Motivic Decomposition of Projective Pseudo-Homogeneous Varieties

Srimathy Srinivasan

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- Else it is of outer type over k

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  Examples: Severi-Brauer Varieties $SB_n(A)$ corresponding to a central simple algebra A

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- What are Hom sets? If X is irreducible,
  $Hom_{Chow(k,\Lambda)}((X, n, p), (Y, m, q)) = q \circ [CH_{dim\ X+n-m} X \times Y \otimes_{\mathbb{Z}} \Lambda] \circ p$

Composition of Morphisms

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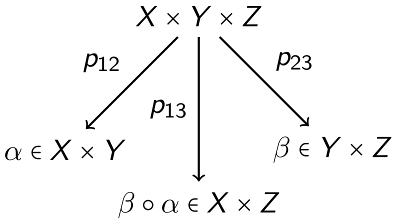

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where $\alpha_{m-n} = [pt \times \mathbb{P}^{m-n}] \in End\ \mathcal{M}(\mathbb{P}^{m-n})$

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- In general,

$$\mathcal{M}(\mathbb{P}^n) \simeq \Lambda \oplus \Lambda(1) \oplus \cdots \oplus \Lambda(n)$$

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- Many interesting consequences. One of them - finding projectors

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- Not known if RN holds in general

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  Yes - Krull-Schmidt

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- Contains lot of information

Parabolic Subgroup Schemes

- Suppose $G = SL_3$. Consider

$$\widetilde{P} = \left\{ \begin{pmatrix} * & * & * \\ x & * & * \\ y & z & * \end{pmatrix} \middle| x^{p^3} = 0, y^{p^3} = 0, z^{p^4} = 0 \right\}$$

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  Notation: $\widetilde{P}$ - parabolic subgroup scheme, P - underlying reduced subscheme of $\widetilde{P}$

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- $\widetilde{X} = G/\widetilde{P}$ is a VUF

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- VUFs behave very differently from flag varieties
- Nothing much known for their twisted forms over non-algebraically closed fields

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I show that their motives are isomorphic in $Chow(k, \Lambda)$

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Theorem: $\mathcal{M}(X) \simeq \mathcal{M}(\widetilde{X})$

Rost Nilpotence and Krull-Schmidt for $\widetilde{X}$

I also show the following

Theorem

Rost nilpotence holds for projective pseudo-homogeneous varieties for G

Corollary

Krull-Schmidt holds for projective pseudo-homogeneous varieties for G

Generic Criterion for Isomorphic Motives

To prove the main theorem first I prove the following

Theorem

Let X be projective G-homogeneous variety any field k of any characteristic . Let Z be any geometrically split projective k-variety satisfying RN such that the following holds in $Chow(k, \Lambda)$:

1. $U_X \simeq U_Z$
2. $\mathcal{M}(X_L) \simeq \mathcal{M}(Z_L)$ *where* $L = k(X)$

Then $\mathcal{M}(X) \simeq \mathcal{M}(Z)$.

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Proof.

- By induction on $n = \mathrm{rank}(G)$. Trivially true for $n = 0$. Assume true for all groups with rank less than n.
- Let $\mathrm{rank}(G) = n$. Let $L = k(X)$ and G' the anisotropic kernel of G_L. Then $\mathrm{rank}(G') < \mathrm{rank}(G)$.
- $\mathcal{M}(\widetilde{X}_L) = \coprod_i \mathcal{M}(\widetilde{Z}_i)(a_i)$ and $\mathcal{M}(X_L) = \coprod_i \mathcal{M}(Z_i)(a_i)$.
- By induction hypothesis, $\mathcal{M}(\widetilde{Z}_i) \simeq \mathcal{M}(Z_i)$
- $\mathcal{M}(\widetilde{X}_L) \simeq \mathcal{M}(X_L)$.
- Moreover, $U_X \simeq U_{\widetilde{X}}$.
- Applying generic criterion for isomorphic motives, we are done.

Examples and Applications

Corollary

Let A be a CSA over k of degree n and let B denote the CSA of degree n that is Brauer equivalent to $A^{\otimes p}$. Then in the category $Chow(k, \Lambda)$, the motives of twisted flag varieties $X(d_1, d_2, \cdots, d_m, A)$ and $X(d_1, d_2, \cdots, d_m, B)$ are isomorphic. That is,

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Taking $m = 1$, we get $\mathcal{M}(SB_d(A)) \simeq \mathcal{M}(SB_d(B))$ for twisted Grassmannians.

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Taking $m = 1$, we get $\mathcal{M}(SB_d(A)) \simeq \mathcal{M}(SB_d(B))$ for twisted Grassmannians. In particular, for the case of Severi-Brauer varieties we have $\mathcal{M}(SB(A)) \simeq \mathcal{M}(SB(B))$.

Examples and Applications

Corollary

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Corollary

There exists examples of varieties whose motives are isomorphic when Λ is any finite field but not when $\Lambda = \mathbb{Z}$

Some open questions

- Are the motives of $\widetilde{X}$ and X isomorphic even when G is outer?

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- Are the motives of $\widetilde{X}$ and X isomorphic even when G is outer?
- Does the Generic criterion for isomorphic motives hold in general i.e., when X and Z are arbitrary varieties?

Thank You