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The Universal Compactification of Topological Convex Sets and Modules

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A topological convex set is a convex set in a topological linear space with the induced topology. There is a universal continuous affine mapping of such a set into a compact convex subset of a locally convex linear space. Actually this compactification is a subset of a base normed Saks space. The results also hold for topological convex modules.

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1. Introduction

Several authors have investigated the problem under which assumptions a convex subset of a topological linear space can be embedded into a locally convex linear space. Jamison with O'Brien and Taylor in [7], Lawson in [14] and Roberts in [21] have only considered the case of compact convex sets. Semadeni in [23], [24] proves the existence of a universal compactification of a bounded convex subset of a Hausdorff locally convex linear space, which he calls the "*affine compactification*". The method he uses can be applied to the far more general case of any convex subset of a topological linear space, as is shown in the following.

For any topological convex set C, i.e. any convex subset of some topological linear space E there exists a base normed ordered Saks space $\mathrm{S}_*(C)$ such that the base of its ordering cone, which is compact in the locally convex topology of $\mathrm{S}_*(C)$, is the universal compactification of C. $\mathrm{S}_*(C)$ can be canonically extended to continuous affine morphisms between topological convex sets and induces a functor S_* from the category of topological convex sets to the category of compact base normed Saks spaces, which furnishes a description of this compactification analogous to the Stone-Čech compactification. This is proved in §4.

These results can even be proved for topological convex modules and contain Semadeni's result as a special case. The notion of a convex module is a canonical generalization of the notion of a convex set. It is used by Gudder [5], Flood [4], Swirszcz [25] and in [20]. The notions of a convex and a topological convex module are introduced in §2 and some basic

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results are proved. Up to now, ordered Saks spaces seem not to have been introduced, at least as far as the author knows. Consequently there are some degrees of freedom or uncertainty in defining them. For the investigations at hand only a special type is needed, namely base normed Saks spaces; they are defined in §3.

2. Topological convex modules

In the following all linear spaces considered will be real. If $X \subset E$ is a convex set in a linear space and $\hat{\alpha} := (\alpha_1, \dots, \alpha_n),\ \alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1,$ then $\hat{\alpha}_X(x_1, \dots, x_n) := \sum_{i=1}^{n} \alpha_i x_i$ defines a mapping $\hat{\alpha}_X : X^n \to X,\ n \in \mathbb{N}$. Let $\Omega_c^n := \left\{ \hat{\alpha} \mid \hat{\alpha} \in [0,1]^n,\ \hat{\alpha} = (\alpha_1, \dots, \alpha_n),\ \sum_{i=1}^{n} \alpha_i = 1 \right\}$ and define $\Omega_c := \bigcup_{n=1}^{\infty} \Omega_c^n$ the set of *formal convex combinations.* The above operations are also described by the mappings

$$\mu_X^n : \Omega_c^n \times X^n \to X \ , \tag{1}$$

$n \in \mathbb{N}$, with $\mu_X^n(\hat{\alpha}, \mathbf{x}) := \sum_{i=1}^{n} a_i x_i,\ \hat{\alpha} \in \Omega_c^n,\ \mathbf{x} \in X^n$. If $j_n : \Omega_c^n \times X^n \to \bigcup_{n=1}^{\infty} (\Omega_c^n \times X^n)$ denotes the inclusion, there is a unique mapping

$$\mu_X : \bigcup_{n=1}^{\infty} (\Omega_c^n \times X^n) \to X \tag{2}$$

with $\mu_X j_n = \mu_X^n,\ n \in \mathbb{N}$, giving an alternative representation of (1). This leads to the

Definition 2.1 ([18], [20]). A *convex module C* is a set C with a sequence of mappings

$$\mu_C^n : \Omega_c^n \times C^n \to C \ ,$$

$n \in \mathbb{N}$. If one writes the effect of the operations as formal sums, $\sum_{i=1}^{n} \alpha_i c_i := \mu_C^n(\hat{\alpha}, \mathbf{c}),\ \hat{\alpha} \in \Omega_c^n,\ \mathbf{c} \in C^n$, the following equations have to be satisfied:

$$\sum_{i=1}^{n} \delta_{ik} c_i = c_k \ , \tag{C 1}$$

$c_i \in C,\ \delta_{ik}$ the Kronecker symbol, $1 \leq i, k \leq n$, and

$$\sum_{i=1}^{n} \alpha_i \Big(\sum_{k \in K_i} \beta_{ik} c_k \Big) = \sum_{k=1}^{m} \Big(\sum_{\substack{i=1 \\ k \in K_i}}^{n} \alpha_i \beta_{ik} \Big) c_k \ . \tag{C 2}$$

In the last equation $(\alpha_1, \dots, \alpha_n) \in \Omega_c^n\ ,\ n \in \mathbb{N}$, and for $1 \leq i \leq n,\ K_i$ is a subset of $\{k \mid 1 \leq k \leq m\} = \mathbb{N}_m$ of cardinality m_i. Moreover, $\bigcup_{i=1}^{n} K_i = \mathbb{N}_m,\ \hat{\beta}_i = (\beta_{ik} \mid k \in K_i) \in \Omega_c^{m_i}$ and in the formal sum $\sum_{k \in K_i} \beta_{ik} x_k$ the summands are supposed to be written in the natural order of the k's.

A number of important computational rules follows from the above axioms (cf. [15], [18], [20]), e.g. the fact that a sum $\sum_{i=1}^{n} \alpha_i c_i$ is not changed by adding or omitting summands with zero coefficients. Hence, for computations, by adding a suitable number of zeroes one may assume that all the $\hat{\beta}_i,\ 1 \le i \le n$, have the same length, $\hat{\beta}_i \in \Omega_c^m$, so that (C 2) takes the more simple form

$$\sum_{i=1}^{n} \alpha_i \Big(\sum_{k=1}^{m} \beta_{ik} c_k \Big) = \sum_{k=1}^{m} \Big(\sum_{i=1}^{n} \alpha_i \beta_{ik} \Big) c_k \ . \tag{C 2'}$$

Obviously any convex set - which will always mean a convex subset in some linear space - is a convex module. The converse does not hold. Let M be a sup-semilattice and define, for $\hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Omega_c,\ x_i \in M,\ 1 \le i \le n$,

$$\sum_{i=1}^{n} \alpha_i x_i := \bigvee \{ x_i \mid 1 \le i \le n \quad \text{and} \quad \alpha_i \ne 0 \} \ .$$

Then M is a convex module but in general not a convex set.

A *morphism of convex modules* $f : C_1 \to C_2$, also called an *affine mapping*, is a mapping preserving convex combinations:

$$f\Big(\sum_{i=1}^{n} \alpha_i c_i^1 \Big) = \sum_{i=1}^{n} \alpha_i f(c_i^1) \ ,$$

$c_i^1 \in C_1,\ 1 \le i \le n$. The convex modules and the affine mappings constitute the category **Conv** of convex modules. For any set X the standard simplex in $\mathbb{R}^{(X)}$ is a free convex module generated by X and any convex module is a quotient of a free one (cf. [20]).

A convex set X in a topological linear space E equipped with the induced topology is called a *topological convex set in E*. The operations $\mu_X^n,\ n \in \mathbb{N}$, in (1) are then continuous if one considers the topology induced by the l_1-norm on Ω_c^n. This leads to the

Definition 2.2. A *topological convex module* C is a convex module C with a topology $\mathfrak{T}_C$ such that

$$\mu_C^n : \Omega_c^n \times C^n \to C$$

is continuous for any $n \in \mathbb{N}$, if C^n carries the product topology of $\mathfrak{T}_C$, Ω_c^n the l_1-norm topology and $\Omega_c^n \times C^n$ the product topology. A convex set C is called a *topological convex set*, if there is a topology $\mathfrak{T}_C$ on C, with which C is a topological convex module.

If one considers the sum topology on the disjoint union $\bigcup_n (\Omega_c^n \times C^n)$ then an equivalent statement is that $\mu_C : \bigcup_n (\Omega_c^n \times C^n) \to C$, defined as in (2), is continuous. The topological convex modules with the continuous affine mappings form the category **TopConv** of topological convex modules.

Countably convex sets were probably first introduced by Jameson [6] as subsets of Hausdorff topological linear spaces closed under countably convex combinations $\sum_{i=1}^{\infty} \alpha_i x_i,\ \alpha_i \ge$

0, $\sum_{i=1}^{\infty}\alpha_i = 1$. He calls a subset $M \subset E$ of a topological linear space *CS-compact* (convex series compact) if $\sum_{i=1}^{\infty}\alpha_i x_i$, for $x_i \in M$, converges for any x_i and any sequence $(\alpha_i \mid i \in \mathbb{N})$, $\alpha_i \geq 0$, $\sum_{i=1}^{\infty}\alpha_i = 1$, and lies in M. Countably convex modules without topology were then investigated under the name *superconvex* spaces by Rodé [22], König and Wittstock [13], Börger [2], Kemper [2], [9] and the author in [18], where they are called *superconvex modules*.

Define $\Omega_{sc} := \left\{\hat{\alpha} \mid \hat{\alpha} = (\alpha_i \mid i \in \mathbb{N}),\ \alpha_i \geq 0,\ i \in \mathbb{N}, \text{ and } \sum_{i=1}^{\infty}\alpha_i = 1\right\}$ the set of *formal superconvex* (or countably convex) *combinations*.

Definition 2.3 ([18]). A *superconvex module* C is a set with a mapping

$$\mu_C : \Omega_{sc} \times C^{\mathbb{N}} \to C\ .$$

If one writes the effect of this mapping as a formal sum, $\sum_{i=1}^{\infty}\alpha_i c_i := \mu_C(\hat{\alpha}, \mathbf{c})$, $\hat{\alpha} \in \Omega_{sc}$, $\mathbf{c} \in C^{\mathbb{N}}$, the following equations have to be satisfied:

$$\sum_{i=1}^{\infty}\delta_{ik}c_i = c_k\ , \tag{SC 1}$$

$$\sum_{i=1}^{\infty}\alpha_i\Big(\sum_{k=1}^{\infty}\beta_{ik}c_k\Big) = \sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty}\alpha_i\beta_{ik}\Big)c_k\ , \tag{SC 2}$$

with $c_k \in C,\ k \in \mathbb{N},\ \delta_{ik}$ the Kronecker symbol and $\hat{\alpha}, \hat{\beta}_i = (\beta_{ik} \mid k \in \mathbb{N}) \in \Omega_{sc}$.

A *morphism* $f : C_1 \to C_2$ *of superconvex modules*, also called a *superaffine mapping*, is a mapping preserving superconvex combinations:

$$f\Big(\sum_{i=1}^{\infty}\alpha_i c_i^1\Big) = \sum_{i=1}^{\infty}\alpha_i f(c_i^1)\ ,$$

$\hat{\alpha} \in \Omega_{sc},\ c_i^1 \in C_1,\ i \in \mathbb{N}$. The superconvex modules with these morphisms form the category **SC** of superconvex modules.

Let C be a superconvex module and consider an $\hat{\alpha} \in \Omega_{sc}$ with finite support, $\alpha_i = 0$ for $i > n$. Then $\sum_{i=1}^{\infty}\alpha_i c_i$, for $\mathbf{c} \in C^{\mathbb{N}}$, does not depend on c_i for $i > n$ ([18], 1.2 (i)) and one defines $\sum_{i=1}^{n}\alpha_i c_i := \sum_{i=1}^{\infty}\alpha_i c_i$, i.e. any superconvex module is also a convex module (cf. [15], 2.10, 2.12). A set X is called a *superconvex set*, if X is a subset of some linear space E, is a superconvex module and the (formal) superconvex sums extend the usual convex combinations in E : For $x_i \in X,\ i \in \mathbb{N}$, and $\hat{\alpha} \in \Omega_{sc}$ with finite support $\mathrm{supp}(\hat{\alpha}) = \{i \mid \alpha_i > 0\}$, i.e. $\alpha_i = 0$ for $i > n$ with a suitable n,

$$\sum_{i=1}^{\infty}\alpha_i x_i = \sum_{i=1}^{n}\alpha_i x_i$$

holds, where the right sum is the usual sum in E. Rodé shows in [22] that there is at most one superconvex structure on a convex set in a linear space. Jameson investigates superconvex sets in Hausdorff topological linear spaces in [6] and mentions that any sequentially complete, bounded convex set is a superconvex set. As the closed and the open unit ball of a Banach space are superconvex sets, the above condition is not necessary. Superconvex sets in topological linear spaces and 2.2 motivate the

Definition 2.4. A superconvex module C with a Topology $\mathfrak{T}_C$ is called a *topological superconvex module* if the mapping

$$\mu_C : \Omega_{sc} \times C^{\mathbb{N}} \to C$$

is continuous. Here, $C^{\mathbb{N}}$ carries the product topology of $\mathfrak{T}_C$ and Ω_{sc} the l_1-norm topology. The topological superconvex modules with the continuous superaffine mappings form the category **TopSC** of topological superconvex modules. A *topological superconvex set* is a superconvex set with a topology which makes it a topological superconvex module.

A superconvex subset $X \subset E$ of a topological linear space is called a *topological superconvex set in* E if X is a topological superconvex module with the topology induced by E. If X is a topological superconvex subset in E, and E is a Hausdorff topological linear space, then, for any $\hat{\alpha} \in \Omega_{sc}$, $\mathbf{x} \in X^{\mathbb{N}}$,

$$\sum_{i=1}^{\infty} \alpha_i x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i x_i$$

holds in E, i.e. $\sum\limits_{i=1}^{\infty} \alpha_i x_i$ is an infinite convergent series in E in the usual sense (cp. [6]). To see this, consider $\hat{\alpha} \in \Omega_{sc}$ with infinite support and put $A_n := \sum\limits_{i=1}^{n} \alpha_i$. In the following, we may always assume $A_n > 0$ by taking n large enough. Define

$$\hat{\alpha}^n := (A_n^{-1}\alpha_1, \ldots, A_n^{-1}\alpha_n,\ 0\ldots 0\ldots)\ ,$$

$\hat{\alpha}^n \in \Omega_{sc}$, $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} \hat{\alpha}^n = \hat{\alpha}$ in Ω_{sc} holds in the l_1-norm. Hence, $(\hat{\alpha}, \mathbf{x}) = \lim\limits_{n\to\infty} (\hat{\alpha}^n, \mathbf{x})$ follows and the continuity of μ_X implies the assertion. Obviously, X is also bounded and convex. Conversely, we have the

Proposition 2.5. *Let E be a Hausdorff topological linear space and $C \subset E$ a convex set. If C is bounded and sequentially complete, then C is a topological superconvex set in E.*

Proof. C is a superconvex set (cf. [6]). It remains to show that μ_C in 2.4 is continuous. For this, we consider an element $(\hat{\alpha}, \mathbf{c}) \in \Omega_{sc} \times C^{\mathbb{N}}$ and a zero neighbourhood U in E. There is a circled zero neighbourhood V with $5V \subset U$ and, because C is bounded, there exists an $\varepsilon_V > 0$ with $\varepsilon_V C \subset V$. Moreover, for V and any $k \in \mathbb{N}$ there is a $\delta(V,k) > 0$ and a zero neighbourhood $W(V,k)$, such that for any

$$(\hat{\beta}, \mathbf{d}) \in \Big[(\hat{\alpha}, \mathbf{c}) + \mathrm{O}_{\delta(V,k)}(l_1(\mathbb{N})) \times \mathop{\mathsf{X}}_{i=1}^{\infty} W_i\Big] \cap (\Omega_{sc} \times C^{\mathbb{N}})\ ,$$

with $W_i := W(V,k)$, far $1 \leq i \leq k$, $W_i := C$, for $i > k$,

$$\sum_{n \leq k} \beta_n (d_n - c_n) \in V$$

holds. Here, $\mathrm{O}_r(l_1(\mathbb{N}))$, $r > 0$, denotes the closed ball with center the origin and radius r in the Banach space $l_1(\mathbb{N})$. Also, there exists a $k(V) \in \mathbb{N}$ with

$$\sum_{n > k(V)} \alpha_n < 2^{-1} \varepsilon_V \ .$$

Define $\varepsilon_0 := \varepsilon_V$, $\delta_0 := \min\{2^{-1}\varepsilon_V \ , \ \delta(V, k(V))\}$, $W_0 := W(V, k(V))$, $k_0 := k(V)$ and consider any

$$(\hat{\beta}, \mathbf{d}) \in \left[(\hat{\alpha}, \mathbf{c}) + \mathrm{O}_{\delta_0}(l_1(\mathbb{N})) \times \mathop{\mathsf{X}}_{i=1}^{\infty} W_i \right] \cap (\Omega_{sc} \times C^{\mathbb{N}}) \ .$$

This implies

$$\sum_{n > k_0} \beta_n < \varepsilon_0 \ , \quad \sum_{n \leq k_0} \beta_n (d_n - c_n) \in V \ .$$

Put $A_+ := \sum_{\beta_n > \alpha_n} (\beta_n - \alpha_n)$, $A_- := \sum_{\beta_n \leq \alpha_n} (\alpha_n - \beta_n)$ and assume, without loss of generality, $A_+ > 0$ and $A_- > 0$.

Then

$$\sum_n (\beta_n - \alpha_n) c_n = A_+ \sum_{\beta_n > \alpha_n} \gamma_n c_n - A_- \sum_{\beta_n \leq \alpha_n} \sigma_n c_n$$

follows with $\gamma_n := A_+^{-1}(\beta_n - \alpha_n)$, for $\beta_n > \alpha_n$, $\sigma_n := A_-^{-1}(\alpha_n - \beta_n)$, for $\beta_n \leq \alpha_n$. This implies

$$\sum_n (\beta_n - \alpha_n) c_n \in A_+ C - A_- C \subset \varepsilon_0^{-1} A_+ V + \varepsilon_0^{-1} A_- V \subset 2V \ ,$$

because $A_\pm < 2^{-1}\varepsilon_0$. Moreover, we may assume that $A_0 := \sum_{n > k_0} \beta_n > 0$, so that, with $\tau_n := A_0^{-1} \beta_n$,

$$\sum_{n > k_0} \beta_n (d_n - c_n) = A_0 \sum_{n > k_0} \tau_n d_n - A_0 \sum_{n > k_0} \tau_n c_n$$

holds, which implies

$$\sum_{n > k_0} \beta_n (d_n - c_n) \in A_0 C - A_0 C \subset \varepsilon_0^{-1} A_0 V + \varepsilon_0^{-1} A_0 V \subset 2V \ .$$

This yields

$$\sum_n \beta_n (d_n - c_n) = \sum_{n \leq k_0} \beta_n (d_n - c_n) + \sum_{n > k_0} \beta_n (d_n - c_n) \in 3V$$

and hence

$$\sum_n \beta_n d_n - \sum_n \alpha_n c_n = \sum_n \beta_n (d_n - c_n) + \sum_n (\beta_n - \alpha_n) c_n \in U \ ,$$

which proves the assertion. □

3. Base normed Saks spaces

A *Saks space* is a triple $(\|\square\|, E, \mathfrak{T})$, where E with the norm $\|\square\|$ is a normed linear space, E with the topology $\mathfrak{T}$ is a (Hausdorff) locally convex linear space and the unit ball $\mathrm{O}(E)$ is bounded and closed in $\mathfrak{T}$ (cf. [3]). In the following, a Saks space will be denoted by a single letter E, its norm by $\|\square\|_E$ and its topology by $\mathfrak{T}_E$. If $E_i,\ i = 1, 2$, are Saks spaces a morphism $f : E_1 \to E_2$ is a linear contraction, such that its restriction to the unit balls is $\mathfrak{T}_{E_1} - \mathfrak{T}_{E_2}$ continuous. The Saks spaces with their morphisms constitute the category $\mathbf{Saks}_1$ of Saks spaces. A Saks space E is called *complete* if $\mathrm{O}(E)$ is $\mathfrak{T}_E$-complete, which implies that E with $\|\square\|_E$ is a Banach space ([3], 1.2). E is called *compact* if $\mathrm{O}(E)$ is $\mathfrak{T}_E$-compact. These special types define the full subcategories $\mathbf{CSaks}_1$ of complete and $\mathbf{CompSaks}_1$ of compact Saks spaces. Examples of Saks spaces abound. The dual E' of any normed linear space is a compact Saks space if it is supplied with the dual norm and the weak $*$-topology. If E is a locally convex linear space and $B \subset E$ an absorbing, bounded, closed and absolutely convex set, then the Minkowski functional of B makes E a Saks space.

Definition 3.1. A *base normed Saks space* is a quadruple $(\|\square\|,\ C, E, \mathfrak{T})$, where E is a linear space ordered by the proper, generating cone C with base B, which induces the norm $\|\square\|$, i.e. E is a base normed linear space (cf. [6], [26]). Moreover, E with the topology $\mathfrak{T}$ is a locally convex space and B is bounded and closed in $\mathfrak{T}$. Hence, C is well-based and closed (cp. [6], 3.8.3).

A base normed Saks space will be denoted by a single letter E, its norm by $\|\square\|_E$, its cone by C_E with base B_E and its topology by $\mathfrak{T}_E$. If $M \subset E$ is a set in a linear space, let $\mathrm{conv}(M)$ denote its convex closure and $\mathrm{absconv}(M)$ its absolutely convex closure. In a base normed Saks space E $\overset{o}{\mathrm{O}}(E) \subset \mathrm{absconv}(B_E) \subset \mathrm{O}(E)$ holds, if $\overset{o}{\mathrm{O}}(E)$ denotes the open (with respect to the norm $\|\square\|_E$) unit ball, and $\mathrm{O}(E) \subset \overline{\mathrm{absconv}}(B_E)$, if $\overline{M}$ denotes the $\mathfrak{T}_E$-closure of a set $M \subset E$. As B_E is $\mathfrak{T}_E$-bounded so is $\mathrm{O}(E)$. But in general $\mathrm{O}(E)$ is not $\mathfrak{T}_E$-closed, hence a base normed Saks space is not necessarily a Saks space in the sense of Cooper [3]. The Minkowski functional $|\square|$ of $\overline{\mathrm{O}(E)}$ makes E a Saks space with $|\square| \le \|\square\|_E$, but $|\square|$ is in general not a base norm.

A *morphism of base normed Saks spaces* $f : E_1 \to E_2$ is a linear mapping with $f(B_{E_1}) \subset B_{E_2}$, which, restricted to the bases $f/B_{E_1} : B_{E_1} \to B_{E_2}$ is $\mathfrak{T}_{E_1}$-$\mathfrak{T}_{E_2}$ continuous. The base normed Saks spaces and their morphisms form the category $\mathbf{BNSaks}_1$ of base normed Saks spaces. A base normed Saks space E is called *complete*, if B_E is $\mathfrak{T}_E$-complete, and *compact*, if B_E is $\mathfrak{T}_E$-compact. The corresponding full subcategories are denoted by $\mathbf{CBNSaks}_1$ and $\mathbf{CompBNSaks}_1$; obviously $\mathbf{CompBNSaks}_1 \subset \mathbf{CBNSaks}_1$.

Proposition 3.2.

(i) *If E is a base normed linear space with norm $\|\square\|_E$, cone C_E and base B_E and B_E is superconvex, then E is a base normed Banach space ([18]).*

(ii) *If E is a complete base normed Saks space, $(E, \|\square\|_E)$ is a base normed Banach space.*

Proof. (i): For $x \in E,\ \|x\|_E = \inf\{\alpha > 0 \mid x \in \alpha\, \mathrm{absconv}(B_E)\}$ and

$$\mathrm{absconv}(B_E) = \{y \mid y = \lambda_1 b_1 - \lambda_2 b_2,\ b_i \in B_E,\ \lambda_i \ge 0,\ i = 1, 2\ \lambda, + \lambda_2 = 1\}$$

hold. A straightforward computation shows that

$$\|x\|_E = \inf\{\alpha + \beta \mid \alpha, \beta \geq 0,\ x = \alpha b_1 - \beta b_2,\ b_1, b_2 \in B_E\} .$$

$-y \leq x \leq y$ implies $y + x = \alpha b_1,\ y - x = \beta b_2,\ \alpha, \beta \geq 0,\ b_i \in B_E,\ i = 1, 2$, hence $y = 2^{-1}(\alpha b_1 + \beta b_2)$ and

$$\|y\|_E = 2^{-1}(\alpha + \beta) \geq \|x\|_E . \tag{$*$}$$

If $\varepsilon > 0$, there are $\alpha, \beta \geq 0$, $b_1, b_2 \in B_E$ with $x = \alpha b_1 - \beta b_2$ and $\|x\|_E \leq \alpha + \beta < \|x\|_E + \varepsilon$. With $y := \alpha b_1 + \beta b_2$ $\|x\|_E \leq \|y\|_E < \|x\|_E + \varepsilon$ and $-y \leq x \leq y$ follows, hence

$$\|x\|_E = \inf\{\|y\|_E \mid -y \leq x \leq y\} . \tag{$**$}$$

($*$) and ($**$) show that $\|\Box\|_E$ is a Riesz norm so that C_E gives an open decomposition of E because a Riesz norm is semi-decomposable ([26], 3.1, p. 30). Now Proposition 3.5.11 in [6] can be used to show that E is complete. Consider an increasing Cauchy sequence $c_n \in C_E$, $n \in \mathbb{N}$. We may assume that $\|c_{n+1} - c_n\|_E < 2^{-n}$ and $c_n < c_{n+1}$, for $n \in \mathbb{N}$. As $c_{n+1} - c_n \geq 0$ there is $\alpha_n > 0$ and $b_n \in B_E$ with $c_{n+1} - c_n = \alpha_n b_n$. $\alpha_n = \|c_{n+1} - c_n\|_E < 2^{-n}$ implies $\|\hat{\alpha}\| := \sum_{n=1}^{\infty} \alpha_n \leq 1$. Moreover

$$c_{n+1} - c_n = \sum_{i=1}^{n} (c_{i+1} - c_i) = \sum_{i=1}^{n} \alpha_i b_i$$

holds and

$$c_0 := \sum_{i=1}^{\infty} \frac{\alpha_i}{\|\hat{\alpha}\|}\ b_i$$

is an element of B_E. Define $\|\hat{\alpha}_n\| := \sum_{i=1}^{n} \alpha_i$, then an application of (SC 2) yields

$$c_0 = \frac{\|\hat{\alpha}_n\|}{\|\hat{\alpha}\|} \sum_{i=1}^{n} \frac{\alpha_i}{\|\hat{\alpha}_n\|}\ b_i + \left(1 - \frac{\|\hat{\alpha}_n\|}{\|\hat{\alpha}\|}\right) \sum_{i=n+1}^{\infty} \frac{\alpha_i}{\|\hat{\alpha}\| - \|\hat{\alpha}_n\|}\ b_i .$$

As the second sum is in B_E its norm is 1 and we get

$$\|\ \|\hat{\alpha}\| c_0 - \sum_{i=1}^{n} \alpha_i b_i\ \|_E = (\|\hat{\alpha}\| - \|\hat{\alpha}_n\|) .$$

This yields $\lim_{n\to\infty} c_n = \|\hat{\alpha}\| c_0 + c_1$ hence $(E, \|\Box\|_E)$ is complete ([6]).

(ii): B_E is a convex, $\mathfrak{T}_E$-bounded and $\mathfrak{T}_E$-complete subset of $(E, \mathfrak{T}_E)$ hence is a topological superconvex subset of E because of 1.5. Now the assertion follows from (i). □

If E is a compact base normed Saks space, absconv(B_E) is $\mathfrak{T}_E$-compact, hence absconv(B_E) $= \mathrm{O}(E)$ holds, i.e. E is a compact Saks space in the sense of Cooper [3]. There are numerous examples of base normed Saks spaces. The dual E' of any order unit normed space is a compact base normed Saks space with the weak $*$-topology ([6], [26]). If E is a Saks space in the sense of Cooper and $\|\Box\|_E$ is a base norm for a base B of a generating cone C in E with $0 \notin \overline{B}$, then $\overline{B}$ is a base for $\overline{C}$ ([6]) and, with $\overline{B}$ and $\overline{C}$, E is a base normed Saks space.

4. The compactification

From now on, the base of a base normed Saks space E with the topology induced by $\mathfrak{T}_E$ will be denoted by $\mathrm{Bs}_*(E)$.

Proposition 4.1.
(i) *The* $\mathrm{Bs}_*(E), E$ *a base normed Saks space, induce a functor* $\mathrm{Bs}_* : \mathbf{BNSaks}_1 \to \mathbf{TopConv}$.
(ii) *The* $\mathrm{Bs}_*(E), E$ *a complete, base normed Saks space, induce a functor* $\mathrm{Bs}_* : \mathbf{CBNSaks}_1 \to \mathbf{TopSC}$.

Proof. (i): For a morphism $f : E_1 \to E_2$ of base normed Saks spaces the restriction to the bases is denoted by $\mathrm{Bs}_*(f)$ and is trivially continuous and affine. $\mathrm{Bs}_*(E)$ is also obviously a topological convex module.

(ii): 2.5 implies that $Bs_*(E)$ is a topological superconvex module if E is a complete base normed Saks space. □

Let $C \in \mathbf{SC}$ and E be a Hausdorff, linear topological space. Then $f : C \to E$ is called *superaffine*, if $f(C) \subset E$ is a topological superconvex set and the restriction of f to $C \to f(C)$ is a superaffine mapping.

Lemma 4.2. *Let E be a Hausdorff topological linear space, which is sequentially complete, C a superconvex module and $f : C \to E$ a mapping. Then the following statements are equivalent:*

(i) *f is superaffine,*
(ii) *f is affine and bounded.*

Proof. If f is superaffine it is trivially affine and $f(C)$ is bounded. If f is affine and bounded, take an $\hat{\alpha} \in \Omega_{sc}$ with infinite support, so that one may assume $A_n := \sum_{i=1}^{n} \alpha_i > 0$ for n large enough. Then, for $c_i \in C,\ i \in \mathbb{N}$,

$$f\Big(\sum_{i=1}^{\infty} \alpha_i c_i\Big) - \sum_{i=1}^{n} \alpha_i f(c_i) = (1 - A_n) f\Big(\sum_{i=n+1}^{\infty} (1 - A_n)^{-1} \alpha_i c_i\Big).$$

If U is any zero neighborhood, which we may assume to be circled, there is an $\varepsilon_U > 0$ with $f(C) \subset \varepsilon_U^{-1} U$. There exists $n_0(\varepsilon_U)$ such that $0 < 1 - A_n < \varepsilon_U$ for $n \geq n_0(\varepsilon_U)$. This implies

$$f\Big(\sum_{i=1}^{\infty} \alpha_i c_i\Big) - \sum_{i=1}^{n} \alpha_i f(c_i) \in U\ ,$$

i.e. (i). □

Definition 4.3. For $C \in \mathbf{TopConv}$, one defines $\mathrm{Aff}_c(C) := \{f \mid f : C \to \mathbb{R}$ affine, bounded and continuous$\}$, $\mathbb{R}$ with its usual topology.

All constant mappings belong to $\mathrm{Aff}_c(C)$, the constant mapping with value 1 is denoted by 1_C. With the pointwise operations $\mathrm{Aff}_c(C)$ is a real linear space, it is also ordered by the pointwise order, $f_1 \leq f_2$ if $f_1(c) \leq f_2(c)$ for $c \in C$. If E is a topological linear space E' will denote the topological dual, i.e. the space of all continuous linear forms on E.

Proposition 4.4. *If $C \in$ **TopConv**, then:*

(i) *With the supremum norm $\mathrm{Aff}_c(C)$ is an order unit Banach space with order unit 1_C.*
(ii) *With the dual norm and the weak $*$-topology $\mathrm{Aff}'_c(C)$ is a compact base normed Saks space.*
(iii) *The evalution mapping $\tau_C(c)(f) := f(c),\ c \in C,\ f \in \mathrm{Aff}_c(C)$, is affine, $\mathfrak{T}_C$-weakly $*$-continuous and induces a natural transformation.*

Proof. The proof of (i) is straightforward. The assertion of (ii) is a well-known theorem ([6], [26]). The weakly $*$-compact base is

$$\begin{aligned}\mathrm{Bs}(\mathrm{Aff}'_c(C)) &= \{\Lambda \mid \Lambda \geq 0,\ \|\Lambda\| \leq 1 \text{ and } \Lambda(1_C) = 1\}\\ &= \{\Lambda \mid \Lambda \geq 0,\ \|\Lambda\| = 1\}\end{aligned}$$

with the dual norm.

(iii) $\tau_C : C \to \mathrm{Aff}'_c(C)$ is obviously affine. Put

$$\hat{f}(\Lambda) := \Lambda(f)$$

for $f \in \mathrm{Aff}_c(C),\ \Lambda \in \mathrm{Aff}'_c(C)$. Then the linear forms $\hat{f} : \mathrm{Aff}'_c(C) \to \mathbb{R}$ induce the weak $*$-topology. For $f \in \mathrm{Aff}_c(C),\ c \in C$,

$$\hat{f}(\tau_C(c)) = \tau_C(c)(f) = f(c)$$

holds, i.e. $\hat{f}\tau_C = f$, hence τ_C is $\mathfrak{T}_C$-weakly $*$-continuous. If one regards $C \mapsto \mathrm{Aff}'_c(C)$ as an endofunctor of **TopConv**, τ_C induces trivially a natural transformation. □

For a subset $M \subset \mathrm{Aff}'_c(C)$, $\mathrm{cl}_*(M)$ will denote the weak $*$-closure of M. Define

$$\mathrm{S}_*(C) := \mathbb{R}_+ \,\mathrm{cl}_*(\tau_C(C)) - \mathbb{R}_+ \,\mathrm{cl}_*(\tau_C(C)) \ ,$$

with $\mathbb{R}_+ := \{x \mid x \in \mathbb{R},\ x \geq 0\}$, $C \in$ **TopConv** and

$$\|z\|_* := \inf\{\alpha > 0 \mid z \in \alpha \,\mathrm{absconv}(\mathrm{cl}_*(\tau_C(C)))\}$$

for $z \in \mathrm{S}_*(C)$. As $\mathrm{cl}_*(\tau_C(C)) \subset \mathrm{Bs}(\mathrm{Aff}'_c(C))$

$$\|z\| \leq \|z\|_*$$

holds, for $z \in \mathrm{S}_*(C)$, with $\|\Box\|$ the dual norm in $\mathrm{Aff}'_c(C)$. Hence, $\|\Box\|_*$ is a norm in $\mathrm{S}_*(C)$. Let $\mathfrak{T}_*(C)$ or simply $\mathfrak{T}_*$ denote the locally convex topology induced in $\mathrm{S}_*(C)$ by the weak $*$-topology. □

Theorem 4.5. *The following statements hold for $C \in$ **TopConv**:*

(i) *$\mathrm{S}_*(C)$ is a compact base normed Saks space, with norm $\|\Box\|_*$, order cone $\mathbb{R}_+ \,\mathrm{cl}_*(\tau_C(C))$ with base $\mathrm{cl}_*(\tau_C(C))$ and topology $\mathfrak{T}_*(C)$. The $\mathrm{S}_*(C), C \in$ **TopConv**, induce a functor S_*: **TopConv** $\to$ **CompBNSaks**$_1$.*

(ii) *Let* $\tau_*(C)$ *denote the restriction of* τ_C *to* $\mathrm{Bs}_*(\mathrm{S}_*(C)) = \mathrm{cl}_*(\tau_C(C))$ *with the topology induced by* $\mathfrak{T}_*(C)$, $C \in$ **TopConv**. *Then the* $\tau_*(C)$ *induce a natural transformation* τ_*: **TopConv** $\to \mathrm{Bs}_* \circ \mathrm{S}_*$ *and* S_* *is left adjoint to* Bs_* *with adjunction morphism* τ_*.

Proof. (i) As $\mathrm{cl}_*(\tau_C(C)) \subset \mathrm{Bs}(\mathrm{Aff}'_c(C))$, $\mathrm{cl}_*(\tau_C(C))$ is weakly $*$-compact and $P_0 := \mathbb{R}_+ \,\mathrm{cl}_*(\tau_C(C))$ is a proper, generating, weakly $*$-closed, well-based cone ([6], 3.8.3). P_0 is also $\|\Box\|_*$-closed and $\mathrm{cl}_*(\tau_C(C))$ is $\|\Box\|_*$-closed and $\|\Box\|_*$-bounded. Hence $\|\Box\|_*$ is a base norm and, if $\mathrm{O}_*(\mathrm{S}_*(C))$ is the unit ball with respect to $\|\Box\|_*$,

$$\mathrm{O}_*(\mathrm{S}_*(C)) = \mathrm{absconv}(\mathrm{cl}_*(\tau_C(C)))$$

because $\mathrm{absconv}(\mathrm{cl}_*(\tau_C(C)))$ is weakly $*$-compact ([8], 6.7.3). The proof of the last assertion in (i) is obvious.

(ii) We first show that the functor Bs_* : $\mathbf{CompBNSaks}_1 \to$ **TopConv** (cf. 4.1, (i)) is full and faithful. Let $\varphi : \mathrm{Bs}_*(E_1) \to \mathrm{Bs}_*(E_2)$ be a continuous affine mapping. φ can be uniquely extended to a linear mapping $\hat{\varphi} : E_1 \to E_2$, which is continuous because φ is, hence $\hat{\varphi}$ is a morphism in $\mathbf{BNSaks}_1$. That $\tau_*(C)$, $C \in$ **TopConv**, induce a natural transformation is obvious.

To prove the universal property of $\tau_*(C)$ let $\varphi : C \to \mathrm{Bs}_*(E)$ be a morphism in **TopConv** and $E \in \mathbf{CompBNSaks}_1$. For simple notation put $\tau_0 := \tau_*(\mathrm{Bs}_*(E))$, $\tau_0 : \mathrm{Bs}_*(E) \to \mathrm{Bs}_*(\mathrm{S}_*(\mathrm{Bs}_*(E)))$. τ_0 is continuous hence $\tau_0(\mathrm{Bs}_*(E))$ is weakly $*$-compact, i.e. τ_0 is surjective. If $\tau_0(x_1) = \tau_0(x_2)$ for $x_1, x_2 \in \mathrm{Bs}_*(E)$, then, for any $f \in \mathrm{Aff}_c(\mathrm{Bs}_*(E))$, $f(x_1) = f(x_2)$ follows. Moreover, any linear, $\mathfrak{T}_E$-continuous λ: $E \to \mathbb{R}$ is an element of $\mathrm{Aff}_c(\mathrm{Bs}_*(E))$ when restricted to $\mathrm{Bs}_*(E)$. This implies that τ_0 is injective, hence an isomorphism in **TopConv**. As Bs_* is full and faithful, there is exactly one extension $\psi_0 : E \to \mathrm{S}_*(\mathrm{Bs}_*(E))$ of τ_0 and ψ_0 is an isomorphism in $\mathbf{CompBNSaks}_1$. Then $\varphi_0 := \psi_0^{-1}\mathrm{S}_*(\varphi) : \mathrm{S}_*(C) \to E$ is the unique morphism in $\mathbf{CompBNSaks}_1$ with the property $\varphi = \mathrm{Bs}_*(\varphi_0)\tau_*(C)$, which proves the left adjointness of S_* . □

If we now consider the case $C \in$ **TopSC**, we see that because of 4.2 the above constructions do not change. So, restricting S_* to the full subcategory **TopSC** of **TopConv** 4.5 yields the

Corollary 4.6. *The functor* S_* : **TopSC** $\to \mathbf{CompBNSaks}_1$ *is a left adjoint of* Bs_* : $\mathbf{CompBNSaks}_1 \to$ **TopSC** *with adjunction morphism* τ_*: **TopSC** $\to \mathrm{Bs}_* \circ \mathrm{S}_*$.

It should be noted that Bs_* maps $\mathbf{CompBNSaks}_1$ to **TopSC** in any case. Also $\mathrm{S}_*(C)$ is a compact base normed Saks space for $C \in$ **TopConv**. If C has the discrete topology, i.e. the trivial topology, 4.5 and 4.6 yield, in both cases, a universal compactification, too. If one then forgets the weak topology on $\mathrm{S}_*(C)$ one has in $\tau_*(C)$ a morphism into the base of a base normed Banach space, which is different from the universal one in [18].

A convex set $B \subset E$ in a linear space, $B \neq \emptyset$, is called a *base set*, if, for all $\alpha, \beta > 0$ and $b_1, b_2 \in B$, $\alpha b_1 = \beta b_2$ implies $\alpha = \beta$. If E is a locally convex linear space and a convex subset $B \neq \emptyset$ is not a base set, define

$$E_0 := \mathbb{R} \times E \ ,$$

which is a locally convex linear space with the product topology and $B_0 := \{1\} \times B$ is a base set in E_0. $E_1 := \mathbb{R}_+ B_0 - \mathbb{R}_+ B_0$ with the induced topology is a locally convex linear space, with the proper generating cone $\mathbb{R}_+ B_0$ with base B_0 and $0 \neq B_0$. If B is compact, B_0 is compact, too, and $\mathbb{R}_+ B_0$ is closed and well-based ([6], 3.8.5). This implies that E_1 with the Minkowski functional of $\operatorname{absconv}(B_0)$ as norm is a compact base normed Saks space. Hence, for any compact convex subset $B \neq \emptyset$ of a locally convex linear space E, we may always assume that B is the base of the base normed Saks space E.

Now, let $C \subset E$ be a convex, bounded subset of the locally convex linear space E. C is a topological convex set so, in particular, a topological convex module. Let $\tau_*(C)(c_1) = \tau_*(C)(c_2)$, then, for any $\lambda \in E'$, $\lambda(C) \subset \mathbb{R}$ is bounded, hence $\lambda/C \in \operatorname{Aff}_c(C)$ and $\lambda(c_1) = \lambda(c_2)$, which implies $c_1 = c_2$. This means that $\tau_*(C) : C \to \operatorname{Bs}_*(\operatorname{S}_*(C))$ is injective. Combining the preceding considerations with 4.5 shows that $\tau_*(C)$ is exactly the universal compactification of Semadeni in [23], [24].

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