text{'s}} = \alpha_m.$$

Table 4 above shows the various proportions. It includes an expression for $P_{m-1\text{'s}}$, which is obtained from $P_{m-1\text{'s}} = 1-(m-2)P_{0\text{'s}} - P_{m-2\text{'s}}$. Note that, as m grows, the proportion of digits that are i for $0 \le i \le m-1$ becomes nearly equal.

ACKNOWLEDGMENTS

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-826 ***Proposed by the editor***

Find a recurrence consisting of positive integers such that each positive integer n occurs exactly n times.

B-827 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland***

Find a solution to the recurrence $A_{n+3} = A_n - 2A_{n+2}$, $A_0 = 0$, $A_1 = 1$, $A_2 = -2$, in terms of F_n and L_n.

B-828 ***Proposed by Piero Filipponi, Rome, Italy***

For n a positive integer, prove that $\sum_{r=0}^{\lfloor \frac{n-1}{4} \rfloor} \binom{n-1-2r}{2r}$ is within 1 of $F_n/2$.

B-829 ***Proposed by Jack G. Segers, Liège, Belgium***

For n a positive integer, let $P_n = F_{n+1}F_n$, $A_n = P_{n+1} - P_n$, $B_n = A_n - A_{n-1}$, $C_n = B_{n+1} - B_n$, $D_n = C_n - C_{n-1}$, and $E_n = D_{n+1} - D_n$. Show that $|P_n - B_n|$, $|A_n - C_n|$, $|B_n - D_n|$, and $|C_n - E_n|$ are successive powers of 2.

B-830 ***Proposed by Al Dorp, Edgemere, NY***

(a) Prove that if $n = 84$ then $(n+3) | F_n$.
(b) Find a positive integer n such that $(n+19) | F_n$.
(c) Is there an integer a such that $n+a$ never divides F_n?

SOLUTIONS

Mr. Feta's Lost Theorem

B-808 ***Proposed by Paul S. Bruckman, Jalmiya, Kuwait***
(Vol. 34, no. 2, May 1996)

Years after Mr. Feta's demise at Bellevue Sanitarium, a chance inspection of his personal effects led to the discovery of the following note, scribbled in the margin of a well-worn copy of Professor E. P. Umbugio's "22/7 Calculated to One Million Decimal Places":

> To divide "n-choose-one" into two other non-trivial "choose one's", "n-choose-two", or in general, "n-choose-m" into two non-trivial "choose-m's", for any natural m is always possible, and I have assuredly found for this a truly wonderful proof, but the margin is too narrow to contain it.

Because of the importance of this result, it has come to be known as Mr. Feta's Lost Theorem. We may restate it in the following form:

Solve the Diophantine equation $x^{\underline{m}} + y^{\underline{m}} = z^{\underline{m}}$, for $m \le x \le y \le z, m = 1, 2, 3, \ldots$, where $X^{\underline{m}} = X(X-1)(X-2)\cdots(X-m+1)$. Was Mr. Feta crazy?

Solution by Gerald A. Heuer, Concordia College, Moorhead, MN & Karl W. Heuer, The Free Software Foundation, Cambridge, MA

Well, owning a copy of Umbugio's "22/7 Calculated . . ." perhaps casts a slight shadow over Mr. Feta, but his statement is correct. Presumably the book had extremely small margins, for the solution, $x = y = 2m-1$, $z = 2m$ for every natural number m, does not require a substantial margin.

For each of the cases $m = 1$ and $m = 2$ there are infinitely many solutions, and we may give the general solution. With $m = 1$, things are rather simple: $x \le y$ arbitrary, and $z = x + y$.

With $m = 2$, we have the following family of solutions: Choose $x \ge 3$ arbitrary, and choose integers s, t, u, v even, $uv > st$, and $su \le (uv - st + 1)/2$. Then one routinely verifies that

$$\left(su, \frac{uv - st + 1}{2}, \frac{uv + st + 1}{2}\right) \tag{1}$$

is a solution. Note that for every x the choice of $s = t = 1$ satisfies the remaining conditions, so at least one solution exists. Moreover, every solution is of this form, for if $x(x-1) + y(y-1) = z(z-1)$ and

$$z = y + r, \tag{2}$$

then one finds at once that

$$x(x-1) = r(r + 2y - 1), \tag{3}$$

so every prime factor of r divides either x or $x-1$. Thus, we may write $r = st$, where $s|x$ and $t|(x-1)$. Then u and v may be defined satisfying $x = su$ and $x - 1 = tv$, and solving (3) for y and using (2), we obtain the solution (1). That just one of s, t, u, v is even follows from the facts that x and $x-1$ have opposite parity and that y is an integer, so that $uv - st$ is odd. The inequalities assumed must hold in order that $2 \le x \le y$.

With $m = 3$, in addition to the above solution, we find $(10,16,17)$, $(22,56,57)$, $(36,120,121)$ and there seem to be no more with $z = y + 1$, but we do not have a proof. A computer search with $m = 3$ yields many solutions and suggests probably an infinite family exists. With $m = 4$ there is at least one more solution, $(132,190,200)$, and with $m = 6$ at least two more: $(14,15,16)$ and

$(19,19,21)$. Each of these two seems to begin an infinite family: Whenever m is a solution of the Fermat-Pell equation $8m^2+1=n^2$ (i.e., $m=6,35,204,\ldots$), we have a solution

$$\left(\frac{4m+n-3}{2}, \frac{4m+n-3}{2}, \frac{4m+n+1}{2}\right),$$

and whenever m is a solution of $5m^2-2m+1=n^2$ ($m=6,40,273,1870,\ldots$),

$$\left(\frac{3m+n-3}{2}, \frac{3m+n-1}{2}, \frac{3m+n+1}{2}\right)$$

is a solution.

Also solved by Leonard A. G. Dresel, Piero Filipponi, David E. Manes, H.-J. Seiffert, and the proposer.

It Keeps on Going

B-809 ***Proposed by Pentti Haukkanen, University of Tampere, Tampere, Finland (Vol. 34, no. 2, May 1996)***

Let k be a fixed positive integer. Find a recurrence consisting of positive integers such that each positive integer occurs exactly k times.

Solution 1 by David E. Manes, SUNY College at Oneonta, NY; H.-J. Seiffert, Berlin, Germany; Lawrence Somer, The Catholic University of America, Washington, DC; and David Zeitlin, Minneapolis, MN (independently)

$$w_{n+2k}=2w_{n+k}-w_n, \quad n\geq 0,$$

with initial conditions $w_0=w_1=w_2=\cdots=w_{k-1}=1$; $w_k=w_{k+1}=w_{k+2}=\cdots=w_{2k-1}=2$.

Solution 2 by the proposer

$$w_{n+k+1}=w_{n+k}+w_{n+1}-w_n, \quad n\geq 0,$$

with initial conditions $w_0=w_1=w_2=\cdots=w_{k-1}=1$; $w_k=2$.

Solution 3 by Gerald A. Heuer, Concordia College, Moorhead, MN, and Russell Jay Hendel, Drexel University, Philadelphia, PA (independently)

$$w_{n+k}=w_n+1, \quad n\geq 0,$$

with initial conditions $w_0=w_1=w_2=\cdots=w_{k-1}=1$.

Solution 4 by Murray S. Klamkin, University of Alberta, Canada

$$w_{n+1}=1+\left\lfloor\frac{w_n+n}{k+1}\right\rfloor, \quad n>0,$$

with initial condition $w_1=1$.

Several solvers gave the reference G. Meyerson & A. J. van der Poorten, "Some Problems Concerning Recurrence Sequences," Amer. Math. Monthly ***102*** *(1995):698-705, which contains related problems. See Problem B-826 in this issue for a related problem.*

Also solved by Graham Lord.

Divisible Determinant

B-810 ***Proposed by Herta T. Freitag, Roanoke, VA***
(Vol. 34, no. 2, May 1996)

Let $\langle H_n \rangle$ be a generalized Fibonacci sequence defined by $H_{n+2} = H_{n+1} + H_n$ for $n > 0$ with initial conditions $H_1 = a$ and $H_2 = b$, where a and b are integers. Let k be a positive integer. Show that

$$A_n = \begin{vmatrix} H_n & H_{n+1} \\ H_{n+k+1} & H_{n+k+2} \end{vmatrix}$$

is always divisible by a Fibonacci number.

Proof by Steve Edwards, Southern College of Technology, Marietta, GA

We use equation (8) in [1]: $H_{n+m} = F_{m-1}H_n + F_m H_{n+1}$. Then

$$\begin{aligned} A_n &= H_n H_{n+k+2} - H_{n+1}H_{n+k+1} \\ &= H_n[F_{k+1}H_n + F_{k+2}H_{n+1}] - H_{n+1}[F_k H_n + F_{k+1}H_{n+1}] \\ &= F_{k+1}[H_n^2 - H_{n+1}^2] + H_n H_{n+1}[F_{k+2} - F_k] \\ &= F_{k+1}[H_n^2 - H_{n+1}^2] + H_n H_{n+1} F_{k+1} = F_{k+1}[H_n^2 - H_{n+1}^2 + H_n H_{n+1}]. \end{aligned}$$

Thus, A_n is always divisible by F_{k+1}.

Disproof by Russell Jay Hendel, Drexel University, Philadelphia, PA

Since $F_1 = 1$, every integer is divisible by a Fibonacci number. Thus, the proposer probably intended to ask us to show that A_n is always divisible by a Fibonacci number larger than 1. But, in that case, the proposition is false. If $a = 2$, $b = 5$, and $k = 1$, then $A_1 = -11$, which is not divisible by any Fibonacci number larger than 1.

A more correct statement of the problem would have been: "Show that A_n is always divisible by F_{k+1}."

Generalization by Pentti Haukkanen, University of Tampere, Tampere, Finland

Let $\langle G_n \rangle$ and $\langle H_n \rangle$ be any two generalized Fibonacci sequences satisfying $G_{n+2} = G_{n+1} + G_n$ and $H_{n+2} = H_{n+1} + H_n$ for $n > 0$. Vajda [1, p. 27] proves that

$$G_{n+k+1}H_{n+h} - G_n H_{n+h+k+1} = F_{k+1}(G_{n+1}H_{n+h} - G_n H_{n+h+1}).$$

Thus,

$$\begin{vmatrix} G_n & H_{n+h} \\ G_{n+k+1} & H_{n+h+k+1} \end{vmatrix}$$

is always divisible by F_{k+1}.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood Ltd., 1989.

Most solvers noted that $A_n = (-1)^{n-1}(a^2 - b^2 + ab)F_{k+1}$. Redmond and Somer showed (independently) that if $\langle W_n \rangle$ is a sequence of integers that satisfies the recurrence $W_{n+2} = PW_{n+1} + QW_n$ with initial conditions $W_1 = a$ and $W_2 = b$, then

$$J_k = \begin{vmatrix} W_n & W_{n+1} \\ W_{n+k+1} & W_{n+k+2} \end{vmatrix}$$

is equal to $U_{k+1}J_0$ *and, hence, is divisible by* U_{k+1}, *where* $\langle U_n \rangle$ *denotes the sequence satisfying the same recurrence as* $\langle W_n \rangle$ *with initial conditions* $U_0 = 0$, $U_1 = 1$.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Russell Jay Hendel, Murray S. Klamkin, Harris Kwong, Carl Libis, David Manes, Don Redmond, H.-J. Seiffert, Lawrence Somer, David Zeitlin, and the proposer.

Alternating Lucas

B-811 ***Proposed by Russell Euler, Maryville, MO (Vol. 34, no. 2, May 1996)***

Let n be a positive integer. Show that:

(a) if $n \equiv 0 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots - L_2 + 1$;

(b) if $n \equiv 1 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots - L_3 + 1$;

(c) if $n \equiv 2 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots + L_2 - 1$;

(d) if $n \equiv 3 \pmod 4$, then $F_{n+1} = L_n - L_{n-2} + L_{n-4} - \cdots + L_3 - 1$.

Solution by L. A. G. Dresel, Reading, England

We use the well-known formula $F_{n-1} + F_{n+1} = L_n$, which is formula (6) in [1]. When n is even, consider the sum

$$\begin{aligned} S_n &= L_n - L_{n-2} + L_{n-4} - \cdots + (-1)^{(n-2)/2} L_2 \\ &= (F_{n+1} + F_{n-1}) - (F_{n-1} + F_{n-3}) + \cdots + (-1)^{(n-2)/2}(F_3 + F_1) \\ &= F_{n+1} + (-1)^{(n-2)/2} F_1. \end{aligned}$$

Since $F_1 = 1$, this proves (a) and (c).

When n is odd, $n \geq 3$, consider the sum

$$\begin{aligned} T_n &= L_n - L_{n-2} + L_{n-4} - \cdots + (-1)^{(n-3)/2} L_3 \\ &= (F_{n+1} + F_{n-1}) - (F_{n-1} + F_{n-3}) + \cdots + (-1)^{(n-3)/2}(F_4 + F_2) \\ &= F_{n+1} + (-1)^{(n-3)/2} F_2. \end{aligned}$$

Since $F_2 = 1$, this proves (b) and (d).

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications*. Chichester: Ellis Horwood Ltd., 1989.

Also solved by Paul S. Bruckman, Herta T. Freitag, Pentti Haukkanen, Russell Jay Hendel, Harris Kwong, Carl Libis, David E. Manes, Bob Prielipp, Don Redmond, H.-J. Seiffert, Lawrence Somer, and the proposer.

A Triangle in Space

B-812 ***Proposed by John C. Turner, University of Waikato, Hamilton, New Zealand (Vol. 34, no. 2, May 1996)***

Let P, Q, R be three points in space with coordinates $(F_{n-1}, 0, 0)$, $(0, F_n, 0)$, $(0, 0, F_{n+1})$, respectively. Prove that twice the area of ΔPQR is an integer.

Editorial composite of solutions received from Steve Edwards, Southern College of Technology, Marietta, GA, and Murray S. Klamkin, University of Alberta, Alberta, Canada

We will show that if $P=(x,0,0)$, $Q=(0,y,0)$, and $R=(0,0,z)$, where x, y, and z are positive integers such that $x+y=z$, then twice the area of ΔPQR is an integer.

Heron's Formula [1, p. 12] gives the area of a triangle with sides of lengths a, b, and c as $A=\frac{1}{2}\sqrt{s(s-a)(s-b)(s-c)}$, where $s=(a+b+c)/2$. Using the Pythagorean Theorem to get the sides of ΔPQR, and a straightforward algebraic reduction, gives

$$\begin{aligned}2A &= \sqrt{x^2y^2+y^2z^2+z^2x^2} = \sqrt{x^2y^2+y^2(x+y)^2+(x+y)^2x^2} \\ &= \sqrt{x^4+2x^3y+3x^2y^2+2xy^3+y^4} = x^2+xy+y^2.\end{aligned}$$

Thus, $2A$ is an integer.

Reference

1. H. S. M. Coxeter. *Introduction to Geometry.* 2nd ed. New York: Wiley & Sons, 1989.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Herta T. Freitag, Harris Kwong, David E. Manes, John Oman & Bob Prielipp, H.-J. Seiffert, Lawrence Somer, David Zeitlin, and the proposer.

A Very General Determinant

B-813 ***Proposed by Peter Jeuck, Mahwah, NJ***
(Vol. 34, no. 2, May 1996)

Let $\langle X_n\rangle$, $\langle Y_n\rangle$, and $\langle Z_n\rangle$ be three sequences that each satisfy the recurrence $W_n=pW_{n-1}+qW_{n-2}$ for $n>1$, where p and q are fixed integers. (The initial conditions need not be the same for the three sequences.) Let a, b, and c be any three positive integers. Prove that

$$\begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = 0.$$

Solution by Paul S. Bruckman, Highwood, IL

Let $\langle U_n\rangle$ be the sequence that satisfies the same recurrence, but with initial values $U_0=0$ and $U_1=1$. The sequence $\langle qX_1U_{n-2}+X_2U_{n-1}\rangle$ also satisfies this same recurrence and has the same values as $\langle X_n\rangle$ when $n=1$ and $n=2$. Hence, these sequences are identical. In a similar manner, we see that $Y_n=qY_1U_{n-2}+Y_2U_{n-1}$ and $Z_n=qZ_1U_{n-2}+Z_2U_{n-1}$. Thus,

$$\begin{pmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{pmatrix} = \begin{pmatrix} qX_1 & X_2 & 0 \\ qY_1 & Y_2 & 0 \\ qZ_1 & Z_2 & 0 \end{pmatrix} \begin{pmatrix} U_{a-2} & U_{b-2} & U_{c-2} \\ U_{a-1} & U_{b-1} & U_{c-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, the determinant of each matrix on the right is 0. Hence, the determinant of the matrix on the left is 0.

Also solved by Leonard A. G. Dresel, Russell Jay Hendel, Murray S. Klamkin, Harris Kwong, David E. Manes, H.-J. Seiffert, Lawrence Somer, and the proposer.

Note: The Elementary Problems Column is in need of more *easy*, yet elegant and nonroutine problems.

ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-526 ***Proposed by Paul S. Bruckman, Highwood, IL***

Following H-465, let r_1, r_2, and r_3 be natural integers such that

$$\sum_{k=1}^{3} kr_k = n, \text{ where } n \text{ is a given natural integer.} \tag{1}$$

Let

$$B_{r_1, r_2, r_3} = \frac{1}{r_1 + r_2 + r_3} \frac{(r_1 + r_2 + r_3)!}{r_1! r_2! r_3!}. \tag{2}$$

Also, let

$$C_n = \sum B_{r_1, r_2, r_3}, \text{ summed over all possible } r_1, r_2, r_3. \tag{3}$$

Define the generating function

$$F(x) = \sum_{n=6}^{\infty} C_n x^n. \tag{4}$$

(a) Find a closed form for $F(x)$;

(b) Obtain an explicit expression for C_n;

(c) Show that C_n is a positive integer for all $n \geq 7$, n prime.

H-527 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let q, a, and b be positive integers, with $(a, b) = 1$. Prove or disprove the following:

$$\textit{a)} \quad \sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1} \sum_{s=0}^{b-1} (-1)^{q(br+as)} L_{2q(br+as)} = \frac{F_{q(a+b-ab)} F_{qab}}{F_{qa} F_{qb}} + (-1)^{q(1-ab)} \frac{F_{q(2ab-1)}}{F_q};$$

$$\textit{b)} \quad 5 \sum_{\substack{r=0 \\ (br+as<ab)}}^{a-1} \sum_{s=0}^{b-1} (-1)^{q(br+as)} F_{2q(br+as)} = (-1)^{q(1-ab)} \frac{L_{q(2ab-1)}}{F_q} - \frac{F_{qab} L_{q(a+b-ab)}}{F_{qa} F_{qb}}.$$

H-528 ***Proposed by Paul S. Bruckman, Highwood, IL***

Let $\Omega(n) = \sum_{p^e \| n} e$, given the prime decomposition of a natural number $n = \prod p^e$. Prove the following:

(A) $$\sum_{d|n}(-1)^{\Omega(d)}F_{\Omega(n/d)-\Omega(d)}=0;$$

(B) $$\sum_{d|n}(-1)^{\Omega(d)}L_{\Omega(n/d)-\Omega(d)}=2U_n,\text{ where }U_n=\prod_{p^e\|n}F_{e+1}.$$

SOLUTIONS

Poly Forms

H-508 (Corrected) ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 34, no. 1, February 1995)***

Define the Fibonacci polynomials by $F_0(x)=0$, $F_1(x)=1$, $F_n(x)=xF_{n-1}(x)+F_{n-2}(x)$, for $n\geq 2$. Show that, for all complex numbers x and y and all positive integers n,

$$F_n(x)F_n(y)=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}(x+y)^kF_{k+1}\left(\frac{xy-4}{x+y}\right). \tag{1}$$

As special cases of (1), obtain the following identities:

$$F_n(x)F_n(x+1)=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(x^2+x+4); \tag{2}$$

$$F_n(x)F_n(4/x)=n\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{1}{2k+1}\binom{n+2k}{4k+1}\left(\frac{x^2+4}{x}\right)^{2k},\quad x\neq 0; \tag{3}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(x^2+4)^k; \tag{4}$$

$$F_n(x)^2=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}\frac{x^{2k+2}-(-4)^{k+1}}{x^2+4}; \tag{5}$$

$$F_{2n-1}(x)=(2n-1)\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}x^kF_{k+1}(4/x). \tag{6}$$

Solution by the proposer

We also consider the Lucas polynomials defined by $L_0(x)=2$, $L_1(x)=x$, $L_n(x)=xL_{n-1}(x)+L_{n-2}(x)$, for $n\geq 2$. It is known that

$$F_n(x)=\frac{\alpha(x)^n-\beta(x)^n}{\sqrt{x^2+4}}\quad\text{and}\quad L_n(x)=\alpha(x)^n+\beta(x)^n, \tag{7}$$

where $\alpha(x)=\frac{1}{2}\left(x+\sqrt{x^2+4}\right)$ and $\beta(x)=\frac{1}{2}\left(x-\sqrt{x^2+4}\right)$, and that

$$F_{2n}(x)=\sum_{k=0}^{n-1}\binom{n+k}{2k+1}x^{2k+1}.$$

Integrating the latter equation and noting that $L_{2n}'(x)=2nF_{2n}(x)$ and $L_{2n}(0)=2$ gives

$$L_{2n}(x)-2=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}x^{2k+2}. \tag{8}$$

Since both sides of the stated equation (1) are analytic functions of x and y, it suffices to prove it for real x and y such that $x \geq y > 0$. Let

$$u=\tfrac{1}{2}\left(\sqrt{(x^2+4)(y^2+4)}+xy-4\right)$$

and

$$v=\tfrac{1}{2}\left(\sqrt{(x^2+4)(y^2+4)}-xy+4\right).$$

Then we have $u>0$ and $v>4$. From (8) it follows that

$$\frac{L_{2n}(\sqrt{u})-L_{2n}(i\sqrt{v})}{u+v}=n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}A_{k+1}, \tag{9}$$

where $i=\sqrt{(-1)}$ and

$$A_j:=\frac{u^j-(i\sqrt{v})^{2j}}{u+v},\quad j\in\mathbb{N}_0.$$

Since $u-v=xy-4$ and $uv=(x+y)^2$, it is easily seen that

$$A_j=(xy-4)A_{j-1}+(x+y)^2A_{j-2},\quad j\geq 2,$$

so that, by $A_0=0$ and $A_1=1$, we must have

$$A_j=(x+y)^{j-1}F_j\left(\frac{xy-4}{x+y}\right),\ j\in\mathbb{N}_0. \tag{10}$$

Simple calculations show that

$$\alpha(\sqrt{u})^2=\tfrac{1}{4}\left(2u+4+2\sqrt{u(u+4)}\right)=\alpha(x)\alpha(y),$$
$$\beta(\sqrt{u})^2=\tfrac{1}{4}\left(2u+4-2\sqrt{u(u+4)}\right)=\beta(x)\beta(y),$$

and, since $x\geq y$,

$$\alpha(i\sqrt{v})^2=-\tfrac{1}{4}\left(2v-4+2\sqrt{v(v-4)}\right)=\alpha(x)\beta(y),$$
$$\beta(i\sqrt{v})^2=-\tfrac{1}{4}\left(2v-4-2\sqrt{v(v-4)}\right)=\beta(x)\alpha(y).$$

From these four equations and (7), it follows that

$$F_n(x)F_n(y)=\frac{L_{2n}(\sqrt{u})-L_{2n}(i\sqrt{v})}{u+v}. \tag{11}$$

Now, the desired identity (1) follows from (9), (10), and (11).

Using the properties $F_j(-x)=(-1)^{j-1}F_j(x)$, $F_{2j}(0)=0$, and $F_{2j+1}(0)=1$, we show that (2)-(6) are all special cases of (1). Since we wish to exhibit some particular cases, we also note that $F_n(4)=F_{3n}/2$, $F_n(3i)=i^{n-1}F_{2n}$, $F_{2n}(\sqrt{5})=\sqrt{5}F_{4n}/3$, $F_{2n-1}(\sqrt{5})=L_{4n-2}/3$, and $5^{(n-1)/2}F_n(4/\sqrt{5})=(5^n-(-1)^n)/6$. Also, let $P_n=F_n(2)$ denote the n^{th} Pell number.

(2): In (1), replace x by $-x$ and then take $y=x+1$. We note the interesting particular case

$$F_nP_n = n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}F_{k+1}(6).$$

(3): Take $y = 4/x, x \neq 0$. For $x = 1$ and $x = 2$, we obtain, respectively,

$$F_nF_{3n} = 2n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}25^k, \text{ and } P_n^2 = n\sum_{k=0}^{[\frac{n-1}{2}]}\frac{1}{2k+1}\binom{n+2k}{4k+1}16^k.$$

With $x = \sqrt{5}$, eq. (3), after replacing n by $2n$ and n by $2n-1$, produces the curious identities:

$$F_{4n} = \frac{36n}{25^n-1}\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{2n+2k}{4k+1}81^k5^{n-1-k};$$

$$L_{4n-2} = \frac{18(2n-1)}{5^{2n-1}+1}\sum_{k=0}^{n-1}\frac{1}{2k+1}\binom{2n+2k-1}{4k+1}81^k5^{n-1-k}.$$

(4): Take $y = -x$. For $x = 1, x = 3i$, and $x = 2$, we obtain, respectively,

$$F_n^2 = n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}5^k,$$

$$F_{2n}^2 = n\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n+k}{2k+1}5^k,$$

and

$$P_n^2 = n\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}8^k.$$

(5): Take $y = x$ and use the Binet form of the Fibonacci polynomials. For $x = 3i$, this gives

$$F_{2n}^2 = \frac{n}{5}\sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}}{k+1}\binom{n+k}{2k+1}(9^{k+1}-4^{k+1}).$$

(6): In (1), replace n by $2n-1$ and then set $y = 0$. For $x = 1$, $x = 4$, $x = \sqrt{5}$, and $x = 2$, we obtain, respectively,

$$F_{2n-1} = \frac{2n-1}{2}\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}F_{3k+3},$$

$$F_{6n-3} = (4n-2)\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}4^kF_{k+1},$$

$$L_{4n-2} = \frac{2n-1}{2}\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}(5^{k+1}-(-1)^{k+1}),$$

and

$$P_{2n-1} = (2n-1)\sum_{k=0}^{2n-2}\frac{(-1)^k}{k+1}\binom{2n+k-1}{2k+1}2^kP_{k+1}.$$

Also solved by P. Bruckman and A. Dujella.

Pell Mell

H-510 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 34, no. 2, May 1996)***

Define the Pell numbers by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. Show that

$$P_n = \sum_{k \in A_n} (-1)^{[(3k-2n-1)/4]} 2^{[3k/2]} \binom{n+k}{2k+1}, \text{ for } n = 1, 2, \ldots,$$

where [ ] denotes the greatest integer function and $A_n = \{k \in \{0, 1, \ldots, n-1\} \mid 3k \not\equiv 2n \pmod 4\}$.

Solution by the proposer

First, we prove two theorems concerning the Fibonacci polynomials defined by

$$(1 - xz - z^2)^{-1} = \sum_{n=0}^{\infty} F_{n+1}(x) z^n, \tag{1}$$

which are also of interest in themselves.

Theorem 1: For all real x one has

$$F_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} i^{n-k} (x - 2i)^k, \text{ where } i^2 = -1.$$

Proof: Consider the special Jacobi polynomials defined by

$$(1 - 2xz + z^2)^{-j} = \sum_{n=0}^{\infty} C_n^j(x) z^n, \; j = 1, 2, \ldots. \tag{2}$$

It is well known [1, p. 374] that C_n^j has the derivatives

$$\frac{d^k}{dx^k} C_n^j(x) = 2^k \frac{(j+k-1)!}{(j-1)!} C_{n-k}^{j+k}(x). \tag{3}$$

If we substitute z by iz in (2) and compare the newly obtained equation with (1), we see that $F_{n+1}(x) = i^n C_n^1(x/2i)$. Thus, we have (3), and simple calculation gives

$$\frac{d^k}{dx^k} F_{n+1}(x) = k! i^{n-k} C_{n-k}^{k+1}(x/2i). \tag{4}$$

Since F_{n+1} is a polynomial of degree n, and since [1, p. 374]

$$C_{n-k}^{k+1}(1) = \binom{n+k+1}{2k+1},$$

the stated equation follows from (4) and Taylor's theorem. Q.E.D.

Theorem 2: For positive reals x one has

$$F_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} \Delta^k \cos \alpha_k$$

where $\Delta = (x^2 + 4)^{1/2}$ and $\alpha_k = (n-k)\pi/2 - k \arccos(x/\Delta)$.

Proof: Since $i = e^{i\pi/2}$ and $x - 2i = \Delta \exp(-i \arccos(x/\Delta))$, Theorem 1 gives

$$F_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} \Delta^k \exp(i\alpha_k),$$

which implies the stated equation by separating the real part. Q.E.D.

Now we are able to prove the proposer's equation. Using $P_{n+1} = F_{n+1}(2)$ and $\cos(\pi/4) = 1/\sqrt{2}$, Theorem 2 gives

$$P_{n+1} = 2^n \sum_{k=0}^{n} \binom{n+k+1}{2k+1} A_{3k-2n}, \tag{5}$$

where $A := 2^{j/2}\cos(j\pi/4)$ for all integers j. Using the addition theorem of the cosine, we easily find that, for all integers r,

$$A_{4r} = (-1)^r 2^{2r},\ A_{4r+1} = (-1)^r 2^{2r},\ A_{4r+2} = 0,\ A_{4r+3} = (-1)^{r+1} 2^{2r+1},$$

or, in a more compact form,

$$A_j = \begin{cases} (-1)^{[(j+1)/4]} 2^{[j/2]}, & \text{if } j \not\equiv 2 \pmod 4, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Observing that $[(3k-2n)/2] = [3k/2] - n$, we see that (5) and (6) prove the stated equation with $n+1$ instead of n.

Reference

1. Ryshik & Gradstein. *Tafeln* $\sum\prod\int$ *Tables*. Berlin: VEB Deutscher Verlag der Wissenschaften, 1963.

Also solved by P. Bruckman

Editorial Note: The editor wishes to acknowledge that H.-J. Seiffert also solved H-504 and H-505.

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