



PROCEEDINGS
OF THE
46th CONFERENCE
of the International Group for the
Psychology of Mathematics Education

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PME-46
MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

Volume 3

Editors: Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel and Michal Tabach



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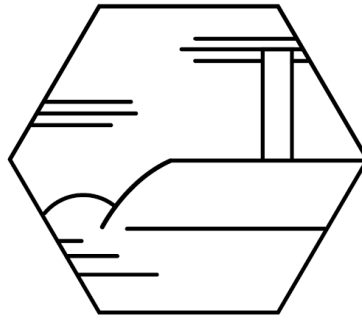
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Volume 3

Research Reports

H – O



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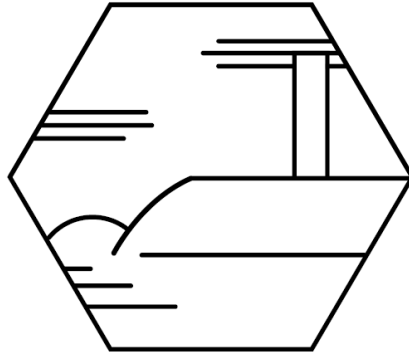
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FACILITATING LEARNERS' APPRECIATION OF THE AESTHETIC QUALITIES OF FORMAL PROOFS: A CASE STUDY ON A PAIR OF JUNIOR HIGH SCHOOL STUDENTS

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The goal of this study is to better understand how students appreciate the particular beauty of formal proofs. This study revealed the following through a single-case study of junior high school students. By contrasting formal proofs with proofs of analytic construction and preformal proofs, the general students can appreciate the aesthetic quality of formal proofs, in which all inferences of intermediate propositions are synthetic. In addition, if the learner has learning experience with formal proofs, the qualitative differences may affect the success or failure of the sensing and the process of it.

INTRODUCTION

The beauty of formal proofs is a particularly fascinating topic for mathematics education research. We are aware that successive new proofs are being developed for propositions that have already been proved. Proofs are expected to have values other than logic, and beauty is one of those values (Aigner & Ziegler, 2018). However, some empirical studies suggest that general learners are unable to appreciate the beauty of mathematical objects (e.g., Dreyfus & Eisenberg, 1986; Tjoe, 2016).

Mathematics education research also includes studies that classify the beauty of proofs (Raman-Sundström & Öhman, 2018), while some try to demonstrate that there is no consensus on criteria for the beauty of proofs (Inglis & Aberdein, 2014, 2020). Educational problems about the beauty of formal proofs, for which no established theory has been attained, need to be pursued through further multifaceted research.

The research question of this study is to clarify, based on an empirical case study, the process by which general learners appreciate the particular beauty of formal proofs.

METHODOLOGY

Theoretical framework

The concept of beauty has branched out with the growth of art. In specific terms, there have emerged perspectives that ascribe beauty only to some particular forms of the perceived object, such as symmetry or the golden ratio, to the perceiver alone, or to both the perceiver and the attributes of the perceived object.

The author views the beauty of mathematical objects as the “aesthetic qualities” based on the theory by Toshio Takeuchi, one of the aestheticians who belongs to the last of the above perspectives (Hanazono, 2021). In Takeuchi (1979), based on the principle

of “unity in variety,” aesthetic qualities are explained by several attributes of the perceived object and the cognitive and sensory activities of the perceiver. In this study, the aesthetic qualities of mathematical objects are explained using the following four viewpoints. The first viewpoint is the “form,” which is the relationship among the components of the mathematical object. This viewpoint can be split into (i) “equivalence relations” (e.g., proportion) and (ii) “quasi-equivalence relations,” which are not full equivalence relations but rather similarity relations (e.g., mappings). While the former reasonably produce unity among the components and therefore gives rise to the aesthetic qualities of perceived objects in a wide range of fields, including mathematics, the latter unifies components according to cultural and personal perception. In mathematics, similarity relations such as mappings are very essential and give rise to value such as simplicity. The second viewpoint is the “essence” of the mathematical object, which is the common properties between the original mathematical object and other related mathematical objects. Sometimes the “form” coincides with the “essence.” The third is the “whole” of the mathematical object, which is the range in which its “essence” is recognized. The fourth is the “vastness” of the mathematical object, which is perceived on the basis of intuition about the “essence” and the “whole” of the mathematical object.

The above four viewpoints and the principle of “unity in variety” are also well suited to the below reference by Poincaré (1908):

What is it that gives us the feeling of elegance ...? It is the harmony of the different parts, their symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts (pp. 30-31).

In Inglis and Aberdein (2014), the results of a survey showing among mathematicians were statistically analyzed and the adjectives related to mathematical proofs were described in four dimensions. They concluded that opposing to the classical view that relates beauty to simplicity, beauty and simplicity are almost entirely unrelated in mathematics. Inglis and Aberdein (2020), on the other hand, focused on the social influence of judgments. That is, they pointed out that in former investigations (e.g., Inglis & Aberdein, 2014) that have argued for the heterogeneity of judgments of beauty by mathematicians, participants were asked to make judgments in conditions isolated from others, and thus there was no social influence on their judgments. The authors then designed a questionnaire survey method that asked mathematicians to select the strength of fitness between the adjectives and a certain proof, pointing out the possibility that the judgment of beauty is not a judgment of “whether it is beautiful or not,” but a social judgment of “whether it is considered beautiful by the community of mathematicians or not.”

The arguments that there is subjectivity and sociality in the criteria by which mathematicians judge the beauty of proofs are very interesting. However, the result that simplicity and the beauty of proofs are unrelated could also be interpreted as that there are simplicities that are related to the beauty of proofs and simplicities that are

not related to it. And the fact that there is a social influence on the criteria of beauty leads to the hope that general learners can also learn the beauty of mathematics.

In Raman-Sundström and Öhman (2018), the relationship between mathematical proofs and beauty is considered through a focus on the sense of fit. Through their classification of a set of approximately 20 proofs, they categorized the sense of fit into three kinds from the viewpoint of the object of fit, and then defined two aspects for each of them. Of the total of six aspects, they identified three as related to the beauty of the proof: level of detail, transparency, and connectedness. The first two are aspects of presentational fit. “Presentational fit refers to the way a proof is communicated and the extent to which the proof is write-up makes the underlying ideas accessible to the reader” (Raman-Sundström & Öhman, 2018, p.187). And “level of detail” is described as “The underlying ideas are presented with the appropriate amount of detail” (ibid., p.187). On the other hand, “transparency” is described as “The structure of the argument is clear” (ibid., p.188). “Connectedness” is an aspect of familial fit. “Familial fit refers to the relationship between a particular proof and a family of proofs” (ibid., p.188). And “connectedness” is described as “The proof idea connects to proof ideas of other theorems” (ibid., p.188).

Raman-Sundström & Öhman’s framework focusing on the sense of fit, which explains beauty from both the object to be fit and the sense of fit, is highly compatible with the theoretical framework of this study. This study, however, will focus on the beauty of proofs not mentioned by them because their “list is not meant to be comprehensive” (ibid., p.185), which will be explained below.

In this study, a single formal proof is considered to be composed of several intermediate propositions. Then, the first viewpoint, “form,” is defined as a relation in which the inferences as components of the intermediate propositions are equal in the sense that they are all synthetic and deductive. The second viewpoint, “essence,” is a property common among several formal proofs in this paper, so it is defined as “the fact that all the antecedents in the intermediate propositions are already admittedly true statements” and “the expression of the components is a mathematical symbol or language,” in addition to “form” as defined above. The third viewpoint, the “whole,” which is the range within which these “essences” are recognized, is all propositions except those for which no formal academic proof is given, or those generally assumed as axioms. The fourth viewpoint, the “vastness,” is defined as the sense of expanse when formal proofs are applied to explanations and empirically acknowledged statements other than geometry. This definition of the “vastness” is based on the learning experiences of Japanese eighth graders, who are the participants in this study.

A single-case study on a pair of junior high school students

This study conducts a single-case study on one pair of Japanese junior high school students. One of the rationales for single-case studies, according to Yin (2014), is when the case is critical to the theory. As described above, the general learners’ sensibility of aesthetic qualities of mathematics has been doubted in mathematics education

research. In this sense, the demonstration of the process of appreciation, even in a single-case, is a critical one. However, a single-case has its constraints to make the process of its appreciation clear. Replication for this latter purpose is a future issue.

The following process was used to carefully choose the case study participants. First, three explanations for the proposition that the opposite angles of any parallelogram are equal were shown to the all 8th-grade students ($n=149$) from the cooperating schools in the study. These explanations included deductive inferences or non-deductive inferences, synthetic construction, or analytical construction. Then, through having students judge the validity of the proofs, 38 students were selected who were able to determine that the explanations were deductive without considering whether they were synthetic or analytic. From these 38 students, two students who were collaborative and able to express their own ideas clearly were selected as participants in the case study with the advice of the cooperating school's teachers. These two are referred to as Fumi and Masa in this paper. They are both excellent students, but in the context of the study contrasting mathematicians and non-mathematicians, they are at the level of the general learners. However, in Japan, formal proofs are introduced in the 8th-grade; two students had already learned them at the time of the case study.

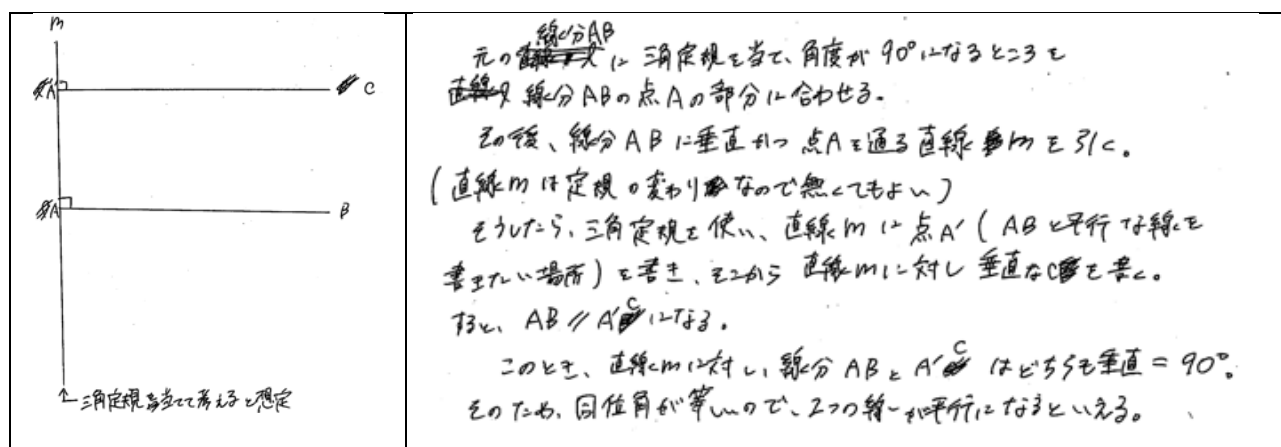
In the main investigation, students were required to provide a proof explaining how to draw a line that is parallel to a given line and passes through a point that is not on the given line. At the time when the construction of the proofs by the students appeared to be complete, the author presented several explanations and asked the students to compare and comment on them. The explanations presented to the students included those in which all inferences were deductive-synthetic, as well as those in which the inferences in some intermediate propositions were deductive-analytic, and preformal proofs (Bloom & Kirsch, 1991). By having them work on the task in pairs, it was hoped that their thoughts would be expressed and their stress on the investigation would be reduced. The entire investigation was recorded with a video camera. The analysis focused on the conversations in these recordings and students' descriptions.

RESULTS

Fumi and Masa, through comparing several explanations presented by the author and answering questions from the author, identified the above "form," intuited the "essence" and partially intuited the "whole," and then felt the "vastness." In other words, they appreciated the aesthetic qualities of formal proofs. The details of this process are as follows.

For the above construction problem, both Fumi and Masa first described an operational, deductive, and synthetic explanation (Figure 1). This operation was about how to construct parallel lines by sliding a triangular ruler, which they learned in elementary school. When the author asked the students after they had completed the description of this explanation, they understood the formal procedure of construction by using a straightedge and compass. Additionally, they were able to distinguish between the proof of this formal construction procedure and the explanation of how to

slide a triangular ruler. In Figure 1, the modifications in the original text are the result of the pursuit of a better description, but the modifications are limited to symbolic expressions.



Place a triangular ruler on the original segment AB and adjust the angle 90° to point A of segment AB.

Then, draw a straight line m that is perpendicular to AB and passes through point A. (Line m is an alternative to a ruler and does not need to be drawn.)

Then, using a triangular ruler, draw a point A' (where we want to draw a line parallel to AB) on line m , and from there draw a C that is perpendicular to line m .

Then, AC is parallel to AB.

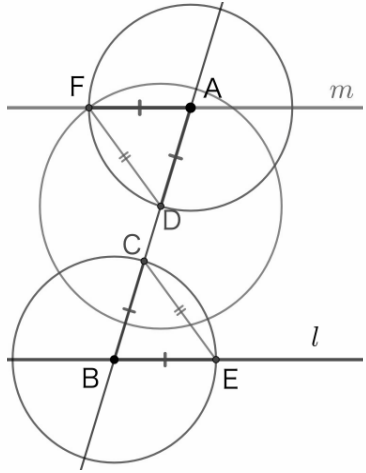
Now, both segments AB and $A'C$ are perpendicular ($=90^\circ$) to line m . Therefore, we can say that the two lines are parallel since the corresponding angles are equal.

Figure 1: Fumi's construction and explanation
(The lower column is a translation by the author.)

The author then presented (a) a deductive-analytic proof, (b) a deductive-synthetic proof, and (c) an operative explanation, and had the students select one that was similar to the proof of the formal procedure of construction they had constructed. These are all based on corresponding angles equality. As a result, Fumi first chose (b), focusing on the basis of the proof, while Masa chose (b) based on the synthesis of the inference. Fumi then agreed with Masa after hearing his explanation.

Next, the author presented alternative basis proofs, three each of (a) and (b), and one proof in which only some intermediate propositions are analytic (Figure 2), and indicated their classification. As a result, they both made the same classification into four categories, but their criteria were different in the following ways. Fumi classified them as (f1) synthetic, (f2) analytic assuming parallelism, (f3) analytic assuming properties other than parallelism, and (f4) other. Masa, on the other hand, classified

them as (m1) synthetic, (m2) analytic, (m3) analytic in the middle, and (m4) other. Convinced that Masa had identified the “form” at this point, the author offered one further proof of (b) on other bases, with the main purpose of inquiring about Fumi’s perception. The author deduced that Fumi also identified the “form” because Fumi had included this proof in (f1).



<Procedures for construction>

- Draw a line passing through point A and intersecting line l . The point of intersection is B.
- Place point C on line AB, and place point D on line AB so that $BC = AD$.
- Place point E on line l so that $BC = BE$, and place point F so that $AF = AD$ and $DF = CE$.
- Draw a straight line AF (let it be line m).

<Proof>

If $\triangle AFD$ and $\triangle BEC$ are congruent, then the corresponding sides are equal.

Now, $AF = BE$, $AD = BC$, $FD = EC$.

Therefore, $\triangle AFD \cong \triangle BEC$.

Since the corresponding angles of congruent figures are equal, $\angle FAD = \angle EBC$.

Since two lines with equal angles of the complex are parallel, m is parallel to l passing through the point A.

Figure 2: A proof in which only some intermediate propositions are analytic

The author asked the students in which category the deductive-synthetic proof for the following proposition about integers would be included: “The sum of a two-digit natural number and the number in which the first and tenth places are switched from the original number is a multiple of eleven.” Then, Fumi focused on the setting written at the beginning of the proof, “Let x be the tens place of the original number and y be the one’s place,” and regarded this statement as the antecedent that assumed the conclusion part in the analytic construction, and included it in (m2). Masa, on the other hand, recognized that this setting was not a conclusion and added this proof to (m1). Fumi then agreed with this explanation by Masa and reincluded this proof in (m1).

Finally, the author asked the students to reflect on their discussion up to this point, and asked for their feelings about “writing proofs.” Some of the responses to this question are depicted in Table 1.

Table 1: The responses from the students

Fumi	Masa
<p>After looking at the different methods of proof, ...I thought it was necessary to follow the order and the method of proof. I thought that you know, clarity is the most essential thing in a proof, and ...I wondered if I should follow the same order as this (referring to the “synthetic” category), and ...when I saw that proof (referring to the proof of the properties of integers), ...I thought that ...just like the proof of geometry, clarity is important ...</p>	<p>I was ...a little surprised to find such a construction (referring to the analytic category) ...well, it is interesting to make discoveries. I also felt that the most important thing in the proof is to be correct, which is a prerequisite, and that it is also crucial to make it easy to comprehend.</p>

DISCUSSION

Fumi, through the author’s intervention and influence from Masa, appreciated the aesthetic qualities of formal proofs. What was characteristic of Fumi’s process of appreciating the aesthetic qualities of formal proofs was that, unlike Masa, who immediately differentiated deductive proofs based on differences in the construction of inferences, Fumi was not able to make this differentiation from the beginning. In fact, she based the categorization of many explanations and proofs on the claims that the proofs were based on. And Masa’s explanations helped to modify this recognition.

Furthermore, Fumi’s feeling of “vastness” based on the realization that the “proof of the property of integers” could be considered a formal proof differed significantly from Masa’s. Masa quickly grasped how inferences were constructed during the investigation and saw that the synthetic construction was the same as the proofs in the mathematics textbooks. Therefore, it is assumed that Masa did not feel the “vastness” as a new expanse, at least in a way that the author could recognize.

This case study was performed with Japanese 8th-grade students who had learning experiences with formal proofs. The empirical findings on the differences between Fumi and Masa imply that the success or failure of their appreciation of the aesthetic qualities and the process of the appreciation are influenced by qualitative differences in learning experiences with mathematical objects.

On the other hand, both Fumi and Masa had acquired a new recognition that the synthetic construction was clearer. At the start of the investigation, Fumi did not exhibit this recognition and instead concentrated primarily on the clarity of the symbolic expressions. Furthermore, for Masa, given the mention that he had never seen an analytical construction before, it is thought that learning about analytical constructions as a reference for comparison led him to recognize the value of a synthetic construction.

The clarity resulting from the unity in the whole is the very aesthetic qualities to which Poincaré refers. Although it does not fulfill the framework of this study, the change in Masa's recognition is very important. In Japan, it is up to the teacher to decide whether to include analytic constructions in the lesson plan for formal proofs. Also, it seems that even when analytic constructions are handled, they are often treated as valid constructions for planning proofs. The findings of this case study imply that analytic constructions of proofs might emphasize the value of synthetically constructed formal proofs, which are not a priori for students.

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CO-LEARNING THE DIFFERENCE MEANING FOR MORE-THAN SITUATIONS WITH/FROM A STRUGGLING STUDENT

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Using a constructivist lens, we re-examine a well-studied phenomenon – three main meanings for more-than situations: more = larger number (L=more), more = sum (S=more), and more = difference (D=More). We address the problem of how teachers can co-learn with/from struggling students to design tasks that foster reorganisation (change) of the common, L=more meaning into the accepted, yet conceptually challenging, D=more meaning. Our framework centres on scheme theory and on the reflection on activity-effect relationship model of cognitive change. We analyse data from teaching episodes indicating Juana’s (pseudonym) advance, from her L=more meaning, through 1-to-1 matching actions to equalize two sets, to the proper D=more meaning, and discuss the importance of that teacher-learner, co-construction process.

INTRODUCTION

We designed this teaching experiment study to explain cognitive roots of a well-studied phenomenon. Consider a child who makes a tower of 7 red cubes and another of 5 blue cubes, and asked: How many more cubes are in that (red) tower? Children were found to respond with the larger number of the two sets (here, 7), the sum of both numbers (here, 12), or the difference between them (here, 2). Our emphasis on the child as a sense making person led us to focus on their meaning for “more” in the situation, respectively: more means the larger number (denoted *L=more*), or the sum of both numbers (*S=more*), or the difference between the two numbers (*D=more*).

Prior lines of research seemed to focus on children’s performance (correct/incorrect answers) when solving different types of more-than situations. One line focused on the language/wording of those tasks (Adetula, 1990; Erik, Lieven, & Luc, 1985; Lean, Clements, & Del Campo, 1990; Nesher, Greeno, & Riley, 1982). Another line focused on arithmetical methods children used to solve them (Carpenter, Hiebert, & Moser, 1981; Fuson, Carroll, & Landis, 1996; Hiebert, Carpenter, & Moser, 1982). In both lines, a key distinction was made between task-wording that indicates actions, such as 1-to-1 matching, and tasks that do not, aka, static (Fuson et al., 1996; Gibb, 1956; Hudson, 1983; Sophian, 1987). However, except for one, early study (Brush, 1976), those studies seem to neither explain students’ responses conceptually nor provide instructional designs to foster a shift from inadequate meanings (*L=more* and *S=More*) to a mathematically accepted one (*D=More*). Next, we provide a constructivist framework of children’s meanings for more-than situation.

A CONSTRUCTIVIST FRAMEWORK FOR TRANSITIONS IN HOW MANY MORE REASONING

This study utilizes Simon et al.'s (2004) *Reflection on Activity-Effect Relationships (Ref*AER)*, which draws on von Glaserfeld's (1995) three-part definition of a scheme (goal > activity > result/effect) and Piaget's (1985) reflective abstraction. Ref*AER refers to a learning situation in which the learner assimilates a given task, sets a goal for the task, brings forth an activity sequence to achieve that goal, and reflects on the effect of the activity sequence to compare the actual result to their anticipated result. Based on such reflection, the learner may adjust their activity sequence so they can next get closer to their goal. For example, a learner asked to find the difference in two towers – a tower of 7 red cubes and a tower of 5 blue cubes – may assimilate the task as how many cubes are in the larger tower (L=more scheme). They will thus set a goal to determine how many cubes are in each tower (possibly counting cubes), compare the two numbers, and state (incorrectly): “There are 7 more cubes in the red tower than the blue tower.” In response, a researcher may orient the learner to a matching action: “How many cubes in the red tower do not have a match in the blue tower?” - possibly causing a perturbation for the learner. The learner will adjust their activity sequence to count the unmatched red cubes and answer with 2 more red cubes. After reflecting on the two activities and their results, the learner may construct a new way of reasoning in more-than situations – anticipating a result of the number of unmatched items (i.e., the difference) rather than the number of items in the larger set.

Simon et al. identified two types of Ref*AER. Type I involves comparisons between the learner's goal and the effect of their activity in a given situation (e.g., anticipated number, “7,” differs from actual matching number, “2”). Type II involves comparisons across activity-effect ‘dyads’ of multiple tasks (e.g., in three different trials, the larger number and the matching number were not the same). This type of reflection leads to a reorganization of the scheme's anticipation of activity-effect relationships from a given situation. For example, a learner solves multiple more-than tasks, consistently being prompted about how many unmatched items are in the larger/smaller set. The learner reflects on the adjustment of responses to be about the count of unmatched items and abstracts it as the effect relevant for how many more/fewer tasks (that is, constructs a D=more scheme).

This study also draws on Simon's (1995) *Hypothetical Learning Trajectory (HLT)* model to help us articulate how Juana may have reorganized her available more-than scheme of L = more into the more advanced scheme of D = more along with our learning to adjust tasks. Simon's HLT model allowed us to link our goals for Juana to hypothesized conceptual changes in her scheme and to tasks for fostering such changes.

METHODOLOGY: CONSTRUCTIVIST TEACHING EXPERIMENTS

This study was part of a larger research and professional development project funded by a small US school district. The project focused on conceptual mathematics for K-12 students and pedagogical and content knowledge for K-12 teachers.

This study involved a teaching experiment (Cobb & Steffe, 1983) with six 7th graders identified as struggling in mathematics (parent consent and student assent were

provided for each student). A teaching experiment is a constructivist, qualitative methodology designed to help researchers create models of student reasoning (schemes). When paired with HLT's, researchers can study transition in students' reasoning coupled with task design to promote it. Each, 40-minute teaching episode (session) began with a task involving a real-world context to determine Juana's current conceptual stage (Tzur, 2007). Each episode was followed by the research team's debrief to analyse and make inferences about: a) Juana's reasoning during the episode, b) plausible transitions in her more-than schemes, c) how tasks have fostered (or not) those transitions, d) what reasoning she may use in the next episode (those gained during the episode or prior reasoning), and e) what adaptations to tasks may be needed to foster further transitions in the following episode(s).

Participant and Context

We chose to focus on Juana (pseudonym, grade 7) as she used the L=more scheme when we began working with her on 'static' more-than tasks and eventually transitioned to the proper, D=more reasoning. Thus, Juana's case provides evidence for a possible HLT for more-than tasks and can serve as a case for understanding how other students may transition from the L=more to the D=more scheme. As in any case study, it is the phenomenon (not Juana) – here, conceptual transition that underlies adequate solutions to 'static' more-than tasks – that is the focus (Creswell, 2013). Therefore, the HLT we outline may apply to other students who initially assimilate more-than tasks into an L=more reasoning.

Data Collection

The teaching experiment consisted of weekly, 40-minute episodes with Juana conducted by our three-person research team. The third author was present, in person, at every episode, the first author worked via Zoom only, and the second author via Zoom or in person. We video-recorded each episode, using a camcorder and the Zoom recording software. Videos were stored in a secure system for later analysis. Intensive field notes were also kept by two of the researchers to inform later analysis.

Data Analysis

We used both ongoing and retrospective analysis of the data. The ongoing analysis occurred during each debrief as explained above. The retrospective analysis consisted of researchers examining all data we have collected, individually and then in small teams, to identify critical events (Powell et al., 2003), transcribing those events, and developing an HLT ("storyline") that couples Juana's case of transition from L=more to D=more scheme and the research team's adaptation of tasks to fit with her reasoning and conceptual change.

RESULTS

In this section, we examine data about how the research team and Juana co-constructed a transition from an L=more to a D=more meaning for more-than situation over a few

episodes. Our twofold analysis includes inferences about Juana's conceptual transitions and how her reasoning guided the research team's task design.

Excerpt 1 begins after Juana had consistently answered static, more-than tasks by stating the number of items in the larger set (L=more). To foster her construction of a D=more scheme, our team first focused on her coordination between object-to-object matching actions to more-than responses by orienting Juana's attention onto unpaired items in the larger set as the difference between the two sets.

Excerpt 1 illustrates one of these early attempts. In it, the researchers presented Juana with a task to infer into her independent way of reasoning in a static, more-than situation. Thus, they engaged her in solving a realistic word problem without using any objects (e.g., no cubes) or prompts for how to go about this.

Excerpt 1

Researcher: *Pretend* you have 7 black cubes and [Mr. R has] 8 blue cubes ... how many more [cubes] do you need to make [yours] equal to Mr. R's?

Juana: (Draws 7 squares in a vertical line. Draws 8 more cubes in a vertical line while, seemingly intentional, starting it one square above the first line).

Researcher: So, how many more black cubes do you need to have the same amount as Mr. R?

Juana: I need one (1) more.

Researcher: How do you know?

Juana: He has 8 (points to the line of 8 squares). I have 7 (points to the other line). I need 1 more (points to the space next to the unmatched square).

Researcher: [So,] How many more blue cubes are there [than black cubes]?

Juana: Eight (8).

Excerpt 1 indicates that Juana assimilated the task to determine how many cubes are unpaired and how many are needed to make the two sets equal (one). This claim is supported by both her drawing of the second line one cube higher than the first line and her pointing to the missing cube in her (black) line. Although independent with matching actions, her answer ("Eight") to the more-than static task indicated it was assimilated into an L=more scheme.

Our first attempt at fostering Juana's transition to a D=more scheme was to adapt the task so she would be engaged in equalizing two given sets by decreasing the number of items in the larger set. In our context of work with towers and cubes, that meant "chopping off" cubes from the larger tower. Our rationale was that such a goal-directed activity may further orient her attention onto the resulting difference (of cubes). However, whereas Juana seemed to understand equalizing by chopping off, she seemed to prefer adding cubes to the smaller set, indicated by her consistent choices, over a few more tasks during that and the following episode, to add cubes to the shorter tower

rather than chop some off from the taller tower. Crucially, she consistently told the “story of the equalizing process” to explain how many cubes she needs to add to make them equal (e.g., “I have 7, you have 8, so I need 1 more to make them equal”). Yet, when then asked how many more cubes were in the larger set, she consistently used L=more reasoning (e.g., “The taller tower has 8 cubes more [than the shorter]”).

Juana’s persistence in using a “story” of the equalizing process while using an L=more meaning led us to adapt the task once again. The first author thus designed a new game to foster *her* perturbation due to realization of a contradiction between two effects of her goal-directed activities. In this game, players start with towers of equal height that she makes (e.g., 5 cubes in each tower). One person plays a role of *Raiser* and adds cubes to their tower (e.g., 2 cubes). The other person plays a role of *Equalizer*, who asks three questions: a) How many cubes do you have now (e.g., 7)? b) How many cubes do I have now (e.g., 5)? c) How many more cubes do you have than me (e.g., responding “7” or “2”)? Critically, the rule of the game is that the Raiser’s answer to the last question directs the Equalizer how many cubes to add to their tower for equalizing them again. If the Raiser is correct (checked with cubes), they keep the cubes; if incorrect, cubes go to the Equalizer. To make the ‘perturbing cost’ of an answer realistic, players eventually trade cubes they accumulated with M&M candies. Excerpt 2 illustrates this game, with Juana (Raiser) adding 1 cube to her 3-cube tower.

Excerpt 2

Researcher (Equalizer): How many[cubes] do you have?

Juana (Raiser): Four.

Researcher: And I have three?

Juana: (Shakes her head, “Yes”).

Researcher: I’m going to ask you a question and your answer is going to tell me how many more M&M’s I need to add to my group of M&M’s to have the same number as you. If your answer is correct, then you’ll get that number of candies to keep. If the answer is wrong, I get to keep the M&M’s. So, how many more M&M’s do you have than me?

Juana: I have four. You have three. You need one cube to be equal.

Researcher: So how many more M&M’s do you have than me?

Juana: I have four.

Researcher: So, Mrs. B, you’re going to add four M&M’s to my pile.

Juana: (Gets very animated) No, no, no, no. You need one more.

Researcher: Why did you change your answer?

Juana: Because you need [just] one more to be equal.

Excerpt 2 data indicate that, for the first time we could witness, the game promoted an intended perturbation in Juana, leading to her “Aha!” moment. To answer the first, static task (“So, how many more M&M’s do you have than me?”) she used her available “equalizing storyline.” As the researcher insisted on answering the static task by re-asking it, Juana’s response indicated her L=more scheme. Then, as soon as the researcher asked Mrs. B to add all four cubes (candies), Juana indicated making the intended coordination between her two responses (“one to make equal,” and “four more”), as she quickly, with clear excitement, changed her L=more answer to a D=more, explaining that “you need [just] one more to be equal.” This indicated to us that Juana reflected on the result of her initial activity sequence (four, L=more) and resolved *her* perturbation (would not equalize) by coordinating the more-than task with a newly anticipated result – the number needed to equalize (i.e., difference).

During this episode, Juana and the researcher played a few more rounds (switching Equalizer and Raiser roles after each round). Juana indicated she was working between her prior, independent (L=more) meaning and her newly constructed (D=more) one, with the latter supported through the researchers’ prompts. That is, each round of the game seemed to foster yet another perturbation in Juana, yielding both types of reflection leading to her construction of the D=more meaning (anticipated and actual effects were not the same; across more-than instances I chose to use the difference).

Over the following two episodes, Juana seemed to gradually transition from needing prompts to change her response, from an L=more to a D=more meaning, until she eventually used the latter meaning spontaneously and independently. Excerpt 3 shows her work at the end of start of the third episode since the first in which we introduced the game. Importantly, Juana no longer used the equalizing storyline to solve more-than tasks. Rather, the data show she could identify and respond with the difference, while using figural items (not tangible cubes) in a count-up-to strategy. Excerpt 3 involved a task in which Juana determined how many more apples one researcher had (eight apples) than the other researcher (twelve apples).

Excerpt 3

Researcher: How many more apples does [Mr. R] have?

Juana: (Put up eight fingers. Counted-up to twelve while putting up a finger for each count) Four.

Juana used a sequential count-up-to strategy to count from eight to twelve and found a difference of four apples. This excerpt provides evidence of her transition from L=more reasoning to D=more reasoning when provided more-than tasks to solve in context and without tangible objects.

DISCUSSION

In this study, we examined both a conceptual transition, from L=more to D=more scheme when solving static, more-than tasks and researchers’ co-construction of an understanding of this transition while adapting tasks that afforded Juana with

opportunities to construct the D=more scheme. The case of her conceptual advance provides evidence into how such a transition may occur, and be fostered, in other students. This co-construction process seemed rooted in *our* perturbation that we, the researchers, experienced. Our unsuccessful attempts led us to better explain Juana's schemes, and thus what she might bring forth as a basis for creating her perturbation. We acknowledge many instances in which we believed that a specific task could help Juana transition to the intended, D=more reasoning, only to find she brought her own (L=more, storyline) reasoning to the task. These perturbing instances (for us) led to articulating what she does have available that could bring forth a constructive perturbation for her. Specifically, our perturbation yielded a new, playful task in which the 'cost' of incorrect answers led Juana to coordinate more-than situations with the difference between two sets that *she* would determine through matching actions.

This study thus contributes a plausible HLT for advancing from L=more to D=more reasoning. This HLT focuses on bringing forth students' goal-directed equalizing (matching) actions to foster the intended transition. As our study shows, researchers' analysis of a student's available schemes opens the door for adapting task design to successfully accomplish that transition in students.

For practice, our study can provide teachers with an understanding of conceptual causes leading students to struggle, and perhaps incorrectly construct, a D=more scheme. We believe the study shows a way forward in which teachers utilize both teacher and student perturbations to develop tasks and engage students in solving static more-than tasks. Teacher perturbations are a critical learning experience which can guide them in adjusting student HLT's. Teachers can use static more-than tasks while explicitly capitalizing on students' available (equalizing) schemes to foster transition from L=more to D=more schemes.

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THE ROLES PRESERVICE TEACHERS ADOPT IN MODELLING-RELATED PROBLEM POSING

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Posing problems that are based on given real-world situations is important for teaching mathematical modelling. However, little is known about the posing process and the corresponding roles that teachers play. To help fill this gap, the current study examined the roles that preservice teachers adopt when they pose problems based on given real-world situations. We analyzed data from seven preservice teachers who posed problems based on a given real-world situation and identified three different roles preservice teachers tend to adopt when they pose a problem: protagonist, teacher, and problem solver. Further, we describe the domains that are addressed in these roles and affect teachers' decision-making when posing problems. Implications for how to teach (preservice) teachers to pose real-world problems are discussed.

INTRODUCTION

In order to teach mathematical modelling, teachers need to be able to pose adequate real-world problems (Blum, 2015; Borromeo Ferri, 2018; Greefrath et al., 2022). Modelling-related problem posing begins with the prompt to pose a mathematical problem based on a given real-world situation and results in a problem that can be solved afterwards (Hartmann et al., 2022). Despite the ongoing emphasis on approaches for teaching mathematical modelling, little is known about the process of modelling-related problem posing and especially the roles that have to be anticipated within this process (Geiger et al., 2021). To support (preservice) teachers in posing problems as they teach mathematical modelling, the aim of the present study is to identify and conceptualize the roles that preservice teachers adopt when posing problems based on given real-world situations, including the domains that are related to the decision-making processes that occur while posing a problem.

THEORETICAL BACKGROUND

Mathematical modelling (i.e., solving real-world problems with the help of mathematics) is one of the key competencies in mathematics education and is included in curricula all over the world (National Council of Teachers of Mathematics [NCTM], 2000; Niss & Blum, 2020). In order to teach mathematical modelling, teachers must have modelling-specific content knowledge, including knowledge about interventions, problems, and perspectives (Greefrath et al., 2022). Modelling-specific content knowledge includes knowledge about how to pose adequate problems (Borromeo Ferri, 2018).

The term problem posing subsumes many different processes. In the context of mathematics education, it refers to problem development that is triggered by a stimulus and results in a mathematical problem that can be solved afterwards (Baumanns & Rott, 2022; Silver, 1994). Different situations and prompts can serve as stimuli (Cai et al., 2022). In order to pose problems based on given real-world situations, the problem poser first has to understand and explore the given situation. Then a problem has to be generated. The self-generated problem has to be evaluated with regard to individual criteria (e.g., solvability, adequacy for a specific learning group or adequacy in the given situation), and then possible solution steps can be planned (Hartmann et al., 2022). Hence, in order to pose modelling-related problems, demanding translation processes between the extramathematical and mathematical domains are needed, and many decisions have to be made.

To support (preservice) teachers in posing real-world problems, Galbraith (2006) described principles for the posing process. These principles serve as structural components that can scaffold the posing process and provide guidance on the important characteristics of the posed problem. One of the principles is that the posed problem should be connected to students' lives. Further, it should be possible to translate the posed problem into a mathematical problem. The solution to the posed problem should be feasible for the students and should require the application of modelling-specific activities (e.g., simplifying and structuring, interpreting, validating). Lastly, from a didactic perspective, it should be possible to divide the problem into subproblems in order to scaffold the solution process. The principles reveal that, when posing real-world problems, the extramathematical domain, the mathematical domain, and the didactic domain have to be kept in mind. To take these domains into consideration, the problem poser might adopt different roles that go along with focusing on important aspects of these domains. However, systematic research on the roles that (preservice) teachers adopt when posing problems based on given real-world situations has thus far been missing.

RESEARCH QUESTIONS

To help fill this gap, the overarching goal of this study was to identify and conceptualize the roles that preservice teachers adopt in modelling-related problem posing based on empirical data. To do so, we asked the following research questions:

- a) What roles do preservice teachers adopt when they are instructed to pose mathematical problems that are based on given real-world situations, and how can these roles be described?
- b) Which of the abovementioned domains are related to the decision-making processes applied in each role, and which aspects of the domains are taken into account?

METHOD

Sample

Data were collected from seven preservice teachers from a large German university (four women, three men) between 20 and 26 years of age. All of them participated in the program for future secondary school teachers: five of them for secondary and high schools (Grades 5-12/13) and two of them for secondary schools (Grades 5-10). Six of them reported having previous experience posing mathematical problems for students.

Procedure and Instruments

The analysis was based on data from a prior study for identifying the cognitive processes involved in modelling-related problem posing (Hartmann et al., 2022). For this purpose, the preservice teachers were instructed to pose a problem that was based on a given real-world situation while thinking aloud and to solve them subsequently. An example of a real-world situation that was presented is given in Figure 1.

Cable Car

For more than 90 years, the *Nebelhorn* cable car has taken numerous guests up into the heights. Now it can go into well-earned retirement. Beginning in the summer of 2021, a new cable car will transport enthusiastic outdoor fans up the Nebelhorn mountain. The aim of the project is to avoid long waiting times, provide seated transportation with an optimal view from every seat, and increase the cable car's carrying capacity.

Technical data of the old cable car:

Model:	Großkabinen-Pendelbahn
Weight empty cabin:	1600 kg
Weight full cabin:	3900 kg
Height valley station:	1933 m
Height top station:	2214.2 m
Horizontal difference:	905.77 m
Speed:	8 m/s
Carrying capacity:	500 people/hr
Power unit:	120 PS



Figure 1: The real-world situation of the cable car

The basis of the current analysis were the videos and the corresponding transcripts of the preservice teachers' posing processes as they posed problems that were based on the given real-world situation of the cable car (see Figure 1).

Data Analysis

In order to uncover the roles that the preservice teachers adopted, we used Mayring's (2015) content analysis. In a first step, we paraphrased the transcripts with regard to content-bearing semantic elements (sequences). In a second step, we developed the

coding scheme inductively by using a collaborative process. This involved going through several rounds. In the first round, we reviewed the posing processes, began with an open coding, and created memos. Through an analytic process, we then developed codes. By using subsumption, we extended and refined the codes. For each of the sequences, we decided whether they fit into one of the existing codes or whether a new code had to be developed. In the next round, we gathered the sequences that were coded with the same code and discussed the descriptions of the codes collaboratively. Then we coded the rest of the data. This involved making several adjustments to the descriptions. Further, we found that a lot of sequences could not be clearly assigned to just one code. Hence, we decided that a sequence could be assigned to more than one code simultaneously. On the basis of the analytic process, the roles of the preservice teachers evolved from the data. The analysis resulted in a coding scheme that was used to analyze the data (see the Research Findings section). In a final step, we summarized the posing process for each preservice teacher with regard to the roles they adopted.

RESEARCH FINDINGS

On the basis of the preservice teachers' posing processes, we were able to develop a coding scheme that is presented in Table 1. The coding scheme includes the identified roles, the descriptions of the roles, and the domains related to decision-making.

Role	Description	Domain
Protagonist	The problem poser has adopted the role of a protagonist who wants to pose a relevant problem for themselves or the given situation.	The extramathematical domain is in the foreground, and decisions are made on the basis of the description of the given real-world situation.
Problem Solver	The problem poser has adopted the role of a problem solver who wants to pose a mathematical problem with an interesting mathematical solution.	The mathematical domain is in the foreground, and decisions are made on the basis of the mathematics (i.e., mathematical operations and structures) that can be used to solve the problem.
Teacher	The problem poser has adopted the role of a teacher who wants to pose a suitable problem for their students.	The didactic domain is in the foreground, and decisions are made on the basis of the potential learning group and their knowledge.

Table 1: Roles involved in modelling-related problem posing

In the posing processes, we were able to identify three different roles (i.e., *protagonist*, *teacher*, and *problem solver*) the preservice teachers adopted when posing problems based on given real-world situations. As evidence that the preservice teachers actually

integrated the roles into their modelling-related problem posing, we present Max's posing process, including the roles he adopted and the domains that he focused on for decision-making in the roles.

Max begins by reading about the given situation in the role of a problem solver by asking about what the mathematical context is.

That's all well and good, (laughs) but what is the actual mathematical context now?

Then he identifies the information the question should refer to in the role of a problem solver and of a protagonist.

Okay and now, my question should probably also refer to the new cable car, which will now be built beginning in summer 2021.

From the information he wants to focus on, he goes on to determine the mathematical content in the role of a problem solver.

And there again I have to ask myself what mathematical concept I want to incorporate, so to speak.

He goes back to the real-world situation by making sense of the given situation in the role of a protagonist but also in the role of a problem solver.

So, it's also about changing something about the old data now, so that, um, the waiting times can also be reduced and, also the optimal view is made possible, for example, and the conveying capacity is increased. Okay.

After making sense of the given situation, he continues in the roles of a protagonist and a problem solver by choosing information that he wants to work with mathematically.

... because there is now a lot of information about the old cable car, I could also theoretically work with it and formulate a question that now refers to the old one. That makes more sense to me right now.

Then he switches to the role of a teacher by looking for mathematics that can be practiced with the self-generated problem.

You could, for example, again theoretically use the Pythagorean theorem, and in principle, check or practice it in this task. Apply. The problem could be, for example, that the length of the actual route of this, um, railway, i.e., that the railway has to cover, is not specified here at all.

The upcoming problem is then evaluated in the role of a teacher by anticipating what mathematical knowledge students need to have in order to solve the problem and think about an appropriate formulation for the self-generated problem.

Above all, it is also used to test... It is also used to... Or the pupils are also required to convert units because this also has to be switched here in any case between seconds and hours and so on. Now I'm wondering how my question should be formulated.

After posing a problem, Max checks the solution to his self-generated problem in the role of a problem solver.

The beauty of the task, I think, is that you have to work through the different steps bit by bit in order to solve it [...]. So, we can't determine from the transport capacity the number of people per – that fit into a gondola. Exactly. We have to determine it from different things together. So, we have to go through different steps to solve those.

Figure 2 presents a schematic overview of Max's posing process with regard to the roles he adopted over time.

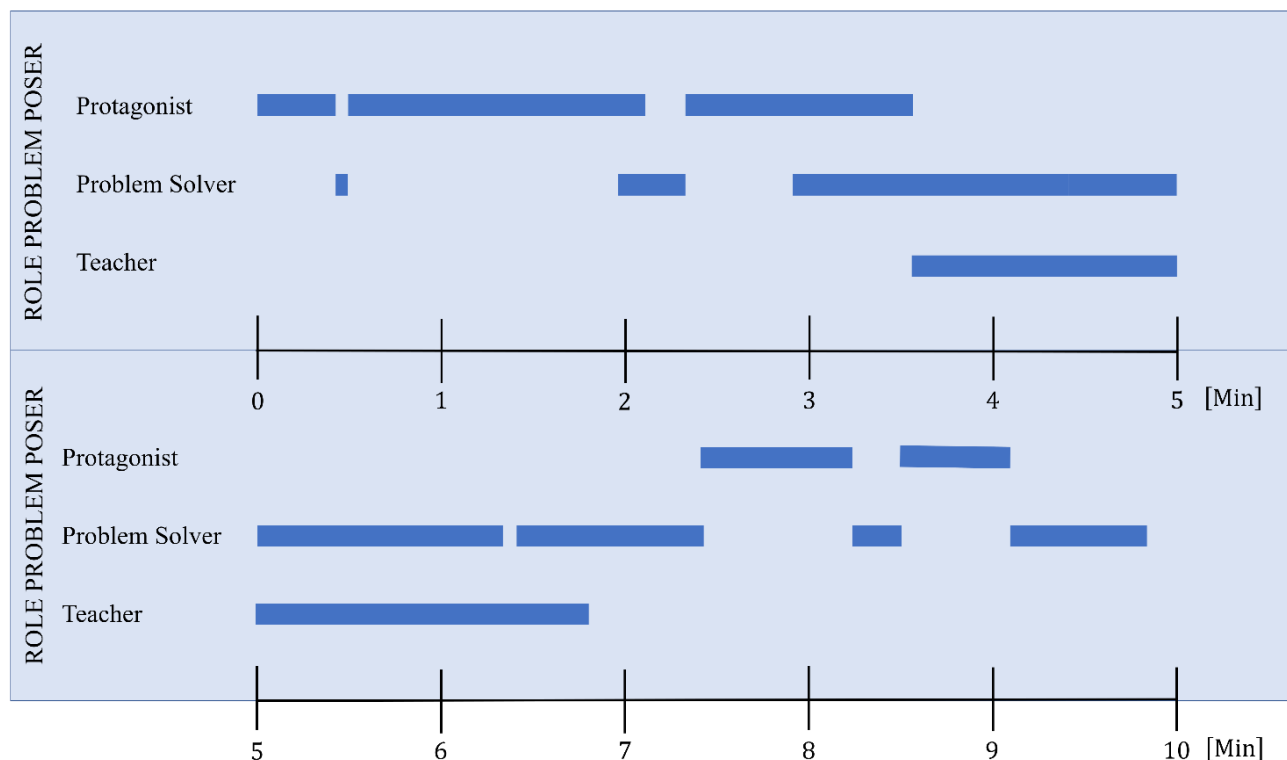


Figure 2: Overview of Max's posing process

Max's posing process (see Figure 2) reveals that the roles do not necessarily occur separate from each other. Rather, a duality of the roles can be observed and especially the role of the problem solver goes along with the other roles. Different profiles for the roles could be observed in the posing processes of the other preservice teachers.

DISCUSSION

In the modelling-related problem posing processes, three different roles could be identified. The preservice teachers adopted the roles of a *protagonist*, a *problem solver*, and a *teacher*. Adopting these roles seems to be relevant for making appropriate decisions when posing problems that are based on given real-world situations. Decisions are made by focusing on aspects of the extramathematical domain (in the role of a protagonist), the mathematical domain (in the role of a problem solver), and the didactic domain (in the role of a teacher). Galbraith's (2006) principles reveal that it is important to take these domains into account when posing real-world problems. Therefore, it is possible that instructing (preservice) teachers to put themselves in a specific role might encourage them to consider important aspects of the

extramathematical, mathematical, and didactic domains and might therefore enhance the posing process. Future research has to reveal whether instructing (preservice) teachers to put themselves in a specific role has a positive effect on the posing process.

Further, the results show that the roles do not necessarily occur separately from each other. Rather, a duality of the roles could be observed in the data. Especially the role of the problem solver tended to accompany the roles of a protagonist and a teacher. This finding indicates that anticipating mathematical operations and structures is also important for making appropriate decisions in the roles of a protagonist and a teacher. Future research has to reveal which role the anticipation of mathematics that is important for the solution of the self-generated problem plays in modelling-related problem posing depending on the roles (preservice) teachers are adopting.

LIMITATIONS

Our study has some limitations that should be kept in mind when interpreting the results. The analysis was based on data from a small sample of preservice teachers who were prompted to pose mathematical problems based on given real-world situations. The roles identified in the problem posing processes seem to be specific to modelling-related problem posing with preservice teachers. Research has yet to determine which of the roles can be identified for different problem posing stimuli (e.g., problem posing based on given intramathematical situations) or other samples (e.g., school students). The transferability of the results needs to be validated in future studies.

CONCLUSION

Overall, our study contributes to research on modelling from a problem posing perspective. The results of our study have theoretical implications for research on modelling and problem posing by underlining the importance of roles and domains for decision making while posing problems based on the real-world situation. In order to pose problems, (preservice) teachers adopt different roles. It might be important to teach (preservice) teachers which roles may occur while posing problems that are based on given real-world situations. Further, stimulating them to reflect on the roles they are adopting can be fruitful for the development of high-quality problems. Therefore, the results of the study have to be kept in mind when teaching (preservice) teachers how to pose real-world problems.

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THE COMPLEXITY OF GRAMMAR IN STUDENTS' TALK: VARIATIONS IN EXPRESSING FUNCTIONAL RELATIONSHIPS BETWEEN TWO QUANTITIES

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Acquiring conceptual understanding of functions challenges students concerning the mathematical content as well as the linguistic compaction. While, among others, the students need to construct meaning of the involved quantities and the dependency in-between, they also face grammatical changes from everyday to school academic language. Based on systemic functional linguistics, we derive an analytic model in this paper to investigate the role of grammar in conceptual learning. The qualitative analysis of 8 students from a design research study helps us draw initial conclusions about the role of verbs expressing quantities and conjunctions for addressing the relation in-between and first recommendations for teaching.

INTRODUCTION

The complex learning content of functional relationships contains topic-specific mathematical and linguistical challenges (e.g., Planas, 2021; Prediger & Zindel, 2017). This paper reports from a design research study aiming at fostering students' functional understanding. In the example task, students shall state and explain the involved quantities as well as the direction of dependency in-between (see Figure 1).

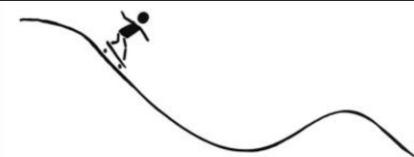
<p>The longboarder is riding down a hill. The 2 quantities are: _____ & _____</p> <ul style="list-style-type: none">• Describe the situation and name the quantities.• Explain: Which quantity is dependent on which? Why?	
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Figure 1: The longboarding task

According to Prediger & Zindel (2017), the functional relationship can be expressed in different variants of linguistic compaction. This paper contributes further theoretical and empirical specifications by analyzing two dimensions of grammatical compaction with the help of systematic functional linguistics: regarding the involved quantities and the relation between them. Therefore, we theoretically introduce (1) the mathematical background and the two main aspects of understanding, and (2) outline the corresponding role of grammatical compaction as well as the related challenges. The empirical part grants qualitative insights into the pathway of two students and, from there, enables us to reconstruct a possible learning trajectory and recommendations for teaching. In this way, the paper contributes to the following research interest:

How can the grammatical compaction of functional relationships be described and structured for learning processes?

THEORETICAL BACKGROUND

Specifying the core of functional understanding

Functional understanding has been widely researched and can be conceptualized in many different ways, e.g., regarding functional perspectives or using different representations (as summarized in Niss, 2014). One of them is the conceptualization of the core of functional relationships by Prediger and Zindel (2017).

This core consists of three conceptual facets of understanding: the *||Involved Quantities||* (e.g., time and speed, visible also when, e.g., assigning 40 km/h to 4 mins); the *||Varying Quantities||* (e.g., the idea that time and/or speed change); and the *||Direction of Dependency||* (e.g., the idea that the speed changes over time and therefore is dependent on time). These facets build on each other as students cannot think about the *||Direction of Dependency||* if they have not yet identified the *||Involved Quantities||*. This conceptualization contains the most important aspects of the two rather rough functional perspectives (correspondence and covariation) in the four most common representations (symbolic, graphic, tabular, and verbal/situational).

Theoretically specifying the language demands

The mathematically complex learning content of functional relationships also contains grammatical demands, which at the same time can serve as a resource for conceptual understanding (Planas, 2021). Prediger and Zindel (2017) already pointed out the high grammatical complexity within sentences (among lexical and discursive demands) students face while working on understanding the core of functional relationships as well as the need to explicitly address the single facets of the core of functional relationships. Both aspects indicate changes from everyday language to school academic or technical language.

In systematic functional linguistics, Halliday (2006) points out ways of expressing meanings in more or less explicit ways. The close connection between language forms and (potential) meanings is grasped in the phenomenon of Lexicogrammar. It refers to so-called word classes: groups of words with the same grammatical function, e.g., verbs typically express processes. In the transition from everyday to school academic and technical language, the mathematical meaning remains almost the same but the use of word classes changes. For instance, the process of ‘assigning something’ (verb) can be nominalized into the more compact object of ‘assignment’ (noun).

Understanding more compacted expressions in unusual word classes for similar meanings (e.g., nouns instead of verbs), is key to higher mathematical topics but a major challenge for many students (e.g., Halliday, 1993; Martin, 1999). For conceptual learning, teachers need to first take less compacted word classes from students’ everyday language into account as these are easier accessible for meaning-making (Butt, 1989). Halliday (2004) describes the most important dimensions of the change in word classes, from which the second one occurs quite late in students’ school careers which makes it especially challenging for them to access meaning (Halliday, 2004):

1. Expressing processes as abstract objects: in English and German, mostly verbs combined with adverbs (e.g., moving fast) turn into nouns and adjectives (e.g., fast movement); here, this change of word classes is relevant for addressing the ||Involved Quantities||.
2. Expressing relations within subclauses: conjunctions (e.g., if) turn into more compacted prepositions (e.g., (depending) on); here, this change of word classes is relevant for addressing the || Direction of Dependency || as a relation between the two quantities.

In Table 1, we present the combination of the two relevant dimensions of word class changes for the learning content of functional relationships as well as examples for possible expressions.

		Everyday language	←	→	School academic language
		Verbs (+adverbs)			Nouns (+adjectives)
		Relation:			
Everyday language	Juxtaposition (J): no relation expressed	It's about how <i>long</i> he <i>rides</i> and how <i>fast</i> he <i>rides</i> .			It's about the <i>minutes</i> and <i>km/h</i> . The quantities are <i>time</i> and <i>speed</i> .
	Conjunctions & adverbs (C)	If you <i>ride longer</i> , you <i>ride faster</i> .			The more minutes you ride, the higher your <i>speed</i> . As <i>time</i> goes by, the <i>speed</i> changes.
School academic language	Prepositional phrases (P): containing verbs / nouns	(prepositions require linguistically at least one nominalized quantity, see right column)			After <i>four minutes</i> , the speed is <i>20 km/h</i> . The <i>speed</i> depends on the <i>time</i> . <i>Speed in dependency of time</i> .
	Verbs (V)	(verbs for the relation require nominalized quantities)			The function assigns <i>time</i> to <i>speed</i> .

Table 1: Two dimensions of expressing functional relationships, categorized regarding the German word classes; relations in **bolt**, quantities in *italics*

In addition to the demands of the word class changes in the individual dimensions, we find that these mutually depend on each other: Using prepositional phrases to express the relation is impossible without nominalizing at least one of the quantities. This further increases the complexity of the interplay of the two dimensions.

Based on the preliminary theoretical considerations and the derived Table 1, we analyze our empirical data and derive a theoretically and empirically grounded learning trajectory in the discussion part to answer the following research questions:

(RQ1) Which word classes do students use to express the quantities and the functional relation in-between within their learning processes?

(RQ2) How can a linguistical learning trajectory be derived based on the theory and the empirical data?

METHODOLOGICAL FRAMEWORK AND RESEARCH DESIGN OF THE STUDY

Design experiments for data collection

For answering the research and design question, we methodologically build on design research (Gravemeijer & Cobb, 2006) and in the overall project focus on the role of language in understanding functional relationships (continuing Prediger & Zindel, 2017). The two main goals are to (a) design an empirically grounded language-responsive teaching-learning arrangement and (b) empirically investigate students' individual learning pathways on functional concepts and language. The design experiments were conducted in small groups of eight vocational students achieving the medium compulsory education certificate. We present insights into the first cycle with 2 sessions of 35 to 45 minutes each, all videotaped (more than 340 minutes of video material) and partially transcribed. In particular, we take parts of Mahir's (17 years old) and Mika's (17 years old) pathways into account while working on the longboarding task (Figure 1).

Methods for qualitative data analysis

The qualitative analysis of the transcripts was carried out by two raters (variations were negotiated) to trace the students' linguistic learning pathways. In Step 1, we identified all sections where students talked about the ||involved quantities|| and the relation between them. In Step 2, all word classes regarding the two grammatical dimensions were coded (see also Hein, submitted). These codings are additionally presented graphically in a miniature version of Table 1 to visualize the grammatical complexity regarding the two dimensions (placed next to the individual utterances). In Step 3, we compared the students' individual pathways to derive first ideas for a grammatical learning trajectory and recommendations for teaching.

EMPIRICAL INSIGHTS

The case of Mahir and Mika – towards grammatically compacted articulation

Before approaching the longboarding task (Figure 1), Mahir and Mika worked on describing a situation in (1) specific ways by assigning pairs of values and describing changes in both quantities and (2) in a more general way by explicitly identifying the ||Involved Quantities|| from the situation. After Mahir already described changes in the quantities, the teacher asks Mika to explicitly identify the two ||Involved Quantities||.

Analysis A: Mika's first situational description			
Translation (T), most important word classes in German (WC) regarding <i>quantities</i> and relation			
	V	N	Die <i>Anzahl der Minuten</i> . [...] <i>Stundenkilometer</i> .
J			T: The <i>number of minutes</i> . [...] <i>Kilometers per hour</i> .
C			WC: <i>nominalization – nominalization</i>
P			

Figure 2: Linguistic analysis A – Mika's first situational description

Looking at the facet of the ||Involved Quantities||, we can see that Mika linguistically compacts them by addressing quantity 1 as the 'number of minutes' (following an earlier example containing the number of bottles as one quantity) and quantity 2 by the corresponding unit 'kilometers per hour'. Both are nominalizations (2nd column in matrix). However, from a mathematical point of view, he does not yet articulate at least the second ||Involved Quantities|| explicitly speed. Regarding the relation in-between, we find that Mika states the quantities independently (Juxtaposition, 1st row in matrix).

When revisiting the longboarding later, the teacher asks Mahir to describe the situation in his own words.

Analysis B: Mahir's first explanation			
Translation (T), most important word classes in German (WC) regarding <i>quantities</i> and relation			
	V	N	Je länger der fährt, desto schneller sollte der ja eigentlich <i>fahren</i>
1	J		T: the longer he rides, the faster he should actually ride
	C		WC: conjunction A – <i>adverb – pronoun – verb –</i>
	P		conjunction B – <i>adverb – pronoun – adverb – verb</i>
	V	N	also von der Geschwindigkeit. Aber nach vier <i>Minuten</i> wird der kurz <i>langsamer</i> und dann halt wieder <i>schneller</i> .
2	J		T: like, from speed. But after four minutes, he becomes more slowly briefly and then faster again.
	C		WC: preposition – <i>nominalization – preposition – nominalization – pronoun – verb –</i>
	P		intensifier – <i>adverb – conjunction – adverb</i>

Figure 3: Linguistic analysis B – Mahir's first explanation in the 2nd session

Therefore, Mahir first combines a conjunction with verbs (1st column, 2nd row in matrix) and then in the second part extends his articulation of the second quantity to the explicit, school academic nominalization 'speed' (2nd column in matrix). From there, he compacts the relation to the preposition 'after' (3rd row in matrix), although Mahir does not address the quantities as explicit nominalizations anymore.

In this small excerpt, we can find Mahir slowly adapting to school academic language. He mixes everyday and school academic phrases within the two dimensions while developing a precise language for articulating the ||Involved Quantities|| and the relation between them. Further, the use of school academic prepositions requires the nominalization of at least one quantity, though not necessarily mathematically explicit.

During the following discussion, Mahir is able to identify the ||Direction of Dependency||. Therefore, the teacher asks him to explain this which makes Mahir

struggle on the conceptual level, although the argumentation is quite productive in a linguistic way.

Analysis C: Mahir's explanation for independency			
Translation (T), most important word classes in German (WC) regarding <i>quantities</i> and relation			
1	V	N	[...] Also, eigentlich ist <i>das</i> gar nicht ganz abhängig so.
	J		T: So, actually <i>it is</i> not completely dependent .
	C		WC: <i>pronoun</i> – verb – adjective
	P		
2	V	N	[...] Dieser Longboarder fährt jetzt [...] zwei Minuten nach unten, dann wird der ja <i>immer schneller</i> .
	J		T: This longboarder now rides [...] <i>two minutes</i> downwards, then he <i>becomes faster and faster</i> .
	C		WC: <i>noun</i> – <i>verb</i> – <i>nominalization</i> – conjunction – <i>verb</i> – <i>intensifier</i> – <i>adverb</i>
	P		
3	V	N	(Ich kann genauso gut ich sage mal jetzt ..) fiünf Minuten fahren, aber ganz unnormal <i>langsam auf die Bremse drücken</i> .
	J		T: (I can just as well say ..) ride <i>five minutes</i> but <i>press the break</i> quite abnormally slowly.
	C		WC: <i>verb</i> – <i>nominalization</i> – conjunction – <i>verb phrase</i> – <i>intensifier</i> – <i>adverb</i>
	P		
4	V	N	[...] Aber da hat die <i>Zeit</i> ja nichts mehr mit der <i>Geschwindigkeit</i> zu tun .
	J		T: But then, <i>time</i> has nothing to do with <i>speed</i> .
	C		WC: <i>nominalization</i> – preposition – <i>nominalization</i> – verb
	P		

Figure 4: Linguistic analysis C – Mahir's explanation for independency

Although the two quantities are not yet nominalized explicitly, Mahir can already address the relationship with the compact prepositional phrase 'depending on', (part 1: 2nd column & 3rd row in matrix). In his attempt to explain the limitation of the dependency, Mahir linguistically draws on everyday language by using a mixture of verb(-phrases) and nominalizations regarding the two quantities (parts 2 & 3: 1st + 2nd column & 2nd row in matrix). In the end, he compacts his explanation and explicitly addresses the quantities by nominalizations and the relation by a preposition (part 4: 2nd column & 3rd row in matrix). Conceptually, we see Mahir struggling to accept mathematical dependencies that differ from the logical everyday meaning of 'depending on', which is especially visible in the last part of his utterance (speed has nothing to do with time but rather with the slope of the mountain). But looking at Mahir's linguistic development, we find that he is now able to use several phrases typical for school academic language (explicit nominalization of the quantities, prepositional phrases for the relation in-between). Still, he draws on everyday language's grammatical phenomena to explain the ||Direction of Dependency|| to articulate these more explicitly.

SUMMARY

Looking at Analysis B of Mika's and Mahir's learning pathway (Figure 3), we can see that students are intuitively able to describe a functional situation by describing changes in both quantities and assigning pairs of values. For addressing the ||Involved Quantities|| they, therefore, use verbs combined with adverbs (close to everyday language, e.g., riding fast) or already nominalized unities combined with concrete values (closer to school academic language, e.g., '4 minutes'). The students usually

express the relation between the quantities by conjunctions like ‘if-then’ or ‘the-the’ (‘je-desto’ in German) which clearly express the ||Direction of Dependency||. These linguistically less compacted combinations of the two dimensions are also visible in a large share of the other students’ utterances. Although the latter variant is not the explicitly nominalized quantity, we could illustrate the potential of students’ use of concrete values from a grammatical point of view: With the help of the nominalization (similar to Mika), Mahir begins to compact the relation between the quantities from conjunctions to prepositions (part 1: ‘the-the’ vs. part 2: ‘after’). Still, the need arises to explicitly nominalize the quantities (here time passed and speed) to precisely articulate the ||Direction of Dependency|| which Mahir did in Analysis C (Figure 4). We find that, due to the design, most of the combinations of nominalizations and prepositions already address the ||Direction of Dependency||.

DISCUSSION AND OUTLOOK

Learning for the design – first implications for a grammatical learning trajectory

In coherence with the theoretical considerations, most of the students at the beginning of the design experiments use words from everyday language for both dimensions, such as Mahir. A hypothetical learning trajectory, which aims at the conceptual understanding of functional relationships and takes into account grammatical challenges should therefore start with conjunctions for relation as well as verbs or nominalized unities combined with concrete values for quantities. For meaning-making, explicit language support by the teachers might be helpful, so that students can learn the unusual word classes, necessary for expressing functional relationships.

Following the empirical results and the theoretical background on word class change, we suggest addressing four grammatical steps of the learning trajectory for fostering students’ understanding when teaching functional relations:

- Step 1 – using students’ everyday language resources: describing functional situations by assigning values and describing changes in both quantities, using verbs + adverbs/concrete values + unities for quantities, conjunctions for relation (e.g., **If** he rides down *4 minutes*, he *becomes faster*);
- Step 2b – nominalizing the quantities: explicitly addressing the quantities with the mathematically desired nominalizations (e.g., time and speed);
- Step 2a – compacting the relation: using concrete values for passing from conjunctions to prepositions (e.g., **After** *four minutes*, ...);
- Step 3 – connecting the two condensed dimensions: precisely articulating the ||direction of dependency|| by using nominalizations for the quantities and prepositions for the relation (e.g., *Speed* is dependent **on** *time*.)

When approaching the complex topic of functional relationships, teachers need to become aware of the grammatical challenges identified and need to flexibly articulate

the different variations of addressing the ||Involved Quantities|| and the relation in-between in more and less compacted ways. For enabling teachers to adaptively use different word classes, teachers such linguistic variants need to be made an issue.

LIMITATIONS AND FURTHER OUTLOOK

As we only show the first insights into our analysis, the results are based on a small set of data consisting of only 8 students. Furthermore, we focus on a specific mathematical topic (expressing functional relationships). Although based on the literature presented earlier, we expect that similar challenges may arise for students while working on other learning contents, further research needs to disentangle grammatical challenges in a topic-specific way. We would like to recommend, supporting teachers through professional development in recognizing the changes in the word classes and verbally modeling variations to express mathematical learning contents. By this, teachers might be enhanced to flexibly address students' linguistic competencies.

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MATHEMATICS TEACHER EDUCATORS IN AN UNKNOWABLE WORLD: TEACHING MATHEMATICS FOR CLIMATE JUSTICE

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How can mathematics teachers (MTs) be supported in teaching mathematics for climate justice (TMfCJ) if mathematics teacher educators (MTEs) lack climate-related expertise? In this paper, we challenge the assumption that such expertise is essential, and consider ways in which MTEs can use their existing expertise to support MTs in TMfCJ. The context of this study is a professional development programme for MTs in England where we examine how one MTE relates to ways of knowing about mathematics, pedagogy, climate justice and their intersections. Results suggest that the MTE's existing expertise in relation to how MTs learn, together with her lack of climate-related expertise, enable an authentic co-inquiry where expertise relating to the intersection of mathematics teaching and climate justice can be co-created.

BACKGROUND, CONTEXT, AND THEORETICAL FRAMING

In a recent large-scale survey (Teach the Future, 2022) conducted in England, secondary school teachers were asked whether a) climate change and b) the ecological crisis (i.e., the decline of biodiversity and the collapse of natural systems that support life on earth), are embedded within their school's subject curriculum. Only a) 7% and b) 3% of mathematics teachers (MTs) responded positively, despite the fact that the vast majority (80%) viewed climate change and the ecological crisis as being relevant to the subject of mathematics. When asked what would support them to embed these issues within their school's mathematics curriculum, the two most popular responses were i) capacity and time for developing relevant curriculum content and ii) inclusion within the national curriculum for mathematics, which currently makes no reference to climate change nor does it suggest links exist between mathematics, the environment, or issues of social justice (Department for Education, 2013).

It seems to be a common assumption in England that education relating to climate change and the associated inequalities should be left to teachers of geography, biology, and citizenship. The Department for Education's (2022) recent *strategy on sustainability and climate change* refers to several school subject disciplines, but makes no reference to the subject of mathematics nor to the relevance of mathematics to climate change. In contrast to this lack of recognition at the policy level, there is an increasing body of research from around the world, primarily from a *critical mathematics education* perspective, providing compelling arguments for why mathematics, climate change, and the associated inequalities are inextricably linked. Not only does mathematics help us to describe, predict and communicate climate change (Barwell, 2013) it has the power to format it in such a way that a particular view of it is formed (Skovsmose, 2021). Through highlighting certain social

dimensions, and concealing others, mathematics and power “become intimately connected” (Skovsmose, 2011, p. 22). For these reasons, we question why the national curriculum pays no attention to the relationship between mathematics and climate.

Teaching mathematics for climate justice

We use the term *climate justice* to capture, in one phrase, issues relating to climate change, the ecological crisis, and the associated implications for societies around the world. The term climate justice draws attention to the impact of climate change on those communities who are most vulnerable to its effects, and to couple *climate* change and the ecological crisis (often seen as scientific in nature) with social *justice* (i.e., principles of human rights including access, participation, and equity, seen primarily as sociological in nature). This is consistent with a *socio-ecological perspective* (Coles, 2022) which recognises that the “social and the ecological cannot be disentangled” (p. 208), a perspective that brings the ecological in to view within mathematics education alongside the various social perspectives that are already commonplace.

Given this context and framing, we ask *how* MTs can develop their teaching in relation to climate justice and, more specifically, *who* has the expertise to support them in this endeavour and *what* does this expertise consist of? Currently, it is the role of mathematics teacher educators (MTEs) to educate practicing MTs through professional development (PD) opportunities. The PD context that forms the basis of this study is a small-scale PD programme in England, was led by an experienced MTE, and involved 10 participating secondary MTs. The PD was designed and facilitated by an interdisciplinary team consisting of three academics (from mathematics, mathematics education (the MTE), and creative writing), as well as a MT and an artist. Though from different backgrounds, the members of this team had a common interest in climate justice, yet none considered themselves to be subject-matter experts. The PD consisted of three full-day workshops over a period of six months, where different team members led a range of activities with the group of MTs, including art-based activities such as creative writing and mapping exercises, mathematical tasks relating to ecosystems, as well as several discussions which were led by the MTE. The aim of the PD was to support participating MTs in developing their own ideas and curriculum materials in relation to teaching mathematics for climate justice (TMfCJ). Between workshops, MTs were invited to try out their ideas and materials at school.

We see this study, and the PD context in which it is situated, as at the “starting point” of the teacher education (TE) and PD strand of Boylan and Coles’ (2017) proposed map of “possible future developments of practice and research for mathematics education and the living world” (p. 13). They describe this starting point as a situation where “small-scale programmes are developed, on isolated sites, linking learning to teach mathematics with a questioning of the role of mathematics in the world” (p. 13). Our hope is that by studying these isolated sites, we may progress towards the next step on the map where TE and PD programmes across the world “begin to include how the study of mathematics can relate to wider ecological issues” (p. 13). To make this

step we first need to understand how MTEs, who themselves lack subject-matter expertise relating to climate justice, can support MTs in TMfCJ.

Mathematics teacher educators' expertise

Recent years have seen a growing interest in research relating to MTEs, who are considered to be key players in the implementation of reform or research-based ideas within mathematics education. Frameworks have been developed to conceptualise the nature of MTE expertise (e.g., Prediger et al., 2022), the majority of these frameworks characterise MTE knowledge and practices as expansions of MTs' knowledge and practices (Goos and Beswick, 2021). Consequently, MTEs are characterised as needing to know what MTs know, and *more*, in order to support their PD.

However, when MTEs facilitate collaborative PD environments, they are often not expected to use their knowledge and expertise directly (by, for example, telling MTs how they should teach), but indirectly, through foregrounding MTs' voices. This facilitation is characterised by a tendency to ask open questions and raise awareness and possibilities. Horn and Garner (2022) describe it as adopting a "co-inquiry stance" (p. 85), that is, being "deliberately humble and uncertain in sharing of our interpretations of teachers' instruction and positioning our expertise as complementary to (not better than) the teachers" (p. 84). This stance contrasts certain views of TE and PD, where MTEs are perceived by MTs as the "epistemic authorities" (Kruglanski, 1989) in the same way that students can perceive their MTs, as the arbiters of knowledge. We challenge this view, by asking what happens when MTEs lead PD when they do *not* have subject-matter expertise, such as that relating to climate justice. Furthermore, since issues relating to climate justice are dependent on time and context and thus characterised as ambiguous, open-ended, and dynamic (Eernstman & Wals, 2013), the notion of climate justice, to an extent, is unknowable, in contrast to a more certain and stable view of school mathematics. To explore these contrasts and intersections, we ask:

How does an MTE (who is not a climate-expert) use her expertise when supporting teachers in teaching mathematics for climate justice?

Next, we briefly set out the research design including our methods of data analysis.

METHODOLOGY

The corpus of data used for this paper are video recordings from the first two PD workshops (approximately 12 hours of video). Video cameras were placed in different positions to capture the various activity, including those sections of the PD led by the MTE. Since we were interested in the different ways the MTE places herself in relation to her existing areas of expertise (i.e., mathematics and pedagogy), and comparing this to the way she places herself in relation to climate justice, we selected episodes where the MTE relates to one or more of these areas and/or their intersections. Moreover, we examined how the MTE places herself in relation to the MTs to explore how these

relationships vary when she is referring to the different foci of the PD. Once the relevant episodes were selected, we fully transcribed them.

To interrogate the selected episodes, we used a method of micro-analysis by drawing on Gee's (2011) tools for discourse analysis where each tool is "a specific question to ask of data" (p. x). According to Coles (2015), a key feature of micro-analysis "is to approach small sections of transcript with a slow and repeated reading, keeping some questions in mind" (p. 241). Specifically, the questions we kept in mind were: i) how does the MTE place herself in relation to the MTs? ii) how does the MTE place herself in relation to mathematics, pedagogy, climate justice and their intersections? In summary, our process of analysis consisted of: 1) viewing the entirety of the video data; 2) selecting episodes where the MTE related to the areas of expertise and/or the MTs; 3) transcribing the episodes; 4) interrogating episodes with our questions in mind.

ANALYSIS OF EPISODES

In the following, we analyse selected episodes of the PD focusing on how the MTE relates to the different foci of the PD and the MTs. We use [...] to indicate where text has been omitted. Words or phrases inserted between square brackets (in italics) indicate additional information (e.g., gestures). All names used are pseudonyms.

Workshop 1

The first workshop began with the MTE, Lara, defining the scope and goals of the PD:

Lara: We will be creating and sharing visions of climate justice earlier on in the day and thinking about how this might inform us as teachers. You might experience different ways of being and doing here, there will be some activities that are potentially quite unusual that we engage in as a way of expanding what we see as being possible, as well as different ways of knowing. [...] What else might be possible is a question we will keep returning to today. And then towards the end of the day we are going to formulate directions of travel, so not to the point where you have a concrete plan, but certainly beginning to formulate your own sense of where you want to take this given the topic is teaching mathematics for climate justice. What it's not is tips for teachers, it's not scrutinising existing materials, it's not the mastering of concepts.

Here, Lara positions herself as an insider ("inform **us** as teachers"; "expanding **our** sense of what's possible"), suggesting that she is part of the inquiry process in terms of incorporating climate justice in mathematics lessons. She does not place herself as an epistemic authority, framing the workshops as aiming at stimulating ideas rather than "the mastering of concepts". Furthermore, in terms of epistemology, Lara highlights that the PD aims at exploring "different ways of knowing", namely, questioning traditional patterns of schooling and supporting the exploration of the dynamic nature of the topics at hand. The new domain – climate justice – together with the creative processes that the MTs engage in, are used as resources for such disruption. Later, when referring to processes of teacher change, Lara places herself differently:

Lara: In thinking about how we learn as teachers, my sense is that to change anything in ourselves, in the way we are, or the way we do things, requires something from within. I cannot change you, I'm not here to change you, that's not my purpose. If you find something from within that you want to work on, that's the only way that you are going to change.

These words suggest that Lara sees her role as assisting MTs in their efforts to deploy changes in their classroom, yet not imposing specific changes on them. In this excerpt she frames herself more as an outsider who is an expert in TE, yet part of her expertise is not presuming she knows what the MTs should do in their own classrooms and contexts. Immediately afterwards, she returns to the authentic co-inquiry stance. When launching an activity where participants present an object that represents how they feel about climate justice, Lara places herself as part of the group, and shares with them her concerns as a mother and a citizen. After the MTs have shared some of their reasons for participating in the PD (participation was voluntary), and what they hoped to achieve from the PD, Lara shares some of her own ideas about the purpose of the PD:

Lara: I suppose for me, these sessions are partly to support you in seeing past some of the constraints you feel that you can't change, so I suppose I might invite you to consider what's contingent and what are the true facts, things you feel can do nothing to change, and even then what might be changeable within that set of constraints.

Lara invites the MTs to critique their practice and examine whether what they know is maybe less certain (“what's **contingent** and what are **true facts**”). She places herself as a knower of TE (though not as a knower of the MTs' contexts), who can identify changeable aspects of teaching. When the MTs share their goals, Lara reflects back:

Lara: What comes to mind in hearing some of you talk about kids who don't like maths and this sense of making maths relevant, for me, it comes down to what we think maths is, our image of mathematics. So, if our image of mathematics is this [*holding out one hand, palm upwards*] how does that intersect with issues of climate justice, how can we bring those things together [*bringing the two hands together*]? What are the intersections of these things?

There is a subtle contrast between how Lara describes mathematics (“if our image of mathematics is **this**”) as something it might be possible to know, and how she describes climate justice, something that is unknowable for her and the group. She is placing herself as a learner, asking a genuine question (“how can **we** bring those things together?”), implying that also for her the intersections are still unformed.

Workshop 2

At the beginning of workshop 2, the MTs worked in subgroups on a task relating to native species of flora and fauna. The aim of the task, led by the mathematician (Amy), was to find the largest network of species that could co-exist based on a variety of information (environment, predators, etc.), presented on a set of cards (one card per species, 50+ cards). Later on, the MTs engaged in a creative mapping activity led by

Ali, the creative writer. For this task the MTs were asked to formulate their ‘utopic goal’ in relation to TMfCJ. They then mapped a ‘traditional’ pathway to their goal (made up of steps they might usually take as MTs), a ‘utopic’ pathway (where *anything* is possible) and then a ‘creative’ pathway, a middle way between the first two, as a way to encourage new approaches and disrupt normal practices. Having formulated their goals, Lara asked the MTs to share their initial ideas. One MT, Evan, offered:

Evan: I like the idea you could teach about something and then the maths would naturally come up as a way to better understand the problem. You could teach about waste, say, and then standard form could come up as a way to talk about big numbers. Rather than the other way around, when you start with the maths and then try to relate it to the real world, you start with the real world and the maths comes up naturally.

Lara: I was really struck by the card activity that Amy did with us. It felt really powerful, just having a go at it. You all had your own goes, then there was this mathematical structure that you were offered, a network, and that gave you a different insight into the problem, it seemed to give you more energy [...] There is this problem that needs solving and then there is some mathematics, at some point when we need it, that comes along allowing you to look at the problem in a different way.

While placing herself emotionally as part of the group inquiry process, Lara also illustrates how modelling an ecological situation can be illuminated using a mathematical representation (a network), by generalising Evan’s idea into a model of TMfCJ, and grounding the model in a shared experience where mathematics was used to explore and understand phenomena earlier in the day. Later in the discussion, another teacher speculated on the viability of TMfCJ in her context. Lara responded:

Lara: It sounds like you are talking about the content, what are the issues that may or may not be appropriate to raise with your class, so a shift you might make is to think, maybe I don’t want to go down that road by taking on issues directly, but, what sort of environment do I want to create in my classroom that maybe supports the children’s confidence, their voice, is there something in the process that you want to think about instead of directly tackling issues of climate justice [...] a way of being that you might want to nurture.

Here, Lara delineates the differences between teaching *about* climate justice as opposed to teaching *for* climate justice. She later unpacks links with social justice:

Lara: Social justice can be about the structures that have become normalised in schools. [...] there is a whole discourse around that, “she’s a top set kid”, “they’re a bottom set group” [...] it is in the structure, in our discourse and potentially in the way that we behave towards others. So, there are many avenues you can take teaching mathematics for climate justice and it could start with questioning the assumptions we make, or ways that we do things, the way that we speak, as well as directly addressing issues that are meaningful. The thing about those global issues that feel too abstract, that

we don't feel them, we don't see them, a different way to think about that could be considering a local issue right here that we can be quite hopeful about, that could be quite empowering. There could be something in the school or in the local community. There are all sorts of things you could do, closer to home that are not abstract and are not necessarily difficult or sensitive. You might not know what those issues are, so you could ask the children what is important to them.

Lara offers possible heuristics for TMfCJ, yet avoids giving concrete answers or examples, since she does not know the specific contexts of the MTs or their students. That is, her expertise is manifested in her formulation of heuristics and mechanisms for finding ways to teach mathematics for climate justice, by generalising ideas offered by the MTs and offering alternatives when faced with resistance, aligning with a co-inquiry stance. By saying “you might not know what those issues are, so you could ask the children”, she is also offering what co-inquiry in a classroom could look like.

DISCUSSION

Our aim in this paper was to inquire about how MTs can be supported in TMfCJ if MTEs lack climate-related expertise, by examining how one MTE, Lara, places herself with relation to MTs and mathematics, pedagogy and climate justice. We identified moments of authentic co-inquiry, as well as moments where Lara's expertise is used to guide processes of teachers' professional growth. We maintain that this expertise does not draw upon epistemic authority (Kruglanski, 1989) on either mathematics, pedagogy, or climate justice, but on Lara's expertise in relation to how MTs learn. Thus, Lara's expertise in this context is at a meta-level to the PD content, through offering structure, provocation, and feedback. Being attuned to what is emerging in conversations allows her to co-create, with the MTs, possibilities for TMfCJ. These observations challenge the assumption that MTEs' subject-matter expertise is essential in supporting MTs' learning. We are not, by any means, disregarding the importance of MTEs' content expertise. We argue, instead, that it is possible for MTEs to work alongside MTs to develop classrooms where TMfCJ can be realised. Given the context of a global emergency, it is worth exploring how MTEs can use their existing expertise to provide PD learning opportunities where what it means to TMfCJ can emerge in the complex reality of Boylan and Coles' (2017) “starting points”.

We contribute to discussions of the role of mathematics TE and PD in relation to TMfCJ, by conceptualising MTE expertise in a context where the MTE is not a subject-matter expert in relation to climate justice. For the future of the planet, it is vital that the international mathematics education community explore how MTs can engage in TMfCJ, assuming that a) becoming experts in climate justice is a long process, and b) many aspects of climate justice are unknowable even for climate experts. This paper aimed to capture, illustrate, and analyse moments where different forms of expertise are observed in the actions and words of one MTE, to demonstrate how MTEs are well placed to support MTs in their endeavours, considering the urgency of taking action.

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EXPLORING DEVELOPING PATTERNS OF MATHEMATICAL IDENTITY WORK BY GIVING ATTENTION TO EMOTIONAL HUE AND TONE OF VOICE IN THE ACT STORYTELLING

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Research on the concept of mathematical identity work is maturing towards a socio-cultural viewpoint, seen as an action not an acquisition. For students who, due to educational policy in England, are compelled to continue studying mathematics in their post-16 setting, the patterns of stories that they tell as identity work can develop over time in terms of plot or storyline. This study focuses on one student participant, that I called Claire, from a broader study into identity work in the context of low attainment in mathematics. The findings explored in this research report suggest that developing patterns of identity work involve facets emotional hue or tone of voice as well as plots or storylines, contributing to discussions around the mathematical identity work in the context of students compelled to continue to study mathematics.

INTRODUCTION

In England, it is educational policy that students who fail their mathematics General Certificate of Secondary Education (GCSE), usually taken at 16 years old, must continue to study mathematics as part of their post-16 education. For many students, the lived experience of continuing to study mathematics is laden with emotion, however, there is some evidence that patterns of mathematical identity work, the stories told by, and about, students from within the lived experience, can develop over time. In this research report, drawn from a broader study, I discuss the stories told, as mathematical identity work, by one student participant called Claire (a pseudonym). I reflect on her developing patterns of mathematical identity work, considering how, as well as the plot of her stories, the emotional hue and tone of voice used in the act of storytelling also developed over time.

MATHEMATICAL IDENTITY WORK

As a unit of analysis, the concept of identity has become prominent in social sciences in the last half century (Brubaker & Cooper, 2000). In the field of identity research, the conceptualisation of identity varies, with some viewing the concept as an acquisition, something you have, and others seeing an action, something you do (Darragh, 2016). In their review of literature on mathematical identity, Graven and Heyd-Metzuyanim (2019) noted that the field of mathematics education was maturing towards a sociocultural view, with mathematical identity seen as an action, being domain specific and situated as well as fluid and relational. The phrase *mathematical identity work* is used in research to indicate the labour involved in constructing various, often contradictory, facets of identity. Many studies that consider students' mathematical

identity work focus on the narratives in relation to teaching and learning, such as, interactions in the classroom (see, for example, Bishop, 2012; Heyd-Metzuyanim, 2013) and self-reported accounts of participants (see, for example, Bartholomew et al., 2011; Foyn et al., 2018). Sfard and Prusak (2005) go further to both conceptualise and operationalise identity, or identifying, as the collection of stories told by, and about, a student. From the point of view of Sfard and Prusak (2005), discursive activity constitutes identity work, with students' identity work not found in recounting stories and experiences, but in the act of storytelling itself, for which I use the phrase *stories-as-identity-work*. The authors state that stories-as-identity-work are both authored by the student themselves and influenced by others in their context, such as teachers and parents, which Sfard and Prusak (2005) describe as significant narrators. Heyd-Metzuyanim and Sfard (2012) posit that:

Mathematics learning is to be seen as interplay between two concomitant activities: that of *mathematizing* – communicating about mathematics objects; and that of *subjectifying*, that is communicating about participants of mathematical discourse. (p.129, *emphasis* in the original)

The authors state that the facet of subjectifying that is identity work can be emotionally charged, interfering with the learning of mathematics. Heyd-Metzuyanim and Sfard (2012) go on to introduce the notion of discursive actions having an emotional hue, features of a person's utterance, such as, combinations of words or tone of voice, that, for the listener, suggest a speaker is communicating certain feelings. Emotional hue, or tone of voice, goes beyond telling stories about feelings, to communicating feelings through the ways of storytelling.

In England, the majority of students take their GCSE examinations at the age of sixteen, with the outcomes graded between a 1 (lowest) and 9 (highest). For those students who achieve less than a grade 4 in mathematics, described as a fail, government policy in England states that they must continue to study mathematics, whilst in their post-16 college, in order to improve their grade (Department of Education, 2021). For students who have to continue to study mathematics in college, there is some research that highlights the developing patterns of mathematical identity work over time, with a number of authors discussing changes in plots or storylines (see, for example, Black, et al., 2010; Helme, 2022). However, for many students, labelled as low attaining through achieving of a grade 1, 2 or 3 at GCSE, studying mathematics can be an affective experience due to, for example, motivational issues from cycles of repeated failure (Norris, 2022; Noyes & Dalby, 2020) or dissatisfaction around the enforced policy of having to continue with mathematics (Bellamy, 2017). I suggest that examining the developing patterns of stories-as-identity-work of students compelled to continue studying mathematics needs to extend beyond considering the plot or storyline towards giving attention to emotional hue. More specifically, not stories of affective experiences, but the emotional hue or tone of voice of the storytelling itself, reflecting on the developing patterns in the way the stories-as-identity-work are told.

METHODOLOGICAL DISCUSSION

The data described in this paper was from a broader study into the mathematical identity work of five students who had to continue studying mathematics to improve their mathematics GCSE outcome. The data collection took place over two phases, phase 1 was pre-pandemic, from October to December 2019, and phase 2 was mid-pandemic, from December 2020 to July 2021. Claire, the 17-year-old student participant in phase 2 of the study, was interviewed over three cycles of data collection, mostly using email but once, in third cycle, using online conferencing software. Due to the restrictions imposed by the government in the United Kingdom, in response to the Covid-19 pandemic, in the summers of 2020 and 2021, students could not sit their examinations, being allocated a grade in each subject based on assessments by their school. In the summer of 2020, Claire had been allocated a grade 3 by her school. In November 2020, students had the opportunity to sit their examinations and Claire, having started to attend college in September 2020, achieved a grade 3 in the November examination.

Research questions

The research questions, developed over the course of the study, considered both the stories-as-identity-work told by the student participants and the stories-as-identity-work about the students told by a significant narrator, in this case the teacher. The final research questions became:

RQ1: What stories-as-identity-work are shared in the context of low attainment in mathematics?

RQ2: What patterns of stories-as-identity-work are perceived when attention is given to the (self)positioning voice through working as part of a teacher-researcher partnership?

In this report, I address the findings in relation to RQ1, focusing on the first-person voice of Claire, the student participant in phase 2 of the study.

Methods

In the study, I employed object-elicitation techniques, based on the tenets of photo-voice (Rose, 2016), to draw out the stories-as-identity-work told by Claire. Introducing an object, in this case images found by Claire and a question level analysis of an assessment, provided by the teacher, afforded the opportunity to centre the dialogue around the object, invoking accounts that may not be spoken in other types of interviewing (Woodward, 2020). Analysing using a voice-centred, relational method called the Listening Guide (Gilligan et al., 2006), I examined the elicited stories-as-identity-work to examine Claire's various voices that are said to co-exist in a person's narrative. The Listening Guide uses a four-step process to bring the researcher-listener into relationship with the data, privileging the voice of the participant over that of the researcher. The four steps of the Listening Guide are:

- Step 1: Listening for the overall plot and recording subjective responses

- Step 2: Constructing and examining I poems
- Step 3: Returning to the narrative data to listen for contrapuntal voices
- Step 4: Composing the final analysis from all the previous listenings

The construction and examination of the pronoun poem in step 2, within the traditions of poetic inquiry, was a means to be sensitive in the analysis to the form, affect and tone of voice in a participant's story telling (Faulkner, 2019). A short extract from Claire's I poem from data collection cycle 2 can be below:

i apologise
that i haven't been
i see
i do still seem to be
i notice
that i still struggle
i will need to
so that i can progress

The use of the lower case "i" for the first-person pronoun was a strategy in the study in order to not privilege one word over any others. I listened and relistened to the constructed pronouns poems to identifying co-existing voices in the narrative by giving attention to, for example, changes in tone of voice or shifts in meaning. Building on the listening in step 1 and 2, in step 3 I returned to the full narrative, defining and redefining the co-existing voices found in the I poem, before bringing together all of the previous listenings to compose a final analysis in the final step 4.

DISCUSSING THE FINDINGS

Using the Listening Guide method of analysis, I identified four different co-existing voices in Claire's narrative data, namely, *struggle/understand* which talked about the state or relationship with learning mathematics; *receiving* which inferred external support from a teacher or online software; *action* which described Claire's own actions, and, *inner* where Claire shared her thoughts, feelings and affect. This research report traces the emotional hue, or tone of voice, in one facet of her stories-as-identity-work, focusing on her struggle/understand voice, from the three cycles of data collection within phase 2 of the study.

Claire's struggle/understand voice, found in her I poem, used of the verbs "struggle" and "understanding" explicitly as well as other verb phrases with similar meaning, such as, "I have progressed" and "I am quite good". Returning to the full narrative, Claire talked about her relationship to learning mathematics, her perception of being in a state of struggle or a state of understanding. In the following extract, from cycle 1 of the data collection, Claire described her choice of an image of a maze to represent her experiences of learning mathematics:

Claire: I struggled to understand maths therefore this image is a perfect example of how I felt about my experience of maths however during the college period that I am in I am finally understanding maths and being able to recognize and interpret maths. Maths did annoy me at times and it made me feel like I couldn't understand maths but as I kept trying and trying to understand maths it got a lot clearer to me and I felt like I could finally answer questions and be able to get maths it just would of (sic) taken me longer to do so.

Claire talked about her perceptions in relation to her previous struggles and current understanding mathematics. She shared that she had moved from “I struggled to understand” to “I am finally understanding”, which she equated to being able to “answer questions and be able to get maths”. However, Claire talked in global terms, a dichotomy where, in her opinion, she either struggled or understood mathematics. Giving attention to the emotional hue, or tone of voice, Claire used phrases that bring in affect “maths did annoy me” and beliefs about herself “it made me feel like I couldn't understand”. Although Claire talked about “trying and trying”, it seemed that learning mathematics might be something that happened to her, that she was somewhat passive in the experience.

Moving on to the following extract, from cycle 2 of the data collection, Claire talked about a question level analysis from a recent assessment on the topics of probability and statistics:

Claire: I notice from the [copy of the spreadsheet] that you have given to me that I still struggle to understand the areas marked in red and that I will need to brush up on my skills so that I can progress further. I also see that I am quite good at maths which are shown (sic) in the colour green. I personally think that I am approving (sic) [improving] massively in maths and that I am pushing myself to get the correct grade that I need to get to pass maths this year. I personally think over the time that I have been taught this that some of the red areas marked will become amber or even green.

Claire commented on the questions for which she received no marks, highlighted in red on the spreadsheet, as well as the questions for which she received full marks, highlighted in green. She continued to talk about her perceptions of herself “I personally think I am approving (sic) [improving]”, beginning to align the state of understanding with getting a pass grade in her mathematics GCSE. There was a developing pattern, compared to the extract from cycle 1, around the way she used her voice when she talked about struggle. The sense of struggle is no longer a sign of globally not understanding mathematics but an occasion to “brush up on my skills”. Understanding has moved beyond answering questions to become equated to getting a pass grade in her mathematics GCSE. Giving attention to Claire's tone of voice in the extract above from cycle 2, she seemed less passive compared to the extract from cycle 1, moving from feeling to thinking with “I personally think” and highlighting her actions, such as, “I am pushing myself”. Claire was actively locating herself in the

process of learning, personally engaged in improving her understanding of mathematics.

In the following extract, from cycle 3 of the data collection, Claire revisited the initial image of a maze from cycle 1 before providing an alternative image of two paths which she discusses in the extract below:

Claire: I am now feeling happy about how much I have progressed in maths and that I am no longer thinking (sic) maths like a maze. I can now clearly see two paths in front of me both of which will help me to progress to my future career. Personally it was difficult to start off with because I didn't have much help at school however now I am at college and I am getting more help from [the teacher] I feel like I have come so much further than I thought I would.

Although Claire did not use the specific verbs struggle and understanding, she referred back to the image of the maze, contrasting with her current chose of two paths as to progress to her future. She stated that she had “progressed in maths” with understanding now being more than gaining a pass grade but also a way to achieve her future career, which, in a later conversation, she shared was training to become a teaching assistant. Giving attention to her tone of voice, Claire seemed to firmly locate herself in the story-as-identity-work, continuing to use the language of perceptions “I feel like I have come so much further”, affect “I am now happy” and discernment “I can now clearly see”. Claire’s narrative suggested strong sense of positivity and awareness, recognising that her opinion of mathematics, as well as of herself as a learner of mathematics, had change over her time in college.

FINAL THOUGHTS

The aim of this research report is to reflect on the developing patterns of emotional hue, or tone of voice, in the telling of stories-as-identity-work by students who are compelled to study mathematics as part of their post-16 education. By examining one of Claire’s co-existing voices, labelled as struggle/understand, I have identified that alongside the development of the plot or storyline, I was able to perceive different patterns in the act of storytelling. For Claire, there was evidence of a move from a passive voice to a comparatively more active voice in her storytelling. However, a challenge for the researcher-listener is the impact of themselves in the research process; to what extend was the developing patterns in tone of voice related to, for example, the emerging research relationship with the listener? What could be the influence of researcher subjectivity on the tone of voice that they perceive? Despite these challenges, the findings warrant further attention to consider, especially for students compelled to continue to study mathematics, both the stories-as-identity-work being told and the role of the tone of voice in the telling.

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STUDENT BEHAVIOR WHILE ENGAGED WITH FEEDBACK-ENHANCED DIGITAL SORTING TASKS

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Sorting tasks are often used in mathematics education to highlight certain features of a set of objects. In this study we strived to make sense of how students use immediate feedback in a two-choice digital sorting task, in the context of elementary school reflective symmetry, using GeoGebra applets we developed. Participants included 29 students (9-12 years old) from Israel and Germany. Calculating variables that describe student actions throughout the sorting process, and using cluster analysis, we were able to identify different behaviors among students for each sorting task. We also highlight some interesting relationships between behaviors across the two tasks.

INTRODUCTION

Feedback is key to learning, as it gives learners the opportunity to improve and maximize their performance. In the context of digital learning environment, feedback can be delivered automatically, immediately and in a way that directly addresses students' responses and behavior. Indeed, a recent literature review suggests that in most cases, automatic feedback boosts student performance (Cavalcanti et al., 2021). Nevertheless, the mere existence of a feedback mechanism does not guarantee that it will be used, as this is mostly up to the learner (Winstone et al., 2016). Yet, research on students' actual use of feedback throughout the learning process is only in its infancy. We take a step forward in bridging this gap by studying the case of two-choice sorting tasks which provides immediate feedback, in the context of elementary school reflective symmetry.

Reflective Symmetry

According to Götz & Gasteiger (2022), reflective symmetry is “the first congruence transformation introduced in primary school, is among the most significant geometry concepts children should learn in mathematics education in primary school” (p. 48). In mathematics education this concept is used to describe a part of the world outside the classroom and as a tool for shaping our surroundings. We are therefore interested in students' recognition of reflective symmetry in real world objects and in mathematical objects. Indeed, the application of mathematical notions to contexts from the real world is a desirable goal of mathematics education (Niss & Højgaard, 2019). Objects outside the realm of mathematics may be more difficult to handle than exclusively mathematical objects (Dapueto & Parenti, 1999), raising the question of whether this difference also affects recognition of reflective symmetry.

Types of Feedback and Feedback Use in Digital Learning Environments

Certain digital learning environments can detect correctness and provide feedback in real-time. Such settings usually distinguish between three main types of feedback: (a) knowledge of correctness; (b) knowledge of correct response; and (c) elaborated feedback, defined as any feedback that goes beyond the previous two types. Studies generally found that elaborated feedback is superior to the other two types of feedback, and that there is no clear distinction between knowledge of correctness and knowledge of correct response (van der Kleij et al., 2015). Still, we cannot conclude that any form of immediate simple feedback can be considered *a priori* to be beneficial to student learning. Feedback does not necessarily foster the desired conceptual development, just as a correct solution does not always coincide with conceptual understanding. Indeed, conceptual understanding is primarily associated with the ways in which learners interact with feedback (Rezat, 2021).

RESEARCH QUESTIONS

From the authors' previous considerations on reflective symmetry and feedback in digital environments the following research questions are addressed:

1. Which types of behaviors can be observed in students' interactions with sorting tasks?
2. What are the relationships between student behavior and success on sorting tasks ?
3. To what extent is student behavior consistent across different sorting tasks?

We will first describe our methodology. Then, in the Findings section, we will answer the research questions, and finally will discuss them from a broader perspective in the Discussion section.

METHODOLOGY

Research participants

Data were collected from 29 participants (12 girls and 17 boys, 9-12 years, $M=10$, $SD=0.9$) in a pilot project carried out in Israel ($n_1 = 12$) and in Germany ($n_2 = 17$).

Research Tools

We designed two applets in the context of reflective symmetry. In the first applet (Polygons), students were presented with seven quadrilaterals to be sorted based on the existence of at least one line of symmetry (Figure 3 left). In the second applet (Traffic), students were presented with ten traffic signs to be sorted based on the existence of a single or multiple lines of symmetry (Figure 3 right). Immediate feedback was available in terms of an updated cumulative count of correct and incorrect classifications. Users could keep dragging objects from any area to any other area on the screen.

Data collection

Data collection took place in March 2022. Members of the research team met with each participant individually. Before interacting with the symmetry-related applets, we made sure of familiarity with the concept of reflective symmetry, and used a test applet for getting used to the interface. While participants used the applets, we captured the screen and used these recordings for our analysis.

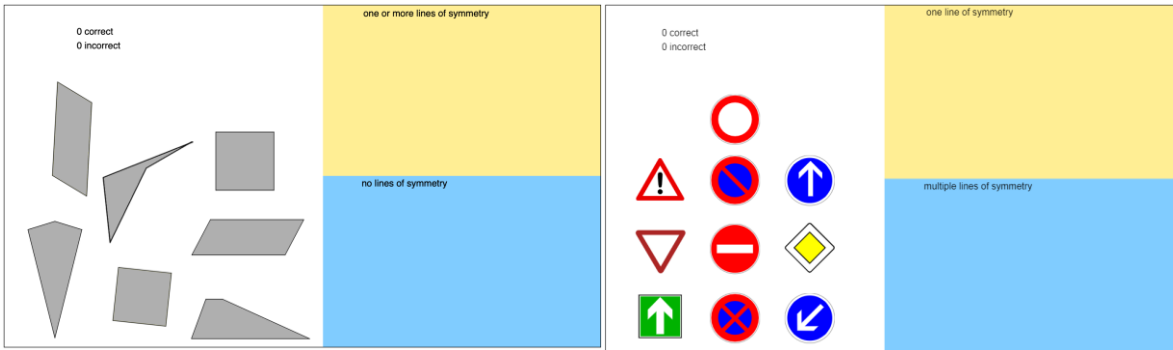


Figure 3: The applets we designed for sorting quadrilaterals according to whether they have no or at least one line of symmetry (left) and traffic signs according to whether they have one or multiple line(s) of symmetry (right)

Research Variables

For each student and each applet, we defined two arrays to describe participants’ cumulative correct/incorrect score along their solution. These arrays can be represented by graphs, where the abscissa represents shape movements, normalized to a (0,1], and the ordinate represents correctness/incorrectness. Figure 4 provides an example of such a graph, showing the cumulative correct score in the Polygons applet for student G42, who made a total of 13 dragging attempts. This student achieved a total correct score of 6 (out of 7), doing so in a non-linear fashion. Note that at some point (between 0.3-0.4) the student dragged objects from the pool area to an incorrect location, thus not adding correct scores; in other times (0.45-0.55, and between 0.6-0.8), they dragged objects out of the correctly classified area, so total correct score decreased.

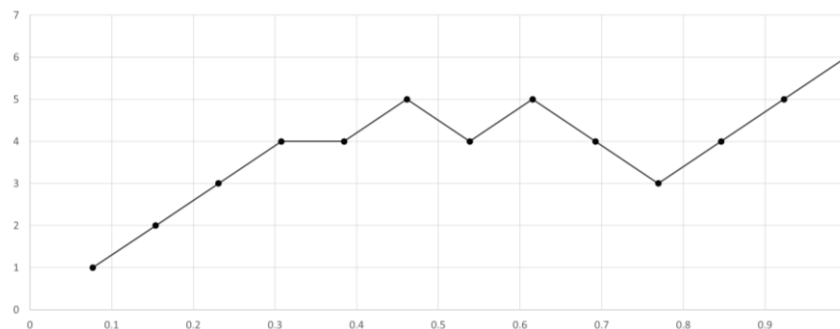


Figure 4. Cumulative correct score graph for student G42 (Polygons applet)

Referring to these graphs, we defined our research variables. **Correctness AUC** measures the area under the curve (AUC) for the graph representing the cumulative correct scores, normalized by 3.5 (polygons applet) or 5 (traffic applet); these scores

denote the AUC of the linear graphs that represent the minimal number of moves required to obtain a full score; the closer this value is to 1, the more optimal the path to success. Similarly, **Incorrectness AUC** measures the AUC of the curve for the graphs representing the cumulative incorrect scores; the closer this value is to 0, the more optimal the path to success. **Correct Decreases** counts how many times the cumulative correct score decreased, normalized by total moves; this value decreases when a shape is moved out of the area to which it belongs, either to the other area or back to the pool, hence it is a proxy for not making proper use of feedback (e.g. by not relating it to the corresponding objects). Furthermore, **Incorrect Increases** counts how many times the cumulative incorrect score increased; the cumulative incorrect score increases when a shape is moved from the pool area to an area to which it does not belong, serving as a proxy for a preconception about reflective symmetry, or when a shape is moved from the correctly classified area to the other area, hence serving as a proxy for not making proper use of feedback. Finally, **Total Score** takes the final correct score normalized by total number of moves, and is our measure of success.

Analysis

We used hierarchical cluster analysis to group students based on their behavior while using the applet, doing so for each applet separately. This bottom-up, unsupervised method partitions the population into groups (clusters) of individuals who are more "similar" to each other than those in other groups, where similarity is based on the values of the research variables (in our case, all variables excluding *Total Score*). We used Pearson correlation to measure distance between individuals, and average linkage to define the distance between clusters. Variables were standardized using Z-scores before clustering. We used the Bayesian Information Criterion (BIC) to find the optimal number of clusters.

FINDINGS

Behavior in the Polygons Applet (RQ1)

The optimal model was based on partitioning the population into three clusters (BIC=104.9). Means of the clusters are presented in Table 1 and summarized here. **Cluster 1** (n=17) is characterized by a *Correctness AUC* average that is very close to the optimal value and by low values—both in absolute terms and in comparison with the other clusters—of *Incorrect AUC*, *Correct Decreases*, and *Incorrect Increases*. The low deviations should also be noted. **Cluster 2** (n=10) is characterized by an average *Correctness AUC* that is smaller than 1—i.e., the graph is situated below the optimal graph—along with higher values of *Incorrectness AUC*, *Correct Decreases*, and *Incorrect Increases*. Finally, **Cluster 3** (n=2) includes two unique cases that are characterized by an average *Correctness AUC* that is close to optimal, along with higher values of *Incorrectness AUC* and *Correct Decreases* and relatively moderate values of *Incorrect Increases*.

To better understand these clusters, we closely examined typical behaviors from each, see Figure 5. Student G31 was allocated to **Cluster 1**. This student made a total of

seven moves. The student's *Correct Decrease* obtained a perfect value of 6, while the *Incorrect Increase* obtained a perfect value of 0. These values represent optimal or near optimal solutions.

Student G40 was allocated to **Cluster 2**. Within a total of eight moves, the student's *Cumulative Correct/Incorrect Score* graphs show that the *Cumulative Correct Score* never decreased. Nevertheless, it reached a plateau (between 0.5-0.9), while demonstrating a corresponding increase in *Incorrect Increase Score*. Hence, Student G40's *Correct Decreases* still gets a value of 0, whereas the *Incorrect Increase* gets a value of 3. On the whole, this behavior is less than optimal, most probably as a result of early wrong classifications that were later corrected.

Finally, student G42 was allocated to **Cluster 3**. We can observe the fluctuations in the student's *Cumulative Correct Score* graph (starting at 0.45). At the same time, the *Cumulative Incorrect Correct Score* increased from 0.3 to 0.75, and then decreased but never got to a 0-value: *Correct Decreases* obtained a value of 3 and *Incorrect Increases* a value of 4. As a whole, this cluster represents paths that included early correct classifications which became non-optimal, along with some corrections, probably due to not making proper use of the feedback.

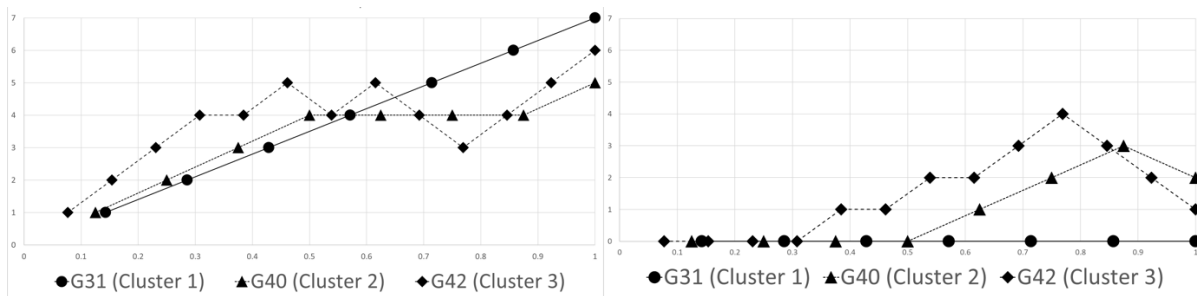


Figure 5. Cumulative Correct (left) and Incorrect (right) Score – examples from each cluster for the Polygon applet

Association between Behavior and Success in the Polygons Applet (RQ2)

We found an overall significant difference in *Total Score* between the clusters, $F(2)=4.45$, at $p<0.02$, with a post-hoc Tukey test revealing that its value in Cluster 1 was statistically significantly higher than in Cluster 2 ($t=2.94$, $p<0.05$); no differences were found in *Total Score* between Cluster 1 and Cluster 3 ($t=1.11$, $p=0.55$), or between Cluster 2 and Cluster 3.

Behavior in the Traffic Applet (RQ1)

Here too, the optimal model was based on three clusters (BIC=118.4). Cluster centers are summarized in Table 1 and described here. **Cluster 1** ($n=14$) is characterized by a *Correctness AUC* average that is very close to the optimal value and by low values—both in absolute terms and in comparison to the other clusters—of *Incorrect AUC*, *Correct Decreases*, and *Incorrect Increases*. The low deviations should also be noted. **Cluster 2** ($n=12$) is characterized by an average *Correctness AUC* that is less than 1—i.e., the graph is situated below the optimal graph—along with rather low values of

Correct Decreases, and medium values of *Incorrectness AUC* and *Incorrect Increases*. Finally, **Cluster 3** (n=3) includes a few unique cases characterized by an average *Correctness AUC* that is higher than 1—i.e., their graph is situated above the optimal graph—along with rather high values of *Incorrectness AUC*, *Correct Decreases*, and *Incorrect Increases*.

Here too, we undertook a close examination of typical behaviors from each cluster. Cluster characteristics are overall similar to those in the Polygons applet, with **Cluster 1** represents optimal or near optimal solutions, **Cluster 2** represents sub-optimal solutions resulting from early wrong classifications that in most cases were relatively easily corrected, and **Cluster 3** represents non-effective solutions that are a result of not acting properly upon feedback.

Association between Behavior and Success in the Traffic Applet (RQ2)

There were no significant differences in *Total Score* when compared across clusters, $F(2)=2.51$, at $p=0.101$. However, a post-hoc Tukey test revealed that *Total Score* in Cluster 1 was marginally significantly higher than in Cluster 3 ($t=2.24$, $p=0.08$), and no significant differences between Cluster 1 and Cluster 2 ($t=0.49$, $p=0.88$) or between Cluster 2 and Cluster 3 ($t=1.91$, $p=0.16$).

Table 1. Means (SD) of the research variable across the resulting clusters

Variable / Cluster	Polygons			Traffic		
	1 n=17	2 n=10	3 n=2	1 n=14	2 n=12	3 n=3
Correctness AUC	1.00 (0.07)	0.80 (0.14)	1.00 (0.03)	1.01 (0.12)	0.86 (0.13)	1.15 (0.26)
Incorrectness AUC	0.07 (0.08)	0.24 (0.09)	0.20 (0.29)	0.06 (0.07)	0.18 (0.13)	0.27 (0.24)
Correct Decreases	0.01 (0.03)	0.24 (0.09)	0.19 (0.06)	0.02 (0.07)	0.05 (0.10)	0.24 (0.21)
Incorrect Increases	0.13 (0.12)	0.22 (0.08)	0.15 (0.22)	0.14 (0.09)	0.23 (0.13)	0.33 (0.22)
Total Score	0.86 (0.14)	0.67 (0.18)	0.73 (0.38)	0.83 (0.15)	0.79 (0.19)	0.57 (0.25)

Associations Between Cluster Allocations across Applets (RQ3)

We observed that participants who were allocated to Cluster 1 in the Polygons applet were divided equally between Cluster 1 (9 of 17) and Cluster 2 (8 of 17) in the Traffic applet. Those who were allocated to Cluster 2 in the Polygons app were distributed to Cluster 1 (5 of 10), Cluster 2 (n=3) and Cluster 3 (n=2) in the Traffic applet. Finally, the two participants allocated to Cluster 3 in the Polygons app were allocated either to Cluster 2 or to Cluster 3 in the Traffic applet. Testing these associations, using Fisher's

Exact Test, resulted with a marginally significant result, at $p=0.089$. This indicates on some associations between the cluster allocations in the two applets.

SUMMARY AND DISCUSSION

The goal of the current study was to examine students' behavior while they worked on two-choice sorting tasks in the context of reflective symmetry, in a digital platform that supplied them with simple, immediate feedback regarding correctness or incorrectness. Recall that in the applets we developed, the provided immediate feedback could have helped all students completing the task with a perfect score in a nearly optimal way, if they had closely followed the feedback. Nevertheless, our empirical findings reveal that this was not always the case, as the final score for some students was not perfect; that is, they had finished the task with some figures which were classified incorrectly. How can this behavior be interpreted? There are several possible explanations. Perhaps these students did not notice the feedback at all – a behavior that was previously observed (e.g., Timmers & Veldkamp, 2011). Other students may have noticed the feedback message about incorrect classifications, but did not make any effort to make changes. Still others may have noticed the feedback message but did not agree with it, hence preferring to leave the object in the area they thought was correct. Furthermore, students might not have been able to relate the feedback to the corresponding objects or draw valid conclusions from the feedback. It seems that some students missed opportunities to learn from the feedback, an important understanding for designers of digital learning, researchers, and teachers.

Conversely, we found evidence of students who did consider the immediate, simple feedback (Cluster 2) for adjusting their conceptualization. This is in line with previous studies in different mathematical domains, illustrating the usefulness of simple feedback (e.g., Frey, 2022). The shift of students from Cluster 2 in the Polygons applet to Cluster 1 in the more complex Traffic applet, as well as from Cluster 3 to Cluster 2, is encouraging; it may be a sign that these students learned from the first task about either reflective symmetry or feedback-use (or both), and applied that knowledge in the second task. Besides these findings, in the context of reflective symmetry, our approach for studying student behavior could be extended to other mathematical domains and to other designs of sorting tasks.

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COMPARING STUDENT VALUES AND WELLBEING ACROSS MATHEMATICS AND SCIENCE EDUCATION

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Recognising and supporting student wellbeing in schools has become a global priority. Wellbeing is value dependent and differs across contexts, including school subjects. Even as a growing number of studies, curricula, and programs focus on student wellbeing, applications to specific school subjects are scant. Here we compare 292 grade eight students' conceptualisations of wellbeing in their mathematics with their science classrooms. Findings point to similar values supporting wellbeing in mathematics and science, yet with differences in the hierarchy of these values. This study strengthens understanding of optimal feeling and functioning in specific subjects and points to areas to target to improve students' feelings and functioning.

INTRODUCTION

Mathematics and science education often have poor participation in advanced subjects at schools and universities, under-representation of females and minority groups, and pervasive academic disengagement (English, 2016), suggesting that there is poor student wellbeing in these subjects. Maths and science subjects typically focus on academic performance and functioning, with little consideration of student wellbeing, defined here as the fulfilment of core values within the learning process, accompanied by positive feelings (e.g., enjoyment) and functioning (e.g., engagement) specific to the subject (Hill et al., 2022). Higher student wellbeing is associated with better academic achievement, greater participation, and improved retention in STEM subjects (Watt et al., 2019). We suggest that mathematics and science education require a paradigm shift, moving from the over-emphasis on academic performance to include a greater focus on student wellbeing.

Whilst the cultivation of student wellbeing in schools has become a global priority (e.g., United Nations, n.d.; Kern & Wehmeyer, 2021), how wellbeing looks and operates in individual subjects remains poorly understood. A whole school wellbeing approach can be problematic because just like a grade point average masks subject differences in academic performance, global measures of wellbeing at school hides subjects in which students are flourishing or languishing. When the language of and principles of wellbeing are targeted and contextualised, students can better articulate and adapt wellbeing into their learning context (Oades et al., 2021).

According to value fulfillment theory of wellbeing (Tiberius, 2018), students' values point to the conditions required for them to thrive within specific contexts. Just like grades can vary in one subject versus another, what a student values in one subject (e.g., mathematics) can differ from another (e.g., science) – thus wellbeing might also vary across subjects. Whilst several recent studies have explored the values supporting

students' wellbeing in mathematics (e.g., Hill et al., 2021; 2022), how these values differ across subjects is yet to be determined and thus is the focus of this study.

BACKGROUND AND THEORETICAL FRAMEWORK

We define student wellbeing according to value fulfilment theory, where individuals' experiences of wellbeing depend on their values (Tiberius, 2018), which can differ across personal, cultural, and/or contextual conditions (Alexandrova, 2017). At the highest level are 'ultimate' values (UV), the things valued for their own sake (e.g., close personal relationships), which are also the most impactful indicators of wellbeing (Tiberius, 2018). These ultimate values can be fulfilled in many ways through various 'instrumental' values (IV), like valuing friendships, group work, or respect to promote close relationships (an UV). Instrumental values differentiate individuals, whilst UVs tend to be consistent across individuals (Tiberius, 2018).

When values are fulfilled, it coincides with subjective experiences of feeling good (i.e., hedonia) and functioning well (i.e., eudemonia) (Huppert and So, 2013). Applied to mathematics education, a student who values working collaboratively would likely feel happy and engage more during collaborative opportunities, whereas the student would disengage with singular activities. Applied to science education, a student who values active activities would feel happy and engage during interactive sessions, whereas the student would disengage with lecture-based pedagogies. Most modern wellbeing models include both hedonic and eudemonic dimensions (e.g., Huppert & So, 2013; Kern et al., 2016; Seligman, 2011). Here we specifically focus on Seligman's PERMA model (Positive emotions, Engagement, Relationships, Meaning, and Accomplishment; Seligman 2011); Kern's EPOCH model (Engagement, Perseverance, Optimism, Connectedness, Happiness; Kern et al., 2016); Ryan and Deci's self-determination theory (2017); and Mathematical Wellbeing (MWB; Clarkson et al., 2010; Hill et al., 2022).

Several recent studies have explored student wellbeing in mathematics. A scoping review revealed seven ultimate values (UVs) supporting student MWB (accomplishments, cognitions, engagement, meaning, perseverance, positive emotions, and relationships; Hill et al., 2022). Australian and Chinese students mentioned these same seven UVs when describing factors and learning moments contributing to their MWB (Hill et al., 2021; Hill & Seah, 2022). A survey with New Zealand students discovered students' mathematical wellbeing declined over the primary to secondary school years (Hill, Bowmar et al., 2022). In mathematics, these seven UVs appeared across diverse student groups and countries. However, it remains unclear if students' values and wellbeing in mathematics appears in other subjects. We begin with science education because of its interrelatedness with mathematics. Along with the seven UVs identified in mathematics education, we add autonomy, which has also been identified in the literature as a core wellbeing construct. Scoping reviews (e.g., Hill, 2023) revealed science students valued autonomous learning. Additionally, self-determination theory posits autonomy as one of three basic psychological needs

supporting wellbeing (Ryan & Deci, 2017). Thus it made sense to include autonomy as a dimension. We ask: *How do the same students' conceptualisations of their wellbeing in mathematics differ to science across these eight dimensions?*

METHODS

Participants included 292 grade eight students (51% females), aged 13–14 years attending 1 of 8 urban and regional schools in and around Melbourne, Australia. Ethnicities included 214 Australian, 21 European, 33 Asian, 19 Indian/Pakistani, 2 Indigenous Australian, 2 South American, and 1 Middle Eastern student.

Students completed two surveys: one on their mathematics wellbeing during a mathematics lesson, the other on their science wellbeing during a science lesson. The wording of each survey was identical except for referencing either mathematics or science. Here we focus on the open-ended questions included in the surveys: 1) What makes you feel good and function well in [maths/science], and why? 2) What is the most important thing for you when learning [maths/science], and why?

Earlier studies (e.g., Hill et al., 2021) found the first survey questions, aligned with Huppert and So's (2013) definition of wellbeing, prompted students to reflect on more holistic components of their learning and wellbeing (e.g., friendships, enjoyment, family). The second questions, which were derived from value fulfillment theory (Tiberius, 2018), explored values (i.e., anything that is important) and provoked more academically related reflections (e.g., achievement, academic skills). By combining and coding these two survey questions together, we hoped to capture a holistic conceptualisation of students' wellbeing across both subjects.

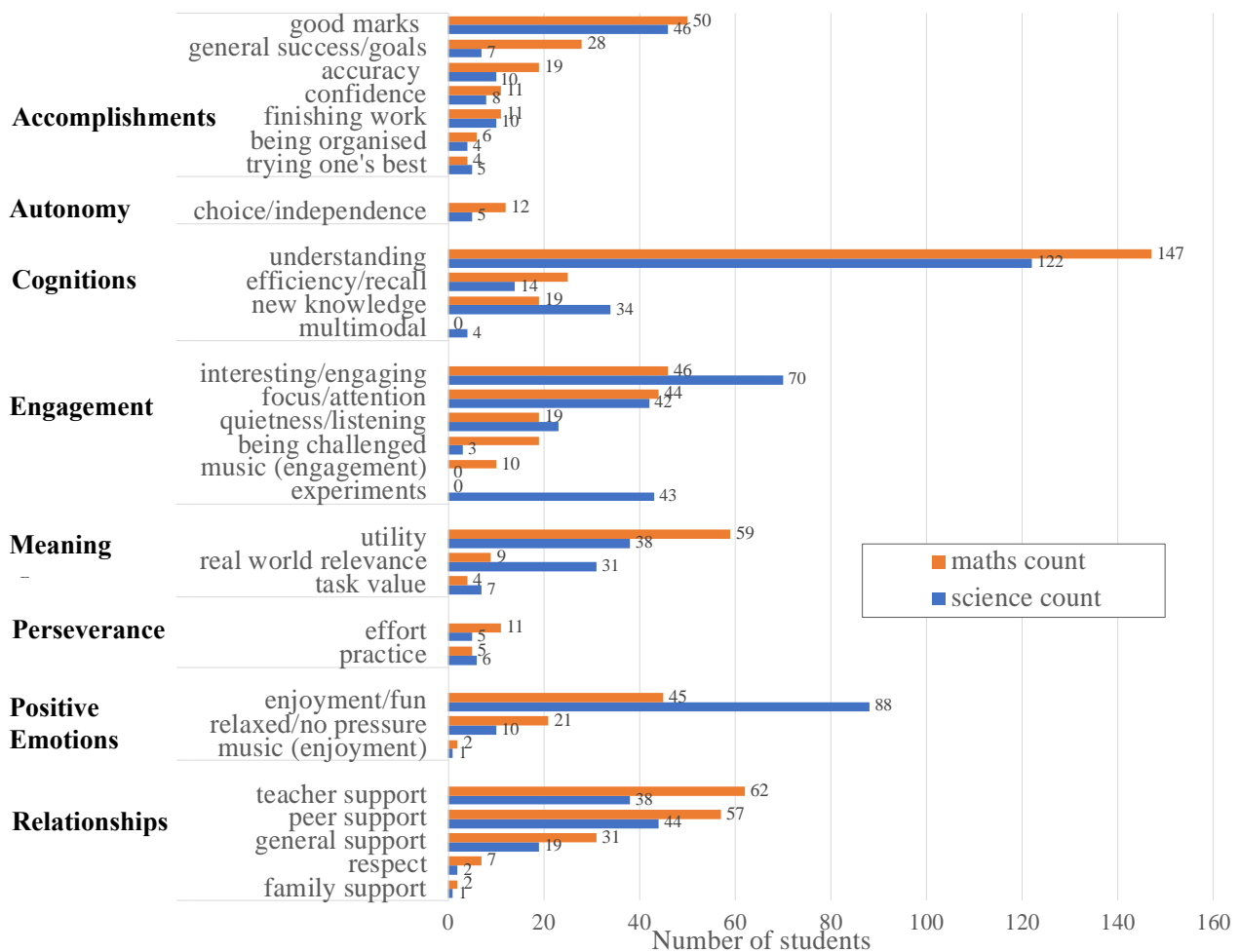
Student responses were analysed using NVivo12 using reflexive thematic analysis (Braun & Clarke, 2019). Initial inductive codes were generated using a data driven approach, for instance "*it will help me in the future*" was coded as *utility*. Next, these inductive codes were categorised deductively into the eight wellbeing themes identified in the literature. Figure 1 summarise the process of categorising inductive codes into the deductive wellbeing components.

RESULTS

Figure 1 summarises the overall frequency of inductive codes in mathematics (orange bars) and science (blue bars). Importantly, the majority of codes appeared in students' conceptions of their wellbeing in both mathematics and science, except for multi-modal representations and experiments – mentioned for science only. At the group level, the same eight UVs emerged across both disciplines, summarised in Figure 1. At the individual student level, there were statistically significant associations between students mentioning each UV in both mathematics and science education, except for engagement, because students associated engagement more with their wellbeing in science than in mathematics. The closest associations across subjects was for relationships, perseverance, and meaning.

The students valued peer support more than teacher support when studying science, but this relative valuing was reversed when these same students were engaged with mathematics. Students valued being interested in their science learning more than in their mathematics learning— often because students equated engaging experiments with their wellbeing. Likewise, enjoyment and fun were noted more often when learning science than mathematics. When learning mathematics, students equated being challenged, understanding their learning, and feeling successful/achieving goals with their wellbeing more than in science. Whilst valuing utility and links to employment were noted more in mathematics than science, students valued real world relevance more for their science than their mathematics learning.

Figure 1. Inductive value frequencies. Bars and data labels represent the number of students mentioning each inductive value for mathematics (orange bars) and science (blue bars).



The hierarchy of UVs differed slightly across subjects (see Table 1 for student frequency counts), with students in science valuing cognitions most frequently, then engagement, relationships, accomplishment, meaning, perseverance, positive emotions, then autonomy. The same students valued cognitions most when learning mathematics, followed by relationships, engagement, accomplishments, meaning, perseverance, positive emotions, then autonomy.

Table 1. Frequency of students mentioning themes in science, maths, and both subjects. Both M + S = Number of students mentioning theme for both maths and science; χ^2 = difference of mentions in science versus maths.

Ultimate value	Science #	Maths #	Both M + S	χ^2	<i>p</i>	ϕ
Accomplishment	75	108	46	25.7	<.001	.30
Autonomy	5	12	2	16.63	<.001	.24
Cognitions	144	182	107	17.36	<.001	.24
Engagement	139	109	58	2.19	.14	.02
Meaning	71	67	34	33.01	<.001	.34
Perseverance	11	16	5	35.27	<.001	.35
Positive emotions	11	16	5	4.90	.03	.13
Relationships	89	140	67	38.32	<.001	.36

DISCUSSION

Earlier studies (e.g., Hill, Kern et al., 2022) identified seven UVs supporting MWB and here we examined if these seven UVs, plus autonomy, appeared across both mathematics and science. Students tended to mention the same eight UVs across both subjects. However, engagement was more associated with students' science than mathematics wellbeing. Taken together students' values and conceptions of their wellbeing across mathematics and science appear to be similar.

However, our results also point to subtle subject-related differences in IV. For example, students valued interesting and engaging pedagogies (especially experiments), enjoyment, and real-world relevance more for science than in mathematics. This is not surprising, considering that science is often more hand-on, humanised, and embedded in real-world contexts than mathematics (Bishop et al., 2006). Also, here we focus on Year 8 students – a period when experiments in science are still novel (Abrahams, 2011), potentially contributing to the higher valuing of interesting pedagogies in science than in mathematics. Students (and teachers) often perceive experiments as the fundamental appeal of science (Abrahams, 2011), exemplified by a student in our study noting “*science is the only subject where I get to blow stuff up*”.

Whilst science experiments may offer immediate engagement, students often remember only what they did, rather than remembering the intended learning outcomes (Abrahams, 2011). Students in our study valued mathematics more for utilitarian purposes, and for success in life compared to science, perhaps because mathematics was perceived as essential for everyday life and employment more than something to enjoy and be interested in. Also, in other studies, students often linked science with

entertainment and fun, such as museums, exhibitions, and science documentaries, which differs from mathematics (Bishop et al., 2006) – potentially contributing to higher interest values in science than mathematics.

Conversely, students valued cognitive aspects more for their mathematics than their science learning, thus mathematical understanding was especially important for students' MWB. The relationship aspects, especially teacher support, were also more associated with wellbeing in mathematics than in science. Conversely, in science, the students valued support from peers, perhaps because science pedagogies are often more collaborative in nature. Students often see mathematics as progressive, abstract, and linear in nature and fear being left behind in a fast-paced curriculum (Hill et al., 2021). In our study, students' desire to understand their learning may have contributed to the higher valuing of teacher support and a relaxed class climate to feel safe making mistakes. Typically, mathematics teaching focuses on developing competency and achievement, with less attention to relationships. The impact of positive relationships, especially teacher support, on overall wellbeing, school belonging, and academic achievement are well recognised (e.g., Kern & Wehmeyer, 2021). Closer attention to building relationships, especially between teachers and students, may be essential to fulfil student's valuing of mathematical understanding and support their MWB.

IMPLICATIONS & CONCLUSION

The first step to building wellbeing in classrooms is to make wellbeing visible (Waters, 2021). A central aim of this study was to make wellbeing visible in mathematics and science by capturing how students communicate the values supporting their wellbeing in these subjects. Making values visible and central to learning can help students to become more engaged, feel calmer and more connected, and improve their (and their teachers') wellbeing (Lovatt et al., 2010). Our findings point to target values to support teachers and students to talk about, assess, and then develop wellbeing across eight broad UVs. Future studies might use methods outlined here to explore student wellbeing in other subjects such as literacy.

Science and mathematics education have strong negativity biases – that is a preoccupation with the failings, anxieties, misunderstandings, 'achievement gaps' and so forth. We suggest a need to incorporate a wellbeing perspective into mathematics and science curricular and pedagogical practices, where students' values and strengths are central to, celebrated, and nurtured in the learning process. This could mean using challenging science/mathematics tasks to teach students about cognitive reappraisal, resilience, and meaningful failure; using project work to encourage collaboration and respect; or identifying students' (and teachers') signature strengths to promote a sense of meaning and accomplishment in mathematics/science. Incorporating wellbeing into the "caught" curriculum, beyond the "taught" curriculum, may make wellbeing knowledge and skills more explicit (White & Kern, 2018), whilst giving teachers a positive language to communicate about wellbeing and thus giving teachers greater agency to make a difference for their students' lives (Kern & Wehmeyer, 2021).

Lastly, given the shortage of mathematics teachers in many countries (English, 2016) our findings suggest science teachers may be a good fit for the mathematics classroom, since their approaches to enhancing students' science wellbeing may also benefit students' MWB development.

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THE CONNECTION BETWEEN MATHEMATICS AND OTHER FIELDS: THE DISCIPLINE OF MATHEMATICS VS. MATHEMATICS EDUCATION

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This theoretical essay argues that the connection between mathematics and diverse fields and life areas is portrayed differently in the discipline of mathematics and in the mathematics education scholarly work. In the discipline of mathematics, an important characteristic of this connection is that it is bi-directional, reflecting mutual contribution between mathematics and other fields and life areas. In contrast, the mathematics education scholarly work emphasizes mainly one direction, namely the contribution of the discipline of mathematics to other fields, whereas the contribution of different fields to the development of the discipline of mathematics is often obscured. Interpretations and ways to address this discrepancy are proposed in relation to two areas: mathematical applications and modeling, and the history of mathematics.

Knowledge about the nature of mathematics as a scientific discipline comprises knowledge about the connection between mathematics and different scientific domains and life areas (Blum & Niss, 1991; Hoffmann & Even, 2018; Jankvist, 2009b; Kaiser & Sriraman, 2006; Ziegler & Loos, 2014/2017). An important characteristic of this connection is that it is bi-directional, reflecting mutual contribution (e.g., Einstein Institute of Mathematics at The Hebrew University of Jerusalem, 2022; Jankvist, 2009b; Kaiser & Sriraman, 2006). One direction – to which we refer as the *M-application* direction of the mutual contribution – denotes the contribution of the discipline of mathematics to solving problems in different fields. The other direction – to which we refer as the *developing-M* direction of the mutual contribution – denotes the contribution of different fields to the development of the discipline of mathematics by raising new mathematical questions and motivating the development of new concepts, methods, and research areas.

A recent study (Hoffmann & Even, 2022) examined what teachers may learn about the nature of the above-mentioned mutual contribution in an academic program that offered opportunities to learn about the connection between the discipline of mathematics and different fields and life areas, and what mathematicians, who taught in this program, wanted to teach teachers about the nature of this mutual contribution. The results indicated that all the mathematicians wished to teach teachers about both directions of the mutual contribution. For example, one of the mathematicians stated: “It is very important, the connection between mathematics and other fields and the mutual fertilization.” Relating to the *M-application* direction, a mathematician explained:

The idea is... to show some examples of mathematics in things that they [the teachers] know, like GPS, like Google, like RSA... all sorts of things they encounter on a daily basis and there is math... and somehow convince them that like, math is not something abstract. That it really helps in life.

Relating to the *developing-M* direction of the mutual contribution another participant stated:

A great many of the developments in mathematics had real motivation... questions from life, not from mathematics, and the mathematical way helped solve them, and then they developed mathematics.

In contrast to the mathematicians, none of the teachers referred to the connection between mathematics and different fields and life areas as comprising mutual contribution. Instead, they referred to the *M-application* direction only, reporting on considerably advancing their knowledge and appreciation regarding the contribution of mathematics to other fields. For example, one of the teachers remarked:

So, what [the course instructor] actually said, that mathematics... there is a problem in life, and one needs to construct a model, a mathematical model, in order to solve that problem... And it was something that was really new and interesting for me to see mathematics like this... Beforehand, I viewed it as an intellectual field, which is fun to deal with, it is interesting in itself. I did not look at the, at what it actually gives in practice.

This theoretical essay situates these empirical results in a broader context, arguing that they echo the different ways the connection between mathematics and diverse fields and life areas is portrayed in the discipline of mathematics and in the mathematics education scholarly work.

THE MUTUAL CONTRIBUTION BETWEEN MATHEMATICS AND OTHER FIELDS: THE CASE OF THE DISCIPLINE OF MATHEMATICS

The interaction between mathematics and different fields and life areas is realized in the discipline via applied mathematics. As mentioned above, an important characteristic of this interaction is that it is bi-directional, involving mutual contribution. This is clearly expressed in the following description of applied mathematics:

Applied mathematics is concerned with the interaction between mathematics and other sciences, such as physics, computer science, engineering, economics and biology. This interaction is often bi-directional: mathematical concepts and techniques are used to model and solve concrete problems in other fields. Reciprocally, scientific progress raises new mathematical problems, and motivates the development of new mathematical concepts and tools” (Einstein Institute of Mathematics at The Hebrew University of Jerusalem, 2022).

The contribution of mathematics to solving problems in different fields – the *M-application* direction – occurs when mathematical tools and techniques are used to model and analyze different situations in non-mathematical contexts. It includes works on spectral and dynamical problems of quantum mechanics, population genetics, image processing and medical imaging, mathematical finance, quantitative risk management,

and many more. For instance, mathematics contributed greatly to solving the immense problem of mapping the human genome – a prominent biological project initiated at the end of the 20th century – by providing advanced mathematical tools and techniques such as hidden Markov chains and pattern recognition, that enabled researchers to locate genes in the decoded sequence of all nucleotide bases that make up the human genome (Malkevitch, 2002). Also, systems of differential equations that are based on laws of physics are used nowadays to mathematically model current atmosphere and ocean conditions at different locations and altitudes in order to generate a weather forecast (Bauer et al., 2015).

The contribution of different fields to the development of the discipline of mathematics – the *developing-M* direction – occurs when work on solving problems in other fields and life areas raises new mathematical questions which generate new mathematical concepts, methods, theories, and areas of research. Such works characterize quite a few of past and modern-day mathematical developments (but of course not all – mathematical questions arise from both inside and outside mathematical contexts). For instance, the theory of probability and in particular concepts such as randomness, independence and equiprobability have their origin in works that attempted to answer questions raised in the context of gambling (Rowlett, 2011). Also, the need to correct digital information that was damaged during transmission over unreliable or noisy communication channels, and the need to reduce data size in order to more efficiently transform digital information, were the impetus for establishing a new mathematical field, namely information theory, during the first decades of the 20th century (Jankvist, 2009a).

THE MUTUAL CONTRIBUTION BETWEEN MATHEMATICS AND OTHER FIELDS: THE CASE OF MATHEMATICS EDUCATION

The interaction between mathematics and diverse fields and life areas is addressed in the mathematics education scholarly work primarily in relation to the teaching and learning of mathematical modeling. Occasionally, it is addressed also in relation to the teaching and learning of the history of mathematics.

The teaching and learning of mathematical modeling have seen a substantial increase of interest in the past four decades. The literature provides important information regarding two central kinds of using modeling in mathematics education: as a vehicle and as a content (e.g., G. A. Stillman, 2019). The former implies teaching modeling as a means for achieving goals that do not necessarily concern mathematical modeling. For example, a professional development program for teachers that incorporates mathematical modeling activities in order to promote knowledge of statistical numeracy (Lamb & Visnovska, 2015). When mathematical modeling is used as a vehicle, both directions of the mutual contribution between mathematics and other fields are obscured.

In contrast, when mathematical modeling is used as a content, the *M-application* direction is greatly illuminated. Treating mathematical modeling as a content to be

learned is associated in the literature mainly with the development of competencies of using mathematics for better understanding the world around us and for solving problems arising in non-mathematical situations (Cevikbas et al., 2022). This approach commonly incorporates a focus on the complex processes of modeling, often through the diverse frameworks of the modeling cycle. The development of a task design and implementation framework for fostering mathematical modeling competencies (Geiger et al., 2022), and the efforts to identify stumbling blocks and productive metacognitive acts associated with transitions between stages in the modeling process (Jankvist & Niss, 2020; G. Stillman, 2011) illustrate this approach.

A complementary approach to mathematical modeling as a content is associated with the development of knowledge about the nature of mathematics as a scientific discipline of which applications and modeling are an essential part:

The “picture of mathematics” argument insists that it is an important task of mathematics education to establish with students a rich and comprehensive picture of mathematics in all its facets, as a science, as a field of activity in society and culture. Since modeling and applications constitute an essential component in such a picture, this component should be allotted an appropriate position in mathematics curricula. (Blum & Niss, 1991, p. 43)

Our review of the literature suggests that the teaching and learning of mathematical modeling usually explicate the *M-application* direction of the mutual contribution. For example, Kaiser and Schwarz (2006) reported a change in the mathematical beliefs of students and teachers towards a more “application-oriented” view of the discipline following teachers’ participation in university-based mathematical modeling seminars. Maas (2013) reported a similar change with students who participated in an intervention that incorporated mathematical modeling activities. In contrast, the *developing-M* direction, although occasionally mentioned as a component of the modeling process (e.g., Blum & Niss, 1991; Kaiser & Sriraman, 2006), is rarely addressed in the educational literature that concerns mathematical modeling and applications.

The potential to highlight the *developing-M* direction of the mutual contribution is sometimes linked in the literature to the teaching of the history of mathematics (e.g., Jankvist, 2009b; Katz, 1993; Tzanakis & Arcavi, 2002). Similar to the two kinds of using mathematical modelling and applications – as a vehicle and as a content – Jankvist (2009b) suggests two kinds of using the history of mathematics in mathematics education. One is as a tool to support the teaching and learning of specific mathematical contents. For example, to raise students’ interest or to offer a different point of view on a specific mathematical idea. The other kind is as a goal in itself, where the “focus is on the developmental, and evolutionary aspects of mathematics as a discipline” (p. 239).

Theoretical considerations suggest that the developmental and evolutionary aspects that may be illuminated by the history of mathematics include the mutual contribution between mathematics and various fields and life areas in general and the *developing-*

M direction in particular. Exemplifying the latter, Jankvist (2009b) claimed that an important goal for teaching the history of mathematics is to show students that the evolution of mathematics is “driven by internal and *external* forces” (p. 239). Similarly, Tzanakis and Arcavi (2002) suggested that the history of mathematics provides an opportunity to show that mathematical research is “often motivated by questions and problems coming from apparently unrelated disciplines” (p. 205). And Katz (1993) stated in the context of teaching calculus:

An historical approach to calculus helps to provide not only a motivation for its study but also a reason for the students further to explore the connections between their studies and the world around them. By an historical approach to calculus, I do not mean simply giving the historical background for each separate topic or giving a biographical sketch of the developers of various ideas. I do mean the organization of the topics in essentially their historical order of development as well as the discussion of the historical motivations for the development of each of these topics, both those within mathematics and those from other scientific fields. (p. 243)

Yet, missing are empirical studies that examine the contribution of studying the history of mathematics to knowledge about the evolutionary and developmental aspects of the discipline of mathematics (e.g., Clark, 2014), and consequently, about the mutual contribution between mathematics and other fields.

DISCUSSION

This theoretical essay argues that the connection between mathematics and diverse fields and life areas is portrayed differently in the discipline of mathematics and in the mathematics education scholarly work. In the discipline of mathematics, this connection is characterized by mutual contribution: Mathematics contributes to solving problems in different areas (the *M-application* direction), and different fields of science and life areas contribute to the development of the discipline of mathematics by raising new mathematical questions and motivating the development of new mathematical concepts, methods and research areas (the *developing-M* direction). In contrast, in the mathematics education scholarly work, the connection between mathematics and other fields is characterized mainly by a uni-directional contribution, where mathematics contributes to solving problems in other fields, but the contribution of various fields to the development of mathematics is rarely attended to.

These different portrayals of the connection between mathematics and diverse fields and life areas are reflected in the results of an empirical study (Hoffmann & Even, 2022) that examined what teachers may learn about the nature of the mutual contribution in an academic program that offered opportunities to learn about the connection between the discipline of mathematics and different fields and life areas, and what mathematicians, who taught in this program, wanted to teach teachers about the nature of this mutual contribution. Whereas the mathematicians maintained that both directions of the mutual contribution are important components of teachers’ knowledge which they wished to advance, the teachers reported on considerably advancing their knowledge and appreciation only regarding one direction of the mutual

contribution, namely the contribution of the discipline of mathematics to other domains.

A question then arises: Should the connection between mathematics and other fields be portrayed in mathematics education more in line with the way it is presented in the discipline of mathematics? Our position is that it should. Both knowledge of specific mathematics topics, concepts, and procedures, and knowledge about the nature of the discipline, are important for teachers and students. Knowledge about the nature of mathematics as a scientific discipline comprises knowledge about the nature of the connection between the discipline of mathematics and different scientific domains and life areas (Blum & Niss, 1991; Hoffmann & Even, 2018; Jankvist, 2009b; Kaiser & Sriraman, 2006; Ziegler & Loos, 2014/2017), of which an important characteristic is that it is bi-directional, reflecting mutual contribution.

Currently, the connection between mathematics and diverse fields and life areas is addressed in mathematics education scholarly work primarily in relation to the teaching and learning of mathematical modeling. Occasionally, it is addressed also in relation to the teaching and learning of the history of mathematics.

The teaching and learning of mathematical applications and modeling is mainly associated in the literature with advancing the *M-application* direction of the mutual contribution (Cevikbas et al., 2022), revealing that it can enhance teacher knowledge regarding the picture of mathematics as an applied science that contributes to problem-solving in diverse areas of life (e.g., Hoffmann & Even, 2022; Kaiser & Schwarz, 2006; Maass, 2013). There are several potential avenues for advancing knowledge about the *developing-M* direction in the context of teaching mathematical applications and modeling. For example, to extend conceptualizations of the modeling cycle to explicitly include the *developing-M* direction. Additionally, using the teaching and learning of mathematical modeling as a tool for developing knowledge of mathematical concepts, procedures, and theorems that are new for the learner (e.g., Lamb & Visnovska, 2015), in a way, is analogic to the way that diverse areas contribute to the development of the discipline of mathematics. It might be interesting to explore how such mathematics learning experiences may contribute to learners' perception about the ways that the discipline of mathematics develops.

Theoretical measures and some empirical findings (e.g., Jankvist, 2009a, 2009b; Niss et al., 2007; Tzanakis & Arcavi, 2002) indicate that the teaching and learning of the history of mathematics may contribute to developing knowledge regarding the mutual contribution between mathematics and other disciplines. The history of mathematics reveals that mathematics is not a fixed and complete science but is lively and developing, where the development of the discipline lies in mathematical work related to questions that originate in and outside mathematics. Thus, the teaching and learning of the history of mathematics has the potential to enhance knowledge regarding the *developing-M* direction by exposing the central contribution of work in other fields to the development of the discipline of mathematics as a scientific discipline. One

enterprise of future research could be the development of rich and comprehensive teaching materials that explicitly exemplify such contributions throughout the history of mathematics. A promising avenue in this regard is the teaching and learning of the history of modern applied mathematics (Jankvist, 2009a).

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COMPARING TEACHER GOALS FOR STUDENT FOCUSING AND NOTICING WITH STUDENT OUTCOMES FOR FOCUSING AND NOTICING

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The topic of study in this report is student focusing and noticing. Specifically, we examined a teacher's goals for student focusing and noticing and the student outcomes for focusing and noticing. The mathematics context for this research was quadratic functions and covariational reasoning. Two whole-class discussion episodes were analyzed. Results showed ways that the teacher's goals and student outcomes were aligned and three ways that they were misaligned. These results could inform how quadratic functions are taught and how teachers can improve the alignment between their goals for student focusing and noticing and student outcomes for focusing and noticing.

INTRODUCTION

Focusing and noticing are important aspects of learning (“noticing, or perceiving, provides the rich backdrop of experience on which learning depends;” Mason, 2002, p. 33). However, as teachers know, “any human adult who interacts with another has opportunities to notice that in many situations the other perceives and attends to things that are different from those one attends to oneself” (von Glasersfeld, 1995, p. 179). Thus, what mathematics students focus on and notice may not align with what mathematics teachers want them to focus on and notice. In this report, we examine how a teacher's goals for student focusing and noticing compare to student outcomes for focusing and noticing. We situated our study in the context of quadratic functions (QF).

Student Focusing and Student Noticing

According to our conceptualization, student focusing and noticing are two related processes. Student focusing itself has two parts. The first part of student focusing is when students direct one or more senses toward one or more features of a perceptual or conceptual field (“focused attention picks a chunk of experience, isolates it from what came before and from what follows;” von Glasersfeld, 1995; p. 91). The second part of focusing is when students make a mental record of the feature or features their senses are directed toward (“For the mind, then, ‘to posit it as object against itself’, is to re-present it,” von Glasersfeld, p. 91). In other words, student focusing goes beyond simply directing senses toward a feature of a perceptual or conceptual field.

Student noticing follows and builds on student focusing. Specifically, noticing is when students identify properties, regularities and/or irregularities, or concepts about the features of a perceptual or conceptual field that are being focused on (“establish[ing]

regularities in the flow of experience,” von Glasersfeld, p. 144). In other words, student noticing goes beyond students simply focusing on something.

Teacher Goals for Student Focusing and Noticing Versus Student Outcomes for Focusing and Noticing

We define *teacher goals for student focusing and noticing* as those features of a perceptual or conceptual field that the teacher wants students to focus on and the properties, regularities/irregularities, or concepts the teacher wants students to notice. We define *student outcomes for focusing and noticing* as those features of a perceptual or conceptual field that students actually focus on and the properties, regularities/irregularities, or concepts students actually notice. Prior research on student noticing has looked at what students focus on and how the classroom interactions bring about what gets focused on and noticed (Lobato et al., 2013).

Importance of Student Focusing and Noticing for Reasoning About Mathematics

We conceptualize student focusing and noticing as of key importance for reasoning about mathematics. Previously we stated that noticing is dependent on focusing (i.e., only what students focus on can they noticing something about). In a similar way, reasoning is dependent on focusing and noticing, because only what students focus on and notice can they reason about (“what we fail to notice is unlikely to have much influence upon our [mental] actions;” Mason, 2002, p. 29, parenthetical added). In other words, student focusing and noticing “provide the perceptual and/or conceptual material on which learning processes operate” (Hohensee, 2016, p. 71). Thus, student focusing and noticing are at the leading edge of cognition.

Student Focusing and Noticing About Quadratic Functions That Supports Covariational Reasoning

Student focusing and noticing are important for reasoning about QFs (Lobato et al., 2012). An important feature of QFs that students could focus on is the quantities involved in QFs. For example, students could direct their eyes toward and make a mental record of the rows of a distance-time (DT) QF table. Important regularities of QFs that students could notice are the regularity that the changes in the dependent variable are changing by constant amounts when the changes in the independent variable are constant (Lobato et al., 2012).

Covariational reasoning is defined as reasoning about “the way the dependent and independent variables [of a function] change together” (Ayalon et al., 2016, p. 381). Student focusing and noticing which, as argued above, could support mathematical reasoning in general, could also support covariational reasoning in particular. For example, students could use the noticed regularity described above to reason covariationally about generating additional values of a DTQF table.

Purpose and Research Question

Our study is the first, to our knowledge, that examines together the teacher goals for student focusing and noticing and the student outcomes for focusing and noticing. If, as explained earlier, focusing and noticing are necessary for reasoning about mathematics, then the lack of research on this topic means our study is poised to make a significant contribution to the field. Our ongoing work for this study is guided by the following question: *In the context of quadratic functions instruction, how do the teacher goals for student focusing and noticing that supports covariational reasoning during instruction compare to student outcomes for focusing and noticing?*

METHODS

Context

This study took place during a summer mathematics program for secondary students in the Mid-Atlantic region of the United States. The program focused on DTQFs and the DTQF instruction provided during the program intentionally promoted students' covariational reasoning. During the program, the students, the teacher, and a research team met for two 1-hour instructional sessions per day. Sessions were held every weekday for two weeks. Each 1-hour instructional session typically focused on a single instructional activity. The instructional activities involved the use of, or referenced, at least one DTQF animation created in SimCalc. Students often explored the DTQF animations on laptop computers through small-group activities and then participated in whole-group discussions about their mathematical reasoning on the activities.

Participants

The participants were students recruited from a youth organization ($N = 18$) that supports students from underrepresented populations academically. A research team of five conducted this study. The team was comprised of the summer program teacher, who was also the first author on this report, two mathematics education graduate students, and two high school mathematics teachers who taught in public secondary schools in the United States and had often taught QFs. The research team met at the end of each day to debrief the day's lessons and to plan for the next day.

Data Collection and Analysis

Data collection also occurred during the 2-week summer program. Specifically, all instructional sessions were video- and audio-recorded. Recordings were made of whole-group discussions and small-group activities. Artifacts (e.g., student responses on instructional activity worksheets) were also collected. However, only recordings of whole-group discussions were analysed for this report.

Transcripts of audio-recordings were analysed using qualitative coding methods. Analysis focused on the whole-group discussions, which we called *episodes*. To code the episodes, we first transcribed the recordings of the whole-group discussions. Then, the first, third, and fourth authors cooperatively developed inductive codes (Strauss & Corbin, 1985) to capture the teacher's goals for student focusing and noticing and

student outcomes for focusing and noticing. Finally, the first and second author used the coded transcripts to answer the research question.

FINDINGS

Initial analysis has revealed that for the QF instruction in our study, the teacher goals for student focusing and noticing and student outcomes for focusing and noticing had some alignment and some misalignment. By *alignment*, we mean the teacher's goals for student focusing and noticing were consistent with student outcomes for focusing and noticing. By *misalignment*, we mean the teacher's goals were not consistent with the student outcomes. Our overarching finding is that ongoing cycles of alignment and misalignment happen as the teacher and students co-construct understandings of QFs that support covariational reasoning. Moreover, we found three kinds of misalignment. To illustrate these findings, we present two back-to-back episodes, Episode 2B and 3A, that show alignment and misalignment.

Episode 2B: Alignment and Misalignment when Discussing Two DTQF Animations

Episode 2B occurred during the second instructional session, on the second day of the summer program. This session involved an activity in which students compared two characters, a clown and a frog, represented in two separate DTQF animations. In this activity, students worked in small groups to record what they noticed about each individual animation and to reason about which character was faster relative to the other animation. The animations could not be played simultaneously. However, each animation included a numbered horizontal axis from which distance measurements could be determined and a clock from which time measurements could be determined.

Teacher goals for student focusing and noticing. During this episode, the feature of the DTQF the teacher primarily tried to direct students to focus on was particular pairs of accumulated quantities of distance and time, one DT pair from each animation. Moreover, the specific DT pairs the teacher wanted students to focus on were those that could be used to determine which animation was faster. Focusing on these features could support covariational reasoning because reasoning covariationally about a DTQF requires reasoning with distances *and* times, not just distances or just times.

The property of those DT pairs that the teacher wanted students to notice was the property (or properties) that indicated which animation was going faster relative to the other animation. The following question from the teacher illustrates their attempts to get students to notice properties of the DT pairs that would indicate which animation was faster:

If Clown and Frog are going the same time, like you had 4 seconds, and you said Clown was faster. What does that mean in terms of the distance? . . . If you find that the distance is the same, what does the time tell you about the two characters?

Noticing properties of DT pairs that indicate which animation was faster relative to the other animation could supported covariational reasoning because noticing that kind of

property (e.g., noticing that the distances are the same but the times are different) could become the perceptual/conceptual material with which students could reason covariationally to decide which animation was going faster (e.g., reasoning covariationally that if both animations ran for 4 second, the animation with the greater distance is faster).

Student focusing and noticing outcomes. During this episode, some features students focused on and properties they noticed were aligned with the teacher's goals for student focusing and noticing. Students did focus on DT pairs that could be used to determine which animation would win a head-to-head race. For example, Bob said:

To get to 70 meters, [Frog] took 4.5, 4 point 50 seconds . . . Now with the [Clown], we tried stopping it around 70 meters and, we got pretty close. And 70 meters, [Clown] took about 4.2, 4 20 seconds . . . It tells us that the Clown from the jump to 70 meters is faster.

In this example, Bob illustrated a focus on DT pairs that had the same distance (i.e., 70 meters) and noticed that one DT pair had a lower time (i.e., Clown with a time of 4.2 s), which aligned with the teacher's goals for student focusing and noticing.

Other features students focused on and noticed were misaligned with what the teacher wanted students to focus on and notice. For example, sometimes students focused on a different quantity, *changes in distance*. Halima talked about this quantity in the following quote:

Another thing that I noticed with the Clown is, each time it moves, it travels way greater than it did last frame. So I think that's a very important, because if you looked at the last jump, from like 35 to 65 [meters], and then from 65 to 100 . . . It travels greater distance each frame the Clown . . . compared to the Frog.

In this example, Halima focused on the changes in distance from 35 to 65 meters and from 65 to 100 meters, which was misaligned with the teacher's goals for student focusing for the activity. In this case, the misalignment was that the students were focusing on and noticing a quantity the teacher did not have as a goal for students to notice. Therefore, something this episode shows is that one kind of misalignment between the teacher's goals for student focusing and noticing for DTQFs and the student outcomes for focusing and noticing for DTQFs is when students focus on and/or notice something about a different quantity than the quantity or quantities the teacher wants them to focus on and notice. This is relevant for the teaching of functions because functions involve numerous different quantities to notice (e.g., distance, time, change in distance, change in time, etc.).

An additional observation we made was that students who focused on changes in distance did not appear to also focus on changes in time. Thus, this focus did not yet support covariational reasoning, which requires focusing on two quantities.

Episode 3A: Alignment and Misalignment when Discussing One DT Quantity

Episode 3A occurred during the first instructional session on the third day of the summer program. This instructional session was about an activity in which students

viewed a single DTQF animation of a dog entering a forest, turning around, and leaving the forest. In this activity, the distance measurements were hidden from view, but the clock displaying the time remained visible. Additionally, the time took on negative, zero, and positive values.

Teacher goals for student focusing and noticing. In this episode, the feature of the DTQF the teacher primarily tried to direct students to focus on was just the independent variable, namely the time quantity. For QF data, the independent variable values are often presented in a constant pattern, which then means that the corresponding values of the dependent variable will be presented as a growing and/or shrinking pattern. For this reason, the distances in a DTQF may draw more of students' attention than the times. To support a more balanced focus on distance *and* time in later lessons so that students could engage in covariational reasoning, the teacher intended with this lesson to first establish student focusing on and noticing of time in a DTQF context.

Properties of time in a DTQF (i.e., the independent variable of the quadratic function) that the teacher wanted students to focus on and notice were how time was being measured, how time was changing, the sign of the time was (i.e., positive or negative), and the changing sign of the time. The teacher made the following comments during this episode that reveal this goal for student noticing:

He enters the forest when it hits zero, so right here, right at this point here, it's zero? Okay, time is zero, seconds . . . Mmm, so that's good, so it starts at -4.5, this is the start, and then the end, time, is, what is it? 8.70 seconds . . . Ooh, oh, that's interesting. Total time was 13.2 . . . so in this activity we focused on the time, we're trying to attend specifically to the time.

Student Focusing and Noticing Outcomes. In Episode 3A, students did focus on features of time and did notice properties of time in the DTQF, which aligned with the teacher's goals for student focusing and noticing. For example, Natasha commented, "We noticed that the time continues as he turns around to go away from the forest," and Demarcus noticed "So, ah we noticed that time starts at a negative number."

However, in this episode, like Episode 2B, some of the features students focused on and properties they noticed were misaligned with the teacher's goals for student focusing and noting. Specifically, some of the features focused on and properties noticed were not about time. Two types of misalignment emerged.

The first type of misalignment that emerged was that the students' focusing on and noticing of time sometimes co-occurred with a focus on and noticing of the direction of the dog's travel in the animation. For example, Natasha notice "like when he turns around, he pauses, but the time still continues." Similarly, Halima said "But it's only the time before because once the dog goes back out [of the forest], it's still positive. So, we just think it's the time before the dog enters the forest first." Therefore, something new this episode shows is that another kind of misalignment between the teacher's goals for student focusing and noticing of a DTQF and the student outcomes for focusing and noticing of a DTQF is that students' focus and noticing of a DTQF

that is aligned with the teacher's goals for of a DTQF (e.g., time) might co-occur with some extra focus and noticing that is not aligned with the teacher's goals (e.g., direction of travel).

The second type of misalignment that emerged was when students sometimes focused on and noticed properties of speed during Episode 3A. This is an example of students being distracted from focusing on and noticing of a more basic features of the perceptual or conceptual field of a DTQF (i.e., time) that the teacher wanted students to focus on and notice properties about in favour of focusing instead on and noticing of a more complex feature of the perceptual or conceptual field of a DTQF (i.e., speed). For example, in the following exchange the teacher asked about time and the student responded with what they noticed about speed:

Teacher: We are talking about time. Okay give us another observation.

Natasha: It seemed like his speed was consistent.

Teacher: What did you mean by that?

Natasha: Like, looking at it, we think he's like moving at a consistent pace

Although the speed in the DTQF animation was not actually constant, the transcript shows that Natasha was focused on speed and noticed a property of speed that appeared to distract them from focusing on and noticing properties of the time. Therefore, something new this episode shows is that another kind of misalignment between the teacher's goals for student focusing and noticing of a DTQF and the student outcomes for focusing and noticing of a DTQF is when students are distracted by a more complex feature of the perceptual or conceptual field of DTQFs (e.g., speed) from focusing on and noticing a more basic feature of the perceptual/conceptual field of DTQFs that the teacher wants students to focus on (e.g., time).

Both types of misalignment could distract students from focusing on and noticing what the teacher wants them to focus on and notice. Moreover, both types of misalignment in the context of DTQFs may not support students with covariational reasoning. In the former case of misalignment, a focus on the direction of travel of the animation may simply distract students from fully focusing on time, which is one of the quantities needed for covariational reasoning in DTQF contexts. In the latter case of misalignment, the focus on speed bypassed a focus on time, which is one of the quantities needed for covariational reasoning in at DTQF context.

DISCUSSION

This study examined the teacher's goals and the student outcomes for focusing and noticing in the context of DTQFs. The motivation for this research was that (a) focusing and noticing are important for mathematical learning, and (b) what teachers want students to focus on and notice may not always in fact be what students focus on and notice. Thus, it seemed important to us to better understand the relationship between what mathematics teachers want students to focus on and notice and what students actually focus on and notice. Also, because we are interested in researching ways to

promote covariational reasoning in quadratic functions contexts, it seemed important for that research to better understand the focusing and noticing that does and does not support covariational reasoning.

Our results showed alignment and misalignment between what the teacher wanted students to focus on and notice and what students actually focused on and noticed. A contribution our research makes is uncovering different ways teacher goals and student outcomes for focusing and noticing might be misaligned. Our study found three types of misalignments, (a) when students focus on and notice different quantities than intended, (b) when student focusing and noticing of the intended features co-occurs with focusing on and noticing unintended features, and (c) when focusing and noticing bypasses more basic intended features in favour of more complex features. Better understanding misalignments could inform the design of instructional activities for teaching QFs and may have relevance for teaching other mathematics concepts. Our ongoing research on other episodes in the current data set, and on other data sets will further explore this issues.

CONCLUSION

There is a need in the field of mathematics education for more research on student focusing and noticing. With a coordinated analysis of teacher goals and student outcomes, we have uncovered ways teacher's goals and student outcomes are aligned and misaligned. This is a line of research that offers promise of new insights for teaching quadratic functions and also for teaching other mathematics concepts.

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WHY MANY CHILDREN PERSIST WITH COUNTING

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In this paper, we propose a new, theoretical account of why many children persist with using counting strategies to solve single-digit addition problems. We hypothesise that, in some contexts, teaching approaches favour a phonological route for strengthening problem-answer associations in memory, which disadvantages children who have weaker skills with phonological memory. Furthermore, we hypothesise that more children will develop retrieval-based strategies if they are provided with opportunities to practice using tools that strengthen problem-answer associations in memory via a visual-spatial processing route. We also describe a new tool that we designed to help test these hypotheses, called the Keyboard. The Keyboard models a mental number line and makes use of children's subitising skills.

THE BENEFITS OF PRACTICE

Practice with solving computational problems is essential for children when learning early arithmetic. Problem-solving practice, together with instructional conversations and opportunities to reason about numbers and operations, facilitate a transitional process whereby counting-based strategies are replaced by fast and accurate retrieval-based strategies. This transition represents a turning point in children's mathematics development with the frequency of children's correct use of retrieval-based strategies in lower primary school predicting their growth in mathematics achievement through to upper primary school (Geary, 2011a). There exists, however, a substantial number of children who do not benefit from practice in this way. In Australia, for example, many children continue to use counting strategies for single-digit addition well beyond a time when curriculum documents suggest fluency be achieved (Gervasoni, 2017; Hopkins & Bayliss, 2017).

In this paper, we argue that the traditional view of retrieval difficulties does not explain more recent research findings highlighting the strategies children use for early arithmetic. We provide a new theoretical account of why, in some curriculum contexts, children are not benefitting from practice in ways that enable them to utilise retrieval-based strategies and describe a new tool that will be used to help test this premise.

Background

As children start to count, they learn foundational knowledge about numbers and develop increasingly sophisticated counting-based strategies for solving single-digit addition problems. Counting-based strategies include: (i) a count-all strategy, where children start the count at one (e.g., $3 + 5 = 1, 2, 3; 4, 5, 6, 7, 8$); (ii) a count-from-first strategy, where they count on from the first operand (e.g., $3 + 5 = 3; 4, 5, 6, 7, 8$); and (iii) the min-counting strategy, where they count on from the larger operand (e.g., $3 +$

$5 = 5$; 6, 7, 8). The min-counting strategy (sometimes called the count-on-from-larger strategy) is considered the most efficient counting strategy since it requires the least number (i.e., minimum number) of counts.

Learning efficient counting-based strategies for basic addition may be emphasised in mathematics curricula in first and second grade, but by at least third grade, children are expected to have achieved fluency and use fast and efficient retrieval-based strategies. Retrieval-based strategies encompass retrieval and decomposition strategies: Retrieval refers to the direct retrieval of an answer from a store of facts held in long term memory and decomposition strategies are strategies that make use of retrieved facts, including the bridging-through-ten strategy (e.g., $7 + 4 = 7 + 3 + 1 = 10 + 1 = 11$) and the near-ties strategy ($7 + 8 = 7 + 7 + 1 = 14 + 1 = 15$).

Traditionally researchers have viewed the protracted use of counting-based strategies for basic addition as being uniquely characteristic of students with a mathematics learning disability (MLD) or students who display persistently low achievement in mathematics (Geary et al., 1991; Geary et al., 2012; Jordan et al., 2003; Ostad, 1997). It has been established that, compared to average achieving peers, children with MLD use inefficient counting-based strategies for longer and are more likely to make counting errors (Geary et al., 2000; Geary et al., 2004; Jordan & Hanich, 2000), and are less likely to encode and/or retrieve basic addition facts (Geary et al., 2011b; Jordan & Hanich, 2000; Ostad, 1997). It is estimated that around 7% of students have MLD and another 10% show persistent low achievement in mathematics (Geary, 2011b).

More recent research suggests that it is not just students in the lowest achievement group who are not retrieving basic addition facts. Cowan et al. (2011) reported that many second and third grade children in the UK were not using retrieval for basic addition. Rhodes et al. (2018) highlighted the prevalent use of min-counting by adolescent students in the United States. Graven and Venkat (2021) described children's frequent use of counting to solve multi-digit problems in South Africa as an "extreme situation requiring urgent attention" (p. 24). Our own research has found that around a third of Australian children frequently use min-counting for single-digit addition beyond the time when fluency is expected (Hopkins & Bayliss, 2017; Hopkins et al., 2022). Importantly, this group of children (called accurate-min-counters) did not encompass children who used inefficient counting strategies or made frequent counting errors. Accurate min counters displayed lower overall achievement when compared to peers who used retrieval-based strategies - but not so low as to suggest a learning difficulty (Hopkins & Bayliss, 2017) - and struggled to identify flexible mental computation strategies for multi-digit addition (Hopkins et al., 2022). These findings suggest the need to generate and test new explanations of why many children exhibit difficulties learning retrieval-based strategies - explanations that possibly encompass weaknesses in how early arithmetic is taught.

Theoretical Framework

There is a significant gap in theory for understanding why so many children continue to use the min-counting strategy if they are accurate when using it. According to Siegler's model of strategy development (Shrager & Siegler, 1998; Siegler & Araya, 2005), retrieval is made possible when problem-answer associations are strengthened in memory as a result of practice using accurate backup strategies (any strategy other than retrieval). Siegler's account of retrieval development and the model's wider application to strategy development, strategy choice and strategy adaptiveness, has made significant and substantial contributions towards explaining the cognitive mechanisms underpinning children's development of basic arithmetic skills (Verschaffel et al., 2009). Yet despite its widespread utility, Siegler's account of retrieval development falls short when it comes to explaining the prevalence and persistence of accurate-min-counting. According to Siegler's model, if children use the min-counting strategy and are accurate when using it then retrieval-based strategies will become more prominent as a result of practice: yet many children persist with accurate-min-counting despite years of practice (Hopkins & Bayliss, 2017).

We propose an extension of Siegler's account of retrieval development, which encompasses the following elaboration: as a result of practice using accurate backup strategies, retrieval is made possible when problem-answer associations are strengthened in memory *via two possible pathways, a phonological (sound-based) pathway and a visual-spatial pathway*. Following on from this extended account, we hypothesise that (in some contexts) approaches for teaching early arithmetic favour a phonological route for strengthening problem-answer associations in memory during practice, which disadvantages children who have weaker skills with phonological memory. Counting-based strategies provide a phonological route for strengthening problem-answer associations in memory leading to retrieval (Imbo & Vandierendonck, 2007; Ostad & Sorensen, 2007). We also hypothesise that more children will develop retrieval-based strategies when expected to do so if they are provided with opportunities to practice solving problems using tools that strengthen problem-answer associations in memory via a visual-spatial processing route.

EXTENDING THE BENEFITS OF PRACTICE WITH TOOLS

Tools have long been used in mathematics education to support children's visual-spatial processing of number. For single-digit addition, children's fingers are considered particularly useful tools (Jordan et al., 2008). For some problems, fingers can provide a visual-spatial processing route for strengthening problem-answer associations in memory. For example, children can hold up five fingers and three fingers and see eight fingers without counting. This strategy is referred to as the fingers strategy (Geary, 1990). Alternatively, they may hold up five fingers and three fingers, and then count from one to calculate eight using a count-all strategy. Either way, children use their fingers to represent the size of both addends simultaneously and therefore the size of the sum. However, as children mature and start solving problems

with sums up to 20, they use the more efficient min-counting strategy. They may still use their fingers to count, but they use them in a different way, to keep track of the count. For example, using the min-counting strategy to solve $8 + 4$, a child counts “9, 10, 11, 12” putting up one finger for each count and stopping counting when four fingers are up – so the size of both addends are not represented simultaneously (i.e., the problem). The visual-spatial processing route for strengthening problem-answer associations in memory breaks down when children use their fingers in this more sophisticated way. Thus, fingers can promote a visual-spatial pathway for encoding number facts but only for problems with sums up to ten.

Children may use other tools for larger sums, including counters, strings of beads, interlocking blocks, a tens frame (a five-by-two array), or an arithmetic rack (two rows of 10 beads). While these tools are widely recommended for classroom use, research investigating how the use of these tools contributes to children’s strategy development is lacking (Mix, 2010).

The Keyboard

We created a new tool called the Keyboard to promote a visual-spatial route for strengthening problem-answer associations in memory for additive problems up to 20 (addition problems with sums up to 20 and corresponding subtraction problems). The Keyboard is a ruler-like object without numbers (see Figure 1).

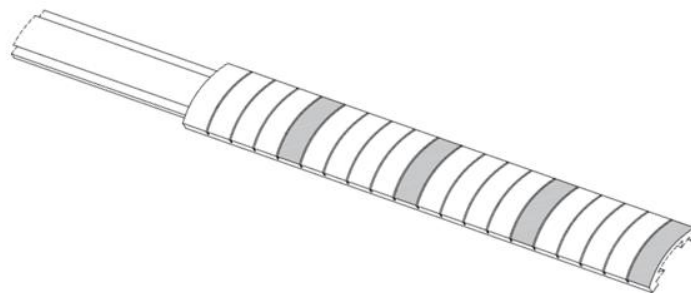


Figure 1. The Keyboard* (*Registered Design)

The Keyboard comprises lines drawn across the width of the object to form 20 rectangles or “keys”, with each fifth key coloured black (or a contrasting colour to other keys). The coloured keys are designed to make use of children’s subitising skills: that is, their ability to enumerate small quantities (up to five) using visual-spatial perception rather than counting (Starkey & Cooper, 1995). On one side of the keys is a space that forms a handle. Two or more Keyboards can be joined together to make one long Keyboard, as the handle of the second Keyboard fits into the end of another, enabling children to extend their knowledge of additive structures to larger numbers.

By making use of subitising skills, the Keyboard encourages children to visualise strategic jumps when learning to solve arithmetic problems with sums of 10 and under, and to utilise the bridging-through-ten strategy for solving problems with sums over ten. For example, to solve $5 + 7 = ?$ a child may point to five on the Keyboard (the first black key) and then make a strategic jump to 10 (the second black key) and then

make a strategic jump of two to arrive at the answer 12 (represented as $5 + 7 = 5 + 5 + 2 = 10 + 2$ and illustrated in Figure 2). Alternatively, a child may start at seven (two keys after the first black key), make a strategic jump of three to 10 (the second black key) and make another jump of two keys to arrive at 12 (represented as $5 + 7 = 7 + 3 + 2 = 10 + 2$). The bridging-through-ten strategy is one of many strategies taught in Australia but it is not emphasised to the same extent as it is in many East-Asian countries (Murata, 2004; Zhou & Peverly, 2005). Similarly the Keyboard encourages children to make and visualise strategic jumps when solving subtraction problems by taking away or using indirect addition (e.g., $5 + ? = 12$).

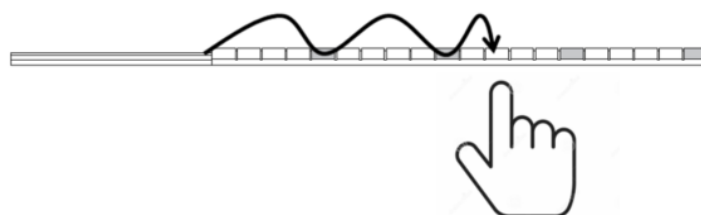


Figure 2. An illustration depicting how the Keyboard can be used to solve $5 + 7$.

The Keyboard shares similar features with existing tools but has some points of difference. For example, both a tens frame and the Keyboard make use of children's subitising skills but the Keyboard allows children to accurately represent additive problems as a linear distance. This is important as teaching tools that share similar features to a mental number line have been shown to promote mental representations of number magnitude (Siegler & Ramani, 2009), which are considered essential in children's mathematical development (Torbeyns et al., 2015). There are also practical advantages for children using the Keyboard. For example, it is less awkward to use than connecting blocks or bead strings, and possibly more appealing.

The idea that children benefit from using tools that promote visual-spatial representations of number when learning early arithmetic is supported by the advanced computational skills demonstrated by children in East Asian Countries (Geary et al., 1996; Vasilyeva et al., 2015). While this advantage may be due to a variety of factors (including the language of instruction and emphasis on mastery), at least in part it may be attributed to use of the abacus – a computational tool that relies on the visual-spatial processing of numbers. Barner et al. (2016) demonstrated clear advantages in teaching children to first use an abacus and then use a mental abacus when solving single-digit and multi-digit arithmetic problems. It is worth noting, however, that children in the study received 100 hours of instruction in using a mental abacus over a three-year period, thus making this approach impractical for teaching children in all contexts. The most direct evidence suggesting the potential efficacy of the Keyboard for promoting retrieval-based strategies comes from a small-scale study (Hopkins & de Villiers, 2016) involving a short subitising intervention that made use of a tens frame. The intervention was found to improve the rate at which problem-solving practice led to retrieval for children who did not benefit from practice alone.

FUTURE RESEARCH

We hope that the ideas presented in this paper will generate discussions in the mathematics education community that will lead to more research investigating why (in certain contexts) many children persist with counting and what can be done about it. We have commenced a three-year project to address three research questions related to the hypotheses presented here: (i) How do children's patterns of proficiency with basic addition relate to differences in phonological memory and visual-spatial memory?, (ii) How does use of the Keyboard improve the benefits of practice for children identified as accurate-min-counters?, and (iii) What outcomes are associated with incorporating the Keyboard and similar tools into classrooms? Our ultimate goal is to extend the benefits of practice so that more children come to utilise retrieval-based strategies when learning early arithmetic, making it easier for them to learn increasingly sophisticated mathematics as they progress in school. We envisage that this will be achieved not by more practice but by increasing the efficiency at which practice leads to the use of retrieval-based strategies for all children.

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INFLUENCE OF FIELD-DEPENDENCE-INDEPENDENCE AND SYMMETRY ON GEOMETRY PROBLEM SOLVING: AN ERP STUDY

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The study adopted an ERP methodology to examine two psychological constructs—field-dependence-independence (FDI) and symmetry (SYM)—on geometry problem solving. Based on a newly developed instrument, analyses of Taiwanese high school students showed that both FDI and SYM significantly influenced students' response accuracy (Acc). The FDI also determined students' reaction time for correct responses (RTc). When deliberately examining geometry problem solving through the stages of introducing problem situation, question presentation, and answer verification, the ERP analyses showed that SYM greatly influenced brain activities in stage of introducing problem situation, while FDI determined stage of question presentation.

INTRODUCTION

Mathematics education research is interdisciplinary and has robust ties to psychology (Silver & Kilpatrick, 1994). Psychological theories and methods, and the interplay with mathematics, have a longstanding history in developing research in mathematics education. The study reports investigation on the influence of the psychological construct—field dependence-independence (FDI) and symmetry (SYM)—on geometry problem solving. FDI refers to the ability to perceive and identify targets situated in a complex environment (Goodenough & Witkin, 1977). SYM is related to how the structure of an object remains unchanged after transformative actions (Wagemans et al., 2012). Psychological literature showed that the FDI and the SYM determine the complexity of cognitive processes related to visual stimuli (Evans et al., 2013; Wagemans et al., 2012). However, roles of the FDI and the SYM on geometry problem solving have rarely been examined. The psychological literature makes it reasonable to hypothesize that the FDI and the SYM influence the ways students perceive a geometry diagram and cognitive complexity of geometry problems. The research question for the study is “*to which extent and in which ways do the FDI and the SYM influence students' cognitive processes when solving geometry problems?*”

LITERATURE REVIEW

Field Dependence-Independence (Fdi)

Psychologists referred FDI as a cognitive ability accounting for individual differences in perceptual and intellectual performance. Field-independent persons have a greater tendency to isolate individual elements from an organized field, whereas field-dependent ones often face difficulties in identifying parts from the whole (Evans et al.,

2013). Literature showed that FDI is strongly tied to brain activities, especially those related to perception (Jia et al., 2014). Riding et al. (1997) reported how individuals with different FDI ability process information and how individual differences are related to the lateralization of specific cognitive strategies. Tinajero et al. (1993) proposed two directions for neuro-cognitive research on relationship between FDI and brain functioning. One is to identify brain functions for perceptual selectivity, especially about lateralization. The other is about the examination of brain activation of individuals with extreme field-dependent individuals as contrasted to that in field-independent ones. It is believed that field-independent and field-dependent individuals function differently in brain organization. In mathematics, Reid and Ali (2020) argued that field-independent skills are essential to mathematics problem solving as it requires identifying important information. Thus, field-independent persons often have advantage of recognizing the keys to search for correct answers in examinations.

Symmetry (SYM)

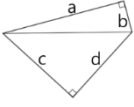
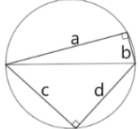
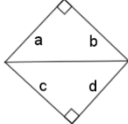
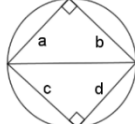
SYM is a crucial component of the visual system and has a neural basis (Treder, 2010). As one of the Gestalt principles, SYM is defined as achieving the maximal level of stability when perceiving visual information (Wagemans et al., 2012). In psychology, research on SYM primarily focused on its effects on perception and memory. Psychologists also clarified how brains function SYM detection. Object SYM makes memory processes easier (Kayaert & Wagemans, 2009). Symmetrical patterns are recalled more easily than asymmetrical ones (Howe & Jung, 1987). Researchers also indicated that SYM detection occurs in a pre-attentive and automated manner, which is denoted as SYM-defined visual search (Olivers & Van Der Helm, 1998). According to Leikin et al. (2000), SYM plays a unique role in mathematical problem solving. Their studies indicated that SYM bridges various branches of mathematics, such as algebra, group theory, geometry, probability, and calculus.

METHODOLOGY

The instrument for the study

A new instrument, namely Geometry Field Dependence-Independence Symmetry-Segment test (GFDIS-segment), developed based on the two psychological constructs was adopted. The GFDIS-segment test includes problems that require reasoning the relationship of segments, and the problems include different diagram configurations and geometry properties. In this study, FD problems are defined as those having more complex geometry diagrams when compared to the paired FI problems. The GFDIS-Segment test includes 21 sets of problems, each set of which has four types: Field-Independence-Symmetry (FI-SYM), Field-Dependence-Symmetry (FD-SYM), Field-Independence-Asymmetry (FI-ASYM), and Field-Dependence-Asymmetry (FD-ASYM) (see Figure 1). The problem situation, question presentation, and answers were identical in each set of problems. The only difference is the diagram configurations. Each problem has two trials, one of which was a true statement. The GFDIS-Segment test totally has 168 trials.

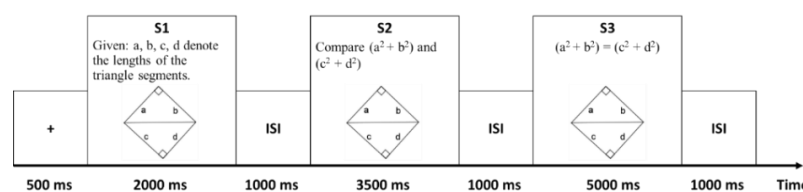
Figure 1: A set of four-type geometry problems designed based on FDI and SYM factors

	Field-Independence Diagrams (FI)		Field-Dependence Diagrams (FD)	
Asymmetrical diagrams (ASYM)	FI-ASYM		FD-ASYM	
Symmetrical diagrams (SYM)	FI-SYM		FD-SYM	

Subjects and data collection

58 (28 boys and 30 girls) Taiwanese 10th- and 11th-grade students participated in the study. All students are right-handed and their electrophysiological data confirmed that no extensive noises were collected. The E-prime software was used to perform the GFDIS-segment test. Students had to practice several geometry problems to familiarize themselves with the testing environment before formally collecting the data. As seen in Figure 2, a three-stage problem-solving model was adopted. S1 refers to the stage of introducing problem situation, while S2 is question presentation. For S3, it is the stage for answer verification. Each individual trail began with a 500 ms fixation and then presented problem situation (S1) for 2000 ms. After a 1000 ms time break, the question presentation (S2) along with the diagram was presented for 3500 ms. When having another 1000 ms time break, answer verification (S3) along with the diagram that students needed to evaluate was presented for 5000 ms. A 1000 ms were designated for students to rest. The 168 trials for the GFDIS-Segment test were randomly presented in the E-prime environment to prevent learning transfer among the four types of geometry problems.

Figure 2: E-prime model for GFDIS-Segment test



The software systems—SynAmps RT 64—channel Amplifier from COMPUMEDICS Neuroscan along with Quick-Cap 64 electrode caps were used to record scalp voltages. Pin-type electrodes were mounted on a Quick-Cap 64 electrode cap arranged based on the 10-20 system. Two flat electrodes were placed on the sides of the eyes to monitor horizontal eye movement. A third flat electrode was placed underneath the left eye to monitor vertical eye movement and blinks. During the session electrode offset was kept below 80 μ V. The EEG signals were amplified and digitized with a 24-bit AD converter. A sampling rate of 1000 Hz (0.5 ms time resolution) was employed.

Data analysis

The behavioural data and electrophysiological data were examined. For behavioural data, students' response to each trial was analyzed in terms of accuracy (Acc) and reaction time for the correct responses (RTc). Both Acc and RTc can be used as indicators to determine cognitive complexity of tasks. Acc was counted based on percentages of correct responses to the total number of trials for each type of geometry problems. The RTc associated with each type of geometry problems was calculated as the mean time spent verifying an answer in all correctly-solved trials. Concerning electrophysiological data, brain activities were analyzed based on event-related potential (ERP) technique using the Brain Vision Analyzer software. ERPs were Zero phase shift filtered offline (bandpass: 0.53-30 HZ) and referenced to the common average of all electrodes. Epochs with amplitude changes exceeding $\pm 80 \mu\text{V}$ on any channel were rejected. Ocular artefacts were corrected. ERP waveforms were time-locked to the onsets of S1, S2, and S3. The average epoch for ERP, including a 200 ms pre-trigger baseline, was determined. Only correct responses were analyzed. The data were baseline-corrected and the grand wave was calculated for each problem-solving stage. Trials in which the subjects failed to respond during the time arranged for answer verification were not analyzed. 168 trials were available in each problem-solving stage.

As the peaks were not easily identified in the average grand waveform after 250 ms, the mean absolute ERP amplitude of late potentials was performed. We determined the time frames 250-500 ms for statistical analysis, which is in line with the literature on the late component of P300. For behavioural data, a repeated-measures ANOVA was used to assess the differences in Acc and RTc. Pair-wise comparisons were performed when a significant interaction was found. For ERP data, a repeated-measures ANOVA was performed on the P100 and on the ERP mean absolute amplitude for the P300 considering two orthogonal factors: Caudality (anterior, central, and posterior) and Laterality (left, middle, and right). The two orthogonal factors divided the brain into nine areas: AL, AM, AR, CL, CM, CR, PL, PM, and PR. Reliability, using Cronbach's alpha, of Acc was examined to report the internal consistency of the newly-developed GFDIS-Segment test. Cronbach's alpha coefficients for the four types of the problems were found to be sufficiently reliable ($0.805 \leq \alpha \leq 0.827$) to examine the effect of FDI and SYM on geometry problem solving.

FINDINGS

Behavioural data of accuracy (Acc) and reaction time for correct responses (RTc) for FDI and SYM factors

Students' behaviour data analyzed based on the FDI and the SYM factors were reported in Table 1. No significant interaction between the two factors was found ($F = 1.524, p > .05$). However, both FDI and SYM factors influenced the Acc. Students performed significantly better on problems with the FI diagrams than the FD diagrams ($F = 8.428, p < .01$) and those with the SYM diagrams than the ASYM diagrams ($F = 35.129, p < .001$). Regarding students' RTc, no significant interaction was

found ($F = .001, p > .05$). The FDI factor caused a significance as the problems with the FD diagrams took students longer time to answer the questions than the FI diagrams ($F = 10.478, p < .01$). No difference in answering problems between the SYM and the ASYM diagrams was found ($F = 3.159, p > .05$).

Table 1: The analysis of Acc and RTc for SYM and FDI factors

Measure	Mean (SD)			SYM factor $F(1,57)$
	FI	FD	Overall	
Acc (%)	SYM	93.4 (8.6)	92.6 (8.6)	93.0 (8.6)
	ASYM	90.6 (10.0)	88.5 (10.3)	89.6 (10.2)
	Overall	92.0 (9.4)	90.5 (9.7)	
	FDI factor= $F(1, 57)$	8.428** $\eta_p^2 = .034$		SYM×FDI=1.524 $p = .222$
RTc (ms)	SYM	1852.3 (867.6)	2004.8 (1104.9)	1928.6 (992.0)
	ASYM	1886.1 (949.0)	2039.6 (1065.4)	1962.8 (1007.4)
	Overall	1869.2 (905.4)	2022.2 (1080.8)	
	FDI factor= $F(1, 57)$	10.478** $\eta_p^2 = .125$		SYM×FDI=0.001 $p = .976$

* $p \leq .05$, ** $p \leq .01$, *** $p \leq .001$

Acc – Accuracy, RTc – Reaction time for correct responses

Analyses of brain activities for FDI and SYM factors

The analyses of brain activities focused on the influence of the FDI and the SYM on cognitive processes. Generally speaking, the SYM factor determined brain activities during the stage of introducing problem situation (S1), whereas the FDI factor influenced during the stage of question presentations (S2). As seen in Figure 3, the P100 component for the SYM factor at S1 caused a significant difference ($F=5.919, p<.01$). Geometry problems with the SYM diagrams significantly caused students higher amplitude than the problems with the ASYM diagrams. The P100 component at S2 had a significant interaction among the factors of FDI, SYM, and Laterality. However, only the FDI had simple main effects. For the left brain area, analysis showed that the FDI factor caused significance for the problems with the SYM diagrams ($F=5.919, p<.01$). When presenting questions for the problems with the FD diagrams, it caused students higher amplitudes when compared to those with the FI diagrams. For the right brain area, analysis indicated that the FDI factor caused significance for problems with the ASYM diagrams. The FD conditions provoked higher amplitudes when compared to the FI conditions.

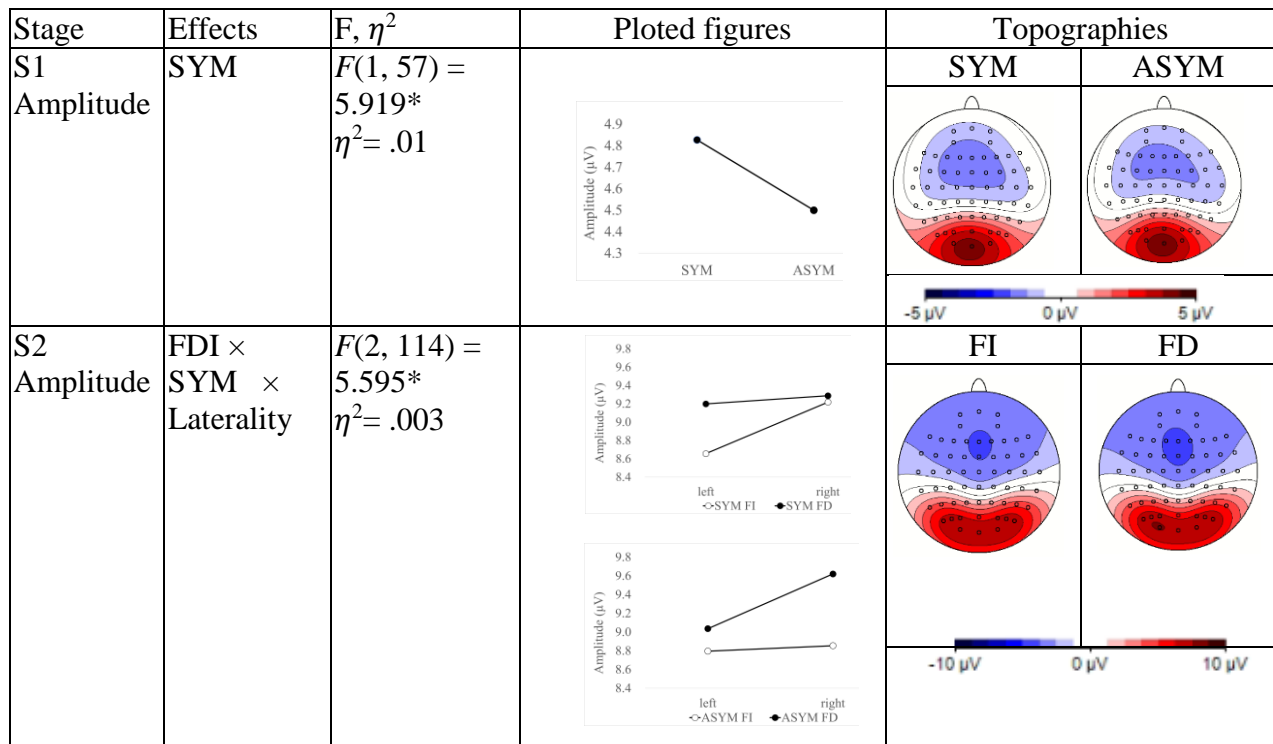


Figure 3: Significant results associated with P100 component

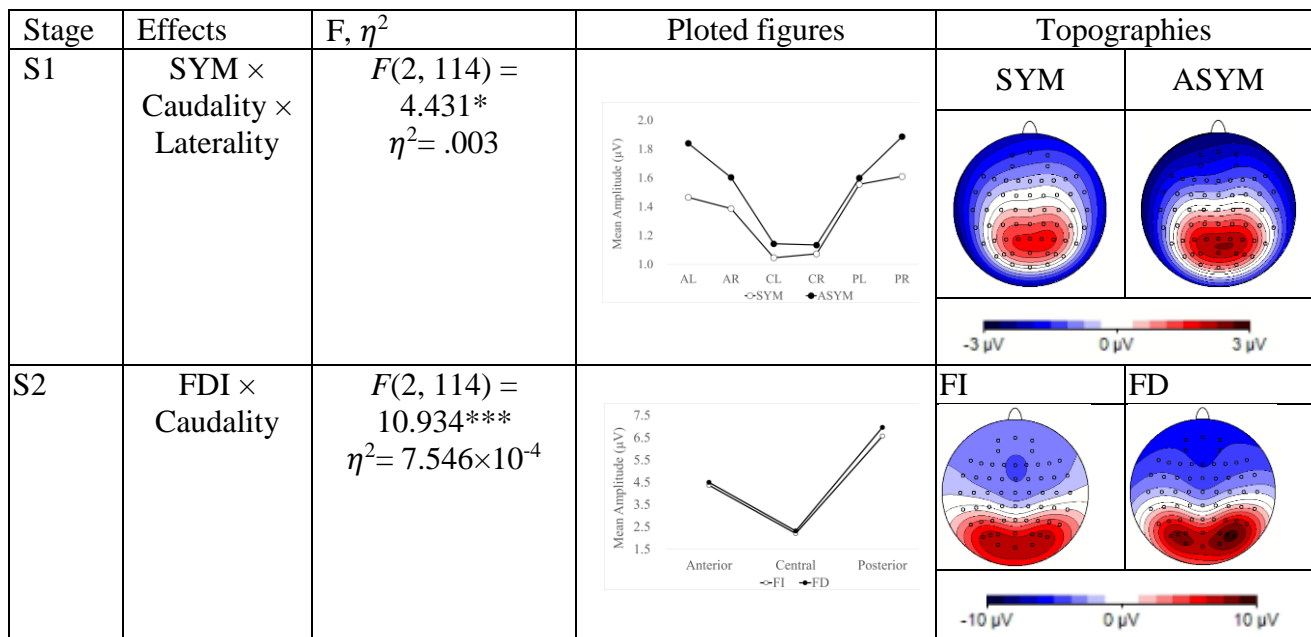


Figure 4: Effects of SYM and FDI factor on P300 component (250–500 ms frame)

Regarding the P300 component, the analysis of mean amplitude with 250-500 ms time frame showed similar results (see Figure 4). The SYM factor caused significance at S1, while the FDI resulted in statistical differences at S2. For S1, a significant interaction among the factors of SYM, Caudality, and Laterality was found ($F=4.431$, $p<.01$). Analysis of simple main effect showed problems with the ASYM diagrams caused higher amplitudes than those with the SYM diagrams in the brain areas of AL, AR, CL, PL and PR ($F=14.636$, $p<.001$ for AL; $F=4.986$, $p<.05$ for AR; $F=5.256$, $p<.05$ for CL; $F=6.935$, $p<.05$ for PR). Only the brain areas of CR and PL did not cause

significance ($F=1.924$, $p>.05$ for CR; $F=0.22$, $p>.05$ for PL). Regarding S2, a significant interaction between the FDI and Caudality was found ($F=10.934$, $p<.001$). Analysis of simple main effects indicated that the amplitude for the FD problems was significantly higher than that of the FI problems in the central and the posterior brain areas ($F=10.222$, $p<.01$ for central area; $F=21.256$, $p<.001$ for posterior area). Only in the anterior brain area, the FDI did not cause significance ($F=1.882$, $p>.05$).

DISCUSSION

We adopted an ERP methodology to examine the extent to which and in which ways the two psychological constructs—FDI and SYM—influence geometry problem solving. Based on the newly developed GFDIS-segment instrument, analyses of 58 Taiwanese high school students showed that both FDI and SYM factors significantly determined students' accuracy of responses (Acc). The FDI factor also influenced students' reaction time for correct responses (RTc), but not the SYM factor. The analysis of ERP data revealed that the SYM factor significantly influences cognitive processes during introducing geometry problem situations. Geometry problems with SYM diagrams caused higher amplitudes than those with ASYM diagrams for the P100 component. However, for the P300 late component, problems with ASYM diagrams significantly caused higher amplitudes when compared to those with SYM diagrams. The analysis of the SYM factor is aligned with psychological literature as SYM detection occurs in a pre-attentive and automated manner (Driver et al., 1992). For the P300 component, the finding anchors the argument from Kayaert and Wagemans (2009) as the SYM factor makes memory retrieval easier, thus decreasing cognitive load in problem solving. The results imply that SYM diagrams can draw students' attention to geometry problems and become an effective heuristic strategy in problem solving. The analysis on the FDI factor showed that it determined the stage of question presentation for both P100 and P300 components. The more complex the diagrams are, the higher perceptual load at early cognitive processes the problems create (Lavie et al., 2004). Additionally, the FDI factor also caused brain lateralization at late cognitive processes (Riding et al., 1997). For the left brain, geometry problems with complex diagrams caused higher cognitive activity in the SYM condition. For the right brain, complex diagrams resulted in higher cognitive activity in the ASYM condition. The result implies that students may apply different cognitive strategies to solve geometry problems with different diagram conditions (SYM vs. ASYM).

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CULTURAL VARIATIONS IN THE QUALITY AND QUANTITY OF STUDENTS' OPPORTUNITIES TO PARTICIPATE IN CLASSROOM DISCOURSE

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Opportunities for students to share their thinking in mathematics lessons has been a focus of mathematics education research for many years. There is now substantial evidence of the benefits to students' learning from participating in discussions around mathematics as well as a growing body of research focusing on what teachers can do to support students in sharing their thinking in meaningful ways. The evidence around the cultural and normative influences on students' opportunities to participate is more limited. In this paper, results from an international video study of mathematics teaching involving eight countries are shared, focusing on what we can learn about variations in students' opportunities to participate in classroom discourse.

INTRODUCTION

The importance of opportunities for students to share their thinking in mathematics lessons for their learning is now widely recognised. There is a wealth of research examining the associations between participation and learning, the different natures of this participation, as well as the teacher moves that can initiate and sustain different types of participation. Classroom norms around participation and interactions also influence both the quantity and quality of student participation. Yet cultural norms also influence these classroom norms and there is limited research focused on the cultural variations in student participation. In this paper, results from an international video study of mathematics teaching involving eight countries are shared, focusing on the extent to which students share their thinking in these eight contexts, alongside the evidence available about the nature of this shared thinking. International studies of mathematics teaching have both illustrated what we can learn from other country contexts, but also the challenges of supporting teachers to develop their practice whilst taking the distinctive cultural setting into account.

BACKGROUND AND THEORETICAL APPROACH

Classroom discussion and mathematical discourse have been a growing focus of mathematics education research for many years. More recently, there has been a growing interest in large-scale quantitative studies focusing on both the nature and the quality of this discourse (Howe et al., 2019; Lim et al., 2019) that builds on the abundance of smaller scale research conceptualising classroom discussions and the roles of teachers and students within these (Erath et al., 2021; Ingram, 2021; O'Connor & Michaels, 1996; Stein et al., 2015). There is now substantial evidence for the benefits of classroom discussions and interactions for supporting mathematics learning. For

example, Resnick and colleague's (2018) recent review of studies focused on classroom discourse identifies four effects of classroom discourse that supports students' learning: increased learning, longer enduring learning gains, better learning of topics not taught using discussions, and stronger reasoning skills.

Mathematical discourse is a broad term that includes "ways of talking, acting, interacting, thinking, believing, reading, writing but also mathematical values, beliefs, and points of view" (Moschkovich, 2003, p. 326) and classroom discussions and interactions make visible the mathematical discourses that are used in classrooms, but can also "ignite and coordinate student thinking" (Chen et al., 2020, p. 644). It is through these discourses that students learn what it means to learn maths and what it means to do mathematics. In light of this, recent research has argued for the need for opportunities for students to discuss and explain mathematics and the necessity of planning and designing these opportunities (Erath et al., 2021; Stein et al., 2008). Other research has identified the importance of teachers responding to and building upon students' ideas in mathematics discussions (Lim et al., 2019).

Within mathematics education specifically, the majority of studies of classroom discourse and interaction have been restricted to a single country context. However international studies of mathematics teaching have revealed that there are distinctive cultural variations in both the frequency and the nature of mathematics teaching and learning practices. For example, the findings from the Programme for International Student Assessment (PISA) in 2012, the most recent cycle where mathematics was the focus, illustrated considerable variation across countries in students' perspectives on how often teachers asked students to present their thinking (OECD, 2013). Similarly the Learner's Perspective Study showed revealed variations in the extent to which students had opportunities to speak mathematics in lessons which Xu and Clarke (2019) argue reflects the inspirations and cultural values attached to particular types of student participation.

METHODS

The analysis described here is part of a larger study that investigated the teaching and learning of quadratic equations across eight countries and jurisdictions, the TALIS Video Study that is part of the Organisation for Economic Co-operation and Development's (OECD) Global Teaching InSights programme (OECD, 2020). This study collected two videos from between 50 and 110 teachers in each context as well as the lesson materials, student and teacher questionnaires, and student assessments. The videos were then rated by trained raters against higher-inference component rating scales and lower-inference indicator rating scales. The study used a two-stage random sampling design however the actual sampling process in each of the country contexts varied meaning that the teaching captured in the videos can only be considered representative of the teaching of quadratic equations in five of the eight contexts. The sample in Germany was a convenience sample.

This paper focuses on the video component analysis across the different country contexts that include a measure of students' contributions within classroom interactions rather than the more usual focus on teacher actions that initiate these contributions. The three video component ratings of interest from the study are the *Nature of discourse* which measured the extent to which the classroom interaction was teacher-directed and the level of detail included in student contributions, *Eliciting student thinking* which measured how much student thinking was present and the extent to which student contributions focused on answers and procedures or ideas and concepts, and *Aligning instruction to present student thinking* which measured how frequently teachers either used student contributions or how frequently they provided support when a student made an error or struggled mathematically. Full details of the video component scales and the coding process for the broader study, including the inter-rater reliability measures, can be found in the international reports (Bell, 2020; Bell et al., 2021; OECD, 2021).

The videos were scored against each of the video component measures on a scale of 1 to 4 every 16 minutes (a lesson segment) by two trained raters. Average ratings across both raters, then lessons and then teachers were included in the international report. In this paper the focus is at the lesson level and the maximum ratings within a lesson for each of the three video component ratings, as well as the coincidence of the three ratings at the lesson segment level. Arguably, for each of these teaching measures we would not necessarily expect to see high levels of detailed classroom interactions for the full duration of a lesson as lessons often include opportunities for teachers to introduce or explain ideas, and opportunities for students to work independently or in small groups rather than as a whole class. The use of the maximum rating in a lesson enables an examination of cultural variations in the extent to which teachers use discussions and whole class interactions as part of their pedagogic repertoire as well as some aspects of the nature of these interactions rather than as an indicator of a particular teaching style.

The analysis below examines the lessons that included an average rating across the two raters of 2.5 or more on the *Nature of Discourse* at some point during the lesson, meaning that there were points in the lesson where the discourse was sometimes or rarely teacher-directed and students' discourse was sometimes or frequently characterised by detailed contributions. Similarly, for *Aligning instruction to present student thinking* the focus was on lessons that included an average rating of 3.5 or more meaning that the teacher frequently used students' contributions or if students made errors or struggled mathematically, the teacher frequently provided cues or hints to support student understanding. For *Eliciting student thinking* a distinction is made between a maximum rating of 2.5 and above and 3.5 and above. These ratings mean that there was a moderate amount or a lot of student thinking present and that the questions, prompts and tasks resulted in detailed student contributions that concerned answers, procedures and the steps necessary for solving a problem, or for the highest

possible rating (3.5 or above) ideas or concepts as defined in the conceptualisation of teaching used in the study (Bell et al., 2021).

At the lesson segment level, i.e., one of the 16-minute episodes that were rated, the analysis focuses on those lessons segments rated as 2.5 or above for *Nature of Discourse* and then examined the extent to which these lesson segments also included a highest ratings (3.5 or more) for *Eliciting student thinking* or *Aligning instruction to present student thinking*. This focuses the analysis on those lesson segments where students were making detailed contributions and the potential nature of these contributions alongside how teachers uses these contributions in their own teaching. Although the data for all the participating contexts is reported, the discussion focuses on four contexts that illustrate the differences between contexts: Colombia, England, Germany and Shanghai.

FINDINGS

Almost all lessons in England and Germany (90% and 95% respectively) included a lesson segment with a maximum rating of 2.5 or more for the *Nature of Discourse*. In contrast only around half of the lessons in Shanghai and Colombia (54% and 45% respectively) included a lesson segment with this rating. Similarly in England and Germany around three quarters (73% and 79%) of lessons included a lesson segment with a maximum rating of 3.5 or more for *Aligning instruction to present student thinking*, whereas in Shanghai (28%) and Colombia (24%) the proportion of lessons is closer to one quarter. The proportions for all countries are given in Table 1. This reveals cultural variations both in the extent to which students contribute to classroom discourse where in England and Germany students making detailed contributions is a common practice and appears to be part of the classroom norms. In contrast, in many other contexts such as Shanghai and Colombia there is more variation between teachers in terms of the level of detail of student contributions within lessons. Similarly, in England and Germany it was typical of lessons to include segments where teachers responded to their student contributions in some way, either by building on their ideas or by supporting them when they encountered difficulties. In other contexts, such as Shanghai and Colombia, this was observed more rarely.

The analysis that focuses on the nature of these student contributions using the highest rating for *Eliciting student thinking* reveals a different picture, irrespective of the detail of a student contribution. In Shanghai 61% of lessons included a lesson segment where a student contribution focused on an idea or a concept at least briefly. This was the only context where this happened in the majority of lessons. In all other contexts the majority of lessons did not include a student contribution focused on ideas or concepts, with student contributions focused instead on ideas, procedures or the steps necessary for solving a problem.

At the lesson segment level, within lesson segments rated as sometimes or frequently including student detailed contributions (2.5 or above for *Nature of Discourse*) 49% of these segments in Shanghai also included the highest rating for *Eliciting student*

thinking, meaning they included a student contribution focused on an idea or concept. While 29% in Japan, 21% in England and 18% in Colombia of these segments included a student contribution focused on an idea or concept. The proportion of lessons for all contexts are reported in Table 2.

Table 1: Proportion of lessons including a segment with a higher rating for *Nature of Discourse* (2.5 or more) or *Aligning instruction to present student thinking* (3.5 or more) or *Eliciting student thinking* (3.5 or more)

Context	Number of lessons	Nature of Discourse (%)	Aligning instruction to present student thinking (%)	Eliciting student thinking (%)
Chile	196	65	42	15
Colombia	166	45	24	17
England	167	90	73	31
Germany	100	95	79	48
Japan	177	80	61	40
Madrid	169	83	38	20
Mexico	206	68	39	23
Shanghai	170	54	28	61

Table 2: Proportion of lesson segments rated as 2.5 or above for Nature of Discourse also including the highest rating (3.5 or above) for *Eliciting Student Thinking* or a high rating (2.5 or above) for *Aligning instruction to present student thinking*.

Context	Rating of 3.5 or more	
	Eliciting student thinking (%)	Aligning instruction to present student thinking (%)
Chile	8	27
Colombia	18	25
England	21	62
Germany	28	46
Japan	29	49
Madrid	18	30
Mexico	22	30
Shanghai	49	16

DISCUSSION

These findings illustrate both within and between country context variations in participation in classroom discourse in interaction. Within England and Germany there is little variation with the context as students contributing in a detailed way occurred in almost all lessons at some point. In both these contexts this may be a feature of

lessons with higher achieving students as the curriculum in these two contexts restricts lower attaining students' access to some of the content that was the focus of the TALIS Video Study, and hence the teaching captured may not be representative of mathematics teaching more generally.

In contrast, in both Shanghai and Columbia there is variation between teachers in that only around half of the lessons included detailed student contributions. However, when students contribute in Shanghai, the nature of these contributions is more likely to include reference to an idea or concept than in the other contexts. In all contexts the majority of student contributions focused on answers, procedures or the steps needed to solve a problem, but only in some classrooms did students also engage with ideas and concepts.

These results reflect much of the recent research in the West that has focused on providing students opportunities to participate in mathematical discourses and on responsive teaching. Yet in Shanghai the quality and nature of student participation is also something that teachers appear to consider, enabling students to go beyond the routine reporting and describing of processes to discussing ideas and concepts, something that is widely advocated for in the literature (Chen et al., 2020; Erath et al., 2021; Stein et al., 2008).

This paper has pointed to cultural variations in both the quantity and quality of student participation in mathematics lessons. However the data is limited by the focus on just one mathematical topic (quadratic equations) and to students at a stage of education where this topic is taught in each contexts (aged between 13 and 17 years old). The analysis is also limited by the specific measures used in TALIS Video Study, which only capture a narrow range of aspects of student participation. They also will not capture sequential aspects of student contributions and also do not distinguish between teachers building and using student ideas and teachers supporting students when they encounter a difficulty. It would be interesting to see if these variations are also apparent in other topics and at other education stages.

The focus in this paper has been on variations and differences, but it is important to note the similarities. The average ratings in the international report from the study point to relatively few opportunities students have to discuss mathematics in lessons, either through detailed contributions to whole class interactions, or through group or pair work. Yet students need something to talk about. The inclusion of opportunities to contribute, in detailed ways or in relation to ideas and concepts as well as procedures and answers, is part of the majority of lessons in all the contexts considered here. Opportunities to contribute are a tool widely used by most teachers, but further research is needed to explore whether these variations in opportunities have different influences on students' learning in different contexts.

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SNAPSHOTS OF CURRICULAR NOTICING: PLANNING A SUBTRACTION ALGORITHM LESSON IN PRIMARY EDUCATION

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Teacher noticing is one of the core practices to be developed in teacher education programs. A particular focus of this competence is curricular noticing conceptualized as recognising relevant aspects of the curricular materials, interpreting the opportunities for instruction, and decide how to use them. 28 pre-service primary school teachers solved a task where they were asked to recommend a resource/material to introduce the subtraction algorithm in 3rd grade, to give reasons for this recommendation and to plan a lesson using this resource. Results show that the majority of these pre-service teachers identified the base ten blocks as a resource and indicate how these PTs interpreted the opportunities of this material for introducing the subtraction algorithm and how they planned the lesson using them.

INTRODUCTION

Professional noticing of mathematics teaching-learning situations is a relevant competence for the teachers' practice that has generated an important research agenda in Mathematics Education during the last years (e.g. special issues Dindyal et al., 2021; Schack, et al., 2017). In fact, Jacobs and Spangler (2017) present the competence of professional noticing as one of the core practices to be developed in teacher education programs in their review of "Research on Core-Practices in K-12 Mathematics Teaching".

The results of previous research have provided information regarding different aspects of this competence. A great part of this research focuses on a particular aspect of this competence, noticing students' mathematical thinking, understood as the three interrelated skills of attending to the mathematical details in students' strategies, interpreting students' mathematical thinking taking into account the details identified and deciding how to respond considering the students' mathematical thinking (Jacobs et al., 2010). This research has provided information about how pre-service teachers identify the mathematically relevant details in students' strategies, how they interpret the students' mathematical thinking and decide how to continue on the basis of students' mathematical thinking (e.g., Fisher et al., 2019; Gupta et al., 2018; Ivars et al., 2020). These previous studies have also indicated that the skill of making decisions is the most difficult and demanding. Pre-service teachers can be very specific about the details they observe in students' mathematical thinking, but they have difficulties in using this information to decide how to continue with the instruction (Barnhart & van Es, 2015; Choy, 2014).

Research has also identified contexts and tools that favour the development of this competence in teacher education programs (e.g., Fernández & Choy, 2020; Schack, et al., 2017). Representations of practice and conceptual tools (such as theoretical information about how students learn specific mathematical topics) have been shown tools to support the development of this competence in initial training programmes (Fernández & Choy, 2020). Nevertheless, this research has also highlighted pre-service teachers' difficulties in providing teaching decisions (Barnhart & van Es, 2015; Tyminski et al., 2020). These difficulties can be explained by the lack of resources, materials and mathematics teaching orientations (Schoenfeld, 2011).

In this context, as far as we know, there is not too much research focused on how pre-service teachers interact with materials and resources to provide teaching decisions. Amador et al. (2017) have shown that comparing different versions of the same activity allows pre-service teachers to identify differences in the design of activities and to perceive new opportunities for teaching of the materials. Furthermore, they concluded that pre-service teachers may benefit from task exploration and also from the use of analytic tools for interpreting materials. From their studies, they also concluded that pre-service teachers attend differently to different materials (digital and printed textbooks; written lessons and single tasks), and often based their decisions on their own understanding or knowledge. Our study contributes to this line of research examining how pre-service teachers notice different resources/materials when they are planning a lesson for introducing the subtraction algorithm in 3rd grade.

Curricular noticing

Curricular Noticing is a construct defined as how teachers make sense of the complexity of content and pedagogical opportunities in printed or digital curriculum materials (Amador et al. 2017; Dietiker et al., 2018). This construct focuses on the relationships between teachers and curricular materials considering the students' learning. In our study, we understand as curricular materials not only printed or digital textbooks, written lessons, or single tasks, but also any technological (software, etc.) or manipulative resource (such as abacus, base ten blocks... etc.) that a teacher has available.

Following the conceptualization of curricular noticing from Dietiker et al. (2018) and considering the objective of this study, this competence would imply (i) recognising relevant aspects of the curricular materials available to them, (ii) interpreting the opportunities for instruction provided by each curricular material, and (iii) deciding how to use them considering the opportunities they provide in the design of a lesson or a sequence of activities.

Objective and research questions

The objective of the study is to examine how pre-service teachers notice different resources/materials when they are planning a lesson for introducing the subtraction algorithm in 3rd grade. The following research questions have been formulated: (i) What materials/resources would use pre-service teachers when they have to plan a

lesson for introducing the subtraction algorithm in 3rd grade? (ii) Why would they use these materials/resources? (iii) How would they use these materials/resources in the lesson?

METHOD

Participants and instrument

Participants were 28 pre-service primary school teachers (PTs) enrolled in their third year of the degree to become a primary school teacher. Previously, PTs had completed two mathematics education courses related to numerical and geometrical sense. They were attending the course “the learning and teaching of mathematics in primary school” as part of their degree. This course consisted of different learning environments aimed at developing teacher competences such as the curricular noticing competence.

As a part of one of the learning environments, participants, individually, had to solve the following task:

Xavi is a 3rd grade primary school teacher, and he is planning a lesson to introduce the subtraction algorithm with 3 digits. Could you help him? Answer the next questions:

- *What resources/materials would you recommend Xavi to use in the lesson? Justify your answer*
- *If you were Xavi, how would you design the lesson? Justify the design explaining how the resource/material are contributing to the learning of the subtraction algorithm.*

PTs had two hours to solve the task. In the learning environment PTs were provided with theoretical documents that contain information from Mathematics Education research on how primary school students understand the decimal numbering system, including the algorithms and information about different material and resources that can support students’ understanding of the decimal numbering system (e.g., Beattie, 1986). Furthermore, they had participated in other tasks such as analysis of textbooks (or sequences of activities) and analysis of student’s answers to different activities.

In Spain, the subtraction algorithm is introduced through the “borrowing method”. This method is based on the ideas behind of the Decimal Numbering System: place value and grouping and ungrouping. In this method, a higher order unit is exchanged for 10 units of an immediately lower order, obtaining, whenever necessary, different non-canonical decompositions. In other words, the minuend is transformed, based on its non-canonical decomposition.

Analysis

Data are the answers of PTs to the task. In the analysis, we have focused on: (i) what are the materials/resources PTs would recommend to introduce the subtraction algorithm; (ii) how PTs relevant aspects of the resources/materials and how they

interpret the opportunities for instruction of the materials/resources recommended; and (iii) how they decide to use them in the lesson.

RESULTS

In general terms, the great majority of the participants recommended the use of manipulative materials, particularly, base ten blocks (n=26). Seven out of these 26 PTs also recommended the use of other materials like the abacus, worksheets, and the textbook for introducing the subtraction algorithm. There were two PTs who did not recommend any material. The two PTs who did not recommend any material or resource provided general comments such as “I would recommend a precise and deep explanation of the subtraction algorithm” [PT11].

However, results show differences in how PTs recognised relevant aspects of the materials/resources and in their interpretations according to the opportunities for instruction. Twenty out of these 26 PTs who recommended base ten blocks as a resource, identified relevant aspects of this resource and opportunities for the learning of the subtraction algorithm, such as the role of this resource in the understanding of the unit transformations (non-canonical decompositions) or the idea of place value. We exemplified it with excerpts of PT02 and PT22:

[...] Some of the resources I would recommend are the manipulatives, for instance, the base ten blocks, that are key in the understanding of the algorithms, and the abacus. With these resources we can work the idea of grouping and the idea of place value [PT02].

[...] The resource that I would recommend to Xavi is the base ten blocks, since this resource can help students see more clearly the subtraction algorithm and the transformations from hundreds to tens and from tens to units [PT22].

Nevertheless, six out of the 26 PTs did not identified relevant aspects of the materials recommended. These PTs provided vague or general explanations (or no explanation at all) regarding the opportunities of these material/resources for the learning of the subtraction algorithm. In the next excerpt, PT17 seems to be aware of the benefits of using manipulatives for the understanding of the algorithms but he provided a general explanation without describing how it can favour the understanding of the subtraction algorithm.

[...] I will recommend the use of manipulatives since they are key for the understanding of the algorithms since they are objects that represent de mathematical ideas that can be abstracted through the direct manipulation of objects [PT17].

Regarding how PTs used the resources recommended to plan the lesson, a group of 18 PTs provided a sequence of different subtractions that were ordered according to the number of transformations needed in the minuend (therefore, these subtractions were sequenced according to their level of difficulty). Furthermore, these PTs provided explanations about how to link the different modes of representation for helping students to understand the subtraction algorithm: what is done with the material (ten base blocks), what is orally said and what is written symbolically in the algorithm. This is the case of PT15 who provided a sequence of four subtractions (Figure 1) and

explained that the first one would be used as an example to show the different transformation of the units in the minuend, and the rest of subtractions would be done by students:

The designed lesson consists of 4 subtractions. The objective of the first subtraction is to help students understand the subtraction algorithm, particularly, the transformations of the units of different order performed in the subtraction algorithm. The rest will be done by the students.

1st Subtraction

$$221 - 218$$

3rd Subtraction (248 – 109)

Alex has 248 soccer cards. In a yard he has lost 109. How many cards does he have now?

2nd Subtraction

$$342 - 218$$

4th Subtraction (620 -245)

In a football stadium there are 620 people. But when the game starts it starts to rain and 245 people decide to go home. How many people are there in the stadium?

Figure 1. The sequence of four subtractions designed by PT15.

She continued justifying why she has proposed the subtractions in this order.

The level of difficulty will increase after each subtraction. In the 2nd subtraction: a transformation will be needed, in the 3rd subtraction a transformation will be needed, and it is presented within a problem, and, in the 4th subtraction, two transformations will be needed, and it is also presented within a problem.

Moreover, she was aware of the necessity of using different representations during her explanation (1st subtraction) and the necessity of relating the different modes of representations to help students understand the ideas behind the algorithm (Figure 2). She wrote:

The explanation of the first subtraction will be made [by me] by relating the four representation modes: concrete (multi-base blocks), graphic (drawings that represent multi-base blocks), symbolic (written operations) and oral (“what is said”, underlining the transformations accordingly).

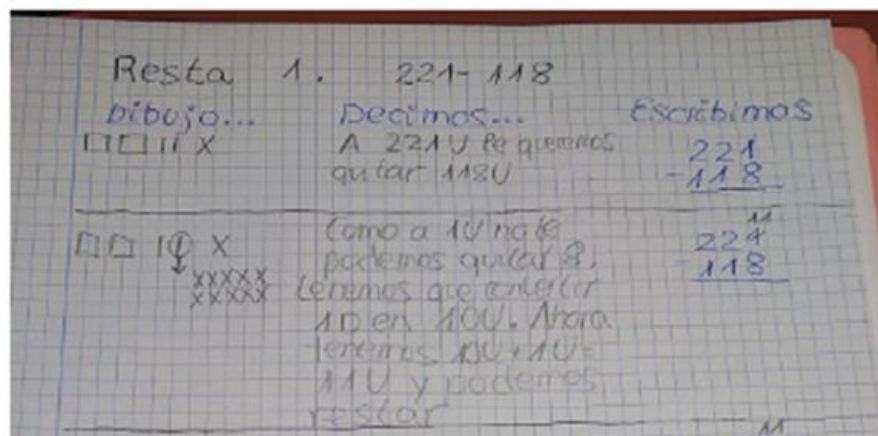


Figure 2. Excerpt from PT15 exemplifying how she would relate the modes of representations (what it is done with ten base blocks, what it is said and what it is written in the algorithm) ["Since 8U cannot be taken from 1U, then we need to transform 1T (long) into 10U (little cubes)". Now that I have 18U we can subtract".]

There were 4 PTs who did not provide a particular sequence of activities (or subtractions), but they provided explanations about how it would be this sequence. For instance, PT16 wrote.

It is very important to consider a correct order of the activities, so that the students learn from the easiest activities to the more complicated ones. In the case of the subtraction algorithm, the sequence that the teacher should plan for the understanding of this algorithm would be the following: 1. Subtraction without carrying. 2. Subtraction with carrying. 3. Subtraction with a 0 in the minuend.

Although PT16 was aware of the importance of providing a sequence of subtractions according to an increasing level of difficulty, she did not provide a specific sequence of subtractions. This lack of specificity hides, on the one hand, the way the PTs are going to introduce the subtraction algorithm, and on the other hand, whether PTs were able to provide this sequence of division according with the characteristics they have considered. In the example of PT16, the condition “having a 0 in the minuend” does not necessarily imply the need of more transformations, and therefore can or cannot increase the level of difficulty. These PTs did not provide comments about how they would use the material during the lesson.

Finally, the last 6 PTs did not provide any explanation neither a sequence of subtractions. Some of them solved only a subtraction.

CONCLUSION AND DISCUSSION

Our study examines how pre-service teachers notice different resources/materials when they are planning a lesson for introducing the subtraction algorithm in 3rd grade, particularly, we have focused on the materials/resources pre-service teachers would recommend, their interpretations regarding the opportunities of these materials/resources for instruction, and how they would use them in the lesson.

Results show that most of the PTs would recommend base ten blocks to introduce the subtraction algorithm in 3rd grade. Other materials were mentioned but always as additional ones to the base ten blocks. Regarding their interpretations about the opportunities of these materials for instruction, most of them (20 out of 28) were able to establish a relationship between the material and how it can help students understand the main ideas behind the algorithm: the different transformation of units – non-canonical decompositions of the minuend and the idea of place value. Nevertheless, some PTs provided general comments showing difficulties in explaining the opportunities for instruction of the material selected.

Considering how they planned to use the materials, 18 PTs not only provided a detailed lesson including a sequence of subtractions that were sequenced according to their

level of difficulty (number of transformation needed in the minuend) but also were able to explain how they would use the resource to help students understand the ideas that are behind the algorithm, linking the different modes of representation: what is done with the material (ten base blocks), what is orally said and what is written symbolically in the algorithm. 10 PTs did not provide a detailed lesson. Some of them provided general comments about the sequence of activities but without any specificity related to the way they would introduce the subtraction algorithm or how they would use the resource. Nevertheless, these general comments can be seen as a resource for professional developers (Jacobs et al., 2011) since these PTs are learning in a teacher education course.

Our results provide us with a snapshot of how these PTs are connecting the theoretical information given in the course with the practice, in this case, planning a lesson to introduce the subtraction algorithm. The relative success of these PTs in identifying a resource to introduce the subtraction algorithm, in interpreting the learning opportunities of this resource and in deciding how to use it in their lesson, underlines the potential of this type of tasks in teacher education programs. Results seem to show that the task and the theoretical information provided in the learning environment helped pre-service teachers to start giving meaning to the complexity of content and pedagogical opportunities of different resources considering the students' learning (Dietiker et al., 2018). Nevertheless, more research is needed to have more insight of how these PTs develop their curricular noticing during the course when they plan lessons for introducing different mathematical contents.

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THE DEVELOPMENT OF CONCEPTIONS OF FUNCTION - A QUALITATIVE LONGITUDINAL STUDY ON THE TRANSITION FROM SCHOOL TO UNIVERSITY

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Within the framework of a qualitative longitudinal study, we interviewed 15 students about their conceptions of function at two points in time. For the first time, we interviewed the students in the upper school. A second interview took place at the end of the students' first semester at university. The study pursues a case study approach. This paper focuses on the case of Tom and his of conception of function. At university, Tom reflects on his conception of function constructed in school. He contrasts and relates the content covered in school on the concept of function to that covered in the lecture on discrete structures.

CONCEPTIONS OF FUNCTION AT SCHOOL AND AT UNIVERSITY

The concept of function is addressed in mathematics lessons at school and in many mathematics-related courses at university. Despite this thematic overlap, there are indications that first-year students have difficulties constructing conceptions of function in which the university learning content is linked with the content addressed at school (e.g., Juter, 2010; Vandebrouck, 2011). A conception of function is "the individual's idea or understanding" (McDonald et al. 2000, p. 78) of function, while a concept of function is "the collective understanding of that content by the community of mathematicians" (Arnon et al. 2014, p. 18).

Different studies attribute the above-mentioned difficulties in linking school and university learning content, among other reasons, to changes of level of conceptualization (e.g., Artigue, 1999; Tall, 1995; Vandebrouck, 2011). These changes include a higher degree of formalisation of mathematical notations in university courses than in high school mathematics classes. This higher degree of formalisation generally makes it more difficult for first-year students to understand the mathematical content. While a higher degree of formalisation applies to many different content areas, Vandebrouck (2011) attributes the problems specifically with understanding the concept of function to the fact that many first-year students are unable to link, on the one hand, a pointwise and, on the other hand, a global perspective on functions. In addition, many students reduce the work with functions to the manipulation of algebraic expressions at the end of their school career or at the beginning of their university career (e.g., Carlson, 1998; Vandebrouck, 2011). The APOS Theory refers to such manipulations as the "action conception of function" and distinguishes this action conception from a process and an object conception (e.g., Dubinsky & McDonald, 2001; Arnon et al., 2014). In the context of functions, an action conception is constructed, for example, when a student inserts a concrete value into a function term

to calculate the result. With a process conception, the student can imagine transformations of unspecified inputs into outputs without having to insert concrete values. The process can then be encapsulated into a mental object, which constitutes the transition to object conception. Composing functions or recognising properties of entire function classes are examples of such object conceptions of function. When the student establishes connections between the different conceptions, the individual has constructed a schema for function. Empirical results illustrate that a coherent schema of the concept of function "evolves over a period of many years and requires an effort of 'sense making' to understand and orchestrate individual function components to work in concert" (Carlson 1998, p. 115). To develop such an understanding, it is crucial to acquire a "language of functions" and apply this language to different representations (Carlson 1998, p. 138).

An overview of the state of research reveals that, for the most part, studies examining conceptions of function at the transition from school to higher education refer to surveys conducted either once at school or once at university (e.g., Balacheff & Gaudin, 2009; Vandebrouck, 2011). There is only one longitudinal study known to the author, by Juter (2010). Juter (2010) studies conceptions of functions, limits, derivatives, integrals, and continuity at the beginning of an analysis course and for one year thereafter. The study is framed by the conceptual-change theory (Posner et al. 1982), which has been established internationally as a theoretical framework for studies on the development of conceptions. Current psychological and mathematics didactic studies understand conceptual change as a reorganisation of existing knowledge or conceptions (e.g., Juter, 2010; Vosniadou & Verschaffel, 2004). According to the theory of conceptual change, reorganising existing cognitive structures is challenging. In keeping with this, Juter (2010) shows to what extent conceptions of function can be stable over a certain period. Conceptions expressed by the students that have endured are the notion of a function as an "equation" to which a corresponding graph can be drawn or the notion as a "sequence of events presented by a formula or a coordinate system" (Juter, 2010, p. 2279 & 2280).

In addition to Juter's study (2010), further longitudinal studies would be desirable to describe and explain more detailed how conceptions of function develop at the transition from school to university. In particular, a study would have a lot of explanatory potential that examines the conceptions of function at school and again at university, i.e., a study that explicitly focuses on the transition from school to higher education. This paper refers to such a study we conducted. The results should provide a much more insightful analysis of which conceptions of function students actually constructed at school and to what extent these conceptions have potential to link content addressed at mathematics-related courses at university. With this knowledge, we could explain difficulties in the first semester better. These explanations are fundamental to improve courses both at school and at university. This paper aims to address the above-mentioned research desiderata by reporting results that give a first answer to the main question of the longitudinal study:

How do conceptions of functions develop at the transition from school to university?

METHODS

Procedure

In order to analyse how conceptions of function develop at the transition from school to university, we take a case study approach (e.g., Thomas, 2011). We examine several cases under the same aspects using a multiple case study. We have chosen this approach because we assume that conceptions of function develop very differently. Analysing each case separately is the best way account for the uniqueness of development processes. Furthermore, this approach makes it possible to include different schools and study programs. In order to be able to trace the development of the students' conceptions, we have chosen a qualitative longitudinal approach with interviews at two points in time (e.g., Hermanowicz, 2013).

First interview: The first interviews took place at German schools three to six months before the Abitur examinations (highest graduation in Germany) in 2020. The teachers from these schools informed the students about the study. The students voluntarily participated in the interviews, which were video recorded. The length of the interviews ranged from 45 to 75 minutes. During the interviews, we encouraged the students to verbalise their thought processes. They could take notes at any time. In these first interviews, the focus was on questions and tasks that addressed the concepts of function, derivative, and integral and their interconnections. In the following, some questions and tasks related to the function concept will be discussed. The first question was, "What comes to mind about the concept of function?" Afterwards, we asked the students to explain the function concept to a fictitious classmate. Later, the interview considered different representations of functions (e.g., terms, tables, graphs) and addressed different perspectives on function, which relate to a possible action, process, or object conception of function.

Second interview: The second interviews took place at German universities at the end of the winter semester 2020-2021. The interviews lasted 45 to 90 minutes. Due to the COVID pandemic, we conducted the interviews via video conference. The students could take notes digitally. Before an interview, the interviewer studied the documents of the students' mathematics courses (e.g., a script from the lecture, accompanying tasks, and solutions) – if available – in order to refer to them. The second interview revisited the questions asked at the first one and added questions about the difference between mathematics at school and mathematics at university (Hermanowicz, 2013). For example, we asked to relate the concept of function discussed at university with the concept of function discussed at school. The representations of function were supplemented by other representations addressed at university.

Sample

The first measurement point consists of interviews with 30 upper secondary school students, aged 17-19, from 12 German schools. The students took mathematics as a

basic course (11 students) or as an advanced course (19 students). The range of the students' report grades in mathematics was from 5 to 15 (the maximum) points. All students interviewed stated, at the time of the interviews, their willingness to take up studies in a mathematics-related subject (e.g., mathematics, physics, business studies, etc.). The second measurement point comprises interviews with the 15 students who have taken up a mathematics-related course of study. (Twelve students have changed their plan of study and entered a non-mathematics program, started a voluntary social year, or other. Contact has been broken off with three students.) These 15 students studied logistics or aerospace, physics, biology, chemistry, (business) informatics, business administration, industrial engineering, and mathematics as a teacher or specialised mathematics. All students took at least one mathematical course (e.g., mathematics for natural scientists or economists, analysis, linear algebra, discrete structures) in the first semester. The students are spread across seven universities and three universities of applied sciences in Germany.

Analysis

The 15 longitudinal cases are the subject of the data analysis. In the sense of the case study approach, we include all information available for the case (existing information about the study programme, if applicable, documents from the study, the interviews) in the data analysis (Thomas, 2011). In order to obtain an overview of the responses and be able to better compare the two interviews conducted with one student at two different times, the transcripts were transcribed and coded (Hermanowicz, 2013). The coding concentrated on the extent to which different conceptions of function (according to the APOS-Theory) can be identified in the learners' utterances. In addition, it was analysed which representations were mentioned by the learners (e.g., graph, term). Up until now, we have mainly compared the two measurement points for almost all interviews. An interview's comparison of different students is ongoing research. Answers to questions asked in a similar format at both the first and second measurement points are compared in a table for each study participant in order to identify and discuss consistent and changing components of the conceptions of function (Hermanowicz, 2013). In order to be able to grasp a case in all its complexity, and in particular to take different perspectives on the case, we analysed and discussed the cases at least in pairs with regard to the research question.

RESULTS – THE CASE OF TOM

In this section, excerpts from the interview with Tom (a pseudonym) are presented. The case of Tom was chosen for this paper because it can be used to analyse both similarities and differences between the conception of function constructed at school and later at university. After graduating from school, Tom started studying business informatics at a German university. In his first semester, he took the course "Discrete Structures," in which the concept of function was discussed in detail. The function concept was introduced in the course as follows: A relation is a function (mapping) if: For every $a \in A$ there is exactly one $b \in B$ with $(a, b) \in R$. Functions were represented

mainly by bipartite graphs and set notation such as: $f = \{(a, 2), (b, 3), (c, 1)\} \subseteq A \times B$ for $A = \{a, b, c\}$ und $B = \{1, 2, 3\}$. At the end of Tom's school career, he describes the concept of function as follows:

Interviewer: What comes to mind about the concept of function?

Tom: The function is typically $f(x)$. The y -result is somehow dependent on x . There are different types of functions, whether it is a straight line, a quadratic function, or an e -function.

Interviewer: Can you give examples of functions?

Tom: Classically, $f(x) = mx + b$, the equation of a straight line. There are quadratic equations. That is something with x^2 , thus $f(x) = 3x^2 + 7x - 3$. Then there were e -functions, as previously stated. For example, $f(x) = 3e^{5x} - 3$.

Interviewer: Can you represent a function in another way?

Tom: As a graph and as a table of values. So that you insert single values $f(x)$ and x . Then you set up such a table.

Later in the interview, Tom should explain the meaning of functions to a classmate:

Tom: It's easiest to explain with an example. You have so many buns; how expensive is that in total? So that you then consider that you don't have to calculate it every time but rather set up something general for it, where you then see, every time you go to the bakery, how expensive it is in total.

In summary, we can identify an action conception, a process conception, as well as an object conception in Tom's statements. The action conception becomes especially clear when Tom talks about setting up a table of values and inserting single values into the function term. His explanation of the meaning of functions, in which he points out that it is characteristic for a function "not to calculate a value every time" but to "set up something general," shows that Tom has interiorized this action into a process, as is shown by his explanation of the meaning of functions. The object conception is indicated in the interview passage by Tom's comments on function classes. This object conception is also highlighted later in interview, for example, when Tom discusses function derivation or integration.

The interview passage from the school is in contrast with the following interview passages from the interview after the first semester:

Interviewer: What comes to mind about the concept of function?

Tom: For me, a function is the mapping of certain values onto other values. It is known from school that a function assigns a y -value to each x -value. This fact is expressed either way, so I'm more familiar with it from school, by a mathematical term. For example, x^2 . In contrast, in the lecture on discrete structures, concrete quantities were given and it was indicated which values map onto which other values. You can always represent functions graphically, either in a coordinate system or as we had it at the university:

You have the set with the numbers 1 to 5, which map onto another set. We always had that with arrows, which then mapped onto each other. You can assign certain properties to functions. In school, we had something like symmetry and slope. We had the three terms injective, surjective and bijective at the university.

Tom structures his answer according to the content covered at school and the content covered at university. Taking this answer, Tom is asked how he "relates what was discussed at university about the concept of function to what was thematised at school," to which Tom says:

Tom: I changed my basic idea of the term function. I now understand it in such a way that a value maps onto a certain other value and is connected to it. In school, you always have this x and y . What was certainly different was simply how functions were defined. In school, this is often done more generally by means of a mathematical expression. We never really had that in university, but we always had those really concrete values mapped onto other concrete values. However, you can also understand the functions from school in the sense that certain values are mapped onto other values.

In summary, the following interpretations of the interviews with Tom were elaborated:

1. Already at the beginning of the second interview, it becomes clear that Tom, without explicit demand, very reflectively contrasts and partly relates the contents of the concept of function dealt with at school with the contents dealt with in the lecture on discrete structures. We reconstructed that by comparing the contents, he divides the function concept into functions that map from \mathbb{R} to \mathbb{R} (as functions were primarily treated in school) and functions that map from \mathbb{N} to \mathbb{N} (as functions were primarily treated in the lecture on discrete structures).
2. It seems that the lecture on discrete structures caused Tom to reflect on the conception of function he constructed in school. This reflection is a de-encapsulation of the object conception to a process conception. This de-encapsulation is supported by the fact that, with respect to both functions from \mathbb{R} to \mathbb{R} and functions from \mathbb{N} to \mathbb{N} , Tom can imagine unspecified inputs being transformed to produce outputs. However, he does not recognise that functions from \mathbb{N} to \mathbb{N} can also follow a function rule, such as x^2 . He associates such a "general" function rule expressed as a function only with the concept of function as he learned it at school.
3. An object conception of function, as can be seen in several places in the first interview, becomes briefly clear in the second interview when Tom talks about properties of functions like symmetry and gradient, or injective, surjective, and bijective. This object conception, which can only be shown briefly, and the overall separation of the concept of function into functions that map from \mathbb{R} to \mathbb{R} and from \mathbb{N} to \mathbb{N} , respectively, call into question Tom's ability to dynamically combine his action conception, process conception, and object conception into a coherent schema of the concept of function independent of the domain and codomain. Tom also underlines this

separation linguistically while using the term "school function" in the further interview when talking about the functions from \mathbb{R} to \mathbb{R} .

DISCUSSION & PERSPECTIVES

Discussion and outlook for cross-case analyses

As previous studies have shown (e.g., Carlson, 1998; Vandebrouck, 2011), the case of Tom highlights that numerous students very strongly associate the concept of function with a function term both at the end of their school career and at the end of the first semester. In this respect, this study, like the study by Juter (2010), shows, in the sense of conceptual change theory that mental structures constructed at school are stable to a certain extent. The first interview with Tom and other cases in the study show that the work with these function terms is not limited to the use of concrete values in the sense of an action conception. Rather, we could also reconstruct a process and object conception in Tom's case, particularly near the end of his school career.

The cross-case analyses reveal that it is highly dependent on the content of the specific course in which way first-year students draw on the conceptions they constructed in school. The fact that Tom is very reflective in contrasting the content addressed at school with the content addressed at university sets him apart from other cases in this study. Other students, especially those who took a course in calculus or mathematics for physicists, chemists, economists, or similar, hardly recognise any breaks in content between what was thematised at school and at university regarding the concept of function. This becomes especially clear in the case of Ben and his statements about the concept of function in the second interview. In contrast to Tom, when he is asked to "relate what was discussed at university about the concept of function to what was thematised at school," Ben says: "A function is now, at university, nothing different from what it was still in school. The only difference now is that you may encounter jumps and kinks more frequently."

Methodological reflection and perspectives

The case study approach has proven successful in analysing different developments of functional concepts in detail. Although we cannot make any quantitative statements with this case study design, individual developments of ideas could be traced very precisely. In particular, the case study design made it possible to investigate these developments with respect to different courses of study. Looking at different courses of study is important insofar as mathematics instruction at school should also prepare students for different courses of study. The fact that we conducted the interviews digitally in the second part of the study did not negatively affect the data collection. Due to the digital teaching, the students were used to the format of a video conference and had the possibility to take notes digitally or use the camera for gestural underpinnings. The presented findings have to be regarded as a partial result of our study. We still need to analyse a few cases in detail. The comparison of different people's interviews, in particular, is ongoing research.

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LEARNING ABOUT STUDENT'S STRATEGIES BASED ON AUTOMATED ANALYSIS: THE CASE OF FRACTIONS

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We argue that learners' generated examples can do more than answer mathematical task requirements. Studies show that examples may also reflect students' strategical thinking, which is fundamentally related to how students solve problems. Our argument is relevant to the use of automated assessment platforms. A central challenge of such platforms is to provide automated data that enables learning about students' strategies based on their examples. We report here on a study that examined data analysis of submitted examples of a 7th-grade student engaged in two activities comprising six example-eliciting tasks in the context of fractions. Characterization of the examples' properties, provided by automated assessment of the student's submissions, enabled us to learn about his strategies to answer the tasks, such as the expanding strategy.

INTRODUCTION

Mathematical strategies are students' methods to solve problems with mathematical content, whether their answers are correct or not (Hegedus & Otálora, 2022). Ellis et al. (2019) offered an analytical tool to examine aspects of students' thinking by examining their self-constructed examples when they involve a process of exploration, development, and justification of mathematical conjecture. According to their study—and unlike 'criteria', which refer to students' reasons for choosing individual examples—students' strategies refer to students' reasons for choosing *sets of examples* that: (1) represent a diversity of cases across a range of features; (2) represent systematic variation (e.g., shifting each successive example by varying one or more of its elements); and (3) represent a particular mathematical property or set of properties.

Following the study by Ellis et al. (2019), this study investigated the innovative possibility of using automated assessment to learn about students' strategies, as reflected in students' examples constructed as answers to a sequence of tasks. Based on known strategies of comparing fractions, the aim of our study was to explain students' mathematical reasoning based on the analysis of the submitted self-constructed examples. We asked, how might we learn about students' strategies based on automated analysis of their examples submitted as answers to a sequence of example-eliciting-tasks in the context of fractions? Our data is based on examples of one student engaged in two activities comprising six tasks in the context of fractions. We use the automated analysis provided by the assessment platform STEP to collect evidence of the student's strategies while comparing fractions and choosing equivalent fractions. We designed example-eliciting-tasks (EETs) that ask students to construct examples by an interactive diagram to support or contradict a mathematical claim or to create examples under given constraints (Yerushalmy & Olsher, 2020).

THEORETICAL BACKGROUND

Fractions are a central topic in the mathematical curriculum. Previous research has found that students often have difficulties considering fractions (Pedersen & Bjerre, 2021). Some of the common (mis)conceptions mentioned in the mathematical education literature are these: (a) some students consider fractions as necessarily being less than 1; (b) students are usually taught to find equivalent fractions by multiplying both the numerator and denominator of a fraction by the same integer— some students erroneously use additive instead of multiplicative reasoning; (c) some students consider fractions as equivalent only when they look identical or have the same numerator or denominator; otherwise, (d) in the comparison of fractions, students tend to interpret a/b as two independent whole numbers, which leads them to conclude that a fraction will be larger when its numerator, its denominator, or both are larger. The roots for the above conceptions could be found in misunderstanding the strategies taught for comparing fractions, as in the case of finding equivalent fractions by using the expanding method or when comparing fractions by using a benchmark of one whole. Other such conceptions are students' strategies that they develop on their own and that lead them to common mistakes (Schneider & Siegler, 2010). Studies have shown that analysing the mathematical properties of students' answers over sets of examples can reflect reasoning (Lithner, 2003).

In this study, the term 'characteristic' refers to each of the predefined properties of the *first fractions* that the student chose to deal with (e.g., a fraction that is less than one), these first fractions are important because they are considered the first step in problem solving; or the predefined relations between the properties of the first fraction (a given/chosen one) and the properties of the fractions with which the student chose to fulfill the requirements of the tasks (e.g., fractions that have the same denominator as the given/chosen fraction). The term 'strategy' refers to the presence of characteristics that the student chose to use over sets of examples throughout a sequence of EETs.

We defined specific characteristics based on the common (mis)conceptions mentioned in the mathematical curriculum and the literature. STEP provided us with automated analysis indicating whether these characteristics are present in the students' submissions or not. The automated analysis represents the presence of the characteristics of each example differently in two activities: by words in the first activity on equivalent fractions (Table 1), and by a combination of words and visual representations that reflect a conceptual landscape that "is a landscape in the sense that it gives a visual planar density complex representing student perceptions" (Leung & Lee, 2013, p. 366) in the second activity of comparing fractions (Table 2).

In two of the tasks, these characteristics were formulated in the form of statements as part of the task. The student was asked to identify and mark the statements as existing in his examples in order to characterize them prior to submission. In addition, the requirement of these tasks was to use *as few statements as possible*, which we called LoN tasks (Kadan-Tabaja & Yerushalmy, 2022), with the aim of making the student

rethink the characteristics of his examples and in an attempt to examine if the student is able to critique (students could revise their answers) the choices he made.

In the **interactive diagram** (Fig. 1) that is used in this study fractions are represented in the Cartesian coordinate system by a point whose vertical coordinate is the numerator and whose horizontal coordinate is the denominator (Arnon, Nesher, & Nirenburg, 2001). All equivalent fractions are represented on a straight line passing through the origin, which appears automatically when the fractions are equivalent. The origin and points on the vertical axis do not represent any fraction. Points that exist on the line with a larger slope represent larger fractions (Fig. 9). The red point represents the first fraction that the student chose to deal with, and the green/ blue points represent fractions that the student chose in order to fulfill the requirement of the task.

METHODOLOGICAL CONSIDERATIONS

We examine a case of a 7th-grade student's submitted examples related to two activities on the term of equivalence and comparable fractions, which were subjected to an automated analysis. The goal of the analysis was to identify his strategies. Each activity had three tasks. The student worked on the activities using his computer individually, 90 minutes for each activity. The student was selected following his and his parents' agreement to participate in the study.

The first activity contained three tasks. In every task, the student was asked to choose a fraction by dragging the red point (Fig. 1), then to construct other fraction/s that are equivalent to the one he chose. The student was asked to construct three examples that are as different as possible, and in another task to characterize his examples before submitting them, using a given set of four statements that formulate possible characterization of the examples (Table 1). The activity was formulated as follows: Tasks 1 and 3 were, "Choose a fraction, then construct three fractions that are equivalent to the fraction you chose" (<https://www.geogebra.org/m/bvzqpnhb>). Task 2 (Fig. 2) was "Construct two equivalent fractions that have *as few statements as possible* from the given set of statements." (<https://www.geogebra.org/m/yehmzt5q>). The first characteristic should be found in any example that answers the requirements (Ch1).

Table 1. Characteristics/statements of the first activity (equivalent fractions)

Ch1) The visual representation line crosses all points at the same time (Fig. 1).

Ch2) You chose the first fraction to be less than one.

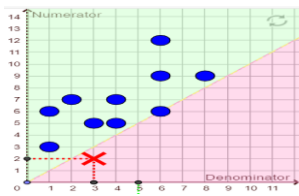
Ch3) One fraction is an expansion or reduction of the red fraction.

Ch4) The numerator and the denominator of one fraction are larger additively by the same number than the numerator and the denominator of the other fraction.

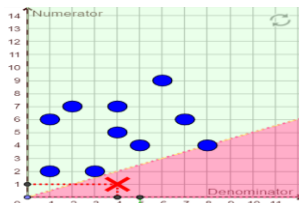
The second activity (<https://www.geogebra.org/m/rwxxpnbw>) contained three tasks of comparing fractions. The activity requirement was to construct (green/blue) fraction/s that are larger than a chosen or given (red) fraction. During tasks 2 and 3, the student was asked to characterize his example/s before submitting them according to a set of statements that formulate the characteristics of comparable fractions. The choice

of statement was reflected immediately in a visual landscape of the specific characteristic that the statement formulated (Table 2).

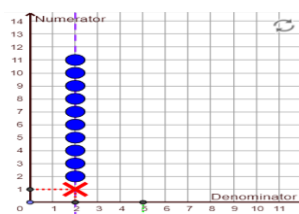
Table 2. Characteristics/statements of the second activity (comparing fractions) and the visual landscape of each (as response to the student's choice).



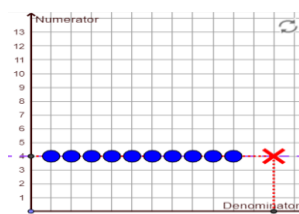
M1) Comparing fractions by using a benchmark of one whole. The green/red area represents all fractions that are larger/smaller than one, respectively. The line between them represents fractions equivalent to one whole.



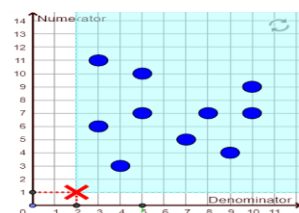
M2) Comparing fractions by using a benchmark of one half. The green/red area represents all fractions that are larger/ smaller than one half, respectively. The line between them represents fractions equivalent to one half.



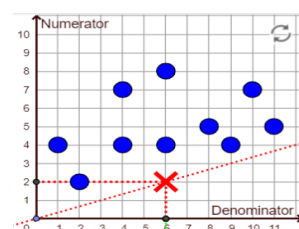
M3) Comparing blue fractions by using the same numerator as the red fraction.



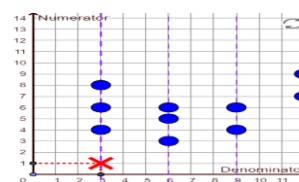
M4) Comparing blue fractions by using the same denominator as the red fraction.



M5) The numerator and the denominator of each blue fraction are larger than the numerator and the denominator of the red fraction, respectively (misconception).



M6) The visual representation line of the blue fraction is above the visual representation line of the red fraction. (This characteristic is parallel in its essence to Ch1 in the first activity; both are based on the visual representation line). This is true when the student's example fulfills the requirements of the tasks.



M7) The denominator of each blue fraction is a multiple of an integer of the denominator of the red fraction (This characteristic is parallel in its essence to Ch3 in the first activity, both are based on expanding the first fraction).

The activity was formulated as follows: Task 1: “Choose 10 fractions that are larger than the fraction $\frac{2}{5}$ ”; Task 2: “Choose a fraction. Then choose 10 fractions that are larger than the fraction you chose. From the set statements of comparing fractions, choose which statement you used while constructing your examples.” Task 3: “Construct two fractions, one larger than the other, that apply as few statements as possible from the given set of statements.” In this task, the interactive diagram automatically presents

which fraction is larger. The characteristic M6 should be found in any example that answers the requirements.

Data sources and analysis.

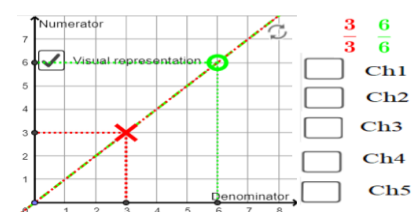
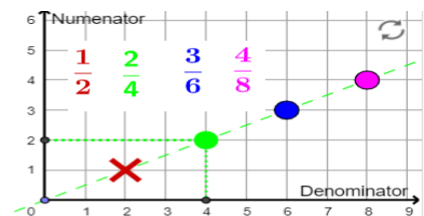
To answer the research question, we designed sequences of EETs in the context of fractions. The student was asked to submit examples of these tasks. For each submitted example, STEP provided us with an automated analysis of the characteristics presented in it. The automated analysis provided us with data about the following: (1) the presence of the characteristics of the first fraction that the student chose to deal with; (2) the presence of the characteristics related to the relations between the first fraction (the chosen/given fraction) and the fraction/s that the student chose to use to fulfill the requirements of the tasks; (3) the presence of characteristics in the sequence of six EETs. Based on this data, we tried to identify (4) the student's strategies that are consistent with strategies that are mentioned in the literature. Then, (5) for strategies that we identified, we tried to assume the reason that led the student to choose them, according to Ellis et al. (2019).

RESULTS

To answer the research question, we represented the characteristics that were present in the examples of the student in the two activities (six tasks) based on the automated analysis. Characteristics are marked *in italic case*. We focused on the LoN tasks which are marked with a *. Based on this data, we try to point out the student's strategies of choosing fractions.

Student examples in the first activity

Task 1	$\left(\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}\right)$ $\left(\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}\right)$ $\left(\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16}\right)$	Fig. 1 The interactive diagram and an example submission from Task 1
Task 2 *	$\left(\frac{6}{3} = \frac{18}{9}\right), \left(\frac{2}{1} = \frac{4}{2}\right),$ $\left(\frac{3}{3} = \frac{6}{6}\right)$	Fig. 2 An example from Task 2
Task 3	$\left(\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}\right), \left(\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}\right), \left(\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16}\right)$	



In Tasks 1 and 3, the automated analysis provided us with the information that the student's examples were *correct* (Ch1). The characteristics of the first fraction he chose to deal with (the fractions in bold) were *less than one* (Ch2), and we noticed that he chose specifically a *unit fraction*. In addition, the characteristic of finding equivalent fractions based on *expanding* was used (Ch3).

Based on the characteristics of the student's examples in the LoN task (Task 2), the automated analysis provided us with a new characteristic of the first fraction the student chose to deal with (fractions that are not less than one). We assume that the task requirement of constructing examples using *as few statements as possible* made the student rethink the characteristics of his examples. This may indicate that the task requirement for constructing counter-examples of the statements enabled the student to critique Ch2 while choosing the first fraction. However, the automated analysis showed that the student still based his answer on expanding (Ch3) in order to find equivalent fractions in his examples.

Student examples of the second activity

<p>Task 1</p>	<p>The student chose these fractions as larger than the first fraction:</p> $\frac{3}{5}, \frac{4}{5}, \frac{5}{5}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{15}, \frac{9}{20}, \frac{11}{20}$	<p>Fig. 4, the first fraction was $(\frac{2}{5})$ given.</p>	
<p>Task 2</p>	<p>The student chose these fractions as larger than the first fraction:</p> $\frac{2}{3}, \frac{3}{3}, \frac{9}{6}, \frac{4}{9}, \frac{5}{9}, \frac{7}{12}, \frac{8}{12}, \frac{9}{15}, \frac{11}{15}, \frac{10}{21}$	<p>Fig. 5, the student chose $(\frac{1}{3})$ as first fraction</p>	
<p>Task 3 *</p>	<p>$\frac{1}{5} < \frac{4}{10}$ (M5, M6, M7).</p>	<p>Fig. 9, the student chose $(\frac{1}{5})$ as first fraction</p>	

Throughout the activity, the automated analysis showed that the examples of the student were *correct* (M6). In addition, in the first task, it showed that the characteristic M7 was identified in his examples (Fig. 4). In Task 2, the student had the opportunity to choose the first fraction, and the automated analysis showed that the characteristic of the first fraction was identified as *less than one* (Ch2) and *a unit fraction*. Furthermore, it showed that characteristic (M7) was identified; this characteristic is based on expanding the first fraction in order to find larger fractions (Fig. 5).

In the LoN task (Task 3), when the student was asked to construct an example that has as few statements as possible, the automated analysis identified characteristics M5, M6 and M7 (Fig. 9). The student could not provide an example with fewer characteristics, and he was unable to critique the characteristic M7 that was present in his examples throughout the activities.

In summary, based on identifying characteristics that the automated analysis provided us over sets of student's examples throughout a sequence of tasks and activities, we

were able to identify a student's strategy of choosing the first fraction to deal with, and the student's strategy of choosing examples that fulfill the requirements of the tasks. The student's strategy of choosing the first fractions was based on unit fractions which were less than one. In addition, the strategy of finding fractions to fulfill the requirements of the tasks was based on expanding the chosen/given fraction. We assume that the student's strategies of choosing the set of examples was based on the representation of relevant mathematical properties that fulfill the requirements of the task according to Ellis et al. (2019).

DISCUSSION AND CONCLUSIONS

The empirical results show that we were able to identify characteristics of the student's examples throughout the activities, which helped us learn about his strategical thinking. Based on the automated analysis of the characteristics of the student's examples, we were able to identify a strategy of choosing the first fraction to deal with, namely the student chose fractions that are less than one, specifically unit fractions. His strategy throughout the two activities was to find examples that fulfill the requirements of the tasks based on expanding the given/chosen fraction. The strategies of this student for choosing fractions were based on common examples with known properties that are compatible with common conceptions. The student was able to critique his strategy thinking regarding the type of fractions that he chose when the tasks required it. However, he was unable to critique the strategy that he chose in order to fulfill the requirements of the task.

The findings are consistent with the literature, which has reported that technological assessment platforms make it possible to supply reports based on automated analysis that reveals trends in performance over time, and enables tracking learners' learning progress, especially their strategies, concepts, and structures, rather than mechanical processes (Stacey & Wiliam, 2012). This is especially true when the analysis is based on the students' examples in a collection of tasks performed sequentially in an exploration process (Luz & Yerushalmy, 2019). A central challenge of technological assessment platforms is to develop ways that enable automated analysis of rich, complex, and big data throughout sequences of EETs. The automated analysis shown in this study provided information about the characteristics of submitted examples throughout a sequence of tasks, which helped us identify the learner's strategies. Our argument is that such automated information can be used more efficiently if the technological platform enables automated analysis of a sequence of tasks and activities in order to support planning differential teaching or adapted learning. The study was limited by one participant and should be reproduced with larger groups. In addition, we did not correlate between the statements the student chose and the characteristics that the automated analysis identified, and the correctness of the student submissions was not crucial.

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HOW A TEACHER'S PROFESSIONAL IDENTITY SHAPES PRACTICE: A CASE STUDY IN UNIVERSITY MATHEMATICS

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This study explores the nature of a teacher's professional identity and how that shapes the nature of lecturing. The participant is a purposefully selected mathematics lecturer and the context of the study is a set of introductory university mathematics lectures. The study is theoretically framed within the identifying activity framework (Heyd-Metzuyanim, 2013). The analysis showed that the identifications of the lecturer about himself (1st Person) and about the students (3rd Person) and the comments on the mathematizing that takes place in research influenced the comments on mathematizing that happened in lectures. For the purpose of this study, two examples are presented under the theme "mathematics as a building". The findings could be used as input for the design of professional development programs for university lecturers.

INTRODUCTION

In the past four years two review studies have been published that explored professional, mathematics teacher identity and emphasized the need to focus on practice (Lutovac & Kaasila, 2019) as well as the nature of mathematics (Graven & Heyd-Metzuyanim, 2019). The exploration of professional identity has been receiving increased attention and several studies aimed at its characterization. For example, Bjuland and colleagues (2012) collected reflective narratives over two years that shaped the development of the identity of an experienced mathematics primary teacher, using the definition of identity from Sfard and Prusak (2005). From this study, identity indicators occurred, reflecting the relationship of the teacher with the community of participants. Defining identity as "shifting experiences of being, becoming and belonging related to the profession" (p. 469), Skott (2019) explored the identity development of a participant while engaging in different practices, like professional development programs. The study followed the changes in the identity of the participant from being a mathematics teacher to becoming a mathematics teacher. Another study that was undertaken with the development of a mathematics teacher's professional identity was by Losano et al. (2018). Collecting data from the first teaching year of the participant, the study showcased the complexity of this period for both the personal and professional life of the teacher. In these studies, mathematics is the context and addressed as the background, agreeing with Graven and Heyd-Metzuyanim (2019) earlier observation. Additionally, studies that focus on professional identity rarely examined the identity of lecturers—research mathematicians who teach either introductory or advanced mathematics courses in mathematics bachelor programs. The majority of these studies focused on pre/in-service elementary or mathematics teachers, or teacher educators (Graven & Heyd-Metzuyanim, 2019;

Lutovac & Kaasila, 2019). An exception is a study by Kensington-Miller et al. (2014) which focused on two lecturers, a mathematician and a mathematics educator, and the shifts in their academic identity while making changes in their teaching practice.

For the purpose of this study, identity is defined as “collections of stories about persons or, more specifically, as those narratives about individuals that are reifying, endorsable, and significant” (Sfard & Prusak, 2005, p. 16). In university mathematics classrooms, the lecturers bring with them stories-narratives about their multiple roles that give rise to an interplay of different identities (i.e., mathematics teacher/learner/researcher). These identities are not stable but constantly developing and evolving in parallel to the participation in the mathematical discourses. The different identities reflect stories about the ways the lecturers learn mathematics, were taught mathematics, do research in mathematics, their vision about teaching, the forms of participation in the mathematical discourse, and their assumptions about “to whom” they are teaching to.

In the context that defined this study, lectures are held by lecturers with the primary responsibility to do research in mathematics. The lecturers are responsible to teach students signed to mathematics bachelor degrees with little or no prior teaching training. It is still unknown how a lecturer's identity informs the nature of lectures. Thus, the research question of this study is “*How does a purposefully selected lecturer's professional identity inform the nature of the lecturing?*”.

Identifying activity

Despite the increasing interest in identity-based research, identity has received much criticism mostly associated with the fact that its conceptualization remains not clear (Graven & Heyd-Metzuyanim, 2019). Aiming a precise conceptualization, the investigation of the development of identity in the engagement with the mathematical discourse can be done by analyzing the identifying activity (Heyd-Metzuyanim, 2013). Identifying activity uses the signal $_{Author}Subject_{Audience}$ from Sfard and Prusak (2005) to exemplify “*who identifies whom and to which audience*” (Heyd-Metzuyanim, 2013, p. 345, emphasis in original). This signal may be valuable to give access to the stories told by the same person about oneself or others. Therefore, 1st Person identifying has a signal $_{AA_c}$ (A is identifying A to C) whereas 3rd Person identifying has a signal $_{AB_C}$ (A is identifying B to C). To address the identifying activity, a separation between mathematizing (communication about mathematical objects) and subjectifying (communication about the participants of the discourse) could facilitate the emergence of the stories (Heyd-Metzuyanim, 2013). The direct subjectifying instances are of three layers moving from the specific level - “actions in a specific context”, to the general level that concerns “general evaluations of one’s participation in the discourse” and last to the most general level that is directly related to identifying activity (Heyd-Metzuyanim, 2013, p. 345). Moreover, given that the talk of a person does not always convey direct subjectifying instances, identity can be interpreted through implicit identifying instances where the talk conveys “implicitly message X about a person, while explicitly stating the message Y” (Heyd-Metzuyanim, 2013, p. 346).

METHODOLOGY

This study is part of an ongoing project that investigates lecturing in university mathematics. The case in this study is defined by a mid-career, highly perceived among students, lecturer (he/him/his) with 10 years of teaching experience. The lecturer started teaching in lectures during his graduate studies as a teaching assistant and then as a guest lecturer. Later in his professional career, he coordinated his own courses. For a year, he was hired in a teaching position where he got, as he said, his “real teaching experience”. At the time of the data collection, he had a research position with a small percentage of teaching per academic year.

During the time of the data collection, the lecturer was assigned to teach an introductory, proof-oriented analysis course in an online format for 103 students signed to mathematics bachelor degrees. I collected data for eight weeks (duration of the teaching block) through different sources: eight, weekly, semi-structured interviews, 17 video-recorded lectures, and multiple types of informal communication with the lecturer. Interviews served as the primary source of data for this study. Each interview was designed to focus on a topic relevant to his teaching, learning, or research experience. His views about the students and mathematics were also discussed. In order to explore the topic, the interviews were around specific teaching episodes (relevant to the presentation of a theorem and its proof) that occurred from classroom observations. In some cases, quotes were isolated from the observations (i.e., “it is tricky”, “mathematics are messy”) for further discussion. Each interview took place in an online environment, with attendants the lecturer, and me. The duration of each interview was approximately 30 minutes and all the interviews were recorded and transcribed.

Data analysis

For the purpose of the analysis, I carried out a content analysis of each interview (Merriam, 2019). The goal was to identify the main themes of each interview and the sub-themes under them, through constant comparisons. For example, one of the themes across the interviews was *mathematics as a building* with sub-themes including among others the following: *using previous definitions and theorems*, *structuring a proof*, and *producing mathematics*. Then, for each interview, I moved to identifying tables for the mathematizing and subjectifying instances (Heyd-Metzuyanin, 2013). Moreover, for each instance in the table, I added signal for 1st Person (stories about the lecturer from the lecturer to the interview attendees) and 3rd Person (stories about the students from the lecturer to the interview attendees) identifications. In this phase, I realized that the mathematizing was not direct but rather a meta-commentary of the mathematizing that happened in either the lectures or research. The phase concluded with complete identifying tables that revealed the complexity of the lecturer’s talk and the interplay of the signals in his talk. For the last step of the analysis, I sought for a detailed description of each of the identified themes and I chose to code the interviews. I added in-vivo and descriptive codes, as well as theoretically informed codes. The codes were

categorized under each theme offering descriptions of the themes. The analysis finished with the connection of the themes with the identifying activity and the signals proposing insights into the interplay and influence of the latter for the formulation of the former.

RESULTS

Lecturer's position focused mainly on doing research in his field. He valued teaching and expressed his interest to reflect on his teaching but he did not identify himself primarily as a teacher. When he was asked to comment on his teaching experience, he said:

It's basically strange that we were getting hired, they said, "do you teach this course?" ... It's all based on my own experience as a student and my own ideas of how I would like to be taught if I was a student right now.

In this excerpt, the lecturer implicitly identified himself as a teacher who brings into the lectures his experiences as a student and his ideas of how he "would like to be taught" if he was "a student right now". He emphasized that he was not hired to teach but he was asked to teach, distancing himself from teaching as his primate responsibility. In the following paragraphs, a close analysis of two examples from the interviews is reported, relating teaching in the lectures with his research experiences.

The first example has as a starting point an observation from the lectures where the lecturer used frequently the phrase "what does it mean?" during the proving processes. In the following excerpt, he was asked to reflect on the meaning of this phrase in the episode from the lecture around the teaching of the Mean value theorem [Figure 1]:

Yeah, so I think in mathematics, as the way I see it, you can think about things in two different ways. There's basically a very intuitive way of thinking, which is very playful, and... which is all about intuition and getting a sort of basic understanding. And then there's a more formal way of thinking. And what I noticed when I do a proof myself for something that I don't know yet, for instance, an exercise I have to solve, a proof or a paper doesn't matter, really, there's always this interplay between thinking about a result in terms of intuition, getting a feeling for the problem. And then also thinking about the theorem in a more formal way. So, it also has something to do with how can you remember stuff in the easiest possible way.

Figure 1: Mean value theorem presentation from the slides of the lecture.

Theorem: if

- $f : [a, b] \rightarrow \mathbb{R}$ is continuous
- f is differentiable on (a, b)

Then there exists $c \in (a, b)$ such that

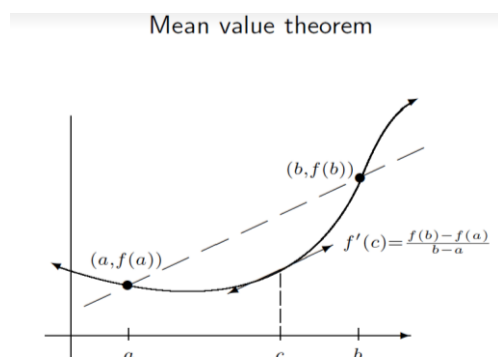
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In this excerpt, a meta-commentary on the mathematizing alongside with general subjectifying appeared. The lecturer commented on the mathematizing that takes place in his research when he deals with tasks like solving exercises, proving a theorem, or writing a research paper. He identified two ways of thinking about mathematics, the intuitive and the formal way. He characterized the intuitive way as playful aiming for a basic understanding and “a feeling” of the task. In his research “there's always this interplay” between the two ways of thinking, characterizing in that way his general participation in the mathematical discourse. Thus, the 1st Person signal of the lecturer’s identity referred to stories of him producing mathematics within this interplay. The comments on mathematizing that happens in the research raised the need to remember the theory “in the easiest possible way”. Following this need, he continued as follows:

So what students typically do is they try to learn by heart, the exact formulation of the theory. But that's a lot of information that you have to store in your mind. For instance the Mean value theorem, you have the condition f is continuous at a closed interval, f is differentiable at the open interval and then, etc. [referring to the conclusion of the theorem]. That's a lot of information to store. But if you remember the picture, I think if you remember the picture or the intuitive meaning behind a theorem, it's easy to, yeah, retrieve the conditions that were needed.

The lecturer moved to an implicit, 3rd Person identification of the students (that attend the course) as ritual participants in the mathematical discourse (e.g., Heyd-Metzuyanin, 2013) learning the theory by heart arguing that the information they need to remember is much. He then referred to the conditions of the Mean value theorem, discussing his choice to present the picture [Figure 2] and commenting on the mathematizing that happened in the lecture. Thus, from his point of view, the presentation of the picture could support the intuitive way of thinking, functioning as a way to remember the conditions of the theorem. Notably, both the mathematizing that happened in his research and the implicit identification of the students brought the picture in the lecture as a way to potentially support students’ mathematizing.

Figure 2: Picture used for the presentation of the Mean value theorem from the slides of the lecture.



The second example is around the discussion of the added value of the lectures. In the excerpts, the lecturer commented on the differences in reasoning in proofs in the textbook of the course, in the lectures, and research.

I mean, getting to prove in the first place is a very messy process. You have an assumption, and you have a conclusion that you want to reach. Somehow you have to bridge that gap. What I typically do myself is sometimes I start from the assumption and I try to work my way towards the conclusion. But sometimes I also do it the other way around: I start from the conclusion and work my way backward and hopefully, in the middle, you will find a way to close the gap... And that's something that I do try to emphasize a little bit in the lecture, that the thought process is different from, yeah, what you have in the very end.

The lecturer commented on the mathematizing in his research arguing that coming up with proof is “a very messy process”, describing the strategies he follows in an effort to “bridge the gap” between assumptions and conclusions. Within mathematizing, an implicit identification of lecturer’s identity as researcher appeared where his goal is the mathematical production (i.e., “getting to prove in the first place is a very messy process”) adding in the 1st Person identification. The comments on the mathematizing in research informed his comments about lecturing when he tried to emphasize that the thinking process (coming up with the proof) was different from the final proof that was presented in the lecture. As continued:

Yeah, so what I do try to emphasize in the lectures I often try to make remarks about is that once you read a proof, it doesn't reflect in any way how people came up with the proof in the first place. I think that's something that they have to understand that really coming up with a proof yourself is a very messy process.

In this excerpt, the lecturer argued that the students have to understand the difference between the mathematical production and the learning of mathematics. He claimed that proof production is “a very messy process” and reading the proof does not reflect how one comes up with the proof. Last, connecting with the textbook, the following excerpt appeared:

So in that sense, suppose that they will skip lectures, I suppose that a student would only read the textbook and not attend any lecture. I think this would give a very false impression that mathematicians think in a very straightforward manner from A to B and in research, this is absolutely not the case. And I think it's one advantage of teaching in a lecture that you can at least comment on the difference between what's been written down at the very end and how the thought processes are being done.

The lecturer discussed the case of the student who chose to read only the textbook, identifying the student as misdirected about mathematician’s thinking processes (3rd Person). Hence, reading the textbook and not attending the lectures hinders the thinking processes that are taking place in mathematical production. The lecturer implicitly identified himself as a researcher who during the lectures has the opportunity to share his experience from research, supporting students to address the differences between doing research and learning.

Both examples were categorized under the theme of *mathematics as a building* and the sub-theme *producing mathematics* where the lecturer commented about his practices from his research and their influence in his teaching. More specifically, the sub-theme included commentary on the mathematizing of research and reflections on the

mathematizing from the lectures. The mathematizing in research adapted for the teaching in relation to 3rd Person identity narratives authored by the lecturer about the students who attended the course when he talked to the author during the interview sessions. Moreover, the 1st Person identification of the lecturer about himself influenced the comments of mathematizing in the lectures as he intended to exemplify how a mathematician works when producing mathematics offering the students an insider's look into the mathematical production.

CONCLUDING REMARKS

This study aimed to explore a teacher's professional identity in university lectures and the ways identity informs the nature of his lecturing. I presented two examples that are under the theme of *mathematics as a building* and the sub-theme *producing mathematics*. In the two examples, the 1st Person instances about the lecturer's identity had as a common ground the practices of the lecturer when he engaged with the mathematical production. Interestingly, identifying (direct or implicit) about himself occurred through his commentary on the mathematizing in his research. Having as a starting point the mathematizing instances that would enable mathematical production, the lecturer shared stories about himself. These stories reflected his views about mathematics as an interplay of intuitive and formal ways of thinking and about mathematical production as a messy process shaping his identity as a lecturer. Each of these views gave rise to the comments about mathematizing that takes place in research (ways of remembering the theory, ways of bridging the gap between assumption and conclusion).

The comments about the mathematizing in research appeared to inform the commentary about the mathematizing that happened in the lectures. Indeed, in the first example, the lecturer addressing the need to “remember stuff in the easiest possible way” that appeared in research, he chose to present a picture in his lecture that would reveal the conditions of the Mean value theorem [Figure 1]. Accordingly, in the second example, he chose to emphasize in the lectures that mathematicians do not produce mathematics in the way the theorems are presented in the textbooks, revealing the added value of the lectures. Moreover, the comments about the mathematizing in lectures came along with 3rd Person identification of the lecturer about the students. In his talk, the students identified as ritual participants, trying to learn the theory by heart, and as not acknowledging a mathematician's thinking processes. Within these identifications, the comments about mathematizing in research adapted and informed the comments about mathematizing in lectures.

This study serves as the starting point for the exploration of the lecturer's professional identity and the ways this identity informs the nature of lecturing using the identifying activity framework (Heyd-Metzuyanim, 2013). I showed the influence of the identifications (1st and 3rd Person) and of the comments of mathematizing in research to the comments of the mathematizing in lectures. The findings are important because they offer useful insights into how a teacher's professional identity might shape the

nature of teaching in university lectures. Thus, these findings could be used as input for the design of professional development programs for university lecturers. However, still, a need to dive into the extended data set is needed to create an in-depth understanding of the ways lecturer's identity might inform the nature of lecturing.

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NETWORKING THE VARIATION THEORETICAL PRINCIPLES IN A PROBLEM-SOLVING BASED MATHEMATICS INSTRUCTION TASK DESIGN STUDY

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In this paper, we discuss how two different theories of variation are coordinated to underpin the problem-solving based mathematics instruction task design study for developing students' conceptual understanding and problem-solving abilities in mathematics. We argue here that the interrelationship between the two theories of variation and the classroom design study can provide a powerful task design perspective and hence sheds a different light on problem-solving instruction.

RESEARCH RATIONALES

In the last several decades, a considerable amount of research on mathematical problem-solving, originated from the seminal work of Pólya and Schoenfeld, has been conducted and provided valuable suggestions for reform in mathematics education. The major focus of the reforms in many countries has been on incorporating problem-solving into classroom instruction as a tool for promoting mathematics content understanding (e.g., Lester & Cai, 2016). At the same time, it is also treated as part of curricular content-as a core competence that students need to acquire. As part of these reforms, Ethiopian primary school mathematics curriculum has promoted problem-solving as a primary goal of classroom instruction (e.g., Ministry of Education, 2021). It is treated as an ability for students to develop, targeting to enhance their transferable skills to be applied in real-life contexts. However, the curriculum does not promote the use problem-solving as a tool for developing student's conceptual understanding. To promote mathematical concept understanding and problem-solving abilities, many researchers have suggested that students must be provided with opportunities to engage in cognitively demanding tasks that embody key mathematical concepts and skills (e.g., English & Gainsburg, 2016). However, some cognitive load theorists have criticized that launching a lesson with cognitively demanding tasks, which is not explicitly linked to prior instruction is problematic, and therefore undermine student learning (e.g., Sweller, 2010). This argument is based on the idea that human working memory has limited capacity to process new information that has not been stored in long-term memory, and is therefore easily overloaded when required to solve unfamiliar and challenging problems with various interacting elements (Sweller, 2010). This approach is also pedagogically demanding for teachers as they might be uncertain about how to structure the classroom learning sessions (e.g., Russo & Hopkins, 2019).

In our problem-solving based mathematics instruction (PS-based MI) task design study, we aimed to contribute to the existing knowledge of problem-solving with the

consideration of the issue of cognitive load. We illustrate how variation theory of learning (VToL) and teaching with variation (TwV) are incorporated to help to identify three key dimensions of the PS-based MI task design study: (1) Simultaneously addressing conceptual understanding and problem-solving abilities; (2) Reducing the unnecessary cognitive load on students learning; and (3) Minimizing teachers' pedagogical challenge in structuring the PS-based MI. The PS-based MI task design study is conducted in grade 6 (12-year-old students) in Ethiopian primary school classroom context. In this paper, our research question is: *How can VToL and TwV be coordinated to underpin the three dimensions of the design of sequences of PS-based MI tasks?*

TASK DESIGN RESEARCH IN MATHEMATICS EDUCATION

The mathematical tasks with which students engage in the classroom are crucial vehicles for developing students' mathematical learning (Stein, et al., 1996; Watson & Ohtani, 2015). Stein et al. (1996, p. 459) stated that, tasks determine 'not only what substance [students] learn but also how they come to think about, develop, use, and make sense of mathematics'. Since mathematical tasks play such key roles in the effectiveness of mathematics instruction, attention to mathematical task design in research and in classroom practice is therefore called for and continues to be an important area of research in mathematics education (Watson & Ohtani, 2015).

As highlighted in Kieran et al. (2015), we are inspired by the serious efforts made in task design community over the decades on striking the integration of theory into task design studies in mathematics education, which was provided in the form of design frameworks and principles. Kieran et al. (2015) classified these task design frameworks and principles into two categories: the scope of the theoretical frames informing the design (grand, middle-range, or domain-specific), and whether the design is considered as design as intention or design as implementation. Design as intention addresses "the initial formulation of the design", which makes use of developed theoretical frames (Kieran et al., 2015, p. 28). In contrast, design as implementation focuses on "the process by which a designed sequence is integrated into the classroom environment and subsequently is progressively refined" (Kieran et al., 2015, p. 28). Using Kieran et al.'s classification, our PS-based MI task design study addresses the three key dimensions in both design as intention and design as implementation by using the two middle-range theories of variation: VToL and TwV.

NETWORKING THEORIES OF VARIATION IN A PS-BASED MI TASK DESIGN STUDY

Networking theories can take a variety of forms (combining, coordinating or integrating), depending on the level in which the theories are connected or intertwined, and the researcher's goals in making the connection (Bikner-Ahsbabs & Prediger, 2010). In our PS-based MI task design study, we coordinate VToL (Marton, 2015) and TwV (Gu et al, 2017) to help us in "better capturing instructional complexity" (Charalambous & Praetorius, 2018, p. 359) in relation to PS-based MI, with the

instructional goal of developing students' conceptual understanding and problem-solving abilities in mathematics.

VToL is developed by Ference Marton and colleagues to understand how a learner might come to experience and discern a variation of aspects of a given phenomenon against a background of invariance (e.g., Marton, 2015). In parallel with VToL, a pedagogic theory in teaching mathematics, called TwV (*biانشi jiaoxue* in Chinese), has been developed by Gu Ling-yuan and colleagues (e.g., Gu et al., 2017). The strength of both theories, which is different from other theoretical frameworks, is their main focus on object of learning (Pang et al., 2017). VToL and TwV are reported to be effective in analysing the systematic use of patterns of variation and invariance in the teaching and learning of mathematics (Pang et al., 2016). Multiple analyses of mathematics lessons using both frameworks of variation indicated that while there is a difference in the intended object of learning with regard to how they emphasize sameness and difference, there is much similarity in the enacted object of learning (Marton & Haggstrom, 2017; Pang et al., 2017). According to Marton and Haggstrom (2017), both frameworks “agree on the principle that novel and essential aspects of the object of learning in mathematics can only be appropriated by the learners by means of separating those essential aspects from non-essential aspects” (p. 404).

Although there is an overlap of interests, Pang et al. (2017) concluded that VToL and TwV are mutually exclusive and “useful in developing a collective understanding” (p. 66). According to Pang et al. (2017), TwV focuses more on the principles and intentions of mathematics instructional task design, while VToL provides a lens for the details of what is enacted and what it might enable learners to discern the critical aspects of the object of learning. TwV develops the concept of variation pedagogy by illustrating procedural variation that focuses on developing problem-solving ability and building a well-structured knowledge base (Gu et al., 2017) and it guides the PS-based MI task design by giving emphasis on how a problem situation may be transformed into a number of different ways so that it becomes solvable by different strategies. VToL guides our task design study by giving emphasis on how patterns of variation, and then invariance is involved in the sequence of tasks that draws students' focus towards each of the critical aspects (Marton, 2015). Learners focus on contrast by seeing how something is different, which leads to the ability to fuse multiple aspects and generalize concepts. With contrast, we as task designer vary essential aspects; with generalization, we vary non-essential aspects; and with fusion, we vary both aspects of the object of learning.

THE PS-BASED MI TASK DESIGN STUDY IN THE CONTEXT OF ETHIOPIAN ELEMENTARY SCHOOL

The study was conducted in four regional public primary schools, serving a community with middle to low class families. The schools are located in a city in the north-west of Ethiopia. Two researchers (one is the first author), and four sixth-grade mathematics teachers from the schools formed the PS-based MI task design study group. The

teachers were certified to teach mathematics in primary schools. They took part in professional development workshops on PS-based MI task design. In the workshops, which were conducted throughout a year, lessons on different mathematics topics were developed through the iterative processes of teaching, observing, revising, and re-teaching targeting to help students to experience and discern different objects of learning. The topics were chosen due to the recommendations from the participating teachers. The design of the instructional sequence follows three iterative phases of classroom design study (Cobb, et al., 2016): (i) preparation and design; (ii) implementation; and (iii) retrospective analysis, that can lead to revisions and a new iteration. Given the research question of this paper, we focus on explaining how VToL and TwV are coordinated to underpin the three dimensions of the design of sequences of PS-based MI tasks.

ILLUSTRATING THE NETWORKING OF THEORIES OF VARIATION IN PS_BASED MI TASK DESIGN STUDY

Using VToL to identify the object of learning and each of the critical aspects

The first step in preparing for our task design study is to use the VToL principles to identify the object of learning and critical aspects of it. The object of learning should be sufficiently well specified to inform the design of the entire instructional support for students learning and indicate the types of evidence required to determine if it has been satisfied. In our study, we aimed at developing students' conceptual understanding and problem-solving abilities in mathematics as objects of learning. The critical aspects that define different ways of understanding the object of learning that need to be pointed out in the instructional sequence are identified through analysing classroom observation data of one mathematics lesson. The lesson was designed and taught by the demonstrating teacher, in a non-intervention class of grade 6. The topic was on solving linear equations and inequalities. At the initial phase of preparation and design (Cobb et al., 2016), two objects of learning and five corresponding critical aspects were identified to address the first two key dimensions of the PS-based MI task design study: (1) simultaneously addressing conceptual understanding and problem-solving abilities; and (2) reducing the unnecessary cognitive load on students' learning (see Table 1).

Table 1: The object of learning and critical aspects to highlight the key dimensions of PS-based MI

Principles of Variation	The use of the principles in the task design process
Direct object of learning (the content to be learnt)	Developing students' conceptual understanding in solving linear equations and inequalities.
Critical aspects of the direct object of learning	Solving linear equation problems involving real-life contexts (CA1); Solving linear inequality problems that involve addition (CA2); Solving linear inequality problems that involve subtraction (CA3); Developing algebraic representations of linear inequalities from the patterns of arithmetic representations of linear equations (CA4).
Indirect object of learning (capabilities to be developed)	Developing students' problem-solving abilities in solving linear equations and inequalities.
Critical aspects of the indirect object of learning	Utilizing multiple solution strategies in the process of solving problems (CA5); Making connections among several strategies, and the mathematical concept of solving linear equations and inequalities (CA6).

Using VToL and TwV to design sequence of PS-based MI tasks

The sequence of PS-based MI tasks for solving linear inequalities was developed collaboratively with the teachers and researchers by referring to the theoretical principles of VToL and TwV. For example, sequence of conceptual and procedural variation tasks shown in Table 2 were designed deliberately and systematically to bring the five critical aspects of the objects of learning outlined in Table 1 into learner's focal awareness. CA1, CA2, and CA5 were addressed by tasks 3a and 3b, CA3 was addressed by task 3c, and CA4 and CA6 were addressed by task 3d. Other sets of tasks (4a-4e), which are similarly cognitive to the previous tasks, were also included at the end of the sequence that aimed at helping students to generalize and consolidate their learning. To help students activate their anchoring knowledge point and connect it to the new problems, the sequence included tasks of solving linear equations (task 1 and 2). In the sequence, each task is used as scaffolding (*pudian* in TwV terms) for the next tasks in the sequence. The structure of the sequence of tasks and the variations in the tasks indicates how procedural variation can be applied as described in the TwV framework to promote students' problem-solving experiences. The tasks were also designed to encourage students to use and make connections between multiple strategies (CA5 and CA6) such as repeated addition/subtraction and multiplication, and representations such as tables, variables, and rules to solve the tasks.

Table 2: The network of VTOL and TwV to prepare teachers to tackle the pedagogical challenge in structuring the PS-based MI.

Variation theoretical principles	Sequence of PS-based MI tasks
Activating student's anchoring knowledge point (simplifying algebraic expression, solving linear equations algebraically).	Task 1: Simplify the algebraic expression $2x + 7 + 5x - 15 + 6 - x$ into the lowest terms.
	Task 2: Solve each of the following linear equations. a) $z + 5 = 9$ b) $2m - 4 = 6$
Solving linear equations that involve real-life contexts (CA1).	Task 3: Mahlet had 400 Birr in her bank account. She saves additional 5 birr each week in her account. Her brother Yonas had 582 birr in his account. He withdraws 8 birr each week from his saving.
Using multiple solution strategies in solving problems involving linear equations (CA5).	Task 3a: How much money does Mahlet have in her account in the fifth week? How about in the tenth week?
Connecting anchoring knowledge point (solving linear equations) to the new knowledge (solving linear inequalities).	Task 3b: In which week does Mahlet have 565 birr in her account? In which week does it be greater than 565 birr?
Solving linear inequalities that involve addition (saving context) (CA2).	
Solving linear inequalities that involve subtraction (withdrawing context) (CA3).	Task 3c: In which week does Yonas have 422 birr in his account? In which week does it be less than 422 birr?
Developing algebraic representations of linear inequalities from the patterns of arithmetic representations of linear equations (CA4).	Task 3d: At what week do Mahlet and Yonas have the same amount of money in their accounts? In which week does Mahlet's money is greater than Yonas's money?
Using multiple solution strategies in solving inequality problems, and making connections (CA5 and CA6).	
Making connections among the critical aspects (fusion).	Task 4: Solve each of the following linear inequalities.
Reaching to generalizations.	a) $x + 5 < 8$ b) $y - 3 > 2$ c) $2z < 10$ d) $2x - 3 > 7$ e) $4y - 1 < 3y + 8$

DISCUSSION

While these two frameworks of variation seem to suggest different ways to apply variation to task design, elements from both frameworks can work to complement each other in guiding our task design study. Here, we argue that one of the frameworks alone may not have the capacity to capture PS-based MI task design in its entirety. We argue here that the theories have been interwoven and have played crucial roles in task design

for simultaneously addressing the issue of developing students' conceptual understanding (conceptual variation in both VToL and TwV), and problem-solving abilities (procedural variation in TwV) in mathematics. In addition, their coordination was useful not only to design tasks of PS-based MI that are at an appropriate level of challenge for students' expertise (mainly VToL), it is also fruitful as a pedagogical tool to structure the sequence of tasks in a way that directs students' attention towards the intended object of learning and critical aspects of it in a hierarchical and step-by-step way (mainly TwV). The diagram in Figure 1 illustrates how the two frameworks of variation are coordinated to help to address the three key dimensions of our PS-based MI task design study.

To conclude, by coordinating the two theories of variation in our PS-based MI task design study, we take a possible new step in the tradition of networking theories: conducting classroom design study in which the design is informed by the coordination of different theories. In addition, such coordination of existing frameworks of variation can lead to the development of a task design framework that goes beyond describing and understanding the key dimensions of instructional practices of PS-based MI, as it contributes to developing a new integrated framework (Prediger, et al., 2008) as a professional learning framework for PS-based MI practices in schools and teacher education programs at university-based training.

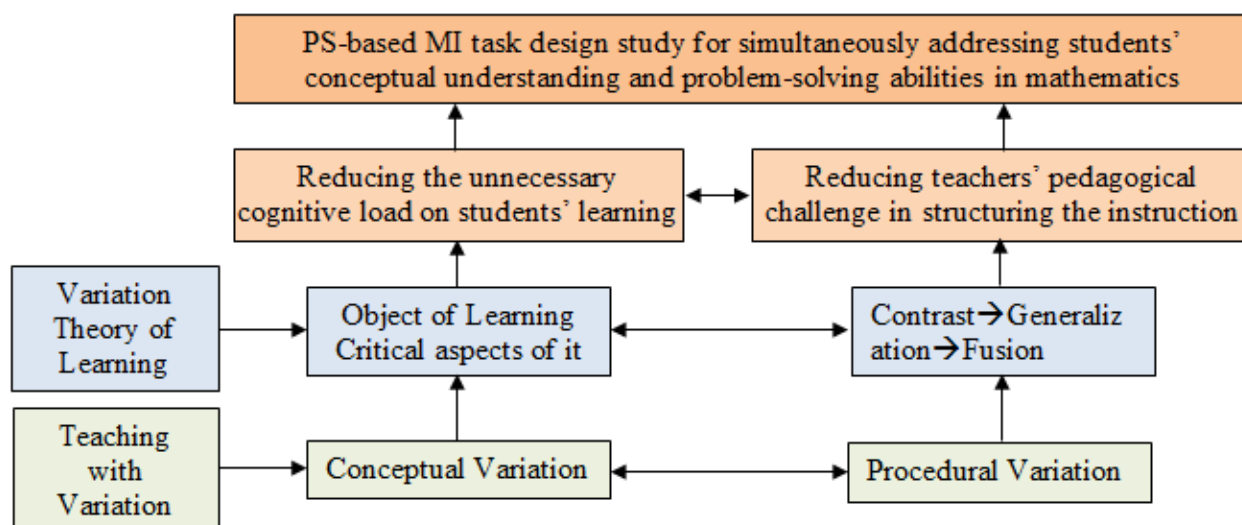


Figure 1: The interrelationship of VToL and TwV to address the three key dimensions of our PS-based MI task design study

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MATHEMATICAL PROVING FOR SUBVERSIVE CRITICAL THINKING

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In this study, we propose an interdisciplinary activity combining archaeology with mathematical proving developed during a five-iterations design-based research (DBR). The activity is aimed to bring students to the enactment of subversive critical thinking (SCT) – a radical type of critical thinking that consists of (i) opposition to an expert's opinion supported by a backing argument or (ii) an original alternative – an idea, interpretation, or opinion strictly divergent from all experts' views. Results from the three last DBR iterations analyzing the answers of 45 peer groups (104 students) show that most groups (87%) constructed SCT arguments. Notably, 51% opposed both experts, and 42% generated original alternatives. Based on our findings, we suggest three tentative design principles for promoting SCT in STEM education.

INTRODUCTION AND THEORETICAL BACKGROUND

Critical thinking (CT) is considered by many an essential 21st-century skill in society as it allows people to construct a profound understanding and rational judgment of any information presented to them (Dwyer et al., 2012). Basing conclusions on robust and verifiable evidence makes CT a desirable skill in many workplaces (Rear, 2019). Accordingly, the development of CT among school students is becoming an increasingly important educational goal (Halpern, 2014). Erikson and Erikson (2019) and Rear (2019) stress that promoting CT does not always go hand in hand with increasing standardized learning outcomes. Still, we embrace the vision of CT as enabling students to become nonconformist citizens, capable of opposing authoritative dogmas by creatively constructing alternative opinions and bravely presenting them publicly. In this paper, we propose the operational construct of *subversive critical thinking* (SCT) – a manifestation of CT that contains at least one of the following: (i) explicit justified opposition to an expert's opinion or (ii) articulation of an original alternative: an idea, interpretation, or opinion divergent from all previously presented experts' views. Do secondary students capable of expressing SCT? How can we support them in doing so? The current paper aims to propose SCT as a pedagogical aspiration and report on a mathematics classroom activity promoting it.

Critical thinking (CT)

In an educational context, CT has been given various, sometimes conflicting, definitions. Facione and Facione (1992) see CT as a process in which a person makes a purposeful, reflective judgment about what to believe or do. According to Ennis (2011, p.17), the ideal critical thinker is disposed to reach a "right" decision, present that position reasonably and clearly, consider others' points of view, seek to be well-

informed, and avoid intimidating or confusing others. Lai (2011) concludes that CT overlaps with several specific abilities, such as: analyzing arguments, making inferences, making decisions, solving problems, judging, and evaluating. Ennis (2011) separated CT into two categories: dispositions and abilities. In this paper, we avoid raising any claims about students' dispositions. We suggest the theoretical construct of SCT as an observable behavior that is a unique and radical kind of CT in action.

CT in mathematics and interdisciplinary education

Educational scholars emphasize the importance of providing students with opportunities to solve authentic, open real-life problems integrating different STEM disciplines to enhance CT-related abilities (e.g. Lai, 2011). Erikson and Erikson (2019) argue that CT can be seen as the ability to reflect on personal experiences and world knowledge to go beyond methodological and disciplinary boundaries. Savard (2018, p.5) maintains that "CT might contribute to the learning of mathematics by supporting the thinking process, but also mathematics might contribute to the development of CT by supporting the judgments made using quantification, measures, or data." Aizikovitsh-Udi and Cheng (2015) showed that when teachers consistently encourage CT by applying mathematics to real-life problems, students are likely to practice CT skills. However, combining mathematics with other disciplines may bear some challenges. For instance, Buldt and colleagues (2008) note that the epistemology in mathematics essentially differs from the other disciplines' epistemologies if only because it is based on mathematical proofs. Kazakevich and Marco (2022) maintain that these epistemological differences may hinder students' ability to apply mathematics in real-life contexts. Sfard (2012) concedes that in modern society, the mathematical discourse is considered highly persuasive, suitable for decision-making, and essential for one's agency and independence. She, however, warns about overemphasizing its importance: "In this positivistically-minded world, whatever is stated in mathematical terms tends to override any other type of argument." (p.7). Hence, as we elaborate on below, this research's activity also aims that students will harmoniously combine mathematics with life experience and other disciplines.

A SCT activity and operational research questions

The Two Jewels activity is designed to promote SCT by engaging students in analytic reasoning, problem-solving, and collaborative learning. The activity is designed based on STEM pedagogical approach incorporating interdisciplinary viewpoints, problem-based learning, and collaborative learning (Kennedy & Odell, 2014). The general idea of the activity is to present students with two conflicting archeology experts' opinions on a dilemma appearing disassociated with mathematics (see Figure 1) and gradually trigger them to develop a mathematical proof with which they can construct opposition to both experts (i.e. express SCT).

Our research questions are: What kinds of SCT (if at all) do students exhibit in the Two Jewels activity? How do they base their arguments when constructing SCT arguments?

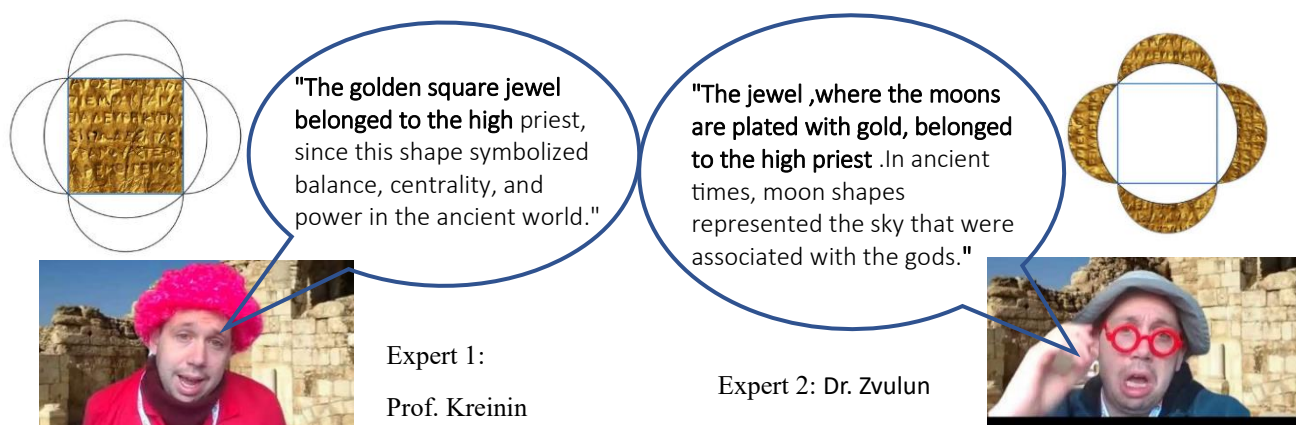


Figure 1: The archeological dilemma: which jewel belonged to the highest priest?

METHOD

Research context: The Two Jewels activity

We developed the Two Jewels activity in five DBR cycles in which the activity was tested on high-school and undergraduate students. In the first two cycles of DBR, not reported here, the structure and content of the activity were developed and tested. Analyzing these cycles' results, we established and refined the construct of SCT presented above. Here we report on the results from the last three DBR cycles. The activity is organized in an online questionnaire divided into four stages (Marco, 2023). In each stage, students are presented with archeological evidence/interpretations and requested to decide which of the two excavated jewels belongs to the highest priest (i.e. more valuable). In the first stage, we used two videos of pretended experts (played by the second author) expressing contradicting opinions without mentioning quantities or mathematical terms (Figure 1). The equality of the golden areas is known as Hippocrates of Chios' Four Lunas Theorem (Nelsen, 2015), which is extracurricular and esoteric in Israel. Students familiar with the Pythagorean theorem and the area formulas of circles and squares can independently prove it – though not effortlessly.

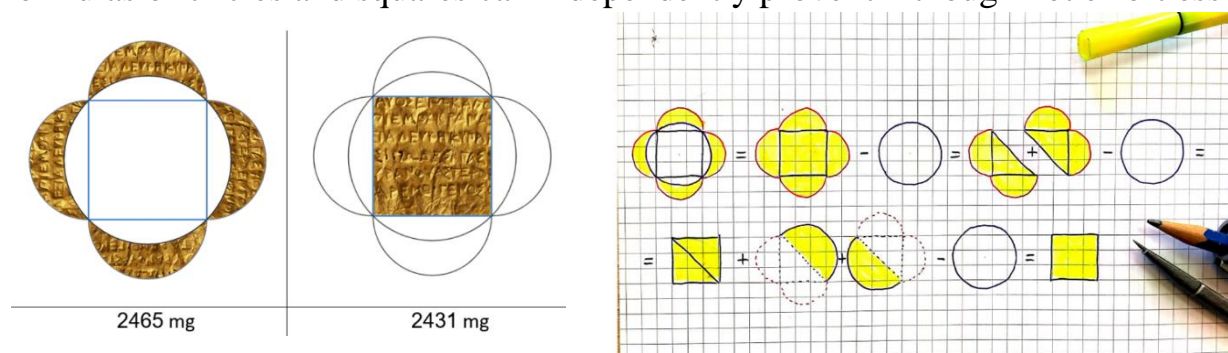


Figure 2: Left, jewels' weighing results. Right, a visual proof of areas equality.

In the second stage, students are presented with the jewels' weighing results, which show that they are negligibly different (Figure 2, left). In the third stage, students are presented with the visual proof of the Four Lunas theorem as if created by two 9th Grade students (i.e. nonexperts; Figure 2, right). The visual proof was redesigned with some principles taken from Marco et al.'s (2022) design principles for Proofs Without Words. In the fourth stage, students are asked to write a letter to the archeologists

describing their position and arguments on the issue to whom the jewels belonged. During all these stages, the mathematics teacher attended the groups making her/himself available for inquiries and making clarifications while providing as minimal guidance as possible. The activity concludes in a classroom teacher-led discussion after all the students submit their answers to the online questionnaire.

Participants

One hundred four students participated in the activity in these three last DBR cycles. Students worked in small peer groups of 2-5 students. The third, fourth and fifth DBR cycles took place at the same secondary school in Grades 9, 11, and 10 with 13, 12, and 20 groups, respectively.

Data collection and analysis

Students were instructed to submit personal answers to the questionnaire. However, many groups opted to submit a single form for all participants. Therefore, our unit of analysis is a submitted answer – regardless of the number of students submitting it (45 submitted answers in total). The data analysis aimed to identify and characterize students' SCT arguments that consisted of (i) explicit opposition to the experts' opinions with a backing argument or (ii) an alternative opinion/interpretation strictly divergent from the experts' views. First, we asked if the students opposed one of the experts, both, or none of them ("level" of opposition). Then we classified their backings and checked if their answer consisted of an alternative. Note that when we coded SCT for a group (submitted answer), we considered the highest level of opposition they expressed in all four stages of the activity. We stress that we only examined the presence of SCT and did not assess its quality (including the quality of mathematical proofs if they existed). To exemplify the analysis, consider the following two excerpts:

We disagree with both of your views because there was mathematical proof that proved both of you are wrong (Grade 10, group 19).

After deep thought about your argument and attempts to prove or disprove one of the claims, we concluded that Expert 2 is right. We were able to prove that the area of the moons is equal to the area of the square, and considering the fact that the moons weigh more, it is understandable that they have more gold (Grade 10, group 15).

As we defined above, SCT necessitates intellectual and social courage to express explicit opposition in public. So, of these two excerpts, we consider only the first to contain SCT. The first student opposed both experts, and the second opposed none of them and did not suggest an original alternative. Considering backing arguments – both students based their answers on mathematical proof. The following excerpt exemplifies an original alternative: "The jewels belong to two priests of the same priesthood status" (Grade 11, group 13).

RESULTS

Our data provide empirical evidence that the Two Jewels activity brings students to construct subversive critical thinking (SCT) arguments that oppose experts' opinions or suggest an original alternative strictly divergent from the experts' views. As can be seen in Table 1, we distinguish between six categories of SCT in students' answers. We identified opposition to experts' opinions in most submitted answers (60%). Three "levels" of opposition were identified: to none of the experts, to one expert, and both. The answers opposing both experts were divided into three types of backing – due to *lack of data*, *math proof*, and *math proof with an alternative*. All the students opposing one expert based their opposition on mathematical proof. Answers without explicit opposition were divided into two subcategories: with and without an alternative. Overall, we found that 87% of submitted answers contained SCT arguments. The following excerpts from students' answers demonstrate the six SCT categories identified (for convenience, we changed the names of the experts to "Expert 1/2"):

Absence of opposition without alternative (no SCT):

In our opinion, Expert 2 is right, and the moon-shaped jewel belongs to the high priest. We think so because the moons have more gold, and they were built in a more complex way than the square (Grade 9, group 2).

First, Expert 1 is a professor instead of a doctor, which means she has more knowledge in the field. Second, what the professor says makes sense because a square can be considered a concentric and "central" shape (Grade 9, group 13).

Absence of opposition with alternative:

Even after weighing and discovering that the [golden] parts are equal, the two claims still stand... But there is no way to check the role played by the jewels. It is equally possible that the jewels belonged to two believers buried in the temple (Grade 9, group 12).

There is no correct interpretation... In my view, the questions we need to ask ourselves are not about hierarchy and senior people. We must look at these jewels and understand their spiritual and historical significance and adopt it to our hearts as a criticism of the social classes' paradigm indoctrinated to us in this era. These jewels symbolize parts of a whole and our unique individuality in a society where the things that matter are amounts of gold (Grade 11, group 10).

Opposition to one of the experts:

Expert 1 was wrong in her basic claim that there is more gold in the right jewel (Grade 10, group 16).

Opposition to both experts due to lack of data

I cannot reach a definite opinion because there is not enough information about the period and the society from which the jewels have come. Therefore, no one can determine which of them is right (Grade 9, group 4).

Opposition to both experts based only on math proof (without alternative):

"Both of them [experts] present solid statements, but from the math perspective, both are wrong... According to the calculations we made, the areas plated with gold are equal" (Grade 11, group 3)

Opposition to both experts with mathematical proof and an alternative:

Your claims are nice, but they are not mathematically based. Of course, not every historical claim needs to be mathematically based, but considering weighing and area calculations, it can be concluded that the larger jewel of the two probably belonged to the high priest. We shouldn't forget that for three-dimensional shapes we should consider not only the area but also the volume" (Grade 11, group 6)

Your symbolism is really stupid, and we think the best way to understand it is through math. It now seems the math is correct, and the areas are equal. You probably have some measurement error because it is only a milligram. You should check yourself again. It could be corrosion, dirt, or such nonsense (Grade 11, group 11).

Table 1 concludes the number of groups in each SCT category:

Table 1: Number of groups in each category of SCT.

SCT category	# of groups in Grades			Total
	9	10	11	
Absence of opposition without alternative (no SCT)	3	0	3	6 (13%)
Absence of opposition with alternative	3	6	3	12 (27%)
Opposition to one expert	3	1	0	4 (9%)
Opposition to both experts due to lack of data	3	0	0	3 (7%)
Opposition to both experts based on math proof	1	9	3	13 (29%)
Opposition to both experts with math proof and alternative	0	4	3	7 (15%)

We also found a wide variety of alternatives in students' answers. In addition to the alternative in the excerpts brought above, we also mention the following ones:

In our opinion, the jewel fell and broke apart for some reason, and at first, there was one jewel (which belonged to the high priest) (Grade 10, group 11).

Definitely, they didn't belong to anyone... There is an unfounded assumption that both jewels are made of 100% pure gold (Grade 10, group 18).

Humans are never perfect and completely accurate, and the difference between the weights is tiny, so it can be concluded that the differences were created as part of the gold crafting process (Grade 11, group 10).

DISCUSSION

Our data provide empirical evidence that the Two Jewels activity brought most students (87%) to construct SCT arguments containing opposition to experts' opinions or presenting original alternatives strictly divergent from the experts' opinions. As

presented in Table 1, 51% of submitted answers included SCT arguments opposing both experts, and 42% included original alternatives. As we anticipated, mathematical proofs were found to be a powerful resource for SCT. Almost all students opposing the experts' opinion based their argument on mathematical proof that the golden areas are equal. However, some groups did develop a proof (or comprehended the visual proof presented in the third stage) and still conformed with the experts. This finding may suggest that even students equipped with a powerful epistemological tool such as mathematical proof (Buldt et al., 2008) may hesitate before using it for opposing authorities. This reveals an opportunity for mathematics educators to establish a classroom norm of expressing respectful opposition in situations demanding it. Some examples of disrespectful opposition in our data ("Your symbolism is really stupid") further stress the need to guide students on opposing someone else's views.

The Two Jewels activity has some features we can generalize into three hypothetical design principles supporting SCT. (1) *Present experts' conflict* – When contradiction is presented to students from the start, they will likely have an inner urge to settle it. They are also more primed to disagree in an environment that already involves disagreement. As we showed, students must generate assumptions, develop alternative narratives and opinions, and mathematical arguments to resolve such conflicts. In the Two Jewels activity, we used pretended experts that made students' opposition more likely. Using real experts' videos may be more suitable for students well-trained in SCT. (2) *Let students gradually reveal the mathematics hinted by the context* – leave students room to bring the mathematics on their own accord. So, they will perceive mathematics as a helpful tool for real-life problems and not merely a means to "get high grades." In addition, as we showed, mathematical arguments and proofs are potent resources for students constructing SCT arguments (c.f. Sfard, 2012; p.7). (3) *Create opportunities for disciplinary boundary crossing* – The activity tries to bring students to decide when to apply mathematics and when to put it aside and use their world knowledge and rationales taken from other disciplines. The activity brings together the epistemologies of mathematics and other STEM disciplines. Mathematically, 2465 and 2431 milligrams are not the same quantity. However, real-life considerations (e.g. measurement error, manufacturing variance) and the slight weight difference should cause the students to speculate that their creators designed the two jewels to weigh the same.

We suggest these hypothetical design principles be tested and refined in further educational research for promoting SCT, especially among students from different countries and cultures. In this paper, we only searched for the presence of SCT in all four stages of the activity. We disregarded learning trajectories depicting how students developed SCT during the Two Jewels activity. For us, individual and group learning evolving SCT is an exciting path for future research.

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STRATEGIES FOR PROOF CONSTRUCTION (SELF-REPORTS VS PERFORMANCE) - IS PRIOR KNOWLEDGE IMPORTANT?

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The construction of proofs, and in particular the understanding of the statement to be proved, can be supported by the use of certain strategies. However, from a methodological perspective, there is a critical debate about how such strategies can be measured in quantitative studies (self-report vs performance). Moreover, from a pedagogical perspective, little is known about the prerequisites, such as the prior knowledge required for strategy use. In this paper, we therefore investigate (1) the relationship between self-reported strategy use and actual performance on five comprehension strategies in the context of proving and (2) the predictive power of prior knowledge for strategy performance. Our sample consists of 152 second-semester university students. Theoretical and practical implications are discussed.

INTRODUCTION

Many students have great difficulty constructing proofs (e.g. Selden, 2012). Some of these difficulties may arise from the fact that students put little time and effort in understanding the statement being shown (Harel & Sowder, 1996; Schoenfeld, 1992). However, in previous studies, the first author of this contribution revealed that what is relevant to successful proof construction is not the duration of comprehension processes, but their specific implementation (Kirsten, 2018). In this context, some potentially helpful comprehension strategies could already be identified in qualitative studies (Kirsten, 2019; Weber, 2015). In our quantitative study, we asked students to perform these strategies before constructing a proof to a given (correct) statement. Besides methodological aspects, this article deals with the question which of these strategies are prerequisite-rich than others and require prior knowledge.

THEORETICAL FRAMEWORK

It is generally agreed that analysing a proving task before attempting to prove it is necessary for successful proof construction (Selden, 2012). The aim of this process is to build a mental problem representation of the statement to be shown that supports flexible and creative thinking. According to Kintsch and Greeno (1985) an appropriate problem representation consists of two components: the propositional text base and the situation model. The propositional text base describes a task-specific representation. It contains the relevant information from the task and its conceptual meaning, but remains task-specific. Thus, a propositional text base allows the problem solver to think conceptually, but is limited to the task-specific structure. In contrast, the situation model focuses on the concepts and relationships themselves and overcomes the given structure. This makes it possible to add additional information about the relevant concepts and to think creatively and agilely. Thus, the situation model can be viewed

as an abstract and extended representation of the mathematical problem that enables the problem solver to select appropriate solution approaches. In order to successfully develop a mental representation to a proving task, different manuals and individual research reports suggest various strategies for understanding a mathematical statement.

Strategies for understanding a mathematical statement

Useful strategies for understanding a statement to be proved are reported in research on both proof construction and proof comprehension. Accordingly, the following strategies may support a rich understanding: *paraphrasing* the statement in one's own words (Weber, 2015), linking different representations *using visualizations* or *examples* (Alcock & Weber, 2010; Mejia-Ramos & Weber, 2019), *enriching* the task context by repeating definitions of the central concepts (Mejia-Ramos et al., 2012) and *inferring* further information such as implicit preconditions (Kirsten, 2019). Although all of the strategies mentioned are considered helpful, it is likely that they address different components of a mental problem representation: While paraphrasing and enriching are possible on a propositional text base, inferring and generating examples require a reorganization of the content and thus promote an extended representation.

That first-year students actually use these strategies during proof construction was demonstrated in a previous study by the first author (Kirsten, 2019). However, the analysis of comprehension activities from eleven proving processes also showed that the use of strategies did not necessarily lead to a rich mental problem representation and is not sufficient for a successful proof construction. Consistent with this, several studies reported that experts and novices differ in their use of strategies and that undergraduates have difficulty elaborating examples or producing a helpful visualization (e.g. Alcock & Weber, 2010; Mejia-Ramos & Weber, 2019). We therefore suggest that difficulties in developing a rich mental representation should be distinguished according to whether they are due to a lack of comprehension strategies or to difficulties in implementing these strategies.

Different kinds of implementing comprehension strategies

First empirical evidence for the proposed distinction between production and implementation difficulties is provided by a qualitative study on comprehension processes in proof construction (Kirsten, 2019). Here, a comparison of successful and less successful strategy applications led to the hypothesis that both lack of strategy knowledge, i.e. knowledge about the characteristics of an accurate strategy application, and lack of conceptual knowledge can lead to implementation difficulties. The strong predictive power of conceptual knowledge for constructing one's own proof as well as for understanding and validating presented proofs is already known (Neuhaus & Rach, 2019; Sommerhoff, 2017). That this general relationship between prior knowledge and proving is also reflected in the use of comprehension strategies is natural, but hardly systematically researched. At least for *generating examples*, Alcock and Weber (2010) reported that students often fail using examples successfully because they generate examples by trial and error and do not check whether their example meets the required

properties. Similarly, in their study of nine proving processes, Sandefur et al. (2013) identified rich experience with the use of examples and a broad example space as prerequisites for constructive strategy use. Regarding diagram use, Mejia-Ramos and Weber (2019) reported weak correlations between diagram use and successful proof construction as a result of a large-scale study. Although they do not refer specifically to the understanding of the statement to be shown, these findings also suggest that effective strategy use is linked to available resources.

While the findings described here come mainly from qualitative studies, research on learning and reading strategies in general often relies on large-scale studies to measure the relationship between strategy use, prior knowledge, and achievement levels (e.g., Neuhaus & Rach, 2019). Student self-reports are typically used for this purpose. As the validity of such self-reports is often criticised, self-reports are increasingly supported by performance data (Neuhaus et al., 2022). However, because these often do not match in comparative studies (e.g., Zhou & Winne, 2012), self-reported strategy use and actual strategy performance may represent two different constructs. To our knowledge, there have not yet been any studies, either via self-report or using performance data, that specifically examine the comprehension strategies used in building a mental representation in the context of proof construction.

RESEARCH QUESTIONS

In order to address the research gap described above, this study investigates the use of comprehension strategies by first-year university students. From a methodology perspective, we first raise the question of appropriate instruments for measuring student strategy use:

RQ1: To what extent is students' self-reported strategy use related to performance data on strategy use?

From a pedagogical perspective, we ask about the prerequisites for strategy use. Since previous studies suggest a dependence of strategy use on prior knowledge, we also investigate whether there are more or less preconditioned strategies. In this context, we focus on students' actual performance in using comprehension strategies.

RQ2: To what extent does prior knowledge predict performance on comprehension strategies?

METHOD

Sample and material

To investigate the relationship between strategy use and prior conceptual knowledge, we turned to second-semester students. These students have already taken two proof-based courses (Linear Algebra and Analysis) and thus have had their first experience with proof construction. We therefore assume that at this stage of the study the relationships to be investigated would become apparent. Data collection took place at the very beginning of the second semester and resulted in data from a total of 152 students, including mathematics majors and future teachers for upper secondary level.

To measure *comprehension strategies*, we combined self-report and performance data. In line with the state of research, both approaches referred to the five strategies of *paraphrasing*, *enriching*, *inferring*, *generating examples* and *visualization*. The self-report questionnaire of strategy use included 16 self-developed items in which the individual use of a particular strategy was to be assessed in a task-unspecific manner in a four-point response format. Since only the scales of *inferring* (4 items), *generating examples* (3 items) and *visualization* (2 items) had satisfactory reliability ($\alpha > .60$) and an exploratory factor analysis did not yield an additional scale, we only calculated the means for self-reported use of *inferring* ($M = 3.03, SD = 0.54$), *generating examples* ($M = 2.46, SD = 0.70$), and *visualization* ($M = 2.45, SD = 0.81$).

In contrast to the self-reported use of strategies, the performance items referred specifically to a proving task related to the intermediate value theorem (see Figure 1). Based on typical student errors as reported in previous studies (Kirsten, 2019), we formulated four authentic applications for each strategy that students were asked to rate in terms of their correctness. At least one was correct and one incorrect in each case. For each strategy, the sum of correct ratings was determined: *paraphrasing* ($M = 2.61, SD = 0.89$), *enriching* ($M = 2.78, SD = 0.85$), *inferring* ($M = 2.48, SD = 1.26$), *generating examples* ($M = 2.80, SD = 0.92$) and *visualization* ($M = 2.29, SD = 0.93$).

Figure 1: Sample items measuring the use of comprehension strategies, self-report on the left and performance on the right

Task 1 Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. Show that f has a fixed point, that is, there exists a $x \in [0, 1]$ with $f(x) = x$.				
For each activity, please indicate the extent to which it reflects your approaches in understanding a statement to be proven.	Not true at all	Rather not true	Rather true	Completely true
I illustrate the given conditions of the proving task with an example I constructed myself.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
To better illustrate the statement to be shown, I draw a sketch.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
I look at the given conditions and consider what further information can be derived from them.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
I consider theorems and definitions from the lecture that might be related to the task.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Evaluate whether the following examples are correct in light of the proving task posed.				
a)	The function $g(x) = 2x - 3$ meets the required properties for $a = 1$ and $b = 4$.	Correct	Not correct	<input type="checkbox"/>
b)	The function $g(x) = x $ meets the required properties for $a = 2$ and $b = 4$.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
c)	The function $g(x) = -\frac{1}{2}(x - 2)^3 + 2$ meets the required properties for $a = 1$ and $b = 3$.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
d)	The function $g(x) = x $ meets the required properties for $a = -4$ and $b = -2$.	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

Prior knowledge was assessed on a short scale due to time constraints. The items were self-developed but based on the items of MOAS/KUM (Rach et al., 2021). We used open and multiple-choice items and tested each item with experienced students from higher semesters. Each item was coded 0 (false) to 1 (correct) point. As the seven items refer to different contents in the area of continuous and differentiable functions, we interpret the items as formative indicators (Stadler et al., 2021) for prior knowledge. Therefore, we do not report values like Cronbach's alpha but the variance inflation factor (VIF), which should be lower than 3.3 (Stadler et al., 2021). Given this for each

item ($VIF < 1.09$), we calculated the sum of those seven items to measure prior knowledge ($M = 3.06, SD = 1.14, Min = 0, Max = 5.50$).

Data analysis

R was used for data analysis. We estimated missing values with package lavaan (Rosseel, 2012) and the full-information-maximum-likelihood method. For RQ 1, we performed correlation analyses to gain insight into the relationship between self-reported use of strategies and the actual performance on those strategies. We report Pearson's r . For RQ 2, we assume a directional relationship based on theoretical aspects: Prior knowledge should influence performance in strategy use and not vice versa. Therefore, we conducted regression analyses using the performance of a strategy as the latent variable and prior knowledge as the predictor in each case. We report the regression coefficients B , the standard error (SE), standardized regression coefficients β , and how much variance in the performance of the strategies can be explained by the possibly predictor (R^2).

RESULTS

RQ1: The correlation analysis ($n = 144$) between the self-reported use of strategies and performance data revealed no significant ($p < .05$) values (see Table 1).

Table 1: Correlation analysis between self-reported use and performance of the comprehension strategies

	Infer. (SR)	G. Ex. (SR)	Vis. (SR)	Para. (P)	Enr. (P)	Infer. (P)	G. Ex. (P)
Self-report (SR)							
Inferring	-						
Generating examples	-.03	-					
Visualization	.04	.29**	-				
Performance (P)							
Paraphrasing	.12	-.06	.05	-			
Enriching	.07	-.03	-.06	-.04	-		
Inferring	.11	.02	.12	.16	-.12	-	
Generating examples	.17	.11	.14	.09	-.05	.23*	-
Visualization	-.06	.06	-.14	-.06	-.03	.11	-.07

Note: $n = 144$; * $p < .05$; ** $p < .01$

This indicates that self-reported strategy use is not correlated with performance data on the same strategy. Significant correlations occurred only for self-reported use of examples and self-reported use of visualizations ($r = .29, p < .01$) and for performance on the strategies inferring and generating examples ($r = .23, p < .05$).

RQ2: For the strategies *paraphrasing* and *enriching*, the prior knowledge of the students did not explain any variance in the performance ($R^2 = 0.00$). However, prior knowledge was a significant predictor of performance for *inferring* ($\beta = .35, p < .01, R^2 = .12$) and *generating examples* ($\beta = .35, p < .01, R^2 = .12$). Judging whether a *visualization* is correct was also not significantly predicted by prior knowledge ($\beta = .07, p > .1, R^2 = .01$).

Table 2: Regression analyses of strategy performance (Predictor: prior knowledge)

	B	SE	β	p	R^2
Paraphrasing	-0.04	0.07	-.06	.51	.00
Enriching	-0.04	0.06	-.05	.58	.00
Inferring	0.38	0.09	.35	.00	.12
Generating Examples	0.28	0.07	.35	.00	.12
Visualization	0.07	0.07	.09	.33	.01

Note: $n = 144$

DISCUSSION

Some of the difficulties students have in constructing proofs may be related to an insufficiently developed mental problem representation of the statement to be proved (Harel & Sowder, 1996; Schoenfeld, 1992). In this context, it can be helpful to draw on certain comprehension strategies and use them properly in the proof construction process (e.g. Alcock & Weber, 2010; Kirsten, 2019). However, it is an open question how to measure the use of these strategies and what individual prerequisites are necessary to use them effectively.

With respect to RQ 1, we did not find significant correlations between the self-reported use of strategies and students' actual performance. Consistent with previous studies, we therefore assume that self-report and performance data refer to two different constructs (Neuenhaus et al., 2022; Zhou & Winne, 2012). Since students indicated a relatively high use of comprehension strategies in the self-report, we suggest that students are overestimating themselves or that simply knowing a strategy is sufficient to indicate its use. Therefore, it is important to think carefully about the type of instrument to be used before starting a study. Furthermore, the results suggest that in the instructional context we need to provide feedback to students on their actual strategy use to support their self-assessment.

The results for RQ 2 showed that prior knowledge predicts the performance of the strategies *inferring* and *generating examples* while no variance could be explained for the strategies *paraphrasing*, *enriching* and *visualization*. With reference to Kintsch & Greeno's (1985) theory of mental problem representation, this finding indicates that *inferring* and *generating examples* represent two strategies in which the propositional text base is abandoned and a situation model is built via linkage to prior knowledge.

Thus, the strategies enriching and generating examples seem to be prerequisite-richer and should possibly be taught later than the other strategies. At least before students' have acquired the necessary prior knowledge, they may have difficulty to generate helpful examples or inferring further information such as implicit preconditions based on a proving task. However, this does not mean that prior knowledge is not required to use the other strategies. Especially in the case of *visualization*, the results may depend on the design of the items (Mejia-Ramos & Weber, 2019).

Caution should also be taken with sample size, although we used methods that are robust to violations of the prerequisites. In addition, we only measured the strategy performance on a single proving task. Therefore, the results provide some initial insight into the relationships, but cannot be generalized without replication.

In further studies, we intend to pay more attention to strategic knowledge and measure the correlations between knowledge about a strategy, self-reported use, and strategy performance to further investigate the relationships between these constructs. Furthermore, based on the additional data collected in this study, we intend to investigate the extent to which high strategy performance also affects performance on proof construction. This could provide new implications for the design of strategy trainings to help students overcome their difficulties with proofs.

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EFFECT OF REPRESENTATION FORMATS ON STUDENTS' SOLVING PROPORTION PROBLEMS

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In this study, we investigate at how two types of visual representations -the Double number line, and the Ratio table affect students' performance on percentage issues. In this study, 572 Japanese primary school students in Grades 4, 5, and 6 were given proportion problems with representations such as the "Double number line" and the "Ratio table diagram." As result, the effect of them does not exist in Grades 4 and 6. However, re-orientations have the greatest impact in Grade 5. All grade students could solve the Unit problem regardless of the presence of representations. The Double number line was a more effective representation than Ratio table diagrams. When students were shown these, they often solved problems by using a scalar or functional connection involved in the features of representation.

INTRODUCTION

Proportionality is a key concept in mathematics education from elementary school to university. Proportional reasoning is a precondition for successful subsequent studies in mathematics, because multiplicative relations support practically all number-related concepts learned in primary school. Despite the nature of proportional reasoning throughout the school years, many pupils have considerable difficulty understanding conceptions of proportional reasoning.

Proportional reasoning is a type of thinking that characterizes essential structural relationships in mathematics as well as in everyday life. This involves understanding multiplicative relationships between rational quantities ($a/b = c/d$) (Cramer and Post, 1993). A proportion is assumed that two ratios are equal in terms within both convey the same relationship.

According to Piaget & Inhelder (1958), children are incapable of proportional reasoning until about 11 years old. Proportional reasoning involves understanding the "relation between relations" and is a hallmark of formal operations.

The usage of "Double number lines" in the teaching of various mathematics topics has been proposed by researchers (e.g., Orrill & Brown, 2012). In Japanese mathematics textbooks, Double number lines are widely employed (Watanabe et al., 2010). There is a wealth of research on the cognitive benefits of different types of external representations, including the addition of relevant diagrams to text, which improves learning (Mayer, 2005). Several theories of diagrammatic reasoning theories, however, observe that the utility of diagrams is contingent on their relevance to the task itself, the context of the representation, and the user (e.g., Nistal et al., 2009).

In this study, we investigate at how two types of visual representations -the Double number line, and the Ratio table, affect students' performance on percentage issues.

FRAMEWORK

Some factors to affect proportional reasoning

It has been demonstrated that the characteristics of proportion problems influence students' solution strategies and success rates. First, all proportion problems can be identified as either missing value problems or comparison problems. In missing value problems, the solver must locate the missing value when given three others. A comparison problem is one in which two given ratios are compared, which may or may not be proportionally related. According to Tournaire & Pulos's (1985)'s evaluation of the literature, comparison problems are more difficult than missing value problems.

Second, Riehl & Steinhorsdottir (2017) found that in the presence of an integer ratio (i.e., the ratio between two numbers forms an integer number) instead of a decimal ratio, children frequently reached a correct solution on proportional missing value problems (i.e., the ratio between to numbers forms a non-integer number).

Third, when the measure problems have familiar associations (e.g., miles to hours, dollars to ounces), middle school children who are forming an initial concept of proportionality have the highest success (Kaput & West, 1994). However, similarity problems (Lamon, 2007) and mixture problems (Tournaire, 1986) were the most difficult environments for students to understand proportional relationships in.

Fourth, Vergnaud (1983) has found that students were far more likely to establish a relationship that was within a measure space (what he calls a *scalar relation*) than between measure spaces (a *function relation*) in solving multiplicative problems.

Representations for proportional relationships

The Double number line consists of two single number lines with corresponding pairs of values lined up (see Fig 1). The Double number line is a representation that may be used to visually compare two quantities and can be applied to a variety of mathematical situations contexts. The Double number line is a powerful way of representing multiplicative relationships and can help students to visualize equivalent forms of the same ratio. The Double number line also has the advantage of providing a feeling of scale.

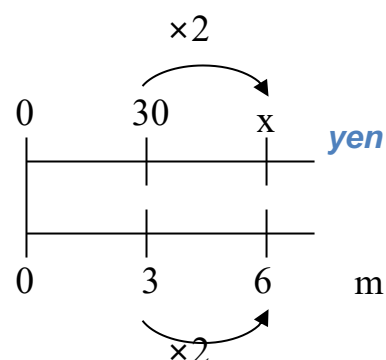


Figure 1: Double number line

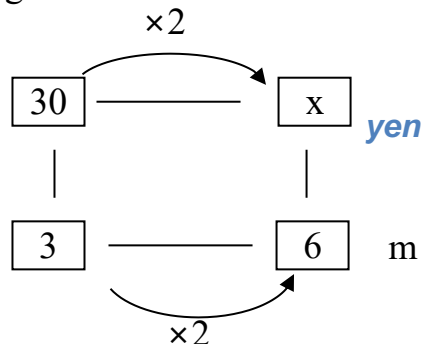


Figure 2: Ratio table diagram

The Double number line links well to other representations such as the Ratio table diagram. The Ratio table graphic depicts two specific pairs of Double number line values (see Fig 2). While the Ratio table diagram is simpler than the Double number line, some of the structure may get lost in the compression.

METHOD

Subjects

Three grades of children were used as participants to explore problem solving abilities. 572 subjects who were 184 in Grade 4, 188 in Grade 5, and 200 in Grade 6 were drawn from an elementary school in the 2022 school year in Japan. They were chosen because of the years of transition to full proportional reasoning (Inhelder & Piaget, 1958) - a period in which students think relationally about proportion problems but have not yet fully mastered the multiplicative relationships required for true proportionality understanding. According to the Japanese curriculum, students in Grade 4 of elementary school have learned fractions. Students in Grade 5 have completed decimal multiplication and division, rate, and ratio. In Grade 6, students have done multiplication and division with fractions.

Subjects in each grade were randomly allocated into three groups: double number line group (double number line + word problem), Ratio table (ratio table diagram + word problem), and Word problem (word problem only).

Instruments

All experimental problems were controlled for problem type under four conditions; missing value problem, *integer* or *decimal* rate, measure problems, and *scalar* or *function* relation. Each subject was required to solve five types of proportional problems; Unit, Scalar(Integer), Scalar(Decimal), Function(Integer), and Function(Decimal). These were constructed under the condition given in Table 1.

The text of all word problems was identical in all same conditions as follows;

[Relation] A ribbon of m costs b yen. [Result] What is the cost of a ribbon of c m?

Table1: Condition of instruments

Types	Relation		Result	
	a m	b yen	c m	x yen
Unit	3 m	60 yen	9 m	x yen
Scaler(I)	3 m	20 yen	6 m	x yen
Scaler(D)	3 m	20 yen	4.5 m	x yen
Function(I)	7 m	14 yen	11 m	x yen
Function(D)	6 m	9 yen	14 m	x yen

We also included four buffer tasks to prevent students from noticing the experimental design.

Students in the Double number line and Ratio table diagram groups were required to fill in the necessary values \square on the Double number line or the Ratio table diagram (Fig 3, 4) and respond with a valid response.

Students had 50 minutes (i.e., the duration of a regular mathematics lesson) to complete the test.

RESULTS

Correctness of responses

We used Tukey's one-way ANOVA for each type of group in each problem (Double number line, Ratio table, and Word problem group). If there was a substantial difference between conditions, we used Scheffe's Multiple Comparison technique for each type of group. Table 3 displays the mean correct responses score for issues solved by them in Grade 4. There was no significant difference in all types of problems.

There was no difference between Unit and Scalar(I) problems because the mean score of correct responses was so high. In Scalar(D) and Function(D) problems, the mean score in the Double number line group was a few higher than in other groups.

Table 3: Mean Score of Correct Responses in Grade 4

	Unit		Scalar(I)		Scalar(D)		Function(I)		Function(D)	
	%	SD	%	SD	%	SD	%	SD	%	SD
Number line G(n=68)	82.4	0.38	82.4	0.38	29.4	0.46	55.9	0.50	29.4	0.46
Ratio table G(n=52)	84.6	0.36	76.9	0.43	15.4	0.36	57.7	0.50	15.4	0.36
Word problem G(n=64)	81.3	0.39	81.3	0.39	18.8	0.39	56.3	0.50	18.8	0.39

Table 4 displays the mean correct responses score in Grade 5. (1) There is a sign in Unit problem ($F(2,185) = 6.05, p < 0.01$). There is a sign between the Double number line and the Word problem group ($p < 0.05$), and between the Ratio table and the Word problem group ($p < 0.05$). Scalar(I) problem has a substantial ($F(2,185) = 11.10, p < 0.01$). There is a substantial difference ($p < 0.01$) between the Double number line and the Word problem group, as well as between the Ratio table and the Word problem group. (2) There is a significant in Scalar(D) problem ($F(2,185) = 11.61, p < 0.01$). There is a sign between the Double number line and the Word problem group ($p < 0.01$), and between the Ratio table and the Word problem group ($p < 0.01$). (3) There is a significant in Function(I) problem ($F(2,185) = 6.65, p < 0.01$). There is a significant relationship between the Double number line and the Word problem group ($p < 0.01$), as well as the Ratio table and the Word problem group ($p < 0.05$). (4) There is a

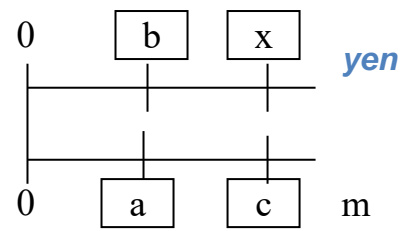


Figure 3: Double number line

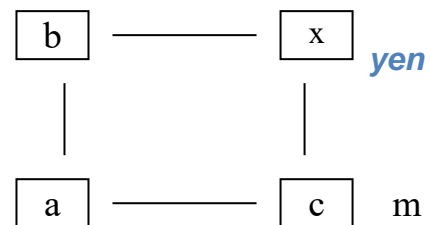


Figure 4: Ratio table diagram

significant in Function(D) problem ($F(2,185) = 4.70, p < 0.05$). There is a significant difference ($p < 0.05$) between the Double number line and the Word problem group, as well as between the Ratio table and the Word problem group.

The mean scores of the Double number line and Ratio table group were higher than that in the Word problem group. The Ratio table group's mean score was the same as the Double number line group's.

Table 4: Mean Score of Correct Responses in Grade 5

	Unit**		Scalar(I)**		Scalar(D)**		Function(I)**		Function(D)*	
	%	SD	%	SD	%	SD	%	SD	%	SD
Number line G(n=64)	93.8	0.24	98.4	0.12	68.8	0.46	81.3	0.39	50.0	0.50
Ratio table G(n=64)	93.8	0.24	98.4	0.12	68.8	0.46	75.0	0.43	50.0	0.50
Word problem G(n=60)	76.7	0.42	80.0	0.40	33.3	0.47	53.3	0.50	26.7	0.44

** : $p < 0.01$, * : $p < 0.05$

Table 5 displays the mean correct responses score in Grade 6. (1) There is a significant in Scalar(D) problem ($F(2,196) = 3.33, p < 0.05$). There is a significant difference ($p < 0.05$) between the Ratio table and the Word issue group ($p < 0.05$). (2) There is a significant in Function(I) problem ($F(2,196) = 3.66, p < 0.05$). There is a significant difference ($p < 0.05$) between the Double number line and the Word problem groups. (3) There is a significant in Function(D) problem ($F(2,196) = 11.52, p < 0.01$). There is a substantial difference ($p < 0.01$) between the Double number line and the Word problem group, as well as between the Ratio table and the Word problem group.

The mean scores were high in all problems. However, the mean score of the Word problem group was lower than that of other groups in Function(D) problems.

Table 5: Mean Score of Correct Responses in Grade 6

	Unit		Scalar(I)		Scalar(D)*		Function(I)*		Function(D)**	
	%	SD	%	SD	%	SD	%	SD	%	SD
Number line G(n=60)	96.7	0.18	98.3	0.13	90.0	0.30	98.3	0.13	90.0	0.30
Ratio table G(n=72)	98.7	0.11	94.7	0.22	94.8	0.22	89.5	0.31	89.5	0.31
Word problem G(n=64)	98.4	0.12	93.8	0.24	81.3	0.39	84.4	0.36	62.5	0.48

** : $p < 0.01$, * : $p < 0.05$

Solution strategy

Many pupils found it challenging to answer Scalar(D) and Function(D) difficulties in all problems. So we decided to pick up two problems. Tables 6, 7, and 8 show the classification of strategies for addressing the Scalar(D) problem. In Grade 4, students approached solving problems by using various strategies. In Grade 5, the scalar

technique was employed by 63% of students in the Double number line group, 63% in the Ratio table group, and 33% in the Word problem group. In Grade 6, students of all groups used more of the scalar strategy.

Table 6: Solution strategies of Scalar (D) problem in Grade 4

Strategy & Example	Number line(n=68)		Ratio table(n=52)		Word problem(n=64)	
	Correct	Error	Correct	Error	Correct	Error
Scalar(20×1.5)	17.6%	5.9%	-	-	6.3%	-
Distribution(3m+1.5m, 20 yen+10 yen)	-	-	7.7%	30.8%	6.3%	-
Unit(1.5)(1.5m \times 3, 10 yen \times 3)	-	-	-	7.7%	6.3%	6.3%
Unit(20/3)($20/3 \times 4.5$ or $6.66... \times 4.5$)	-	17.6%	7.7%	7.7%	-	-
Others	11.8%	47.1%	-	38.5%	-	75.0%

Table 7: Solution strategies of Scalar (D) problem in Grade 5

Strategy	Number line(n=64)		Ratio table(n=64)		Word problem (n=60)	
	Correct	Error	Correct	Error	Correct	Error
Scalar	62.5%	-	56.3%	6.3%	33.3%	-
Distribution	-	18.8%	12.5%	-	-	26.7%
Unit(1.5)	-	-	-	-	-	-
Unit(20/3)	6.3%	-	-	-	-	-
Others	-	12.5%	-	25.0%	-	40.0%

Table 8: Solution strategies of Scalar (D) problem in Grade 6

Scalar Strategy	Number line(n=60)		Ratio table(n=68)		Word problem(n=64)	
	Correct	Error	Correct	Error	Correct	Error
Scalar	86.7%	-	94.1%	-	75.0%	-
Distribution	-	-	-	-	-	-
Unit(1.5)	6.7%	-	-	-	6.3%	-
Unit(20/3)	-	6.7%	-	-	-	-
Others	-	-	-	5.9%	-	12.5%

Tables 9, 10, and 11 show the classification of solution strategies for the Function(D) problem. In Grade 4, 24% of pupils in the Double number line group used a working technique to solve the problem correctly. In Grade 5, 44% of students in the Double number line group and 50% of them in the Ratio table group solved correctly by using a functioning strategy. In the Word problem group, 40% used one of the two units (1.5 or 3) strategies. In Grade 6, many students in the Double number line group and Ratio table group used a functioning strategy.

DISCUSSION

Students in all Grades can correctly solve the Unit problem and the Scalar(I) problem. According to their mean scores, they find it more challenging to solve Function

problems than Scalar ones. This finding consists of the finding of Vergnaud (1983). They struggle to solve the Scalar(D) and Function(D) problems because they must understand the multiplicative conception. Students in Grade 6 can correctly solve proportion problems than them in Grade 5. Students in Grade 6 learn rate, ratio, and proportion in addition to decimal multiplication and division. And they in Grade 6 can solve many kinds of proportion problems. These findings also appear to support the concept of cognitive development (Inhelder & Piaget, 1958).

Table 9: Solution strategies of Function (D) problem in Grade 4

Strategy & Example	Number line(n=68)		Ratio table(n=52)		Word problem(n=64)	
	Correct	Error	Correct	Error	Correct	Error
Function($9 \div 6 = 1.5$, 14×1.5)	23.5%	-	7.7%	7.7%	6.3%	-
Unit(1.5)($9 \div 6 = 1.5$, 1.5×14)	5.9%	11.8%	-	-	-	-
Unit(3)($2m = 3\text{yen}$, 3×7)	-	-	7.7%	-	12.5%	-
Scalar ($14/6 = 7/3$, $7/3 \times 14$)	-	17.6%	-	-	-	6.3%
Others	-	41.2%	-	76.9%	-	75.0%

Table 10: Solution strategies of Function (D) problem in Grade 5

Strategy	Number line(n=64)		Ratio table(n=64)		Word problem(n=60)	
	Correct	Error	Correct	Error	Correct	Error
Function	43.8%	-	50.0%	-	13.3%	-
Unit(1.5)	6.3%	-	-	-	6.7%	13.3%
Unit(3)	-	-	-	6.3%	6.7%	13.3%
Scalar	-	-	-	-	-	-
Others	-	50.0%	-	43.8%	-	46.7%

Table 11: Solution strategies of Function (D) problem in Grade 6

Strategy	Number line(n=60)		Ratio table(n=68)		Word problem(n=64)	
	Correct	Error	Correct	Error	Correct	Error
Function	80.0%	-	76.5%	-	43.8%	-
Unit(1.5)	13.3%	-	-	-	12.5%	12.5%
Unit(3)	-	-	5.9%	-	6.3%	-
Scalar	-	-	11.8%	-	-	-
Others	-	6.7%	-	5.9%	-	18.8%

In only Function(D) and Scalar(D) problems, the Double number line is a more effective representation than the Ratio table diagrams in Grade 4. The Double number line and Ratio table diagrams are also more effective representations in Grade 5. In Grade 6, these weren't effective because the mean score of correct responses is very high. While pupils have learned the Double number line since elementary school, they have not learned the Ratio table diagrams. They fully can't use the Ratio table diagrams for problem solving activities.

If given proper representations, students frequently utilize a scalar strategy to tackle a scalar problem and a functional strategy to answer a function problem. Students can be easy to grasp proportional relationships through these representations. If students were not given representations, they frequently employed unification processes (Lamon, 2007) to solve difficulties. So they did not fully develop a multiplicative conception, they needed to make newly composed units for solving problems. For example, in the Scalar(D) problem, they created new units such as “1.5 m per 10 yen” and “2 m per 3 yen” in the Function problem (D).

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OPEN-ENDED TASKS WHICH ARE NOT COMPLETELY OPEN: CHALLENGES AND CREATIVITY

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Mathematics educators argue that open-ended tasks as a powerful tool for the development of students' creativity in mathematics, while it is well known that solving open-ended tasks is challenging for students. Recently we argued that not every open-ended task is fully open, as even when a task has a multiplicity of solution outcomes completeness of the set of solution outcomes is possible. To make the distinction between openness and multiplicity and avoid ambiguity related to the term 'openness' we use the term 'Multiple Outcomes Tasks' (MOTs). In this paper we analyze students' mathematical performance on two MOTs. We consider the completeness of the set of solution outcomes produced by a student as an indicator of his/her creativity due to the unconventionality of MOTs in regular classes. Our findings suggest that MOTs with continuous-infinite set of solution outcomes are more challenging than MOTs with discrete and finite sets.

RATIONALE AND BACKGROUND

The category of 'open mathematical problems' includes a variety of situations and questions to be answered. Back in 1995, the Discussion Group led by Pehkonen at PME-17 in Japan analyzed the use of open problems in different countries (Nohda, 1995; Pehkonen, 1995; Silver, 1995; Stacey, 1995). Silver (1995) identified several different meanings of the term *open problems*: (a) problems open in mathematics, (b) problems allowing multiple interpretations, (c) problems having multiple solution methods, and (d) problems that lead to other problems or generalizations.

Haylock (1987, 1997) connected solving open problems to divergent mathematical production and creative problem-solving processing as opposed to fixation developed through algorithmic problem solving. Often, openness of problems is connected to an "ill-defined" structure of tasks, that is, by missing data or assumptions (Krutetsii, 1976; Leikin, Klein and Waisman, in press) and thus creativity is required while a solver generates missing givens by her/himself. The ill-defined structure of a problem allows generation of various missing givens and thus leads to the production of multiple solution outcomes.

In some cases, the ill-defined problem remains open. That is, the number of solutions is not limited. For example, in an investigation task, due to different approaches, many and varied solution products can be obtained (Leikin and Elgrably, 2022). In other cases, although the missing givens must be completed, doing so leads to a complete set of solution outcomes, and thus the problem remains closed. This, since despite the multitude of solutions and their dynamic range, the number of solutions is still limited. To acknowledge the distinction between these cases, we use the term 'Multiple

Outcomes Tasks' (MOTs) to refer to both open-ended problems and problems requiring the production of a complete set of solution outcomes (for the variety of MOTs see Leikin et al., in press). Both of these MOT types are united with the common feature of non-routine problems. That is, MOTs require a different type of creativity, one which is related to the understanding of an unfamiliar problem and its structure (Leikin, Klein and Waisman, in press), and is expressed as insight in finding the complete set of valid available solutions (Haavold and Sriraman, 2022). Alternatively, thinking about the variety of possibilities towards the complete set of solutions, is what develops creativity.

Based on the above observations and suggestions we argue that MOTs are inherently challenging and require creative thinking. We examine students' problem-solving performance on two MOTs and illustrate these ideas.

THE STUDY

The goal

The goal of the study presented in this paper was to examine students' mathematical performance on MOTs in the terms of the completeness of the set of solution outcomes obtained. In addition, we ask whether the type of the complete set of solution outcomes (discrete vs. continuous) affects their success in attaining said completeness.

The context

The study is a part of a bigger study focusing on the implementation of the Math-Key program, which contains creativity-directed mathematical activities for middle school students, of which solving MOTs is an integral component (Leikin et al., in press).

The participants

Ninety-three middle school students, from 7th and 8th grades in heterogeneous classes, participated in the study. The students were of different levels of mathematical competencies, and for the research experiment were divided into groups based on their school mathematical achievements. Table 1 depicts the distribution of the participants in four groups according to grade and to competency level as determined by achievements.

Table 1: Distribution of the study participants in research groups

	HA: High-achieving students: school math score ≥ 80	MLA: Mid-to-low-achieving students: school math score < 80
7 th grade	15 (16%)	34 (37%)
8 th grade	6 (6%)	38 (41%)

The two tasks, data collection and data analysis

The study participants were asked to solve two MOTs borrowed from the Math-Key program (Leikin et al., in press): (1) Tickets task (Figure 1) with a complete set of multiple discrete solution outcomes; (2) Three-points Task (Figure 1) with a complete

continuous and bounded range of outcomes (infinite set). Figure 1 presents the two tasks that are reported in this paper.

All the students' work was collected and the solution outcome sets were classified and evaluated according to the completeness of the solutions as follows: We considered complete solutions to the two chosen problems as indicating a high level of creativity due to the unconventionality of the solutions. These solutions were scored with 10. Partial multiple solution outcomes were considered as indicators of a low level of creativity and scored with 1. A solution set that included only one solution outcome was scored with 0.1 as indicating mental fixation (cf. Haylock, 1987). This scoring scheme is analogical to the scoring scheme included in the model for the evaluation of creativity using multiple solution strategies tasks (MSTs) suggested by Leikin (2009, 2013).

Tasks: Solve the following problem	Task characterization	
Tickets task The price of admission to a show is 20 shekels for a child and 30 shekels for an adult. A group of 6 people (adults and children) came to the show. How much money does the group need to pay?	Domain: Algebra/Arithmetic Solution: Discrete set of outcomes	
	children, adults	payment
	1; 5	170
	2; 4	160
	3; 3	150
	4; 2	140
	5; 1	130
Three-points task Given three points A, B, and C, such that the distance AB is 5 cm and AC is 3 cm, what is the distance between B and C?	Domain: Geometry Solution: Continuous set of outcomes: $2 \leq x \leq 8$	

Figure 1: Two tasks reported in this paper

Using ANOVA, we examined between group differences in the completeness of the sets of solution outcomes for each of the two tasks, as related to students' mathematical competency levels and the grades in which they studied mathematics. We also examined within-groups differences related to the two tasks. Taking into account significant effects, we performed a pairwise comparison between the mathematical performance of students who studied mathematics at different levels in different grades (using Bonferroni adjustment).

FINDINGS

Completeness of the solutions

Table 2 depicts the mean and standard deviation scores for completeness in the two tasks in the different groups of the research. The results shown in Table 2 demonstrate that the students were more successful when tackling the Tickets task than when solving the Three-points task. Students who did not find the complete set of solution outcomes for the Tickets task imagined the show's visitors as members of one family only, and considered different combinations of the family members. For example, they were thinking about mother, father and 4 children; or 2 parents, 2 grandparents and 2

children without an option of a single parent with 5 children or three adults (2 parents and a friend of them) with 3 children.

Table 2: Mean and SD of the completeness of the solution outcome sets in different research groups.

	Mean (SD)			
	Grade 7	Grade 8	Level HA	Level MLA
	N=49	N=44	N=21	N=72
Tickets	5.82 (4.69)	6.13 (4.72)	6.48 (4.60)	5.82 (4.72)
Three-points	.41 (.45)	.37 (.44)	.57 (.47)	.34 (.43)

	7-HA	7-MLA	8-HA	8-MAL	Overall
	N=15	N=34	N=6	N=38	N=93
Tickets	6.33 (4.65)	5.60 (4.75)	6.85 (4.89)	6.01 (4.75)	5.97 (4.68)
Three-points	.51 (.47)	.37 (.44)	.70 (.46)	.32 (.41)	.39 (.44)

The Three-points task was very difficult for majority of the study participants. Although the triangle inequality is studied in the 7th grade, few students were able to answer the question successfully. Most students employed strategies that led to some discrete outcomes without referring to the triangle inequality. For example, students displayed an isosceles triangle and found two solution outcomes, or considered three points on the same line while adding or subtracting the values of the given segments. Only a few students used dynamic reasoning and found the complete solution. The majority of the students' solution sets included natural numbers only.

The two tasks were unconventional for students and solving both required imagination. However, the structure of the solution set in the Three-points task – infinite, bounded and continuous – determined the higher complexity of this task compared to the Tickets task, in which the solution set is a finite discrete set of natural numbers.

Between group differences

Table 3: Significant effects of achievements on completeness of solutions

	Between groups differences Level-of-mathematics effect HA > MLA* $F(1,91)$	Pairwise differences Level-of-mathematics in the 8 th grade effect: 8 HA - 8 MLA* $F(1,42)$
Three-points task	4.29* $\eta_p^2=.05$	4.26* $\eta_p^2=.09$

No effect of grade on the completeness of the solution sets of the two tasks was found. The level of mathematical competencies significantly affected the completeness of the students' solutions sets when solving the Three-points task only. This effect was caused

mainly by the differences in solutions of the 8th grade students with different levels of mathematical competencies (see Table 3).

Within group differences

A significant overall tasks effect was found on the completeness of the sets of students' solution outcomes. The cause of these differences is related to the knowledge and skills of students. While formal knowledge of the triangle or imagination of dynamic triangle transformation possible due to the ill-defined structure of the problem are required for solving the Three-points task. At the same time, finite and discrete structure of the solution outcomes set of the Tickets task can be attained through trial and error strategy, or using systematic consideration of all the ways in which missing information in the Tickets task can be added. These possibilities reduced the challenge of completing the set of outcomes for this task (see Table 4).

Table 4: Significant task effects of the two tasks on completeness of the solution outcomes sets

Completeness of the solution outcome set is higher for the Tickets tasks than for the Three-points Task (see Table 2 for the descriptive statistics)				
Overall task effect	Grade 7	Grade 8	HA level	MLA level
$F(1,92)$	$F(1,48)$	$F(1,43)$	$F(1,20)$	$F(1,71)$
136.59***	68.2***	67.13***	36.17***	99.7***
$\eta_p^2=.6$	$\eta_p^2=.59$	$\eta_p^2=.61$	$\eta_p^2=.64$	$\eta_p^2=.58$
*** $p<.001$				

DISCUSSION

The study reported in this paper examined students' mathematical performance on solving MOTs as a function of their mathematical competencies, focusing on their ability to find a complete set of solution outcomes. The tasks that we used were based on studied material, however, both tasks were unconventional for the study participants since solving MOTs was irregular for the study participants. Moreover, although the tasks that previously were considered open-end due to the multiplicity of the solution outcomes, in our view they were not completely open, as the complete sets of solution outcomes could be attained for both tasks. The complete solution sets of the two problems were different: infinite, continuous and bounded set for the Three-points tasks, vs. finite and discrete set (of natural numbers) for the Tickets task.

The students' solutions were analyzed from the aspect of completeness of the outcomes. We examined whether students' level of mathematical competencies linked to their achievements and the level of mathematics they studied in school affected students' success in producing complete sets of solutions. The ninety-three participants were divided into four groups according to their mathematical competencies level (HA and MLA) and grades (7th and 8th).

We found that completeness of solutions is connected to the mathematical competencies of participants. Students at a high-achievements level of mathematics (HA level), as well as students in a higher grade (8th vs. 7th grade), provide more complete solutions. These findings match the results of earlier studies, which displayed connections between knowledge, skills, and school achievements (Kattou, Kontoyianni, Pitta-Pantazi and Christou, 2013; Mourgues, Tan, Hein, Elliott and Grigorenko, 2016; Waisman et al., 2022). In connection with that, school mathematical achievements reflect the skills the students have acquired in solving problems, and students' readiness and willingness to look for a complete set of solutions. That is as Mid-low-achieving students were satisfied in providing a few outcomes, even when it was not the full set of solutions.

The Tickets task, which asked for completeness of solutions as whole numbers, was relatively simple for the students to solve. Therefore, no significant effect reflecting the students' creativity was found in association to this task. However, an effect was found related to competency levels of students in the Three-points task, which asked for completeness of an inequality outcome. Although the students understood inequalities as an algebraic method, having a range of results, especially in a word problem, is not a typical task. All the more so for a geometric problem. Leikin et al. (in press) discussed in details complexity of solving MOTs (called 'inequality tasks' in Leikin et al., in press) vs. equality problems that have one particular solution outcome. They emphasized that the way in which a problem is phrased influences the problem's complexity linked to the difficulty of finding the complete set of solution outcomes. In school, students learn to provide a concrete answer to a problem or, sometimes, answer questions of the form "What could be the value", which means "give an example" of a possible answer. But in the two tasks presented to students in our study, they were asked "What is the value of ...", the solution to which is different, as it requires completeness. Moreover, specific examples in the Three-points task do not lead to the complete set of solution outcomes. We also argue that students' failure in attaining complete solution outcomes set indicate a lack of creativity and imagination.

As Yackel and Cobb (1996) pointed out, in order to develop mathematical creativity, it is necessary to change the mathematical norms used in the classroom, both in terms of teaching methods and in terms of the thinking and solution processes. This, because it is through the presentation of challenges that the intellectual independence of students develops. That is, it is necessary to break the existing norms in order to create new norms, which will eventually become routine. Our findings are indicative of a need to change classroom norms and to enhance product creativity (Leikin and Elgrably, 2022), which is related to finding complete solution outcomes. We recommend guiding teachers to apply and integrate challenging tasks with multiple outcomes in the classroom, as these tasks develop thinking through the search for complete sets of solution outcomes.

In the current study we have attempted to connect students' behavior while solving mathematical problems that have multiple solution outcomes, to students'

mathematical competences. Our findings display insignificant differences when the assignments do not conform to the learning norms used in the classroom. Further research is needed to examine more deeply the issue of differences in complexity of different MOTs with attainable complete sets of solution outcomes. The future research can examine both students' ability to approach and solve these tasks, and the effect of experience in solving MOTs of completeness-type on advancement of students' success in tackling these mathematical tasks.

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THE DISCOVERY FUNCTION OF PROVING BY MATHEMATICAL INDUCTION

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Proof plays multiple roles in disciplinary mathematical practice; discovery is one of the functions of proof that remain understudied in mathematics education. In the present study, I addressed this research gap by focusing on proving by mathematical induction and exploring how students generalise a statement after proving it with this proof method. I conducted and analysed a task-based interview with a pair of undergraduate students. The analysis of this interview shows that their process pattern generalisation for the original statement played a crucial role in their subsequent activity where they further generalised this statement and proved the generalisation.

INTRODUCTION

Proof plays multiple roles in disciplinary mathematical practice. Given the significance of authentic mathematical activity in mathematics education (Stylianides et al., 2022), those roles relevant to students' mathematical learning should also be introduced to school mathematics. De Villiers (1990) examined five functions of proof in mathematics: verification, explanation, systematisation, discovery, and communication. While explanatory proofs have been extensively discussed in mathematics education research (e.g., Hanna, 1990; Komatsu et al., 2018; Lockwood et al., 2020; Stylianides et al., 2016), discovery is one of the proof functions that remain understudied (de Villiers, 2012; Komatsu et al., 2014). The discovery function of a proof refers to the invention of new results, including the generalisation of the original statement that the proof has verified (de Villiers, 1990).

In the present study, I explore this discovery function with a particular focus on proof by mathematical induction. Proof by mathematical induction is a specific method for proving that a statement $P(n)$ is true for every natural number n (I focus on this typical case in this study, not considering other cases such as statements for the subsets of natural numbers). This proof method consists of two steps: the base step showing $P(1)$ and the inductive step showing the implication $P(k) \Rightarrow P(k + 1)$ for every k .

While several researchers argue that proofs by mathematical induction are generally not explanatory (e.g., Hanna, 1990; Lange, 2009; Steiner, 1978), others present case studies where proving by mathematical induction is explanatory for the provers (Stylianides et al., 2016). The latter study suggests the possibility that proving by mathematical induction can include the discovery function because the explanatory and discovery functions of proofs are deeply related to each other (de Villiers, 2012). Examining the discovery function of proving by mathematical induction is important for school mathematics because it can provide students with a productive image that this proof method not just prescribes a mechanical procedure that students must follow but offers a useful tool by which they can generalise the original statement.

Nevertheless, to the best of my knowledge, no research to date has examined the discovery function of proving by mathematical induction. In the current study, I address this gap in the literature by exploring the research question: How do students generalise a statement after proving it by mathematical induction?

THEORETICAL FRAMEWORK

Researchers view the meaning of the discovery function of proof differently, and Komatsu et al. (2014) reviewed different illustrations of this function observed in the literature. The present study focuses on one type of activity that consists of three phases: (1) conjecturing and proving a statement (or proving a given statement), (2) changing one of the conditions of the statement by asking a what-if-not question (Brown & Walter, 2005), and (3) using the idea of the proof constructed in phase 1 to invent and prove a generalisation of the original statement. Consider, for example, the statement that the sum of three consecutive integers is three times the second integer. This statement can be proved by representing the three integers as $a - 1$, a , and $a + 1$, and calculating their sum to show it as $3a$ (phase 1). Phase 2 involves asking the question what if the three integers are not consecutive. The proof constructed in phase 1 explains the reason why the sum of the three numbers in the original statement is $3a$: -1 and $+1$ in this sum are cancelled. From this proof idea, provers can generate and prove a new statement that the sum of three integers with equal differences (e.g., $1 + 3 + 5$ and $2 + 5 + 8$) is also three times the second integer. This statement is a generalisation of the original one (phase 3).

Similar to Stylianides et al. (2016), this study distinguishes *proving*, which is an activity or process leading to the proof of a statement, and *proof*, which is the final written product of the proving activity that establishes the truth of the statement. Proving encompasses various activities that may not be represented in the resulting proofs. Those activities include investigating examples to make conjectures and providing informal arguments that suggest the veracity of the conjectures and offer insight into ways to prove them. By the discovery function of proving, this study considers the activity where the prover generalises the original statement by using an idea or ideas included in the proving process or proof of the original statement.

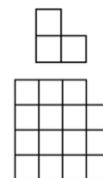
In the context of mathematical induction, several researchers (e.g., Pedemonte, 2007; Stylianides et al., 2016) have considered the distinction made by Harel (2002) between *result pattern generalisation* and *process pattern generalisation*. While result pattern generalisation refers to focusing on regularity in the obtained results, process pattern generalisation refers to focusing on regularity in the processes performed to obtain the results. Process pattern generalisation involves making a conjecture $P(n)$ through observing a common feature in the processes of $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, $P(3) \rightarrow P(4)$, etc. This type of generalisation is relevant to the inductive step in proof by mathematical induction. Pedemonte (2007) presented a case where students succeeded in proof construction from their process pattern generalisation, and Stylianides et al. (2016) analysed students' work where their process pattern generalisation enabled

proving by mathematical induction to be explanatory. Given the deep relationship between the explanatory and discovery functions of proofs (de Villiers, 2012), it is anticipated that process pattern generalisation would also relate to the discovery function of proving by mathematical induction. In this paper, I examine this hypothesis by analysing students' activity on a sequence of tasks.

METHODS

The data analysed in this study are taken from a task-based interview conducted as part of a research project for developing task design principles for the discovery function of proving by mathematical induction. In the initial phase of this project, two sequences of tasks were designed based on certain principles and implemented with two pairs of Japanese undergraduate students. Considering the page limit, in this paper, I present the analysis of an interview using the task sequence shown in Figure 1 with one pair of students, Misaki and Sakura (pseudonyms). The activity of the other pair of students on these tasks was similar to that described in this paper.

Task 1. Let's name a 2×2 square with a 1×1 square removed as an *L-shape figure*. Consider a 4×4 square with one 1×1 square removed from one of its four corners. Can this figure be tiled without gaps with L-shape figures? (If students complete this task, the same question is asked for an 8×8 square, a 16×16 square, and then a $2^n \times 2^n$ square.)



Task 2. Do the same results as Task 1 hold if we change the location of removing a 1×1 square from four corners? From what location can a 1×1 square be removed while the whole figure can still be tiled without gaps with L-shape figures? Let's begin with considering a 4×4 square and then an 8×8 square, a 16×16 square, ..., and a $2^n \times 2^n$ square.

Figure 1: Task sequence implemented in the interview

Regarding these tasks, I hereafter refer to both a $k \times k$ square with a 1×1 square removed and a $k \times k$ square itself as a $k \times k$ square as long as it does not lead to confusion. Task 1 is relevant to phase (1) in the activity discussed in the previous section, and Task 2 is relevant to phases (2) and (3). The students worked on these two tasks for approximately 40 min. They were provided worksheets, grid papers, and a sufficient number of L-shape tiles made from paper. The students were in their third year at a Japanese national university. They majored in mathematics and completed several courses in mathematics education in secondary school. They were introduced to mathematical induction when they were high school students.

The task-based interview was video-recorded and transcribed. The data for analysis included the video recording, the transcripts produced from this recording, and the students' worksheets and grid papers. These data were qualitatively analysed with a particular focus on how the students generated and proved conjectures in Tasks 1 and 2, and whether and how they referred to their work on Task 1 while tackling Task 2. The students' utterances and written proofs in the interview were translated from Japanese into English by the author.

RESULTS

Process pattern generalisation and proof

The students worked on Task 1 and found that a 4×4 square could be tiled with L-shape figures, as shown in Figure 2a. The students then considered an 8×8 square. They tiled this square by placing L-shape figures one by one, not employing their previous finding that a 4×4 square could be tiled.

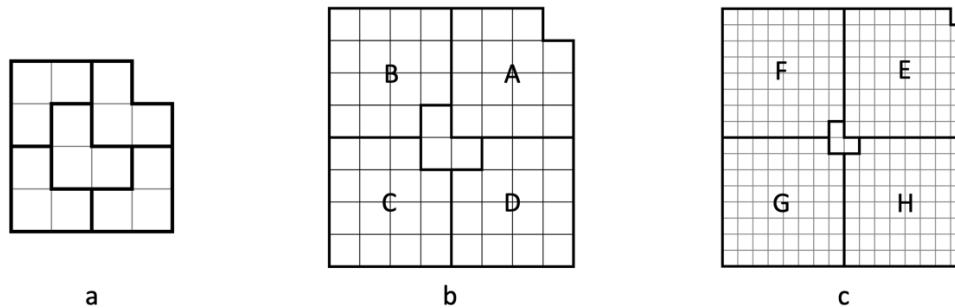


Figure 2: Tiling of 4×4 , 8×8 , and 16×16 squares in Task 1

The students then proceeded to a 16×16 square. During their work, they revisited the 8×8 square they had tiled and noticed that the upper-right quarter of this square (part A in Figure 2b) was identical to the 4×4 square they had tiled. While they re-tiled an 8×8 square, they discovered the structure of tiling:

- Misaki: I've got it. I've probably got it. This, this quadrilateral [part B in Figure 2b] is a quadrilateral with one removed.
- Sakura: Is it connected to this [part A]?
- Misaki: Yes. So, we can make a quadrilateral with one removed here [part B]. [...] We can also make this here [part C].
- Sakura: Well, you mean...
- Misaki: Do you see it?
- Sakura: Yes. [...]
- Misaki: Surely, this [part B] can be tiled without gaps.
- Sakura: Sure, and also here [part D]. So, we can think similarly. It's really the same if we look carefully.

In this interaction, the students divided an 8×8 square into four regions and found that these regions could be tiled with four 4×4 squares and one L-shape figure. Regarding the 16×16 case, they placed an 8×8 square in its upper-right quarter (part E in Figure 2c) and considered that the remaining part could be tiled with 4×4 squares and L-shape figures.

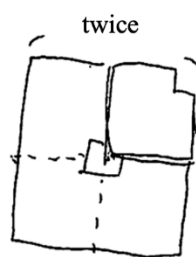
The students then worked with a $2^n \times 2^n$ square. During their work, they revisited the 16×16 case and noticed that they could have employed the 8×8 case, not the 4×4 case:

- Misaki: Well, although we considered this 4×4 , we can also consider it with this [part E in Figure 2c].
- Sakura: Where?
- Misaki: If we consider it with 8×8 , because our conclusion was that 8×8 is possible, at this point, well, this, this, this, this [parts E–H].
- Sakura: Ah. Well, how to prove it?
- Misaki: Um.
- Sakura: It's a numerical sequence, well, using the one before.
- Misaki: Is it mathematical induction?
- Sakura: Induction?
- Misaki: Yes.

At this stage, the students observed a common feature between the 8×8 and 16×16 cases. They made a process pattern generalisation where they divided a $2^n \times 2^n$ square into four regions to relate it to the $2^{n-1} \times 2^{n-1}$ case. They thus conjectured that a $2^n \times 2^n$ square could be tiled with L-shape figures. They proposed to employ mathematical induction to prove this conjecture and constructed the proof shown in Figure 3.

In the case of $n = 1$, when removing one corner from a 2×2 square, it can be tiled (one L-shape figure).

Suppose that a $2^k \times 2^k$ square can be tiled. Can a $2^{k+1} \times 2^{k+1}$ square also be tiled?



The figure becomes like the diagram [on the left] because [the length of each side of] a $2^{k+1} \times 2^{k+1}$ square is twice the length of [each] side of a $2^k \times 2^k$ square.

Here, this figure becomes the shape connecting three $2^k \times 2^k$ squares to the figure in the case of $n = k$, so if we place one L-shape figure so that it covers all the three corners, we can make three $n = k$ case figures, and thus the figure can be tiled from the hypothesis.

Therefore, when removing one 1×1 square from one of the four corners of a $2^n \times 2^n$ square, this figure can be tiled without gaps with L-shape figures.

Figure 3: The students' proof of the original conjecture

Generalisation of the original statement with proof

The students began their work on Task 2 by considering the 4×4 case. They found that the two cases shown in Figure 4a (where a 1×1 square is removed from the grey location in each case) could be tiled with L-shape figures. They also argued that, because these two cases could be rotated and flipped, a 1×1 square could be removed from any location in the 4×4 case.

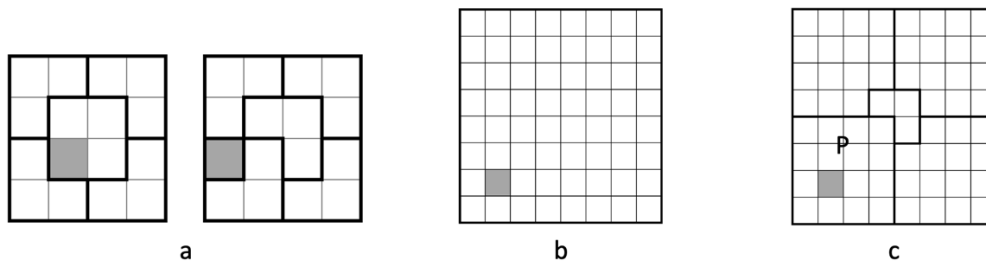


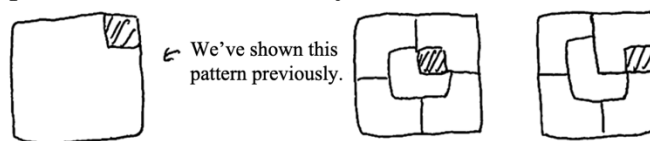
Figure 4: Tiling of 4×4 and 8×8 squares in Task 2

The students then discussed the 8×8 case. They examined the case shown in Figure 4b as an example and found that this case could also be tiled:

- Misaki: This is 8×8 if we look at the whole [case], but if we focus on this [part P in Figure 4c], we can tile it [based on their previous work on the 4×4 case].
- Sakura: We can do it in the same way as previously.
- Misaki: Yes. We can tile here, here [part P] even if we remove [a 1×1 square] from anywhere, anywhere [in part P].
- Sakura: Yes.
- Misaki: Then, the remaining part becomes this [she drew an L-shape figure at the centre of the 8×8 square and suggested that the remaining part could be tiled with three 4×4 squares].
- Sakura: Yes, it's possible.
- Misaki: Completed.
- Sakura: Well, this is also true for another case. If we remove from here, remove here, we can make it here and tile in the same way [she meant that a 1×1 square could be removed from different locations from the case of Figure 4b].
- Misaki: Yes, so...
- Sakura: All are possible.
- Misaki: Our conjecture is that [a 1×1 square can be] removed from anywhere.
- Sakura: [...] Let's prove it.

In this interaction, the students generalised the original statement they had proved in Task 1 and conjectured that a $2^n \times 2^n$ square with a 1×1 square removed *from any location* could be tiled with L-shape figures. In this generalisation, the students employed the previous idea—that is, dividing a $2^n \times 2^n$ square into four regions to relate it to the $2^{n-1} \times 2^{n-1}$ case—they had devised during their process pattern generalisation in Task 1. They then constructed the proof for this generalised conjecture, as shown in Figure 5. Although they considered the base step of their inductive proof with the 4×4 case, they noticed that they could have started with the 2×2 case after completing the proof.

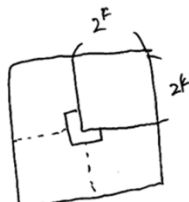
- In the 4×4 case, it is possible to make a hole anywhere.



We've been able to show [the possibility] by rotating and flipping these [two right-hand cases].

- Suppose that it is possible to make a hole anywhere in the $2^{2k} \times 2^{2k}$ case [the student miswrote here; she meant the $2^k \times 2^k$ case; the same applies hereafter].

Let's show that it is possible to make a hole anywhere in the $2^{2k+1} \times 2^{2k+1}$ case.



We do not lose generality by making a hole in the upper-right $2^k \times 2^k$ part.

From the hypothesis, the upper-right $2^k \times 2^k$ part can be tiled even if we make a hole anywhere.

Similar to the previous task, the remaining part can be considered as $3 \times$ a $2^k \times 2^k$ figure with a hole in a corner, so it is possible.

Therefore, even if we make a hole anywhere in the $2^{2n} \times 2^{2n}$ figure, it can be tiled with L-shape figures.

Figure 5: The students' proof of the generalisation of the original conjecture

DISCUSSION

In Task 2, the students referred to two ideas they had devised during their work on Task 1. First, they used their process pattern generalisation for Task 1 and similarly divided a $2^n \times 2^n$ square into four regions to relate it to the $2^{n-1} \times 2^{n-1}$ case in Task 2. Second, they referred to their proof produced in Task 1 to consider that the remaining part in Task 2 was equivalent to three $2^k \times 2^k$ squares and one L-shape figure (Figure 5). In this way, the students succeeded in generating and proving the statement—a $2^n \times 2^n$ square with a 1×1 square removed from any location can be tiled with L-shape figures—that is a generalisation of the original statement considered in Task 1.

In the theoretical framework section, I proposed the hypothesis that process pattern generalisation is relevant to the discovery function of proving by mathematical induction. This hypothesis is corroborated in the interview analysed in this study. Previous research has shown several benefits of process pattern generalisation in the context of mathematical induction: it can be used to introduce mathematical induction to students (Harel, 2002; Harel & Brown, 2008); it facilitates the transition from empirical arguments to the construction of proofs by mathematical induction (Pedemonte, 2007); and it enables proving by mathematical induction to be explanatory for the provers (Stylianides et al., 2016). This study presents another benefit of process pattern generalisation; namely, it can activate the discovery function of proving by mathematical induction. This benefit is significant because it shows that mathematical induction can play a crucial role in not only showing the truth of statements but also further generalising the original statements.

This paper presents the activity of one pair of students on a task sequence, and a similar process was observed in the activity of another student pair. However, the analysis was

based on a single task sequence with two pairs of students; thus, it is necessary to scrutinise the results by implementing different tasks in diverse settings.

ACKNOWLEDGEMENTS

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PRE-SERVICE TEACHER TRAINING WITH AI: USING CHATGPT DISCUSSIONS TO PRACTICE TEACHER-STUDENT DISCOURSE

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The recent general availability of easy-to-use tools for interacting with large-scale generative language models, for example, ChatGPT, has shaken the educational community. Both teachers and students must adapt to a world that offers technology-based assistants that can produce text and other output almost effortlessly. A major problem of the output is that it is not guaranteed to be correct, and both correct and incorrect statements are almost indistinguishable without proper content knowledge. We would like to contribute to the discussion by suggesting a way to use AI as a training tool for teachers, using the fact that the output is similar to students' contributions that need to be checked for correctness. Also, the sophisticated capabilities of ChatGPT allow for advanced training tasks where the flaws in ChatGPT's argumentation must be revealed in a Socratic dialog.

INTRODUCTION

The relationship of Artificial Intelligence (AI) to (mathematics) education has been discussed for several decades now (Balacheff, 1993a, 1993b; Gadanidis, 2017; Hwang & Tu, 2021; Mohamed et al., 2022). A major application of AI is intelligent tutoring systems (ITS), which can guide students through sets of tasks, always finding the best task to proceed with for maximized progress. They are among the most common in applications of AI in education (Holmes et al., 2019, p. 102). These systems bear the potential to individualize teaching and relieve the teachers' workload. However, AI has become even more powerful in recent years, and there exist systems that can answer free-form questions easily, while not always correct. The introduction of ChatGPT on November 30, 2022, (ChatGPT, 2022) has been a major disruption to the education community, and its consequences are still to be seen. A major concern is that ChatGPT might be used for cheating in assignments, which is not far-fetched: Indeed, ChatGPT has been shown to get a score of 1020 on SAT, which corresponds to the 52%-Percentile (teddy [@davidsong], 2022).

In this article, we will propose a way to use this technology in pre-service teacher training and give first results on the implementation in a university mathematics education course. Instead of relying on correct answers from the AI, it will be crucial that the output produced is not correct. Thus, future teachers can acquire diagnostic and communicative skills in a laboratory situation without the involvement of real students.

GENERATIVE AI

For our purposes, only a basic understanding of the underlying technology of modern AI systems is necessary: A neural network connects several, but up to billions of artificial *neurons* to each other in several layers, including an input and an output layer. Each neuron will send a signal to other neurons in the next layer –it will “fire”– if it is activated, for example by the input layer, and the receiving neurons will again send a signal if they receive “enough” signals from the sending neurons. The parameters for the function determining whether a neuron should fire or not are created automatically through training of the neural network: Given a training set of input-output pairs, the parameters are tuned by finding local extrema of the parameters iteratively that give good results for these pairs, i.e., produce output for each input that is close to the known desired result. After the training phase, the neural network can be used on other inputs and the output will be determined by the parameters of the neural network. If the network has been designed properly and had enough training data, the output can match the expected output.

Several techniques are available to turn the recognition of words, sounds, or images around into the generation of such media, leading to Generative AI (see Gozalo-Brizuela & Garrido-Merchan, 2023). Without going into details, this can be compared to the generation of random text by using the suggested next word on the keyboard of your smartphone. By varying between the most probable next word and less probable words, you end up with various random sentences that are meaningful.

Generative AI can be used to create realistic texts, images, sound, but also to transform from one of these to another, for example from a textual description to an image. This can lead to impressive results, as demonstrated by the system DALL·E 2 in Figure 1.

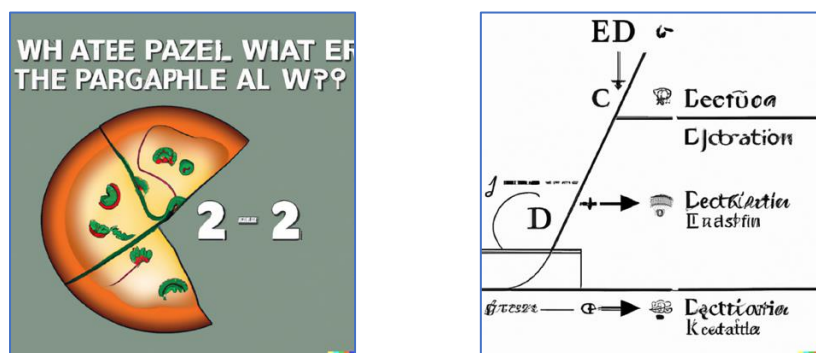


Fig. 1: Two images created with DALL·E, an AI system by OpenAI.

Left: “Two thirds of a pizza explained”; Right: “A drawing explaining the epsilon-delta criterion for continuous functions”

At first glance, the images produced look like they might explain the concept that was given in the prompt. However, on further investigation, they show only a very vague impression of something that is somehow related to the concepts. This reminds us of students who let the mathematics class pass by them, and when asked for explaining anything, they can remember some fragments, but not explain them (because they did

not understand what was going on in the first place). We will use this similarity of AI with inattentive students later.

CHATGPT

Instead of creating a new technology like in (Lee & Yeo, 2022), we rely on an existing and available technology called ChatGPT. ChatGPT is a chatbot that uses a large generative language model (Indeed, the answers of ChatGPT to mathematical questions appear to be real student answers, in fact, of students who can rely on large knowledge. However, it turns out that the answers are not correct many times, known as *hallucination* (ChatGPT, 2022) A recent paper suggests a framework of 101 tasks modeled after the Math Kangaroo competition for the age group 6-8 (Cherian et al., 2023). They use this framework to benchmark the visuo-linguistic algorithmic reasoning abilities of neural networks. Their experimental results show that neural networks still perform poorly on these problems. For us, the interesting part is that they also tested 11 of the tasks that do not use images but are purely in text form, and for each of these problems asked ChatGPT for a solution several times (since ChatGPT is using probabilistic methods, each run yields different results). On average, ChatGPT solved the problems in 36.4% of the trials, with individual solving performance ranging from 0% to 90% (9 out of 10 runs) for the problems.

This is in line with the announcement of OpenAI on Twitter, which states that ChatGPT is not yet ready to be relied on for anything important (Greg Brockman [@gdb], 2022). In this paper, we will show how to use ChatGPT nevertheless for something important: Teacher education.

RESEARCH QUESTION

While ChatGPT is mostly recognized as a threat to traditional ways of teaching (Cotton et al., 2023; Haque et al., 2022; Susnjak, 2022; Zhai, 2022), there might be good ways to include this technology in education, and indeed there are reports of successful teaching accepting the availability of ChatGPT (Roose, 2023; Roose et al., n.d.), connected with the appeal to use it instead of banning it from schools. In our research, we want to find out, *whether and how the output produced with ChatGPT can help teacher students to acquire diagnostic and communicative skills* that are necessary for appropriate cognitive support of students and cognitive activation, two out of three generic dimensions of teaching quality (Praetorius et al., 2018).

METHODOLOGY

Our research took place between December 2022 and January 2023. Our study subjects were about 30 students enrolled in the B. Ed. program who took part in the course Introduction to Mathematics Education. Most of the students were in their 5th semester. The 2h per week lecture is accompanied by small-group tutoring, also 2h per week. Within these tutorials, students reflect on the lecture content and solve tasks that should help them to transform theoretical concepts related to teaching into actual teaching. This course also serves as preparation for their subject-specific school placements in

the next semester when they start to teach mathematics under the supervision of a mentor.

All students received a short introduction to ChatGPT during the lecture. Through the courses' moodle, they received four chat transcripts that have been created by the lecturer, two in English, two in the German language, with the following initial prompts (translated), followed by a longer conversation: (1) "Can you explain integrals to a five-year-old?" (2) "Please act like a five-year-old boy and explain negative numbers to me." (3) "Hello. Can you explain to me what a quadratic function is?" (4) "Please write me an explanation of where the treasure is. This is to use a coordinate system and linear functions and their intersections. I would like to use this in mathematics lessons."

Students were advised to read the texts and mark passages that show that ChatGPT does not *really* understand what it is talking about. They were also asked to pay special attention to the prompting technique used to uncover flaws in ChatGPT's argumentation.

In the next step, students were asked to try to talk to ChatGPT themselves about a mathematical notion of their choice. Working in groups of two or three and asking questions, they should reveal wrong answers. Also, they were asked to collect more signs that the chatbot does not really understand mathematics, but is just able to produce texts that look like it understands mathematics.

Unfortunately, the huge interest in ChatGPT resulted in server downtimes, so only one group of three students did finish the assignment. The other groups will try again but cannot complete the assignment before the PME deadline.

RESULTS

Nevertheless, the dialog created by the remaining group can be used as the first data for our research. We give excerpts of the complete transcript below. The text has been copied verbatim, and some line breaks have been removed for brevity. Bold Numbers have been included to refer to parts of the transcript in the discussion.

Student: Prove that $f(x) = x^2$ is continuous using the Epsilon-Delta criterion

ChatGPT: To prove that the function $f(x) = x^2$ is continuous using the Epsilon-Delta criterion, we must show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all x and y in the domain of f , if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

To prove that $f(x) = x^2$ is continuous, we can start by assuming that $\epsilon > 0$ is given. We must then find a $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

We can start by expanding the expression for $|f(x) - f(y)|$:

$$|f(x) - f(y)| = |x^2 - y^2|$$

(1) Since $|x^2 - y^2|$ is always nonnegative, we can drop the absolute value signs and simplify the expression to: $x^2 - y^2 < \epsilon$

Next, we can add y^2 to both sides to obtain: $x^2 < \epsilon + y^2$

Since ϵ and y^2 are both positive, we can take the square root of both sides to obtain: $|x| < \sqrt{\epsilon + y^2}$

Finally, we can set δ equal to $\sqrt{\epsilon + y^2}$, which gives us:

(2) $|x-y| < \delta$

$$|f(x) - f(y)| = |x^2 - y^2| = x^2 - y^2 < \epsilon$$

This completes the proof that $f(x) = x^2$ is continuous using the Epsilon-Delta criterion.

Student: So you are telling me, that $x^2 - y^2 = |x^2 - y^2|$ is always correct?

ChatGPT: The expression $x^2 - y^2 = |x^2 - y^2|$ is not always correct.

[...]

In general, the expression $x^2 - y^2 = |x^2 - y^2|$ will only be true if $x^2 - y^2$ is nonnegative. If $x^2 - y^2$ is negative, then the expression will not be true.

Student: In your previous proof you noted that $|f(x) - f(y)| = |x^2 - y^2| = x^2 - y^2 < \epsilon$ meaning you said that $|x^2 - y^2| = x^2 - y^2$.

ChatGPT: I apologize for the mistake in my previous response. [...]

To correct the mistake, the correct statement should be:

$$|f(x) - f(y)| = |x^2 - y^2| = |x^2 - y^2| < \epsilon$$

I apologize for any confusion that my mistake may have caused. Thank you for pointing out the error.

Student: Your whole proof is useless now. Could you please rewrite it again without your wrong assumptions?

ChatGPT: I apologize for the mistake in my previous proof. Here is a corrected version of the proof that $f(x) = x^2$ is continuous using the Epsilon-Delta criterion:

[... repeating the same proof again, with the last line changed to...]

$$|f(x) - f(y)| = |x^2 - y^2| < \epsilon$$

This completes the proof that $f(x) = x^2$ is continuous using the Epsilon-Delta criterion.

I apologize again for the mistake in my previous proof and for any confusion it may have caused. Please let me know if you have any questions or if you need further assistance.

Student: (3) I don't think that proof is correct because $|x^2 - y^2|$ isn't greater than $x^2 - y^2$.

ChatGPT: You are correct that $|x^2 - y^2|$ is not always greater than $x^2 - y^2$. I apologize for the mistake in my previous proof. Here is a corrected version of the proof that $f(x) = x^2$ is continuous using the Epsilon-Delta criterion:

[... correct proof using the expansion $|x^2 - y^2| = |x+y||x-y|$...]

I apologize again [...]

DISCUSSION

The transcript above shows the typical nature of ChatGPT answers: They seem to be correct, and they use sophisticated language and subject-specific wording. In (1) we see a common mistake often found in the classroom. For positive a the expression $|a|$ is equal to a , so we can drop the absolute value signs, but ChatGPT uses a wrong version where the (trivial) positivity of $|a|$ is used to drop the absolute value signs. The students realize that problem and focus their further discussion on this fact. ChatGPT is very submissive and apologizes a lot, but the changes in the proof do not suffice to fix the problem, as they are local to a single line of the proof.

The other problem in the proof at (2) goes unnoticed. ChatGPT states boldly that $|x-y|$ is less than δ , while by definition of δ , it is only known that $|x| < \delta$. As this problem appears later in the proof, it is fine not to discuss it too early. But further investigation here shows that the proof is indeed fundamentally flawed.

In (3) the students finally find the right prompt to make ChatGPT come up with a different (and correct) proof. Here we notice that the students are right with their statement, but it is not at all clear why this causes ChatGPT to change to another strategy.

These three items demonstrate three cases for using ChatGPT as a discourse trainer: First, it makes similar mistakes to students in the classroom, even for advanced topics from undergraduate courses. The examples we gave the students were at lower or upper secondary level and they also exhibit the same error creation. Second, it is possible to discuss a problem with the chatbot, and it tries to adapt to the new information it receives, but it is usually trying to fix problems locally – which is very similar to strategies seen in the classroom, where students try to correct single lines in their assignments, without observing the consequences. Third, ChatGPT can switch to a completely other solution if prompted accordingly. While the first two observations give hope that this tool can indeed be used in teacher education, this last observation might make it difficult to discuss a solution from beginning to end. A remedy can be to remind ChatGPT to stick with the old proof (or explanation).

All in all, the first experiments answer the research question positively, thus we identified a good component of a teacher education program, following these lines:

“[...] we believe the core of teaching—interacting with students about the content—is not learned well through automatizing routines or even through acquiring expert strategies during a teacher preparation program. Rather, it is learned through continual and systematic analysis of teaching.” (Hiebert et al., 2007)

LIMITATIONS AND FURTHER RESEARCH

The ChatGPT tool has been available for a few weeks only when we started to use it in our Introduction to Mathematics Education course. Also, during that time it changed

as it is under constant development, and during peak hours (office hours in the U.S.) it was not reliably available, so we could not use it with all students enrolled in the course. As such, this must be considered a first proof of concept showing the potential of using generative language models in teacher education and training.

The rapid progress of AI—we are expecting the next iteration of the underlying language model GPT 3.5 soon—might render the described use of AI as “artificial stupidity” (AS) impossible. Even if the chatbot still might not *understand* what it is talking about: If it does not show any errors in its answer, we cannot encourage teacher students to find flaws in the argument. This can be compared to the Chinese room problem (Searle, 1980) which raises the question of whether a person following rules to produce Chinese answers to Chinese questions following a set of rules does understand the language or not, and if so, where this understanding is located. This can be translated to mathematical notions and concepts easily, reminding us of Benny's case (Erlwanger, 1973). An important skill in teaching mathematics is to reveal the concepts that students use to find answers to questions and tasks in mathematics, and this could also be trained by analyzing an AI.

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TEACHING MATHEMATICS WITH TECHNOLOGIES: PROFILES OF TEACHER CHARACTERISTICS

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Integrating technologies into mathematics learning and teaching is an important, but challenging task. A range of variables are considered relevant to understand why and how teachers vary in their technology use. A person-centred approach allows to identify profiles of teachers regarding this range of teacher characteristics. A latent profile analysis on measures of (technology-related) knowledge and motivational indicators of N=227 pre- and in-service teachers yielded five profiles, that differ in terms of knowledge and technology-related motivational characteristics. The number of pre-service teachers in the profiles varies substantially, indicating a role of post-university teaching experience for prerequisites of technology use in the classroom.

INTRODUCTION

The use of educational technologies in teaching mathematics is promising to enhance learning, especially subject-specific types such as dynamic mathematical tools (Hillmayr et al., 2020). Although technology use varies internationally (Fraillon et al., 2020), there is a call to increasingly use technologies and especially subject-specific tools in the mathematics classroom. The Will-Skill-Tool (WST) model of technology integration emphasizes the important role of teacher characteristics for the productive use of such tools (Knezek & Christensen, 2016). Regarding mathematics education, there is a lack of subject-specific studies investigating those characteristics of teachers, that may contribute to the use of technology and its quality for mathematics learning. Indeed, motivational variables such as value beliefs (Thurm, 2018) and self-efficacy (Thomas & Palmer, 2014) are known to correlate with technology use in Mathematics. Contrary, little is known about the role of professional knowledge in this context, because measures of technology-related professional knowledge are rare (Petko, 2020). The increasing number of variables discussed makes it difficult to theorize and investigate the specific role of each variable. Thus, we apply a person-centred approach to identify different profiles of teachers regarding their professional knowledge and motivational characteristics related to technology use in the mathematics classroom.

FACTORS INFLUENCING TECHNOLOGY USE

To explain the use of technologies the Will-Skill-Tool (WST) model focusses on teachers' attitude towards the use of technology (*will*), their competences regarding the use of these technologies for educational purposes (*skill* and additional factor *pedagogy*), and their access to these technologies (*tool*) (Knezek & Christensen, 2016).

Will and *skill* can be seen as teacher-related variables. Previous studies have subsumed different construct under the *will* facet, that can fall under value beliefs. The *skill* facet

has often been assessed by teacher self-reports, but doubts in their validity have been raised (e.g., Petko, 2012). These self-reports can be assumed to reflect teachers' self-beliefs, rather than skills. As technology-related measures of professional knowledge lack (Petko, 2020) we are not aware of any study including objective measures. *Tool* is mostly assessed as school context variable, referring to the availability of the tools and the ease and flexibility of accessing tools for educational purposes (Petko, 2012).

Beyond these facets, other teacher characteristics, such as professional experience are discussed to have an influence on technology use. Bachfisch and colleagues (2019) found that advanced teachers integrated technologies with higher quality than pre-service teachers. Petko (2012) did not find any effect of gender or age.

Teachers' technology-related professional knowledge

Originally, the *skill* facet is defined as the ability and the self-perceived confidence to use technologies in educational settings. Beyond teachers' self-efficacy, their professional knowledge may be assumed as a central characteristic underlying this ability. While professional knowledge without an explicit relation to technology is known to influence teaching and learning, little is known about the role of technology-related professional knowledge for teaching.

Shulman's differentiation of professional knowledge into CK, PK, and PCK was extended by the TPaCK model of Mishra & Koehler (2006) adding the core facet technological knowledge (TK). This results in two additional content-related intersections: technological content knowledge (TCK) and technological pedagogical content knowledge (TPCK). These may be understood as knowledge about how to use technology to represent mathematical content and work mathematically (TCK) and knowledge about how to use technologies to teach, represent and facilitate learning in mathematics (TPCK). Focussing on subject-specific teacher characteristics, this draws attention to teachers' CK and PCK. Since, some authors hypothesize that Shulman's original knowledge facets, such as PCK, may not predict the quality of technology use for subject-specific learning (Backfisch et al., 2020), also TCK and TPCK should be considered when investigating teachers' technology-related characteristics. Also, Petko (2012) proposes to include TPCK into the *skill* facet of the WST model.

Motivational Teacher Characteristics

In recent years different studies applied expectancy-value theory (EVT) to analyse the role of motivational teacher characteristics on technology use (*will* facet, e.g., Backfisch et al., 2020; Cheng et al., 2020). EVT includes expectation beliefs such as *self-concept*, beliefs about *intrinsic*, *utility*, and *attainment value* as well as (anticipated) *personal cost* of technology use (Eccles & Wigfield, 2002). There are inconsistent results regarding the role of the different value components. For example, Backfisch et al. (2020) identify utility value as relevant mediator between teaching expertise and the quality of use, while Cheng et al. (2020) do not find any relation of utility value with either quantity or quality of use, but that expectancy, intrinsic value and personal cost were related to quantity and expectancy to high quality use.

All these studies are variable-centred focussing on relationships between a subset of relevant variables. Person-centred approaches may help to identify profiles of teachers with respect a broad range of teacher characteristics related to technology use. Thurm (2018) identified four profiles based on personal beliefs and usage reports differentiating between groups of teachers that had either positive or negative beliefs about technology use and that were either frequent or infrequent users. Knowledge measures or covariates such as teaching experience were not included.

PRESENT STUDY

The goal of this study was to identify profiles of teachers based on their expectancy and value reports regarding the use of technology, as well as their (technology-related) professional knowledge. Furthermore, we were interested whether the number of in- and pre-service teachers differs across profiles. We focused on the following questions:

1. Which profiles of secondary mathematics teachers can be identified based on their expectancy and values reports regarding the use of educational technologies and content-related professional knowledge in terms of CK, PCK, TCK and TPCK?
2. Do the profiles differ regarding the frequency of pre-service teachers?

METHOD

Sample

We analyse a sample of pre-service ($N = 67$, median: 9th semester) and in-service ($N = 160$) secondary mathematics teachers in Germany drawn between April 2021 and October 2022. All participants provided data on expectancy and value measures. All data were collected online in a single session (max. 142 min.). In a subsample of $N = 82$ in-service teachers, the session was part of a larger data study. The other participants were from an occasional sample.

Table 1: Demographic data of the sample

	Gender			Age				School Track			
	F	M	D	< 30	30-39	40-49	>50	low	med	high	other
pre	23	41	3	61	5	0	0	35	6	23	3
in	69	79	0	23	63	34	28	37	44	82	3

Note. School track: current school of in- and course of study for pre-service teachers.

Measures

Professional knowledge. We applied a new, self-developed test instrument to measure mathematics-related facets of professional knowledge based on the TPaCK framework (CK, PCK, TCK, TPCK). For PCK and TPCK the items were embedded in typical situations that teachers encounter while preparing, teaching, or reflecting on a mathematics lesson. Technology-related items focus on three predominant subject-specific tools used for teaching and learning mathematics in Germany: computer

algebra systems, dynamic geometry software, and spreadsheets. Mathematical content were linear functions, probability, and spatial geometry as taught in German lower secondary schools. PCK and TPCK included items on instructional strategies and tasks, as well as student cognition. Items were blocked by knowledge facet and participants were given a specific time for each facet in a fixed order (CK, TCK, PCK, TPCK).

The open-answer items were coded based on extensive manuals. Interrater reliabilities determined for 20-30% of the data were satisfying ($kappa$ (weighted for ordinal scales): 0.68-1; mean: 0.90). Using item-response theory (IRT), a four-dimensional structure (CK, TCK, PCK, TPCK) was confirmed with acceptable item fit indices (in- and outfit: 0.5-1.5) and WLE reliabilities (CK: .63, TCK: .64, PCK: .61, TPCK: .74).

Motivational measures. We collected data on self-concept, personal cost, utility, and intrinsic value as well as attainment value. Descriptive data and example items of the scales are shown in table 2.

Table 2: Descriptive Data of the Expectancy and Value Scales

Scale	#	Translation of Example Items	M (SD)	α
self-concept	4	“I know how to use technologies to enhance learners’ understanding of mathematical content.”	2.62 (0.80)	.89
personal cost	8	“To teach mathematics using technologies, I have to invest a lot of preparation time to completely revise my materials.”	1.51 (0.66)	.75
utility value	5	“Teaching mathematics using technologies allows for more persistent and interconnected learning than traditional media.”	2.50 (0.67)	.75
intrinsic value	4	“I enjoy teaching mathematics lessons using technologies.”	2.95 (0.80)	.86
attainment value	6	“Teaching mathematics using technologies, I receive more acknowledgement from learners and parents.”	2.30 (0.73)	.79

Notes. Answers on a five-point Likert scale “Does not apply” (0) – “Applies completely (4), #: Number of items, M : mean, SD : standard deviation; α : Cronbach’s alpha

Data analysis

Data from all scales were z-standardized. As Little’s MCAR test was significant, the EM algorithm was applied to impute missing data (overall 2.9 %). A latent profile analysis (LPA) was conducted using the five motivational (see table 2) and four professional knowledge variables (CK, TCK, PCK, and TPCK) as indicators. We tested local independence models (covariances set to 0) with varying and equal cross-profile variances for 1 to 7 profiles (Bauer, 2022). The selection of the best model and numbers of profiles was based on AIC, BIC, adjusted BIC, Entropy and BLRT p -values, as well as interpretability of the modul. Profile differences in experience (pre- vs. in-service) were investigated with the BCH method (Asparouhov & Muthén, 2020).

RESULTS

Teacher profiles

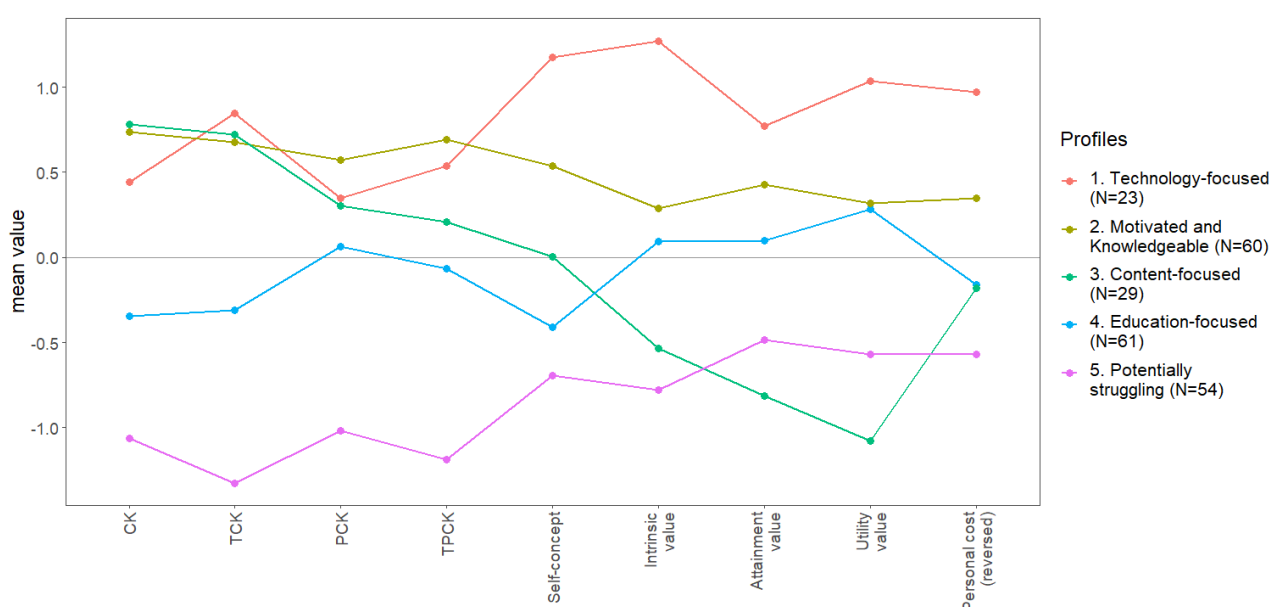
As AIC and BIC were consistently lower for models with varying cross-profile variances, the local independence model with varying variances was chosen for the LPA. The BLRT tests and BIC support a solution with five profiles, while AIC and aBIC indicate that even more profiles could be identified (see table 3). Based on interpretability and the weak decrease of AIC and aBIC for higher numbers of profiles, a five-profile solution was selected.

Table 3: Fit statistics and results of the BLRT (selected in bold)

# Profiles	AIC	BIC	aBIC	Entropy	BLRT p-value
1	5842.6	5904.3	5847.2		
2	5356.0	5482.7	5365.5	0.85	< 0.001
3	5241.9	5433.7	5256.2	0.85	< 0.001
4	5163.6	5420.4	5182.8	0.87	< 0.001
5	5096.3	5418.2	5120.3	0.88	< 0.001
6	5087.4	5474.4	5116.3	0.87	1
7	5057.3	5509.3	5091.0	0.90	1

Based on the mean values of the indicator variables (see figure 1), we interpreted the profiles as follows.

Figure 1: Mean values of the five profiles for z-standardized indicators



Profile 1: Technology-focused (N = 23). The first and smallest profile ranks highest on all motivational indicators and above-average values on all knowledge indicators,

compared to the whole sample. Moreover, the average scores on technology-related knowledge (TCK and TPCK) rank higher than those on CK and PCK.

Profile 2: Motivated and Knowledgeable (N = 60). The second profile has comparably high scores on all knowledge indicators. Also, all motivational characteristics are above average, but lower than for the technology-minded profile.

Profile 3: Content-focused (N = 29). The third profile shows comparably low value traits characteristics, especially regarding utility value. Self-concept and personal cost scores are near the sample average. In contrast, it shows above-average scores on all knowledge indicators, especially for the content-related facets CK and TCK.

Profile 4: Education-focused (N = 61). The fourth profile shows values close to the average on nearly all indicators. Compared to the whole sample, education-related knowledge indicators PCK and TPCK are higher than solely content-related CK and TCK, that are slightly below average. Also, self-concept is slightly below average.

Profile 5: Potentially struggling (N = 54). The last profile shows low values on all indicators, compared to the whole sample. Especially the knowledge indicators are substantially lower than those of all other profiles.

Experience differences between the profiles

To answer RQ2, we included a binary experience measure (0: pre-service, 1: in-service teachers) into the LPA using the BCH method. This method calculates weighted frequencies of in-service teachers in each profile based on the probabilities of assigning participants to the groups.

The *education-focused* and the *potentially struggling* profiles have the lowest weighted frequency of in-service teachers (see table 4). Especially the *education-focused* profile differs significantly from nearly all other profiles except the *potentially struggling* profile, indicated by non-overlapping confidence intervals. The *technology-focused* profile descriptively has the highest weighted frequency of in-service teachers which is higher than in the *potentially struggling* profile.

Table 4: Weighted frequencies (BCH method) of in-service teachers across profiles

Profile	Weighted frequency	95% Confidence Interval	Standard error
Total sample	0.71		
Technology-focused	0.92	[0.81; 1.02]	0.062
Motivated and knowledgeable	0.85	[0.76; 0.93]	0.052
Content-focused	0.88	[0.76; 1.00]	0.070
Education-focused	0.45	[0.32; 0.57]	0.071
Potentially struggling	0.66	[0.54; 0.78]	0.069

DISCUSSION

Because many different technology-related variables are discussed to influence the use of technologies for teaching and learning mathematics, using a person-centered approach is promising to identify typical configurations of teacher characteristics, that can be relevant for quantity and quality of technology use. We found a rich set of five profiles, that are based on objective knowledge measures, rather than self-reports (Petko, 2012). While about the half of teachers were assigned to high knowledge profiles (1-3), the other half assigned to below-average knowledge profiles might require specific professional development to acquire the required knowledge. The *potentially struggling* profile may additionally need motivational support to increase their expectancy and value beliefs while reducing the personal cost to use technology. Interestingly, there is no profile that has high motivational characteristics, but low knowledge, which would require only knowledge interventions, but no motivational support. While some teachers require both, knowledge and motivational support, for the *content-focused* profile a motivational intervention, especially focused on value beliefs might be sufficient as their high knowledge, but low value scores indicate that teachers assigned to this profile have skills that are required to use technology with the aim of enhancing mathematics learning but might not use this potential. As their PCK and TPACK is slightly lower (but still above average), they may also specifically benefit from mathematics education-related contents. The *education-focused* profile may need support regarding mathematical content and skills how to work mathematically with these tools, as lack of TCK may be a barrier increasing personal costs (reversed in fig. 1). Future studies may link profiles to measures of technology use and quality of use to investigate how the obtained profiles differ regarding these outcomes.

As proposed by Thurm (2018), we analyzed experience differences regarding the obtained teacher profiles. Preservice teachers are primarily assigned either to the *potentially struggling* or *education-focused* profile, which are the profiles with lowest scores on knowledge indicators. Interestingly, this also holds for CK, although this is one of the main goals of teacher education in university. This may be since the applied CK items focus on school mathematics and pre-service teachers may not relate the CK acquired at university to mathematics as taught in school. As these profiles have average or below-average values on both motivational and knowledge indicators, this indicates a need for initiatives to further develop motivation as well as professional knowledge related to the use of technologies in university-based teacher education.

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RELATIONSHIPS BETWEEN PROSPECTIVE TEACHERS' HEART RATE VARIATION NOTICING OF CHILDREN'S MATHEMATICS

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Professional teacher noticing is a key skillset for mathematics teaching that is informed by various psychological and physiological factors. The present study investigated the relationship between prospective teachers' heart rate variation and what they described about students after viewing a 360 video of a primary grades mathematics lesson. Results indicated a statistically significant correlation between heart rate variation and noticing students' mathematics.

INTRODUCTION

Professional teacher noticing involves attending to key pedagogical events, interpreting the significance and relevance of such events, and deciding how to shape what happens next (Jacobs et al., 2010; van Es & Sherin, 2021). Teachers' noticing has received increased scholarly attention over the past several decades and across continents (Ding et al., in press; König et al., 2022).

The bulk of such scholarship in mathematics education has concentrated on what and how teachers attend to key events (Santagata et al., 2021), with efforts to explain this phenomenon primarily focusing on psychological constructs such as professional knowledge and beliefs (Jong et al., 2021; Scheiner, 2021). Yet, "such notions of teacher noticing often construe noticing as a disembodied, purely mental form of seeing..." (Scheiner, 2021, p. 92). Conversely, Scheiner (2021) argues that noticing is also influenced by physiological factors that are embodied in teachers' lived experiences. Indeed, recent scholarship suggests teachers' descriptions of what they notice correspond with their physical actions of when and where they look within the classroom (Buchbinder et al., 2021; Huang et al., 2021; Kosko et al., 2021). Adding to such scholarship, Ferdig et al. (2020) found that the nature and directionality of sound also affects teachers' noticing. These findings are important in establishing empirical evidence for the embodied nature of professional noticing. However, noticing involves more than the eyes and ears, and there is a need for scholarship in understanding the role of other physiological data in teacher noticing.

This paper explores the role of the cardiovascular system in teachers' noticing, using heart rate variability—and heart rate variance—as a primary source of data. The cardiovascular system plays a significant role in visuo-spatial orientation, mental processing, and emotional state (Barsalou, 2020). Specifically, change in heart rate is associated with mathematics anxiety (Hunt et al., 2017), as well as teachers' self-efficacy (Schwerdtfeger et al., 2008). Teachers' mathematics anxiety, self-efficacy and

similar psychological factors have also been found to associate with teachers' noticing (Cross Francis et al., in press; Simpson & Haltiwanger, 2016). Despite observed relationships between key psychological factors with both heart rate and noticing, no research studies could be found that examine a direct connection between heart rate and noticing. Thus, the purpose of this study is to explore the role of teachers' heart rate variability in their professional noticing of children's mathematics.

THEORETICAL FRAMEWORK

Professional noticing is an important aspect of mathematics teaching, involving the interrelated skills of attending, interpreting, and shaping (Jacobs et al., 2010; van Es & Sherin, 2021). Attending involves selecting key elements of pedagogical practice to focus. A teacher may perceive many things but attend to a select number of elements (van Est & Sherin, 2021). As teachers attend to events, they simultaneously interpret what they attend to. This is often done via their professional knowledge and beliefs (Jacobs et al., 2010; Jong et al., 2021). Shaping involves the various ways a teacher interacts with a context to access more information of pedagogical significance (van Es & Sherin, 2021). For example, a teacher may choose to move to one group of students over another to better attend to their mathematical thinking (Zolfaghari et al., 2020). The current study focuses primarily on prospective teachers' attending, but we acknowledge the interactive role interpreting and shaping play in what and how teachers attend.

Teachers' attending is influenced by a myriad of factors. This includes psychological factors such as mathematical knowledge for teaching (Jong et al., 2021), mathematics anxiety (Cross Francis et al., in press), mathematics confidence (Simpson & Haltinger, 2016), and so forth. Yet, attending is as much a physiological as it is a psychological experience. Scholars using 360 video, a form of virtual reality that records video omnidirectionally, have examined where teachers choose to look and how this corresponds with what and how they attend. Kosko et al. (2021) observed that prospective teachers who attend to more mathematic-specific aspects of children's reasoning also focused on a larger number of students. Similarly, Buchbinder et al. (2021) found that prospective teachers were able to reflect on more aspects of pedagogy due to 360 video's capacity to record more of the scenario. Seeking to bridge the gap between how psychological and physiological factors inform noticing, Kosko et al. (in press) used eye-tracking data in a 360 video and found that teachers with higher pedagogical content knowledge demonstrated different viewing patterns than those with lower demonstrated knowledge. Such empirical findings corroborate Scheiner's (2021) proposed model for noticing as explained by interactive and reciprocal relationships between psychological and embodied factors.

The study reported in this paper focused on a particular relationship between prospective teachers' heart rate variability and what they attended when viewing a 360 video of a primary grade fractions lesson. As with much of professional noticing literature that acknowledges the complexity of the phenomenon, while attending to a

subset of elements, the study was created to examine how embodied aspects represented by the spatial-visual (where one looks) and cardiovascular (heart rate) systems corresponded to prospective teachers' written noticing.

RELATED LITERATURE

The cardiovascular system is a critical component to living as it distributes blood throughout the human body. However, it also impacts cognitive (e.g., attention, visuo-spatial orientation, processing) and social functions (e.g., emotion, stress) (Barsalou, 2020). Given this connection between the physiological and psychological, researchers have attempted to examine the ways in which heart rate (as a representative of the cardiovascular system) is related to cognition, cognitive processes, or other constructs related to learning (e.g., attention). For instance, researchers found that “slow oscillations in heart rate have the potential to strengthen brain network dynamics” (Mather & Thayer, 2018, p. 98). Liu et al. (2022) also demonstrated a correlation between HRV and cognition, particularly as it related to neurodegenerative disease.

Researchers have been able to apply this more specifically to examining how heart rate is related to mathematics education. Hunt et al. (2017), for example, demonstrated that “self-reported math anxiety was found to be significantly correlated with physiological reactivity to more difficult mental arithmetic” (p. 129). Lee & Lee (2021) showed heart rate changes seemed to be related to some tension in the learner as they read and then reread mathematics text. While such literature has provided a motivation to further examine the impact of heart rate and mathematics instruction, no studies were found that specifically examined heart rate and mathematics noticing.

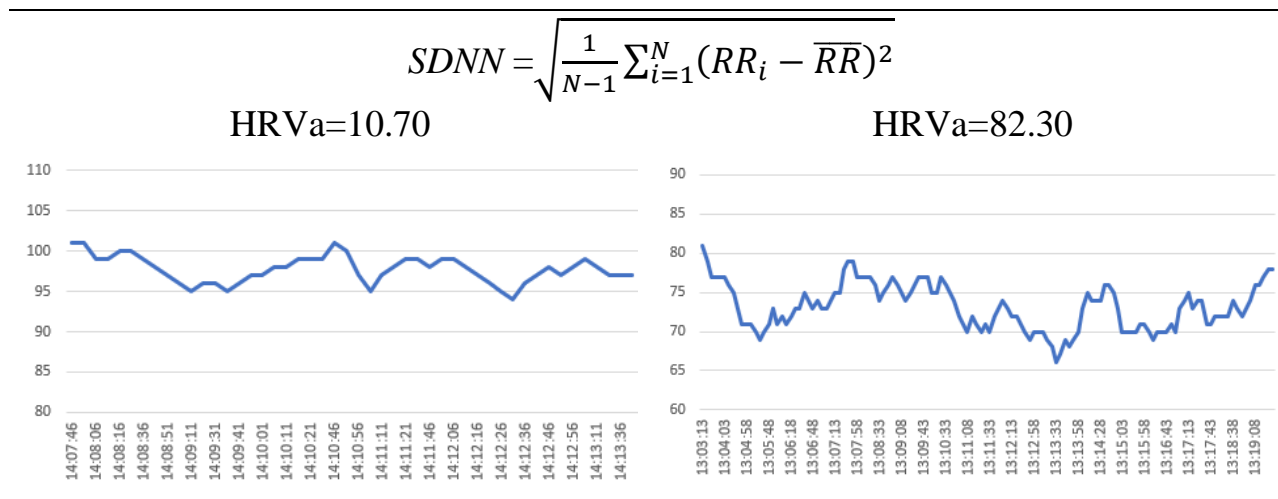
METHOD

Data were collected from a convenience sample of 18 prospective teachers enrolled in an educational technology course in a Midwestern U.S. university. Twelve of the teachers were preparing to teach primary students (ages 5-11), whereas the remaining six were preparing to teach either English or social studies to secondary students (ages 12-18). Prospective teachers received course credit for participating in research studies, with this study being one of several options in Fall 2021.

Prospective teachers used Oculus Go virtual reality headsets to watch a classroom recording of a grade 4 (ages 9-10) lesson on reviewing equivalent fractions (5 minutes, 50 seconds). In the 360 video, students were provided fraction strips and were asked to find equivalent fractions for $\frac{5}{6}$ and then $\frac{3}{8}$, with a brief class discussion after working in small groups with each fraction. Participants viewed the video twice. After each viewing, they described what they noticed related to mathematics teaching and learning. Concurrent with these procedures, participants wore Fitbits (Charge 5) that collect their approximate heart rate every two seconds. They also were asked to screen record their viewing on the VR headsets, and this recording was later synchronized with participant Fitbit data.

Fitbits only approximate heart rate variability (HRV), and only with sustained use by the wearer (a luxury researchers do not often have). Given these concerns, the standard deviation of normal-to-normal (SDNN) heart rates—an accepted formula in cardiology (Silveti, Drago, & Ragonese, 2001)—was used to calculate heart rate variance (HRVa). In this manner, the value was placed on determining high and low HRVa scores rather than focusing on a 1:1 correlation between HRV and HRVa. High HRVa scores represented users who may have had stress during parts of the experiment, but they were also able to return to a normal level. Low HRVa scores represented users that either did not experience stress or tension during the activity, or those users who were at a sustained level of stress throughout the duration of the experiment. Figure 1 contains the SDNN formula as well as an example of both high and low HRVa scores.

Figure 1: Heart rate data and HRVa for two participants during 360 video viewing.



ANALYSIS & RESULTS

A convergent mixed methods analysis was employed where qualitative analysis was conducted to examine prospective teachers’ written noticing and themes were quantitized for statistical analysis (Creswell & Plano Clark, 2018). Teachers’ written noticing were explored by examining how mathematics was referenced using Systemic Functional Linguistics (Halliday & Matthiessen, 2004). Two themes emerged which indicated either employment of generic or mathematical reference chains. Participants employing mathematical reference chains did more than merely mention something mathematical. Rather, they built up meaning from reference to students’ mathematics throughout their writing. Table 1 presents an example where one participant focused on a moment in which students were asked to use fraction strips to find an equivalent fraction to 3/8. They observed that because the strips only go to twelfths (i.e., “the numbers would be too big”), students noted they could not divide because 3 was a “prime number.” By contrast, the participant employing generic reference focused on the teacher’s classroom management (questioning & use of group work). Other participants employing generic reference chains may have included brief mentions of mathematics but did not build up meaning from such isolated references.

Table 1: Example excerpts of generic and mathematical references in noticing.

Theme	Excerpt from Prospective Teacher's Writing.
Generic Reference Chain	[Teacher] asking questions that engaged the class, as well as using the white board. It's important because students have to be engaged and feel that their answers are valued. Seeing other students answer questions will make students speak more. Group work is important also; using their minds together enhances learning.
Mathematical Reference Chain	They [students] could not make the <u>[equivalent fraction]</u> with the <u>fraction strips</u> because <u>it [numerator for 3/8] was a prime number...</u> One moment that I noticed...was when the students had to try to <u>work out 3/8 with the strips</u> . They [students] found out that they were <u>not able to do so</u> . This is valuable because they knew that <u>the numbers would be too big</u> .

Following qualitative analysis, a dichotomous variable was created for whether participants employed mathematical reference chains in their written noticing (0 = no; 1 = yes). A partial Spearman Rho correlation was also calculated to control for whether or not participants were preparing to be primary teachers (0 = not primary grades; 1 = primary grades). We hypothesized that the teaching context (primary grades mathematics) would affect primary grade mathematics teachers more than those, despite being prospective teachers, would not be teaching the content or grade level. Indeed, primary prospective teachers had lower HRVa scores ($M = 29.21$, $SD = 15.14$) than non-primary prospective teachers in our sample ($M = 41.17$, $SD = 21.89$). Results from the partial correlational analysis showed a positive and statistically significant relationship between participants' HRVa and their use of mathematical reference in noticing ($\rho = .492$, $p = .045$).

DISCUSSION

Teachers' professional noticing of children's mathematics is influenced by a myriad of factors (Scheiner, 2021). These include psychological one such as professional knowledge, beliefs, and emotions (Cross Francis et al., in press; Jong et al., 2021; Simpson & Haltiwanger, 2016) and physiological ones such as where one looks at or moves about in the classroom (Buchbinder et al., 2021; Huang et al., 2021; Kosko et al., 2021). In this study, we examined whether one particular physiological factor (heart rate variability) was associated with how teachers notice. Specifically, prospective teachers watched a 360 video of mathematics instruction. After watching the video, they were asked to write about what they saw. Their reference chains were then correlated to heart rate variance scores calculated from heart tracking devices they were

wearing. The results showed a significant and positive correlation between HRVa and mathematical referencing.

Interpretations of these results must be based on an understanding of HRVa scores. Variability in heart rate may indicate stress. However, stress is not always bad; it can, for instance, indicate times of attention, focus, or noticing (Laumann, Gärling, & Stormark, 2003; Yokoyama et al., 1987). As such, increases in heart rate may indicate a time when learners pay attention, notice, or are focused (Barsalou, 2020). Such heart rate would then return to normal when the tension or stress went away (i.e., between moments of pedagogical significance in the video), producing a higher HRVa. This form of HRVa was found to be associated with attending to children's mathematics in the present study. Conversely, prospective teachers with low HRVa might either be chronically stressed or experienced little to no stress, tension, or focus during the study. Such participants tended to describe generic events when writing about what they noticed. Additional research is needed to further understand and extend these results. Following recommendations from Scheiner (2021), further work is needed to understand how embodied (physiological) and psychological factors (i.e., beliefs & knowledge) inform and are informed by teachers' professional noticing. However, these preliminary results combined with past research suggests that HRVa may be a useful factor in better understanding the nature of teachers' mathematics noticing.

ACKNOWLEDGEMENTS

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IN-THE-MOMENT TEACHER DECISION MAKING AND EMOTIONS

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The present study focuses on enlightening the interplay of emotions and decision-making of mathematics teachers in pivotal teaching moments, where dealing with students' mathematical confusion. Using Activity Theory, we study emotions and decision making in two cases of teachers to highlight their similarities and differences. Data from three lessons of each teacher as well as from four semi-structured interviews were analyzed. According to the results, teachers' emotions and in the moment decision making are influenced by teachers' valuation of goal achievement, students' responses and social, anticipatory, moral and systemic dimensions.

INTRODUCTION

Lately, teacher emotions have been in research focus, however, very little research focuses on teachers' emotions during mathematics teaching. Studies in this area concern mainly pre-service and/or elementary mathematics teachers (Martínez-Sierra et al, 2019), adopting mostly quantitative research approaches. Trying to fill this gap, our study focuses on investigating in-service secondary mathematics teachers' emotions during pivotal teaching moments (PTMs) (see Stockero & Van Zoest, 2013) where the teacher makes decisions.

Teachers' decision-making has not been studied much in mathematics education (Potari & Stouraitis, 2019), yet Schoenfeld (2011) thinks that studying teachers' decision-making during teaching can help us even in forecasting it. While, teachers' emotions are related to the quality of teaching and are a key factor in teachers' decision making (Di Martino et al., 2013), teachers' emotions in relation to their decision making have not been the focus of mathematics education research. Through our study we try to enlighten this important gap.

In our paper we attempt to address the complex relation between decision making and emotions during PTMs for two experienced secondary school mathematics teachers. In particular, we address the following research question: How do teachers' emotions and decision-making interplay in PTMs?

THEORETICAL BACKGROUND

Our study adopts a cultural-historical activity theory (CHAT) perspective, in which emotions “reflect the relationship between object/motives and the levels of success that are anticipated as the outcome of a set of actions that concretely realize the activity” (Roth & Radford, 2011, p. 21). For Leontyev (1978) emotions have a very important role in the development of the activity, as “they are not subordinated to the activity but seem to be the result and the 'mechanism' of its movement” (p. 167). Burkitt (2021)

considers emotions as integral components of social interactions, and “they function in complex ways, not only as internal signals to one’s self, but also as signals to others, which are frequently spontaneously expressed in the moment without full consciousness of our intention” (p.13). Shuman and Scherer (2014) see emotions to have multiple components: a subjective feeling component, a motor component, a physiological component, an action tendency component, and an appraisal component. Plutchik’s wheel of emotions (2001) is a tool for identifying and verbalizing complex emotions by illustrating the relationships between primary emotions and their intensity.

CHAT places decision making in the context of object-oriented, collective and artifact-mediated activity systems constantly undergoing developmental transformations (Engeström, 2001). According to Engeström (2001) “decisions are typically steps in a temporally distributed chain of interconnected events” (p.281), which are influenced by other participants of the activity, and they “shape the future of the broader activity system within which they are made” (p. 281). Thus, the subject (mathematics teacher) makes the “necessary” decisions to form the path of meeting the perceived object (students’ learning). These decisions are shaped by the subject’s goals, by the social interaction and needs within the activity, while they are manifested through subject’s actions. Roth and Radford (2011) see actions to be “mediated by emotional valuations that arise from, and reflect/refract the assessment of the current relations and mediate selection of goals and actions that move the activity further long” (p. 22).

In-the-moment teachers’ decision making is usually triggered in a PTM, which is “an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students’ mathematical understanding” (Stocker & Van Zoest, 2013), p. 127). There are five types of PTMs, one of which is mathematical confusion, as well as five teachers’ actions (e.g., ignores/dismisses, acknowledges but continues as planned, etc). Ellis, et al. (2019) use the construct of “teacher move” and provide a more elaborative framework to study how teachers attempt to foster an inquiry-oriented environment. Four main categories of teachers’ moves are described at this framework: Eliciting student reasoning, responding to student reasoning, facilitating student reasoning and extending student reasoning. Stouraitis (2016) examines how decision-making is framed and develops considering social and systemic dimensions, using Engeström’s (2001) four dimensions of expansion: the social-spatial, the anticipatory-temporal, the moral-ideological, and the systemic-developmental.

METHODOLOGY

A case study of two upper secondary school mathematics teachers, Gregory and Aris, in Greece, with 25-30 years teaching experience is reported in the context of a professional development program (PD) EDUCATE (<http://www.ucy.ac.cy/educate/>). The program aimed to support teachers to balance differentiated learning and mathematical challenge, a fertile ground for the emergence of PTMs and emotions.

The data have been generated from three videotaped lessons of each teacher, seven PD sessions and four semi-structured interviews with each teacher. Initially, we analyzed the lessons to identify PTMs, teachers' emotions and actions dealing with each PTM, drawing additional data from teachers' reflections on their lessons during the PD sessions. The interviews were conducted by the first author. The first interview aimed to outline each teacher's overall teaching goals. In the other three video-recall interviews, each teacher was asked to watch his videotaped lessons and identify moments that he considered important and/or that indicate emotions from his side. In the first part of each interview, we replayed the video-extracts and discussed the moments chosen by the teachers, while in the second part the moments chosen by the researchers.

At first, we identified the PTMs in the lessons using Stockero and Van Zoest's (2013) framework. Then, within each PTM we identified teachers' decision points, teachers' actions and emotions, and students' involvement. For teachers' decision points we were inspired by Schoenfeld (2011). For teachers' actions we initially used grounded methods to identify and characterize them and then we contrasted them with the ones used by Ellis, et al (2019) in teachers' moves framework and Stockero and Van Zoest (2013) and formed a final synthesis. Concerning teachers' emotions from the lessons, we focused on identifying emotions' motor component and the action tendency component, from the Shuman and Scherer's (2014) framework. For students' involvement we used grounded methods and open coding. Then we place the above in a chronological order to unfold each PTM. The interview data were analysed initially grounded to verify teacher's emotions (subjective feeling, appraisal and action tendency components) and find relationships between the PTMs, decision-making and teacher's emotion(s) using Engeström's (2001) dimensions. Emotion characterization relied on Plutchik's (2001) wheel of emotions.

We chose to analyze here one PTM on mathematical confusion (MC-PTM) from the lessons of each teacher managed between 29.40 and 33.50 minutes in Gregory's 10th grade class and between 48.00-52.55 minutes in Aris' 11th grade class.

RESULTS

Gregory' MC-PTM

The students, working in groups, have found the roots of several trinomials, using the formula and their factorization. Then, Gregory asks the students to observe their findings and try to see how they can relate the roots of a trinomial to its factorized form. The students do not answer, and one of them say that he is confused. Gregory writes the trinomial on board, and tells them "Here we have $x^2 - 3x + 2 = (x - 1)(x - 2)$ and you told me that its roots are 1 and 2. What is happening here?". The students do not answer and they still seem confused, while discussing with each other. Gregory asks "What do you think that is happening here guys!". The students do not answer and one of them says "but we found everything, what else? I am confused" (31.50). Gregory asks "what is the process of factorization?". Gregory addresses to

Mary (a good student) who says “What process? It is confusing, sir. Maybe you want the form?”. Gregory asks “what form?”, and Mary comes on board and writes $a(x - r_1)(x - r_2)$. Gregory says “Oh! Ok! Is this equal to the trinomial?” and then Mary says “what do you mean?” and Gregory tells her to write $ax^2 + bx + c = a(x - r_1)(x - r_2)$. Then he says “so, I am telling you that this is right, is it verified by what you have found?”, the students answer positively and then Gregory asks them to prove it. Figure 1 captures the chronological MC-PTM’s management.

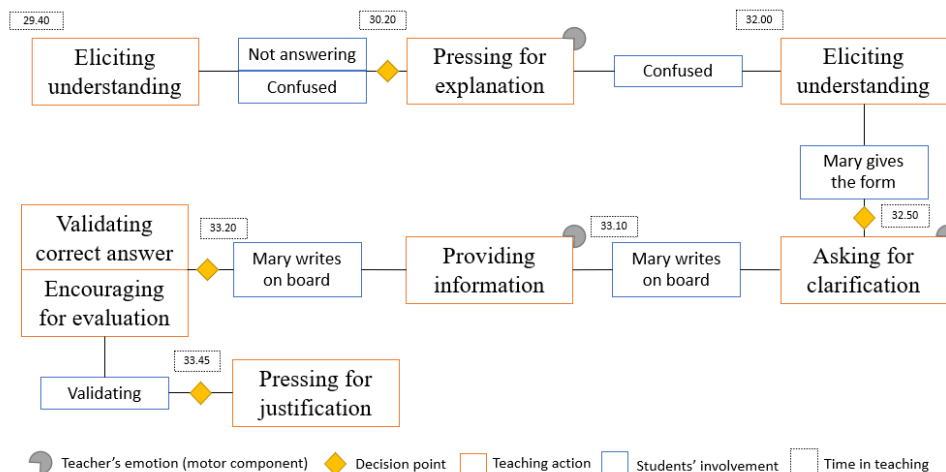


Figure 6: Gregory's MC-PTM

During the third interview we discussed with Gregory about his emotions and his decision making in this MC-PTM:

- 1 G: I got anxious that their work will be lost. They found everything and yet they don't understand what I am asking for. I was wondering why and I wanted someone to give it to me, to describe the process.
- 2 R: What else did you feel here?
- 3 G: You see, the time was running out and there were much to do, we still had to make the claim and then prove it.
- 4 R: And when Mary responded?
- 5 G: I was glad, because finally someone gave me something, that was also correct, but it wasn't what I was looking for.
- 6 R: What do you mean?
- 7 G: I asked them to make a connection, not to parrot me a form. I was afraid that she knew the form from private tutoring.
- 8 R: Yet you chose to continue with it, despite the fact that you wanted the students to conclude that.
- 9 G: Yes. They seemed really confused and she facilitated me to move on with the lesson. We had about five minutes for the ring to bell, so I wanted to show the other students the form, and continue with the proof, which is very important.

Gregory's emotions are negative and are triggered by his failure to promote students' understanding [anxiety (1)], guilt, a secondary emotion being shaped by joy and fear (5,7)] and by time [anxiety (9)]. Dealing with these emotions, his decisions and actions

are driven by the goals he had set for teaching, while sticking at lesson planning, which is shown by addressing a good student when the rest of the students had reached an impasse (32.20), and by choosing to use the algebraic form Mary suggested (32.50), even though he believed that it did not demonstrate her understanding.

Aris' MC-PTM

A student presented on board her group's solution, answering on finding a polynomial which has for roots 2, 3 and 4, and Aris asks the students to evaluate it. Another student, Manos, agrees with the solution presented and proposes to take an extra condition, the polynomial's constant term to be a multiple of twelve. Aris does not respond, and another student asks why. Aris continues to the next question, but Manos insists. Aris asks "can you tell us what you mean?", and Manos explains his thought, which is based on the integer roots theorem. Aris says to the class "I don't understand this. What do you think of what your classmate is saying?". The students seem to be very confused, and a student, Giannis, starts arguing with Manos about his claim. Giannis explains to Manos that they have the polynomial in its factorized form, but Manos insists on using the distributive property in order for the constant term to appear. Aris says "What is it that you are saying? We have $P(x) = (x - 2)(x - 3)(x - 4)$. Has your group found another polynomial?", Manos says "no" and Aris continues "So how are you discussing about the constant? Well, at this one $P(x) = (x - 2)(x - 3)(x - 4)(x - 5)$, you are saying that the constant term is a multiple of twelve, and it has five as a root. So, why do we care about it? About what the constant will be? Why do you discuss only about the constant and not for the x's coefficient"? Students don't answer, some of them are still confused, so the teacher asks for someone to sum up their understanding the task. Figure 2 captures second MC-PTM, in the way described in Gregory's case.

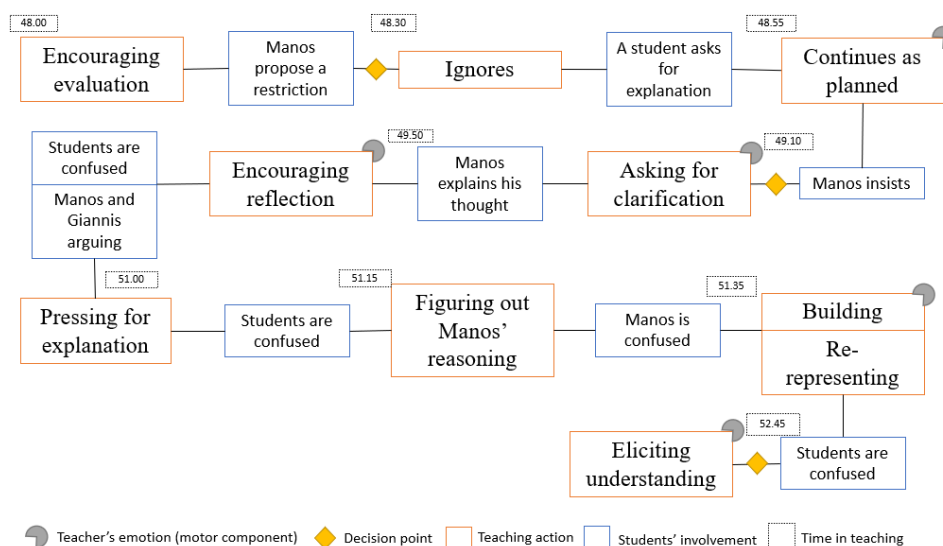


Figure 7: Aris' MC-PTM

During the fourth interview we discussed with Aris about his emotions and his decision making in this MC-PTM:

- 10 A: This happens in class very often, a student to say something and the teacher to not fully understand it, so you have to make a choice. At first, I decided to continue because I thought that what Manos said was totally irrelevant, and because of the time of course. When I saw the video, I said to myself “what have you done?”.
- 11 R: Well then thank for Manos’ persistence (laugh).
- 12 A: (laugh) I understood that he was confused, probably trying to answer something else.
- 13 R: There you are asking the other students to give ideas. Why?
- 14 A: I was buying to myself some time. I was trying to understand what Manos’ claim was and get prepared to answer.
- 15 R: What were you feeling then?
- 16 A: Anxious at first, because I couldn’t understand what he was saying.
- 17 R: When similar incidents happen in class, do you feel the same?
- 18 A: Not always. It’s not about teaching experience, but about how safe the teacher feels in each class. And so, he acts.
- 19 R: What about later on the incident?
- 20 A: I was feeling anxious again (laugh). When I figured out what his claim was about, I tried to discuss it, but then I looked at my watch. We had no time, we had to move on, so I chose a short closure.

Aris’ anxiety is triggered by different situations. At first, he feels anxious because he does not understand Manos’ claim (16) and later on, he gets anxious about time (20). While at first, he decides to continue as planned (48.30 and 48.55), then dealing with his anxiety, he decides to change his lesson plan (49.10), dealing with Manos’ claim and addressing it to other students for ideas, in order for him to gain some time to understand. While feeling anxious about time, he decides to shortly close the discussion (52.45) about Manos’ claim, but he is engaging the students in showing their understanding.

While the two MC-PTMs, presented above, are similar in terms of the mathematical content and the pivotal teaching moments, teachers’ interplay of emotions and decision making indicate some similarities but also some significant differences. Below we try to address the influencing dimensions of these two interplays.

The social-spatial dimension is found in the communities influencing the decisions. Time seems to be a decisive factor for both teachers, as it also affects their emotions. Both teachers felt anxious about the remaining time. Gregory at 33.20 decided to be caught by Mary’s answer and complete the solution. Aris at 51.35 and 52.45 decided to speed up the discussion around Manos’ claim in order to continue to the next task.

The anticipatory-temporal dimension is found in the temporally distributed steps of decisions. Gregory’s plan was for the students to form the claim about the factorized form and the roots of the trinomial, and then to prove it. When Mary mentioned the form, probably knowing it from tutoring, Gregory decided to use it in order to continue in proving it. At these moments Gregory felt guilt, because from one hand someone

finally gave him the answer he was looking for, but on the other hand this answer did not indicate understanding.

The moral-ideological dimension highlights teachers' decisive focus on students' involvement and understanding. Gregory was anxious that students' effort and work would be lost, so he was continuously trying to elicit their understanding. Aris came in a difficult position trying to understand Manos' claim. He was feeling anxious about his readiness to respond, so he decided to reach out to students for ideas.

The systemic-developmental dimension is found in the possibilities for action-based decisions to shape the future of the broader activity. When Gregory's students had reached an impasse, feeling anxious, he decided to address a good student, seeking for an acceptable answer, something that he usually does in similar cases. Aris, on the other hand, dealing with his struggling for understanding Manos' claim, he decides to engage the other students and create a classroom discussion around it, while for him it is of great importance the students to be given opportunities for debating and develop mathematical understanding through discussing mathematics with their classmates.

CONCLUSION

This research contributes to the understanding of emotions and decision-making interplay that mathematics teachers experience in class. In both cases teachers' decisions seem to be developing from students' involvement in relation to their goals' achievement valuation, and by social, anticipatory, moral and systemic dimensions (Engeström's, 2001). Teachers' emotions are triggered by students' understanding (Martínez-Sierra et al, 2019), but also by teachers' valuation of readiness to respond to students' learning, and time management. The differences in the influencing dimensions of teachers' emotions and management of the MC-PTMs, apart from those mentioned above, may also be due to other influential dimensions, such as teachers' identity, so further research is necessary. As teachers' emotions and decision-making are two intertwined forcing units of the activity of teaching, the extension of this study to other PTMs of the same and/or of different teachers, would give us insights for the understanding of this interplay in the context of mathematics teaching.

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ROCK'N'ROLL – EMERGENT AFFORDANCES AND ACTIONS DURING CHILDRENS' EXPLORATION OF TOUCHTIMES

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Drawing on Gibson's notion of affordances and Biesta's notion of 'action' we document the developing relationship between learner and digital application while encountering a new, digital experience through what we call 'emergent affordances'. We report on two elementary school students exploring TouchTimes, an open-ended multiplication application. We detail the students' background and the changing way they take up the beginnings offered by the environment that develops and allows affordances to emerge. We suggest that emergent affordances is a helpful construct in thinking about opportunities for robust engagement with mathematical relations.

INTRODUCTION

Digital technology applications are becoming increasingly integrated into the mathematics classroom (Calder et al., 2018). The applications fulfil different purposes and therefore, their design differs with respect to the kind of mathematical activity they are supposed to afford. Design and usability features depend on specificities of the mathematical content, such as structural aspects the mathematical idea and didactic approaches guiding the design of accompanying tasks. However, in designing a digital learning experience, there seems to be a tenuous balance between guiding students towards viable conceptualization of a mathematical idea and, at the same time, enabling opportunities for playful and undirected exploration of the interface that allows them to observe mathematical structures and relations (Nilsson et al., 2018). Such explorations appear to be a fruitful phase in the working process as it helps students forge relations of the app's functionality and affordances (Gibson, 1977).

Marrying Gibson's notion of affordances (1977) and Biesta's (2022) notion of 'action' – that builds on a view of education that considers the subject-ness of the child (p. 47) – we describe the developing relationship between learner and digital application while encountering a new, digital experience through what we call 'emergent affordances'. Understanding better the affordances of digital applications as dynamic relationship evolving from and with the engagement of the child with the digital environment will, we submit, allow us to better understand students' learning with digital technology.

In this paper, we will elaborate on the framework of emergent affordances. For this, we will be presenting a single case of two elementary school students exploring *TouchTimes* (TT) (Jackiw & Sinclair, 2019), an open-ended multiplication app that is designed to elicit multiplicative relationships through bi-manual touch-based interaction (see Chorney & Sinclair, 2021).

FRAMEWORK AND BACKGROUND

A crucial feature of digital environments is the dynamic aspect of feedback (Joubert, 2017) constantly changing the meaning possibilities of the environment in reaction to students' interaction with the interface. The following sections will present a framework to capture the flexible nature of students' enacted possibilities enabled by such an environment.

Affordances and action

Gibson's (1977) idea of affordances considers the possibilities and meanings that link an agent – or organism – to their environment, describing a relationship between both. With this, the affordance is determined equally by the organism and the environment while not predetermined by either. At the same time, affordances in this sense determine what actions involving organism and environment are possible, based on the configuration and possibilities of both as well as the relationship between them. Gibson gives the example of water, which affords support for a water bug, but not for a person, the affordance of water hence depending on the organism. A chair affords sitting on for most people, with the relationship being based on habit, while other possibilities (standing, building a children's fort, putting your glass down,...) are thinkable. At the same time, other surfaces afford sitting, depending on the background, habits and configurations of an individual. Affordances – in Gibson's sense – are hence inherently relative; they are *action possibilities* (Norman, 1988).

We furthermore draw on the notion of 'action' as suggested by Biesta (2022), building on Arendt (1958). Action here is always *interactional*: it includes a beginning – or initiative – as well as a response in which this initiative is taken up (which we will call 'follow-up'). It denotes "our beginnings *plus* the ways they are taken up by others" (Biesta, 2022, p. 48). Actions, with this, are never predetermined by beginnings – these beginnings merely provide 'action possibilities', similar to the affordances as described above, with the environment becoming subjectified. Importantly, others have the "freedom to take up our beginnings in their own way rather than how we may have wanted them to handle our beginnings" (p. 48). The follow-up then becomes a new beginning, providing new action possibilities.

In the context of digital environments, actions as couples of [beginning – follow-up] have a two-fold nature: One concerns [designing – manipulating], the action possibilities guided by design characteristics. Another one is constituted by feedback and manipulation, with both of these alternating in their role as beginning and follow-up. Action possibilities are then developing with the actions themselves, which is what we call *emergent affordances*. While the design features determining feedback and action possibilities are static in that they do not change in the process, it is the students' background and the changing way they take up the beginnings offered by the environment that develops and allows affordances to emerge.

TouchTimes (TT)

TT is a multi-touch iPad application, designed to develop children's relational understanding of multiplication. There are two sub-applications; the one we focus on in this report is Grasplify (TT-Grasplify). In TT-Grasplify, the iPad screen is split down the middle (Fig. 1a). Whichever side of the screen is touched first results in a set of coloured discs (named *Pips*, see Fig. 1b), which matches the number of fingers touching the screen. The set of Pips represents the multiplicand or unit. While keeping the Pip fingers in contact with the screen, touching the other side of the screen results in the creation of a Pod. A Pod matches the configuration, colour and number of Pips. The number of Pods will also depend on the number of fingers touching the screen. This number represents the multiplier. TT encircles them with a boundary containing a white background (Fig. 1c). (Information on TouchTimes and how to download it for free can be found here: <https://touchcounts.ca/touchtimes/index.html>)

Pip fingers must remain touching the screen while Pod fingers can be lifted (Fig. 1d). The interface is dynamic so any interaction on the Pips will be replicated in each Pod. Pods can be dragged everywhere on the screen. As Pips and Pods are being created, the corresponding number of each side is displayed at the top of the screen and a multiplicative equation is visible (see Figs. 1c or 1d).

One of the multiplicative relationships within the design of TT is the idea of *spread* based on the splitting notion articulated by Confrey (1994), realized in expanding and contrasting shape and size of the Pods based on the manipulation of Pips.

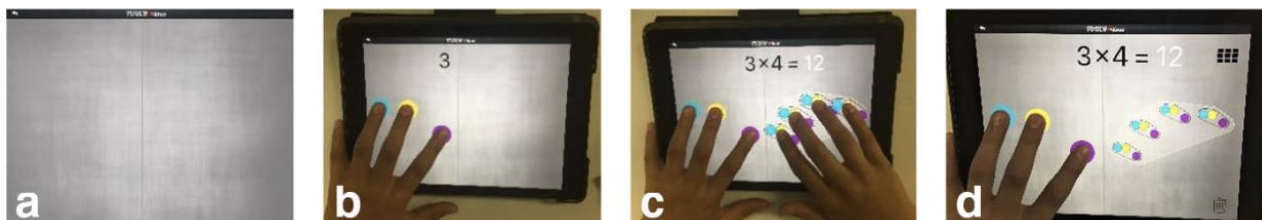


Fig. 1: a) initial screen of TT; b) creating Pips; c) creating Pods; d) lifting Pod fingers

Focus of this study

In our study, we observe the kinds of patterns that result from the various ways that students interact with TT. These patterns are grounded in a multiplicative relationship within the design of the application, which is why they relate to the affordances offered by design; but they are also not reductive to the application or iPad, rather they result from the students' (inter)actions in which further affordances can emerge. Our focus and attention is grounded in the multi-sensory affective dimensions of experiences children have of using TT – experiences that form the basis upon which multiplication is conceptualized. We expand our notion of experiences, however, not to be solely what students do on the screen but the taking up of what happens with their touching. In this paper, we will present the affective dimension of aesthetics that make certain affordances emerge.

ANALYSES: EMERGENT AFFORDANCES IN STUDENTS' EXPLORATION OF TOUCHTIMES

The data we draw on is taken from a pool of data gathered in studies involving TouchTimes over the last couple of years, carried out by the TT groups at Simon Fraser University. It includes elementary school children between 7 and 9 years old. For this study, the authors viewed and discussed the video clips, attending to couples of [beginning – follow-up] as actions and the possibilities emerging within them. In particular, we were interested in the action of *spreading* and how it develops with affordances emerging in the students' interaction with the environment. That is, we seek to identify the students' 'way to engage with spreading as an action'. We will present a case to illustrate how the framework of emerging affordances helps us to better understand how children's mathematical activity is based in subject-ness (affect and social influences) as articulated by Biesta. The case excerpt is based on the interaction of two girls.

Spreading as dancing – emerging affordances grounded in aesthetics

Episode 1.1: From static affordances to affordances emerging through aesthetics

This episode deals with two grade 2 girls exploring TT-Grasplify. In the initial stages of exploration, the girls are making Pips and Pods, not going beyond the action possibility provided by the design but seemingly exploring exactly these with respect to potential feedback. There was feedback within TT, but the girls lifted their hands off the screen after all instances of feedback, not taking up any of these newly offered beginnings. Such actions included, for example, dragging Pips across the boundary or placing Pips on tops of Pods.

In making Pods of 5 on the left side of the screen and dragging them to the right side, one girl (left in Figs. 2) declared after 3:21 minutes: "I- let's make a picture." She starts organizing the Pods on the right side (see Fig. 2a).



Fig. 2: a) making a 'picture'; b) a purple Pip; c) changing the shape of the Pods

They continue to make more Pods by tapping and dragging to the right side of the screen. After about another minute, the girl on the left touched the Pip side while still in the action of dragging Pods across the screen, and an additional sixth purple Pip appeared and – simultaneously – in all the Pods (Fig. 2b). As she started to drag the new Pip and the movement is replicated in all Pods, she says "What the...?;Whoa". Holding the purple Pip, she drags it in large circular movements on the screen. The other girl made another Pip and drags it in smaller circular movements. The girl on the

left then says “Oh wait! I can change the shape!”, putting a finger down to create a new Pip and lifting the right hand, making the first five Pips disappear (Fig. 2c).

Identifying emergent affordances in episode 1.1:

The girls have identified through their interactions in the TT environment – their actions as taking up the design feature offering the creation of Pips and Pods – a new relation between Pips and Pods.

The girls’ take up specific action possibilities offered by being able to drag the Pods making a ‘picture’. By wanting to draw a picture, and dragging the Pods into a configuration, this initiated an action pattern of tapping to make a new Pod, followed by dragging the Pod to the left side of the screen. This action continued until one of the girls inadvertently tapped a purple Pip, a new beginning for a potential new action. A follow-up manifested in the feedback was all the Pods growing in size since they now included the new Pip, together with moving the purple Pip in a circle. Both together shows not only the changing size but also the movement of the Pip in the Pods. Of significance is that in creating a picture of Pods, and establishing a pattern of action, the affordances emerge in the sequence of actions as action possibilities of beginnings, not predicted by the design of TT but afforded by the interest and enjoyment of the girls. This emergent affordance was created through exploration and produced a new kind of action giving the girls an interaction that they came back to many times in their further exploration.

Episode 1.2: Elaborating Pips and Pods as subjects

The first episode was followed by a new form of interaction in which the girls were mainly tapping rather than dragging for about 30 seconds. At 4:55, the girl on the left tells the other girl to “just press – a lot. Press your whole hand”, putting down five fingers to show her partner what she’d like her to do (Figs. 3a & 3b). She then makes six Pods sequentially on the right side of the screen (Fig. 3c). The girl holding the Pips lifts her fingers off and then placed her five fingers down again immediately. The girls laugh and the girl on the left says “They just turned into threes - Wait”. She lifts thumb and pinky finger of the other girls hand off the screen and directs her to do this herself: “Put those and then pick up your pinky and thumb”, modelling her fingers to do what she wants her to do. She creates two Pods on the left side and after the girl creating the Pips lifts thumb and pinky, she giggles “You changed it”. Subsequently, the girl on the right lifts and taps the middle finger alternately and the girl on the left creates two Pods. Recognizing the changes in the Pods, caused by the change of the Pips, the girl on the right recognized “they are dancing”. Giggling and humming melodies, the girls keep on changing the Pods by tapping the middle finger for Pips (right) and moving the Pods around each other (left) (Fig. 3d). Following this, the girls make further changes and observe their consequences, for example lifting different fingers for different configuration of Pips, saying “rock’n’roll” while observing the resulting ‘dance’ in the Pods.

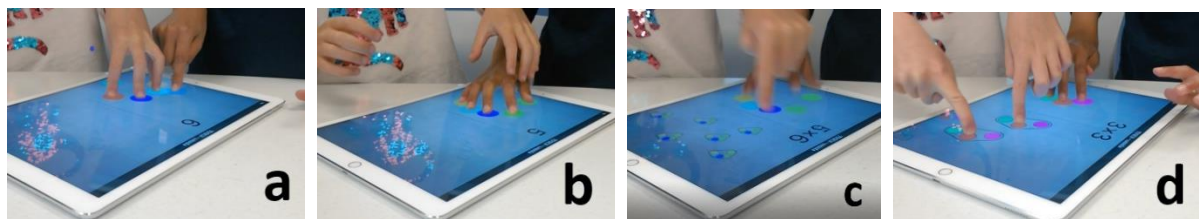


Fig. 3: a) models pressing whole hand; b) guiding partner's hand; c) lifting and tapping fingers; d) dancing Pods

Identifying emergent affordances in episode 1.2:

The girls practice the action of spreading in different ways, making Pods and changing the number and configuration of Pips. Pressing the whole hand offers the possibility to create Pods of 5 on the right side of the screen. Lifting the Pods-hand in the way done by the girl on the right then offered an action possibility recognized by the girl on the left: turning the 5-er Pods into 3-er Pods, an exemplified action of manipulating the pods in size and shape, hence of *spreading*. As a beginning, this is followed up by an action pattern of manipulating the Pips and observing the changing Pods, subjectifying them in saying that they are ‘dancing’.

The emergent affordance of seeing a relationship of spreading through the patterned activity of dancing was not predicted by the researchers, particularly in how it captured the interest of the girls who explored this action pattern further and used phrases as “They’re dancing again”, “rock'n'roll”, “Make a lot of dancing ones” as well they were humming and laughing. Through the actions of [manipulating – feedback] and [feedback – manipulating], which are not predetermined by the environment but emerge as enacted path of possibilities guided by establishing perceived aesthetics, spreading develops into the action pattern of dancing.

SUMMARY, CONCLUSIONS AND OUTLOOK

In this paper, we present the idea of ‘emergent affordances’ to capture the developing nature of action possibilities when working with digital learning environments in math education. Furthermore, we suggest these action possibilities to relate to Arendt’s (1958) notion of ‘action’ as elaborated by Biesta (2022), where action possibilities concern potential ways of following up a beginning/initiative. We illustrate the idea of ‘emergent affordances’ through a case of two elementary school students exploring Grasplify in the TouchTimes application.

In exploration, students stumble across situations that intrigue them, draw them in and from which patterns emerge. We call these emergent affordances because they are non-reductive and they exemplify multiplication in certain ways. Spread is latently positioned within TT’s design but is drawn out through action. The students were guided partially by the static affordances of Pips and Pods which were colourful, easily created, interactive but it is the follow up that brings this multiplicative relation to the forefront of activity. The aesthetic development of rock’n’rolling is one example, from our case study, of a newly framed and constructed context that becomes a new beginning, providing new action possibilities. The significance of this follow-up, in

our theoretical framing, is that it shows that action is not just doing or acting, it is based in how beginnings are taken up. Following Biesta (2022), it grounds the subject-ness of the child, and that they come to see their role in multiplicative action. The affordance is the newly emerging follow-up of noticing one's own position in rock'n'rolling the Pods which ultimately exemplifies the multiplicative relation of spread.

We based the idea of emergent affordances as action possibilities developing in the process on a notion of action as [beginning – follow-up]. The digital environment in this becomes an organism as its own, a subject involved in the action in both components. The students acknowledge that in that they assign an activity to the pods as if they were human – describing them as dancing. Furthermore, this understanding of action brings in the subject-ness of a child (Biesta, p. 47) as goal of education. This is about providing the opportunity for children to encounter a world “that which is not of our own making and that exists independent of us” (p. 48) and through which they establish their role and influence “in its materiality and its sociality” (p. 48). It allows to have them build experiences of this influence in the world of TT.

We suggest that the construct of emergent affordances is helpful for thinking about opportunities for further actions created by an expression of freedom but yet also engages via design with multiplicative relations. The resulting relational interactions of the children show an openness as to what can be explored but also how affordances emerge as a combination of affect (laughing at dancing; singing) and interest, combined with a configuration in TT that can lead to further actions that exemplify multiplication as spreading.

We note that students are excited about the result and repeating the result again and again. These patterns reflect the notion of spreading but are inspired by the way the students have organized the screen. The dancing emerges from the observation of several Pods ‘acting’ simultaneously, hence might not have emerged if there not have been several Pods made. It came up by the girls taking up beginnings offered in their interaction with the digital environment and guided by their orientation towards perceived aesthetics in the feedback given by the environment: The girls were acting on their own interests, they wanted to make a ‘picture’ with Pods. The (probably accidental) creation of an additional Pip invited further curiosity which then led to a follow-up of interactions that included tapping and dragging mirrored in the Pods. They called this dancing as the Pods pulsating with the addition and removal of a Pip, they reminded synchronized dancing where every Pod was doing the same dance move.

We claim that the open-endedness of the application plays an important part in allowing certain affordances to emerge from students’ interactions rather than by design. To understand better the ways and circumstances of affordances to emerge as well as the quality of their affective and social dimensions, we encourage other scholars to contribute in research on emergent affordances in working with digital technology.

Outlook

Further analysis of the video data aims to identify indicators for tracing emergent affordances in action couples of [beginning – follow-up]. So far, we encountered three potential such indicators:

- repeating a pattern; that is, touching the screen in a specific way numerous times;
- naming a relation and/or performing in a way to replicate this relation;
- starting again only to return to a previous situation.

In our ongoing research, we aim to build a bigger framework that captures these indicators from different theoretical perspectives to inform us to better understand the emergent affordances leading to action patterns.

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JUDGEMENT ACCURACY: COMPARING OPEN REPORTS AND RATINGS AS INDICATORS OF DIAGNOSTIC COMPETENCE

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In view of the variety of approaches to investigate diagnostic competence, it seems particularly interesting to consider precisely how the main outcome variables are measured. For instance, judgement accuracy can be assessed by various methods. Aiming to contrast two of these methods, N = 62 pre-service teachers were asked to diagnose simulated 6th graders in an open report followed by a closed-choice rating. Results of this study indicate that both assessment formats yield different measures of judgement accuracy. Especially, participants' content knowledge was related to the accuracy of closed ratings, but not to the accuracy of open reports. This indicates that the assessment format may affect what is measured under the term judgement accuracy, asking for further research to critically investigate effects of used methods.

INTRODUCTION

In view of the heterogeneity of the student population, teachers need diagnostic competence, allowing them to teach adaptively, identifying, and taking into account their students' different needs. Several research projects (e.g., NeDiKo: Herppich et al. 2018; DiaCoM: Loibl et al. 2020; COSIMA: Heitzmann et al. 2019) have investigated (pre-service) teachers' diagnostic competence, aiming to systematize research and to derive recommendations for teacher education programs. Following Heitzmann et al. (2019), diagnosing refers to the purposeful accumulation and integration of evidence to arrive at accurate educational decisions, such as selecting tasks of appropriate difficulty based on an evaluation of a student's understanding (Schreiter et al., 2021). The most frequent measure used as indicator for diagnostic competence is judgement accuracy, which in the past, has been primarily measured by asking teachers to rate student understanding in single subareas, or to estimate students' test scores (Südkamp et al., 2012). Alternative assessment formats, such as open diagnostic reports, may more authentically reflect the way teachers deal with diagnostic information in schools, rather than ranking students or estimating their test scores. This contribution compares the judgement accuracy of written open reports and closed-choice ratings of student understanding in a simulated diagnostic interview, and relations to pre-service teachers' professional knowledge.

(Pre-service) teachers' judgement accuracy

Judgement accuracy is mostly referred as the match between teachers' judgements about students' understanding and the actual understanding as, for example, measured by an independent test (Südkamp et al., 2012). Based on their meta-analysis, Südkamp et al. (2012) report a moderate to large correlation between teachers' judgement and

students' actual performance on a test, concluding that teachers' judgements are "[...] evidently far from perfect [...]" (Südkamp et al., 2012). Other studies have shown interindividual variation in teachers' judgement accuracy, which is related to their students' learning gains (Behrmann & Souvignier, 2013). Research has also established a link between teachers' professional knowledge and their judgement accuracy (e.g., Binder et al., 2018; Kron et al., 2022; Ostermann, 2018).

However, Südkamp et al. (2012) point out that "[...] the different methods used to measure teachers' judgements [...] make a substantial contribution to the degree of accuracy observed". A range of moderators referring to teacher, judgement, test or task, and student or class-level characteristics have been investigated in the past (Südkamp et al., 2012; Urhahne & Wijnia, 2021). In Südkamp et al.'s (2012) analysis, judgement accuracy was higher, if teachers were explicitly informed about the test content, and if the rating targeted the same knowledge domain as the test itself.

How teachers report their judgements of student understanding may also influence judgement accuracy: Asking teachers to evaluate single, given subareas of student understanding might provide them with more information about what should be diagnosed, than just asking for a rank-order in a not further specified mathematics test, or an open report with only general directions for its content. Following Südkamp et al. (2012), providing this information could lead to more accurate judgements. Additionally, Santagata et al. (2021) point out that offering more or less guiding information for a diagnostic report may differently focus teachers' attention and noticing processes while diagnosing. On the other hand, requiring teachers to give judgements also for subareas of student understanding that they are not sure to be able to diagnose correctly, might reduce judgement accuracy of closed answer ratings. In comparison to such closed ratings, open reports might bias judgement accuracy towards those subareas that teachers find more important, or that they are confident to have diagnosed correctly.

Moreover, different assessment formats to report student understanding go along with different methodological implications. While closed ratings are efficient to survey and easy to code, they rely on precise descriptions of what needs to be judged. Open reports require a similarly precise description to ensure reliable coding but take more time to survey. On the other hand, they may provide additional information beyond judgement accuracy such as on teachers' reasoning behind their judgements, or the implications they draw from them (Seidel & Stürmer, 2014).

In prior research, closed ratings of single subareas of student understanding (including the option "cannot be decided based on the observed data") were used in simulations of diagnostic situations in mathematics education (e.g., Kron et al., 2022; Sommerhoff et al., 2023). Other works have extracted teachers' judgements from open reports of student understanding to study judgement accuracy (e.g., Codreanu et al., 2021). Even though a range of formats to measure judgement accuracy is discussed in the literature, methodological comparisons of such formats are scarce.

THE PRESENT STUDY

In this contribution, we compared two different assessment formats (open, self-written report versus closed-choice rating) for pre-service teachers' diagnostic judgements in terms of their *accuracy*, *coverage of relevant subareas* of student understanding, *consistency* of judgements between the two assessment methods, and *relations* of the observed judgements to pre-service teachers' professional knowledge. In particular, we focused on the following questions:

RQ1: To which extent do pre-service teachers cover a range of relevant subareas of student understanding in the two different assessment formats?

RQ2: Do the assessment formats differ in terms of pre-service teachers' average judgement accuracy? To which extent are the pre-service teachers' diagnostic judgements consistent between the two different formats?

RQ3: How does judgement accuracy in the two assessment formats relate to the number of subareas covered in the open report, and to pre-service teachers' content knowledge (CK) and pedagogical content knowledge (PCK)?

METHOD

The diagnostic judgements analyzed in this contribution were collected using a simulated learning environment of diagnostic one-on-one interviews (Kron et al., 2022). During the winter term 2019/2020 $N = 62$ pre-service secondary school mathematics teachers from a large university in Germany participated in two live simulated role-play interviews in the role of a teacher ($N = 124$ interviews in total). After the two simulations, each participant answered a professional knowledge test assessing CK and PCK on decimal fractions.

Simulated diagnostic one-on-one interviews

In each simulated diagnostic one-on-one interviews, the participants took over the role of a teacher, aiming to diagnose a simulated 6th grader's understanding of decimal fractions. The 6th graders were played by trained research assistants. Four different student case profiles were designed and randomly chosen for each interview. Each participant diagnosed two different students consecutively. During the diagnostic interview, the participants were asked to select tasks out of a given set of diagnostic tasks and to derive assumptions about the student's understanding, based on its solutions of the selected tasks. The "teachers" were allowed to ask any additional question. Each interview took up to 30 minutes. Directly after the interview, the participants were asked to diagnose the interviewed student in terms of its understanding of decimal fractions. For the diagnosis, we used two different assessment formats. The participants were asked to write an open report about the student's competences and misconceptions regarding decimal fractions. Moreover, participants were asked to rate the student's understanding in a closed rating. For this rating, nine subareas of decimal fractions (Place-value system: (1) place-value principle, (2) bundling principle, (3) comparison of decimals; Addition & subtraction:

(4) conceptual knowledge, (5) arithmetic abilities, (6) flexible use of strategies; Multiplication & division: (7) conceptual knowledge, (8) arithmetic abilities, (9) flexible use of strategies) were listed, with the response options “student mastered”, “student did not master”, and “diagnosis not possible”. Participants could switch back and forth between the two assessment formats freely. No further information about the expected content of the open report was provided.

Instruments

Diagnostic judgements: For each participant and each interview, two different kinds of diagnostic judgements were assessed. The answers of the closed rating were coded the following: answers of “student did not master” were coded -1, “student mastered” were coded 1, and “diagnosis not possible” were coded 0. Due to technical reasons, there are few missing ratings ($N = 5$ of $N = 1116$). The open reports were coded based on the statements about misconceptions or competences regarding the nine subareas. Answers regarding student’s misconceptions were coded -1, answers regarding student’s competences were coded 1, and statements containing contrary information regarding the same subarea were coded with 0. If the open report did not contain any statement regarding a specific subarea, this was treated as missing. Interrater agreement was moderate ($\kappa_{mean} = 0.49$; $\kappa_{SD} = 0.28$).

Professional knowledge: To assess participants’ professional knowledge, we designed a professional knowledge test (Kron et al., 2022), consisting of twelve items on CK and eight items on PCK, each item referring to the simulation’s topic of decimal fractions (EAP Reliability: CK: 0.60; PCK: 0.58).

Statistical analyses: The dataset contains the participants’ answers of open reports and closed ratings of both interviews and for each of the nine subareas ($N = 62_{participants} \times 2_{interviews} \times 9_{subareas} = 1116$ judgement pairs in total). An expert solution for each student case profile was used to calculate judgement accuracy for both types of judgements: The deviation between the participant judgement (-1, 0, or 1) and the expert solution was averaged over all available judgements for the corresponding interview and judgement type. Similarly, the consistency between the two judgement types was calculated based on the absolute difference. Both measures were reversed and rescaled to values between 0 and 1, so that 0 indicates poorest and 1 indicates perfect agreement.

For all further analyses, we used linear mixed models (Bates et al., 2014), due to the dataset’s nested structure (two interviews per participant). All continuous measures (judgement accuracy, consistency, professional knowledge scores, number of subareas covered in the open report) were z-standardized for these analyses. The position of the interview (first or second) was included as fixed factor in all analyses. The participant and the diagnosed student case profile were integrated as random effects, if they contributed to variance explanation.

RESULTS

Coverage of subareas of student understanding (RQ1)

On average, participants covered a third of the subareas in the open reports and almost all subareas in the closed ratings (see Table 1).

Judgement accuracy and consistency of different diagnostic judgements (RQ2)

Table 1 shows descriptive data for judgement accuracy of open reports, closed ratings, and for the judgements' consistency. On average, judgements in open reports were significantly more accurate ($B = 0.15, p < .001$) than those in closed ratings.

The average consistency of 0.77 indicates a substantial, though not perfect agreement between the two different diagnostic judgements. Only considering those subareas, that were covered in the open reports for each interview, the open reports were still significantly closer to the expert ratings than to the corresponding closed ratings ($B = 0.06, p < .05$).

	<i>M (SD)</i>	min - max
Number of subareas covered in open report	3.08 (1.43)	1.00 – 7.00
Number of subareas covered in closed rating	8.96 (0.20)	8.00 – 9.00
Judgement accuracy of open reports	0.83 (0.21)	0.00 – 1.00
Judgement accuracy of closed rating	0.67 (0.15)	0.33 – 1.00
Consistency of judgements	0.77 (0.23)	0.00 – 1.00

Table 1: Descriptive data of outcome measures.

Explaining differences in judgement accuracy (RQ3)

The number of subareas covered in the open reports was negatively related to the judgement accuracy of the open reports ($B = -0.18, p < .01$), but not significantly related to the judgement accuracy of the closed ratings ($B = 0.03, p = .601$).

<i>Judgement accuracy in...</i>	<i>open report</i>	<i>closed rating</i>
	<i>B</i>	<i>B</i>
Content Knowledge (CK)	-0.04	0.28**
Pedagogical Content Knowledge (PCK)	0.10	0.06
Number of subareas covered in open report	-0.18**	0.03

Table 2: Mixed models for the judgement accuracy of open reports and closed ratings, ** $p < .01$.

Regarding effects of professional knowledge (CK, PCK)¹, the accuracy of the open reports was not significantly related to professional knowledge beyond the number of subareas covered in the open report (see Table 2). Higher CK went, however, along with significantly higher accuracy of closed ratings ($p < .01$). The corresponding relation for PCK was not significant ($p = .516$).

DISCUSSION

The aim of this contribution was to compare two different assessment formats to for the accuracy of diagnostic judgements and to investigate if the assessment format also affects the relations to professional knowledge. Overall, a range of differences between the two assessment methods occurred.

Firstly, open reports contained less diagnostic information than closed ratings, since pre-service teachers only covered a limited range of subareas of student understanding in their reports. This raises doubts about the validity of open reports to provide a broad assessment of judgement accuracy regarding a specific topic of student understanding. Future research needs to investigate which person- and subarea-related factors influence, which judgements are taken up in an open report, and which are omitted.

Taking this restricted amount of diagnostic information as given, the judgements contained in the open reports were, on average, more accurate than those from the closed ratings. These results indicate that participants have prioritized those subareas for the open reports in which they were more accurate, possibly because they were more confident in the corresponding judgements or because they considered them more important. This hypothesis is supported by the observation, that the more subareas a pre-service teacher covered in the open report, the lower was the judgement accuracy of the open report (but not of the closed rating). A selective judgement in open reports is desirable on the one hand, since this seems to provide access to the more accurate judgements of student understanding and indicates which subarea (pre-service) teachers consider more exact or important.

Regarding consistency, differences in pre-service teachers' judgements occurred between the two formats. The open judgement was, on average, even closer to the expert solution than to the closed rating. Even if participants focused on judgements of aspects that they were more confident in, it remains unclear why they would provide a different answer in the closed format. One possibility is that the judgements in the open reports were on a different, possibly more fine-grained level than in the closed format.

Finally, only the accuracy of the closed ratings, but not of the open reports, was significantly related to pre-service teachers' professional knowledge. Even participants with low professional knowledge might have achieved at least one or two accurate and confident judgements, which they included in their open report – leading to a relatively high accuracy, in spite of a narrow scope in their diagnoses. Thus, and since a relation

¹ Similar relations to professional knowledge were observed when removing the number of subareas covered in the open report from the model.

to knowledge is expected from a theoretical perspective, this raises further doubt about the validity of open reports to measure judgement accuracy. It was participants' CK, but not their PCK which was related to closed ratings' judgement accuracy. This is unexpected, since the role of PCK in educational diagnosis is often pronounced (Binder et al., 2018). However, prior empirical work pointed to the role of CK for diagnosis (van den Kieboom et al., 2014). Evaluating the mathematical correctness of students' understanding might primarily depend on school-related mathematical CK, while a deeper interpretation of misconceptions – which was not assessed in this study – might require PCK to a larger degree (Baumert & Kunter, 2013).

Summarizing, our findings indicate that the method used to measure the accuracy of a diagnosis does matter, without necessarily pleading for one of the two presented methods. Closed ratings might be more beneficial to arrive at a valid and critical estimation of pre-service teachers' judgement accuracy. Open reports seem to highlight those judgements that pre-service teachers would prioritize themselves and can possibly provide deeper insights into the noticing and reasoning mechanisms (Seidel & Stürmer, 2014) behind these judgements as well as potential conclusions drawn from them in terms of educational interventions.

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INTERACTIONAL FORCES IN MULTILINGUAL DISCOURSES – A TEACHERS’ PERSPECTIVE ON LEARNERS’ AGENCY

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In multilingual interactions, ‘centrifugal’ as well as ‘centripetal’ forces have to be expected with regard to language form. A deficitary approach to multilingualism might lead to centripetal forces that make a teacher ‘interrupt’ the activation of multilingual learners’ language repertoires. Still, insights into teachers perceptions of multilingual learners’ language choices are not yet sufficient and further insights are needed. This study gives first insights into how a teacher perceives his multilingual students’ language choices and how subtle switches in his reactions lead to a change from more ‘centripetal forces’ to ‘centrifugal forces’ in the planning and videographing of an explanation video of the ‘Auxiliary Task’.

STARTING POINTS AND THEORETICAL BACKGROUNDS

In recent years, educational research in mathematics education has shown the necessity to change the view on multilingualism: It is not a ‘deficit’, but a different and highly creative way of processing languages and thinking-processes (see Barwell 2016; Kuzu & Prediger 2017; Planas 2021). Multilingualism does not only affect the language form, it also might bear nuanced differences in meanings, like the change of thinking-directions, associations, different interpretations of relations etc. (see Kuzu & Prediger 2017; Planas 2021). While there is still more research needed regarding this meaning-related side of multilingual resources, even the *possible* existence of meaning-related differences in the ‘minds’ of multilinguals makes it important enough to demand an *opening of mathematical learning discourses for all language resources* (see Barwell 2018; Planas & Chronaki 2021; Planas 2021). Educators thus should reflect their beliefs and practices with regard to multilingualism: A competence-oriented approach is not a mere recommendation but based on empirical insights being a result of many years of research in the psychology of mathematics education (see Planas 2021).

These insights directly lead to another important facet: *Interaction forces* in multilingual classroom situations emerging when teachers and students interact. Barwell (2016) refers to Bachtins theory of language to describe two main interaction forces: ‘Centrifugal forces’ versus ‘centripetal forces’. In a Bachtinean sense, ‘centrifugal forces’ refer to the production of ‘heteroglossia’, meaning diverse, but legitimate and co-existent forms of language in one utterance or in one persons repertoire *with regard to the requirements of a communicative situation*, for example the usage of different linguistic forms – like dialects, varieties, mixed-utterances etc. – as well as of different socio-political forms – for example contextualized language in form of language registers, languages of social groups, multilingualism etc. (see Barwell 2016; 2018) In this sense, there is no ‘right’ or ‘false’ language form, rather “*it always arises in particular*

situations to accomplish particular social ends” (Barwell 2016, p. 106), thus a speaker aims at a functional inclusion of all of language forms he or she possesses. ‘Centripetal forces’ on the other side stand for the opposite: For a norming process, “*a pressure towards uniformity in language*” (Barwell 2016, p. 107). Being typical for school contexts, this means that mostly institutional people like teachers do ‘control’ how it has to be spoken through norms, rules, sanctioning etc. (see Barwell 2016), f.e. ‘to speak in full sentences’ or ‘to use complex the complex technical terms’. They go beyond looking at the ‘particular social ends’ but rather pretend (or really believe) that only specific linguistic and socio-political forms of language are legitimate. Typically, with regard to institutional ‘centripetal forces’, only high-status languages are allowed or wanted and these languages have to be used in the ‘expected’ form, which is mostly the register of ‘academic language (proficiency)’ in school contexts, meaning a highly decontextualized, abstract and complex language form (see Wessel 2020). Yet, students may express their mathematical thinking also in other languages than the classroom language or high-status languages, in mixed forms, in the everyday register etc., meaning ‘centrifugal forces’. It might be a justifiable didactical goal to teach the academic language register, but in mathematics lessons, it rather becomes problematic when teachers do interrupt mathematical thinking processes of learners for the sake of a ‘good’ language use (see Planas & Setati 2014; Barwell 2016).

Thus, these interaction forces influence teachers’ perception and intervention as well as students’ agency, which can be defined as a “*dynamic capacity of humans to act independently and to make choices [...] it lies in the actions people take in response to particular discourses*” (Noren 2015, p. 173). Such choices can be ‘centrifugal’ or ‘centripetal’, when students decide to use or not to use everyday language means, their multilingualism etc., and these choices are highly influenced by teachers – since they are being “*in response to particular discourses*” (see Noren 2015, p. 173) –, f.e. when they are sanctioned for using ‘other’ languages than the allowed ones. Sanctioning can happen in direct forms like prohibiting and leads to ‘marked’ linguistic practices – meaning “*social ripples because participants [do not wish or] expect [...] a choice*” (Myers-Scotton 2006, p. 159) – or in indirect forms like ignoring, overhearing etc., which would be forms of micro-aggression (see Spanierman, Clark & Kim 2021).

These interaction forces and their effects on students’ agency are integral to every language production, but there are still insights needed on *how exactly* students or teachers perceive these forces – if they perceive it (in most cases, it might be an implicit aspect) – and how they act when they perceive them (see Barwell 2018). This paper focusses on the teacher-side: How do teachers *perceive* these forces when they occur and how do they react, sanction or skip the occurrences? In this study from the German context, a case study being part of a broader study about pre-algebraic thinking processes in the context of the ‘Auxiliary Task’ (see Kuzu 2022) is presented.

METHOD OF THE STUDY

Research context and data corpus of the study. The broader research context is the analysis of pre-algebraic thinking when proceptually reflecting and generalizing the compensation rule behind the ‘Auxiliary Task’, a mental calculation strategy (see Kuzu 2022). $n = 18$ learners from grade 3 to 6 (9- 12 years old) were fostered in a designed learning environment and after that, their pre-algebraic generalizations were focussed. In the analysed sequence of this paper, a pair of students in an age-mixed group (S6 is 12 years old; S7 is 9 years old) were asked to make an explanation video about their understanding of the ‘Auxiliary Task’ (directly after the a proceptual reflection of the strategy). The teacher – the main research-subject in this paper ($n = 6$) – , was male, 35 years old and had five years of teaching experience. He has the same multilingual background as the students (German-Turkish). After the video take, the teacher watched the videographed scene and stopped at scenes he wished to comment. The question he was asked was “Were there scenes where you felt that the students used language in a way that it diverged from your expectations? How and why did you react as you did?”.

Research questions for this paper. Since the research interest of this study was to reconstruct ‘centrifugal’ and ‘centripetal forces’ with regard to the teachers’ perception of the learners’ language production, the research question is two-folded:

Q1: How does the teacher perceive the language production process of the learners?

Q2: What might be possible reasons and effects of an intervention or passing over of the teacher, if he/ she perceives a for him marked language usage?

Methods for qualitative data analysis. The transcripts were successively analysed by conducting a so-called interpretative ‘interaction analysis’ (see Schütte, Friesen & Jung 2019) of the transcript sequences in between design-cycles (since the broader study was part of design-based research). In a turn-by-turn approach, possible interpretations, understood as hypotheses, were formulated with regard to the research interest (the reconstruction of centrifugal and centripetal forces). These turn-by-turn analyses were conducted with an interpretational ‘awareness’, meaning a careful formulation of hypotheses as assumptions. Later turns are not included into the interpretation of a prior turn, but an evaluation of emerged hypotheses is made with every turn so that ‘harder’ hypotheses are marked as ‘explaining hypotheses’ after some turns (see Kuzu 2022)

EMPIRICAL INSIGHTS INTO THE INTERACTION FORCES

In this short sequence, S6 is the ‘moderator’ of the video and S7 is the ‘cameraman’. Later, the roles are changed. It takes place between take one (being recorded in German priorly) and take two, which starts in turn 80 of this scene. In this scene, the students ‘collect’ the language means in Turkish they think they need for the explanation video. On the table, right before the learners, the poster with the task 156 – 28 is visible from the first video take, with a +2 as rounding-up number being written over the number 28 to round it up to 30 (thus, the ‘Auxiliary Task’ the students generate is $156 - 30 = 126$). The task is visible in numerical form and with cardinal manipulatives, which are

placed right under the task and consist of ones-squares, tens-lines and hundreds-squares. The German utterance parts are directly translated and underlined, the Turkish parts are visible in original as well as in transcribed form (in parentheses and in grey).

- 60 S6 Well maybe I'll write down the numbers and just read them.
- 61 Tea Okay, you mean in Turkish, right? Then you have them directly. Yüz elli altı (hundert fifty six).
- 62 S6 [*puts the pen on his document with the language means*] How is it written?
- 63 Tea Yüz (hundred), zzz [*emphasizes the z*] – well you can write it as you hear it. Yüz elli altı (hundred fifty six) [*S6 writes it down and repeats it silently for himself*] – altı (six). Well, yüz elli altı (hundred fifty six), buda yirmi sekiz (and this is twenty eight) [*again, S6 writes it down; the teacher points at the 28 on the poster*]. Nice. İki (Two) [*now the teacher points at the 2 on the poster*], sonra da yüz yirmi altı (, and after that hundred twenty six). Yüz yirmi altı (hundred twenty six) [*emphasized*]. Then we can – oh, the cameraman is already here [*to S7, jovially*] he is at the video take, nice.
- 64 S6 Altı. (Six.)
- 65 Tea Mhm. Yüz yirmi-? (hundred twenty -?)
- 66 S6 And this is yüz yirmi sekiz? (hundred twenty eight?)
- 67 Tea Precisely, look, you can do it! Bence çok güzel Türkçe konuşuyorsun bu arada (By the way, I think that you can speak Turkish nicely), don't let yourself tell from nobody that you cannot speak Turkish well [*S6 smirks*]. S6 you too, right. Süpersiniz (You are great) [*walks away*].
- [*from turn 68- 78, further language means are collected (minus, plus, equal and thirty)*]
- 79 Tea Okay. Three, two, one and go- [*take two of the video in Turkish starts*]
- 80 S6 Hoşbulduk bize(m) matematik video (Welcome to our(s) mathematics video), bugün göstereceğiz nasıl (today we will show how the) Trick, bu (this here) our task [*points at the poster*], biz başlıyoruz yüz elli altı äh aksı(ğ), äh eksik yirmi sekiz (we now start hundred fifty six eh minus/ miss(ing), eh missing twenty eight), eh, ama bu çok ağır (but this is too difficult), şimdi biz yirmi sekize iki artı yapıyoruz (now we make plus two to twenty eight), ehm, otuz oluyo (makes thirty), şimdi biz yüz elli- yüz elli altı, ähm aksı(ğ) otuz yapıyoruz (now we make hundred fifty- hundred fifty six, ehm, minus/ miss(ing) thirty) [*puts away three tens-lines*] bu yüz yirmi altı yapıyo (this makes hundred twenty six) [*now points at the other enactives being left on the poster*] but bu yanlış (this is wrong), because we have to iki artı yapcaz (make/ add plus two), bu (this) äh, yüz yirmi sekiz yapıyo (makes hundred twenty eight). Teşekkür- küm, teşekkür- (Thank- Thank, thanks-) yeah.

Turn 60 start with the idea of S6 to write down the Turkish language means. The idea to record a Turkish video was also his idea. Thus, it is not forced by the teacher. S6 plans to write down the language means and wants to 'just read them' when the video take starts (which seems a bit 'naive' since that is not an easy task, even for adults; but S6 seems confident). In turn 61, the teacher reinforces S6's idea at first ("then you have them directly") and starts to give the first number, presumably since S6 asked for numbers explicitly ("Yüz elli altı"). S6 then asks how it is written in turn 62, which indicates that he might not be proficient in Turkish as written language. Both students had

no Turkish lessons and at the beginning of the intervention, they were highly surprised that it was not ‘forbidden’ since *“their teachers always said that speaking Turkish is forbidden in school”* (a ‘markedness’ which occurs frequently in German schools). Furthermore, the students were shy at the beginning of the intervention and did not dare to use Turkish, and they repeatedly said *“that they cannot speak Turkish well, it is not nice, it is ugly how they speak”*. Here, the teacher wrote down a first memo:

Memo 1: *“At first I thought S6 might be highly proficient in Turkish, but here I realized that he may not be as proficient as it seemed at first, at least in written Turkish. Honestly, I was curious at this point how he would write it because, you know, Turkish letters are highly different from German letters. Perhaps he had learned it at home?”*

The teacher thus identifies that the student might not be as proficient in written Turkish as it seemed at first, but he also speculates if S6 probably had Turkish lessons at home. The emphasis on the written language in the ‘right’ form, with Turkish letters, shows a typical ‘centripetal force’: The expectation of language means in its ‘right’ forms.

Turn 63 then goes on with a repeated language mean offering, but with a specific emphasis: on the “z” of “yüz” (hundred). This might be because the phoneme “z” is written as “z” in Turkish and as a “s” in German (which might be because his assumption that the student might not know the Turkish letters). But then, immediately after emphasizing the phoneme, the teacher says “well, you can write it as you hear it” (see turn 63). Here, the teacher wrote down a second memo:

Memo 2: *“I emphasized the z because I wanted the student to write it down correctly, but then I realized how he looked a bit irritated and how the focus shifted to, well, ‘writing correctly’, and in the end, if he wanted to read them, I thought, it might irritate him and that was not what I wanted. I wanted him to be able to use the language means and if writing them in his own words was faster, he might be able to use them better.”*

This memo is highly interesting. It shows a reflection process of the teacher, a moment of self-control, with regard to his prior expectations. He realizes that a shift into an unnecessary language discourse occurs, at least at this point and for a verbal explanation, and thus he offers and accepts the possibility that the language means might not be ‘clean’. Here, the ‘centripetal forces’ seem to become more dominant, ‘heteroglossia’ seems to emerge since the the discourse is opened up for individual forms of language. Turn 63 then goes on as indicated in the memo: The teacher only repeats the language mean (“Yüz elli altı”) multiple times, but does not emphasize specific phonemes anymore. He gives the language means for the numbers 28 (the subtrahend), 2 (the rounding-up number) and 126 (the interim result). All of these numbers are visible on the poster. A ‘teichoscopia’ of the words S6 noted down, in Turn 63 still being gathered, shows that he indeed wrote all of the Turkish numbers with German letters.

In turn 64 then, S6 reads loudly “six” in Turkish – it is not clear if that was a question or a voicing for himself (later turns rather confirm the latter: S6 often speaks the words silently ‘into the room’) – and in turn 65, the teacher asks back “hundred tewnty-?” in Turkish, probably because he cannot categorize S6s utterance in turn 64. In turn 66

then, interestingly, S6 constructs a new number, which the teacher did not prompt before: The number “yüz yirmi sekiz” (“hundred twenty eight”). That is the result of the Task, which is determined by adding + 2 to the interim result again, thus by compensating the modification of the subtrahend. Here, something interesting happens with regard to the interaction: the ‘initiative’ switches, the teachers does not give the needed words anymore but S6 chooses to construct the necessary language mean *himself*. It is a moment, where S6’ *agency*, his capacity to make choices, becomes visible, and one might assume that this is partly because of the gradual shift of the interaction forces from more ‘centripetal forces’ when highly insisting on right ‘forms’ (see beginning of turn 63) to more ‘centrifugal forces’ when opening the process up for S6’, ‘heteroglossia’, thus opening the language form possibilities with regard to the diverse capabilities of S6. The teacher, seemingly content with S6’ self-directed process of constructing necessary language means in Turkish, reinforces S6 in turn 67 by praising him for being able to do it and by stating that the students “can speak Turkish nicely” (see turn 67). Here, the teacher wrote down a third memo:

Memo 3: “I saw that he wrote the Turkish words with German letters, but that was okay – I thought that it was more important to give him the opportunity to formulate a multilingual video with his language means, which might have been risked when I intervened too much and gave him f.e. the feeling of being a ‘bad’ speaker. I instead praised him since I remembered how he thought that he spoke ‘bad’ Turkish etc. I experienced that too: Often, teachers forbid me to speak in ‘Turkish’ and elders told me to speak ‘nice’ Turkish and not ‘ugly’ or ‘mixed’ language, and that is a very depressing feeling, to be rebuked from both sides.”

In this third memo, the teacher admits that he can anticipate how the students might feel when on the one side, teachers forbid the language usage and elders, on the other side, insist on speaking ‘clean’ Turkish, which he describes with being “rebuked from both sides” (see memo 3). Here, again more ‘centrifugal’ forces become visible: The teacher motivates S6 to speak Turkish – in non-mixed or mixed forms, which he demonstrates in his own utterances – and not to be obstructed by the ‘language form’ (which he does implicitly by not sanctioning the student for his Turkish words with German letters). This is interesting from a sociological perspective: In the sense of the standpoint-theory (see Bourdieu 2000), a presumably *shared experience* makes it possible to emphasize students’ possible feelings and obstructions, leading to a specific form of discourse-opening for diverse forms of language repertoires of the learners – with the aim of using mathematical terms for a mathematical explanation rather than writing down ‘clean’ language means – and thus highly ‘centrifugal forces’ emerge.

From turn 68-74, further language means are collected in a similar pattern as before (S6 asks for the words “minus”, “plus”, “equal” and “thirty” and the teacher answers) The sequence then goes on in turn 80 with the video take, which leads to a viable, mainly Turkish, but also mixed German-Turkish explanation of the ‘Auxiliary Task’ (see Kuzu 2022). In turn 80, the teacher makes a last note:

Memo 4: *“I was suprised and proud at the same time that S6 could [...] use both of his languages, especially Turkish – and I do not think that he read all of the words in the fast-paced video take. When he spoke of “aksı(ğ)”, I realized that he used a dialect and I remembered that the student came from the aegaen region of Turkey and that he had a sliightly aegaen dialect. I found it really nice [...] and did not sanction it.”*

Here, again ‘centrifugal forces’ become more dominant: the possibility to use dialects – beside of Turkish words with German letters – without sanctioning it. As a result, S6 can explain the ‘Auxiliary Task’ in turn 80 in a viable way and with his own language means, including ‘his’ words and spoken with his dialect. He seems to be *empowered* by the openness of the discourse – being influenced by the teachers decisions to skip sanctions and to reinforce the students’ own language production – and thus is able to use all of his multilingual resources, even though he did not think that he could do that at the beginning. At this point, it is a more procedural explanation, but later sequences also show proceptual explanations in both languages (see Kuzu 2022). This sequence, however, has to be regarded as the ‘ice breaking’ moment with regard to interaction forces and for a heteroglossic activation of multilingualism in mathematics education (see Barwell 2018), giving insights into important implementation conditions.

DISCUSSION OF RESULTS AND LIMITATIONS

With regard to the research question Q1, the analysis presented in this paper gave insights into a process, which normally would be implicit: A teachers' perception of learners’ language production process. These insights are important because they show – with regard to Q2 – how specific shifts from ‘centripetal forces’ to ‘centrifugal forces’ by opening the discourse for diverse forms of language means and multilingual students’ individual repertoires might empower them to do something they might not believe to be able to do: For example, to produce an explanation video about the ‘Auxiliary Task’ in a heritage language – in this case Turkish – in combination with the school language – in this case German, thus combining all ‘sources of meaning’ (see Barwell 2018). Yet, these insights go beyond this specific multilingualism context since ‘centrifugal forces’ and ‘centripetal forces’ are universal interaction forces influencing every language production interaction, especially in institutions like schools (see Barwell 2016). An important factor in this shift of interaction forces was the ‘self-control’ of the teacher with regard to an upcoming urge to intervene and correct language forms directly according to a ‘clean’ form. His empathy-based self-restraint, being especially visible in memo 3 and 4, or rather his intention to support the students’ usage of own language means – in *their* preferred forms – , thus with regard to the requirements of the communicative situation, seems to have lead to more ‘centrifugal forces’ by focussing on content and learners agency rather than form. The standpoint-theory seems to be an important aspect, but this does not mean, however, that only multilinguals do understand multilinguals, that would be a false inference. Yet, it shows how important shared experiences can be and herein lies a potential, for example for PD programs: Trainers could use methods to allow participants to understand the effects of contra-productive ‘centripetal forces’ by using role-playing methods, where

it is for example forbidden to think and articulate mathematical ideas in a preferred language. Limitations arise with regard to missing insights into students' perceptions and furthermore, the sample group is too small to formulate more consolidated hypotheses, but still, the detailed analyses show that the teacher behaviour in multilingual discourses has to be reflected also.

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DRAWING ON CULTURAL STRENGTHS FOR COLLECTIVE COLLABORATION

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Negotiating mathematical reasoning through small-group collaboration has potential to provide access and opportunity for all students to experience success in learning mathematics. Here, we demonstrate how Pāsifika students draw on core-cultural values of collectivism and communitarianism while working together on mathematical activity. We present data highlighting the students' perspectives of how their ways of knowing and being shape their mathematical interactions. Results indicate that when mathematics classrooms are constructed in ways that recognise and support all students to draw on the strengths they bring, they can successfully engage in productive mathematical reasoning that leads to a deeper conceptual understanding of mathematics.

INTRODUCTION AND BACKGROUND

Collective collaboration in mathematical reasoning and dialogue holds possibilities for *all* students to have opportunities to engage meaningfully in sensemaking and learning mathematics. Nevertheless, how teachers are to enact mathematics classrooms in which *all* students are gaining such opportunities is challenging. Many researchers (e.g., Cobb et al., 1991; Fernández et al., 2001; Gorgorió & Planas, 2005; Yackel et al., 1991) argue that if students are to learn mathematics in this way, classroom communication and participation patterns need to be specifically constructed. Over many decades, research has shown the effectiveness of teachers explicitly establishing classroom social and sociomathematical norms which promote deep student reasoning (Hunter, 2022). However, as Hunter argues this research has generally focused on supporting students who are from dominant cultural groups rather than being inclusive of *all* students. Less common is consideration given to how non-dominant students can be supported—not to learn how to communicate and participate in the same manner as their peers from dominant cultural groups—but rather, through drawing on strengths they bring into the mathematical setting as part of their social and cultural identity constructed in their home and community (Hunter, 2022). Our aim in this paper is to explore from the perspective of a small group of non-dominant learners, specifically Pāsifika students, how they engaged in collective mathematical dialogue. The question we explored was: What cultural factors shape how Pāsifika students successfully engage in collective mathematical reasoning?

THEORETICAL PERSPECTIVE

This research is situated in a sociocultural framework. Sociocultural theories of learning emphasise the dynamic nature of learning, with key components highlighting how students learn through interaction with others and their setting. Sociocultural

theories identify culture as a system of meaning making that is continually being formed and re-formed in local settings, activity, or cultural practices (Lave & Wenger, 1991). Within mathematics classrooms, a sociocultural framing is characterised by students working collectively on mathematical activity and engaging in mathematical discourse to reason about important concepts (Boaler & Sengupta-Irving, 2016). This view of activity emphasises that how students participate in settings is shaped by how they view themselves as learners and is largely influenced by opportunities provided for them in settings such as schools (Hunter & Civil, 2021). Other studies (e.g., Cobb et al., 1991; Hunter & Hunter, 2018; Yackel, 1995) highlight that when students learn mathematics through collaborative sense-making, they can be supported to develop positive mathematical dispositions, and intellectual autonomy.

Developing student collaboration

Effective student collaboration on mathematical activity relies on the establishment of positive social norms that guide students to interact in mutually respectful ways (Cobb et al., 1991; Gorgorió & Planas, 2005). When social norms are utilised to support students to collectively deepen their mathematical understanding and engage in learning mathematics, they become sociomathematical in nature (Yackel & Cobb, 1996). Sociomathematical norms form the foundation for how students can make sense of mathematical concepts while working collaboratively with others. These norms are developed and maintained when students' attention is focused on the mathematics and negotiation of mathematical meaning as they engage in mathematical activity (Yackel, 1995). Yackel et al.'s (1991) study involved 20 students 7 years old. It examined how classroom norms were developed and the multiple opportunities for learning that arose during student collaboration. During mathematics lessons, the students were expected to actively participate in collective sense-making and mathematical reasoning as opposed to expecting a more knowledgeable peer or the teacher to explain the mathematics to them. In this way, notions of any one individual holding the mathematical knowledge shifted to the collective mathematical strength of the group. Shared cognition was a gateway for the students to develop conceptual understanding of mathematics.

Developing conceptual understanding requires students to seek and comprehend mathematical relationships, reconstruct prior understanding, and connect to new ideas (Hunter & Civil, 2021). Furthermore, when given the opportunity to reason collectively, students can make and justify mathematical claims, and attempt to prove these. To illustrate these ideas, Mueller (2009) conducted a study involving twenty-four diverse students of African American and Latinx backgrounds, exploring how mathematical reasoning emerges. The students worked collaboratively in groups of four to solve open-ended mathematics problems. They co-constructed mathematical solutions and justifications and then explained these to the larger group. What Mueller (2009) noticed over time was that at first, many students merely repeated what others had stated. However, through consistent teacher expectations for deeper mathematical understanding, the students began to listen carefully and consider each other's

mathematical ideas and eventually engage in collective mathematical argumentation and justification.

Pāsifika learners in the New Zealand context

Collectivism and communitarianism in the mathematics classroom context refers to ways of working together in mutually respectful ways. New Zealand is home to the indigenous Māori who are closely connected to their first cousins, Pacific nations peoples. Pāsifika is a homogenous label for peoples belonging to multi-ethnic groups with heritages deeply rooted in their island nations but more of whom currently live in New Zealand. Both Māori and Pāsifika peoples share many common values, including reciprocity, respect, service, inclusion, relationships, spirituality, leadership, love and belonging (Ministry for Pacific Peoples, 2018). Above all, family, communitarianism, and collective responsibility are central to daily life for both Māori and Pāsifika peoples.

Pāsifika learners as non-dominant students in the New Zealand school setting have a long history of failure. Hunter and Hunter (2018) suggest that this can be directly attributed to expectations that these learners adopt the ways of knowing and being of dominant cultural groups, rather than drawing on their own strengths that they bring to the school setting. Through their lengthy and extensive research, these researchers illustrated how teachers could support Māori and Pāsifika students to learn mathematics by engaging in collective mathematical discourse. They showed how teachers modified their mathematics learning environments to respectfully draw on the core cultural values of Māori and Pāsifika students, including reciprocity, respect, service, inclusion, family, relationships, collectivism, and belonging. In collaboration with international colleagues, they drew parallels with non-dominant Hispanic students in the U.S.A. In this research Civil (Civil & Hunter, 2015; Hunter & Civil, 2021) worked with mathematics teachers to develop collaborative student discourse. Civil persistently drew on these students' core cultural values of 'familia' (family), 'confianza' (mutual trust), and relationships. In doing so, she forged strong relationships with these students and their families both in and outside of the school classroom. Like the New Zealand findings, developing these relationships gave the students a sense of belonging and inclusion in mathematics learning resulting in moments of "rich mathematical discussion and argumentation" (Civil & Hunter, 2015, p. 304). However, these studies are all focused on the teacher actions to draw on the non-dominant strengths. In this paper we aim to add to the literature, but from the perspective of the Pāsifika students of how their ways of knowing and being shape their mathematical interactions.

RESEARCH DESIGN

This research reports on a case study of one classroom at a high poverty, urban school in New Zealand, and was nested in a larger study focused on how teachers can teach mathematics reflecting the aims of equity in mathematics education. Mathematics lessons were characterised by students working in collaborative groups, on challenging

contextualised mathematical tasks. The students were grouped heterogeneously from a strength-base. The participants reported on in this paper were four students of Pāsifika descent aged 9-11 years old who participated in two open-ended group interviews. They were selected after extensive interrogation of the group interview data across the class set. Their responses were representative of the wider set of students.

Open ended semi-structured interviews were conducted with all groups to explore their perspectives of their engagement in mathematical dialogue. Utilising this kind of interview structure facilitated students to freely describe their experiences of working in small mathematics groups and positioned them as experiential experts in classroom episodes (Smith & Osborn, 2015). Interviews were video-recorded and wholly transcribed. A grounded approach was utilised to analyse the data. Codes, categories, and themes were established through iterative and both individual and collegial analysis. For reliability of the findings, all members of the research team coded each of the interview data independently and then crosschecked for analysis until common themes were identified which were representative of the whole set.

The findings are presented in the following sections. They aim to explore how collective reasoning embedded within cultural ways of knowing and being support the development of rich mathematical understandings.

RESULTS AND DISCUSSION

Reform classrooms place importance on the co-construction of mathematical reasoning through shared dialogue. Understanding that mathematical meaning can be constructed through dialogue is an important motivational tool for students to engage productively in mathematics talk. From the responses it was evident that these students readily acknowledged collective dialogue as their main source of constructing learning but working collectively in mathematical ways was positioned by the students within their own identity as Pāsifika learners. When questioned about how they perceived group work, they took this as common practice within their cultural ways of being and as one student stated: “we are good at working together”. Their further responses indicated deep understanding of the collective strength of what a group brought to mathematical activity, in contrast to what individuals could achieve alone. For example, one student stated: “I work with a buddy, taking turns of our ideas. When it’s a hard problem there’s only one of me and I might not be able to solve it”. Another described how they all supported each other and ended with: “if you are just by yourself...you will be struggling”. This illustrates what Hunter and Civil (2021) outlined as ways of learning that many non-dominant students, including those of Pāsifika ethnicity use in their home settings. In the home setting, these Pāsifika students think, work, and learn together almost as a single unit and often almost silently. It was clear that in this classroom, these students felt empowered to draw on their own home values and ways of being to work as one to progress their shared mathematical goals.

Pāsifika ways of knowing and being underpinned these student’s perceptions of how they worked collectively. They described how core Pāsifika values were drawn on and

supported their interactions, as evidenced by a student who described the need for: “respecting all by taking turns in our group to share our ideas”. Respect as a core Pāsifika value was the basis of how they described all their mathematical interactions, and this was extended across the other Pāsifika values. For example, when describing who had strengths in mathematics, one of the students identified another student based on her ability to work in respectful and reciprocal ways to ensure all learned, describing: “she always wants to like, can I help with that, and she likes to help with things that makes it logical, and she makes stuff easy...”. Other descriptions implicitly described values embedded in reciprocity, service, and inclusion. It was evident that the cultural aspect of being respectful, keeping others safe and happy (relationship first) “tauhi va” and “ko ho’o me’a ko ‘eku me’a”- what is yours is mine and what is mine is yours (sharing with your friends and living with family members) were their lived values in their mathematical interactions.

Relationships embedded within notions of family and working together as one in ways that were inclusive and supportive were key elements described by the students as productive ways of doing mathematics. As one student said: “I felt really good working in a team...and when Amanaki came in she made it kind of logical and there were all these ideas popping up, it was really good as we were working together”. Others described how they learned through listening to other’s explanations: “by listening to their (mathematical) reasoning”. Another student extended the response to acknowledge how they ensured that their collective dialogue provided space for reconstruction of their mathematical reasoning: “Like, give everyone a chance to change their thinking and actively listening...or I agree with you, or I disagree with you...helpfully”. Cobb and colleagues (1995) noted need for students to learn to work in mutually respectful ways and this was what these students described but they extended our understandings into their own cultural world.

Clearly evident throughout the data was the way in which these Pāsifika students did not view themselves as individuals but rather as a collective—as a family—both responsive to and responsible for each other’s learning. This was illustrated when students were asked whether they considered themselves good at mathematics. One student was asked directly and rather than responding she dropped her head to her chest and remained silent. Other students smiled and then explained their reaction: “We are smiling because we know Hiva is good at maths, whenever she comes up with ideas...and like she can explain, she doesn’t, she’s not shy and keeps it in”. Of significance was the initial response of Hiva when asked if she thought she was good at mathematics and the way in which the other students responded. Within Pāsifika culture, stating one’s own importance is not acceptable, however, acknowledging strengths of others conforms to value placed on family, being of service, and belonging. It was evident that these students were prepared to describe the positive participatory actions others contributed and, if describing actions they took, it was in terms of promoting collective reasoning. For example, one student described: “If someone is stuck, I just help them. But not like give them the answer...help them think, like give

them the main points, or we can just re-read the question, we are all good at mathematics because we are equal”. The use of the word ‘we’ is important here and shows how they considered themselves as one interlinked family unit rather than individual parts of it.

Importantly, the collective activity supported them to negotiate mathematical understandings through use of sociomathematical norms. They all described shared activity including asking questions about the mathematics, sharing their thinking about the solution or explanation, explaining things to help everyone understand, giving everyone a chance to change their thinking, and asking others to repeat something again, and agreeing or disagreeing with other’s solutions. For these students it was evident that engaging in mathematical argumentation conflicted with their own cultural norms related to respect and maintaining other’s mana (pride), but they had co-constructed ways to overcome their reticence. This is illustrated by this exchange with the researcher: “When I work in a group sharing strategies with the group and explaining things sometimes it gets a bit difficult because we disagree with one problem, and we agree with the other”. When prompted by the researcher to explain what happened when the group disagreed the student said: “We have little fights, little mathematical fights about which maths is right. Like if someone doesn’t know one half is bigger than one fifth, they say like explain it...and the other person will say, it is right because one half is bigger than one fifth, look. Then they will show it, like on paper or something. Then we will think and oh yes, we agree, or no we don’t agree with that”. Hunter and Hunter (2018) explained how many Pāsifika students find engaging in mathematical argumentation difficult. However, the way in which these students described their activity they had shifted focus from the individual defending their reasoning to the use of mathematics as the vehicle of justifying and proving collective reasoning. This parallels what Civil (Civil & Hunter, 2015) described in her work with Hispanic students from similar non-dominant communities. A sense of family and mutual trust supported these students to engage in reasoned mathematical dialogue which included mathematical argumentation. These findings also reflect those of Mueller (2009) study where students were provided with multiple opportunities to work together on mathematical activity and in doing so, constructed important ways of learning rich and deeply connected mathematics.

CONCLUSION AND IMPLICATIONS

Of significance in these findings is the acknowledgement of the strength of the collaborative process of working on mathematical activity. The results demonstrate that while the students acknowledged individual strengths of their peers, they recognised that collaborative activity could support mathematical understanding by allowing them to re-examine their individual ideas and build new or additional forms of reasoning from the ideas of others. As other researchers (e.g., Cobb et al., 1991; Yackel et al., 1991) show, these students did not rely on a “more-knowledgeable” other to explain or show the mathematics to them, rather they recognised that working together provided access to learning mathematics. They did not express any reliance

on teacher expertise, nor on any one student. Rather, they recognised each other as essential components in engaging in mathematical activity and co-constructing shared reasoning. Within Pāsifika culture, participating and contributing actively to achieve a common goal is key to a community and these students readily applied their ways of knowing and being to the classroom mathematical setting.

Collectivism and communitarianism are important and accepted ways of working and learning together in the daily home setting for many Pāsifika students. As Hunter and Hunter's (2018) study demonstrated, when the experiences and core cultural beliefs and values of diverse students are drawn from, students can be supported to engage in productive mathematical dialogue. Furthermore, Civil and Hunter's (2015) study emphasised that when the students' idea of family, mutual trust, and relationships were firmly established, access to deep mathematical reasoning occurred. In this study it was evident that working together in mathematical groups created opportunities for these students to work as a community, which ultimately placed their identity at the heart of learning mathematics. As Civil and Hunter (2021) argue, these ways of learning together almost as one for many non-dominant students do not need to be constructed in artificial ways in classrooms because they already act in this way in their families and communities. Rather, teachers need to construct mathematical learning spaces which recognise and support diverse ways of knowing and being so that all students can draw on the strengths they bring. When this happens and such learning environments are created, equity in mathematics education can be fulfilled.

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BUILDING BRIDGES: THE IMPORTANCE OF CONTINUOUS MAGNITUDES IN EARLY MATHEMATICS EDUCATION FROM TWO PERSPECTIVES.

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The importance of mathematical competencies in the western world is well known, both at the personal and the national level. Accordingly, many studies are dedicated to learning about the best conditions to learn and teach mathematics. These efforts appear to run almost in parallel in two fields: mathematics education and cognitive psychology. In this RR, I will bring one example, regarding the role of continuous magnitudes (e.g., area, surface, volume, density, etc) in learning the concept of (natural) numbers. I will highlight the similarities in the findings from the two fields, and how they complement each other. I hope to start a discussion about how to better connect the two fields, spark new questions and find new insights to pave the way to revolutionize how we think about early mathematics education in general.

THE BUILDING BLOCKS OF MATHEMATICS – DISCRETE NUMBERS OR CONTINUOUS MAGNITUDES?

Many studies, spanning decades, investigate how young children learn mathematics. The idea is that knowing how we acquire basic arithmetic skills, will allow us to teach mathematics more efficiently, more like the way we learn mathematics ‘naturally’.

In mathematics education, some suggested that people are ‘born mathematical’ (e.g., Pound, 2006) and that ‘the development of mathematical competencies begins at birth’ (Anthony & Walshaw, 2009, p. 1). The evidence came from behaviours such as detecting patterns, evaluating distance while trying to reach for objects, etc. In cognitive psychology, one of the dominant theories went further and suggested that humans are born with a dedicated system for ‘sensing’ the approximate number of items. According to the Approximate Number System (ANS) theory (e.g., Cantlon et al., 2009; Dehaene, 1997), we are born with a dedicated mechanism allowing us to approximate and discriminate the number of large quantities. Some suggested for example that we can sense ‘sixness’ similarly to the way we sense ‘redness’ (e.g. Burr & Ross, 2008). This system at first allows us to make only crude discriminations and refined with age and experience, and influence further math abilities (Halberda & Feigenson, 2008).

Evidence for this theory comes from behavioural, neuroimaging, and neurophysiological studies. Such studies usually tested the ability of participants to compare the quantity of two groups of items, when the number of the items becomes closer and closer. For example, Xu and Spelke (2000) used the habituation paradigm

to test whether six-months old infants can discriminate between a different number of items. In a habituation paradigm, infants are repeatedly exposed to the same stimuli, in that case, the same number of dots. When the stimuli are no longer novel, the infant is no longer interested and stops looking at the screen. Then, the number of dots changes. If this change was registered, and for the infant, this is a novel stimulus, she will look at the screen again. The infants in the study were able to detect a difference of 8 vs 16 but not 8 vs 12 dots.

The ANS theory influences early mathematics curriculum. One example of such influence is via the triple code model of Dehaene et al (2003). This model assumes that the meaning of the number comes only from non-symbolic quantities (Leibovich-Raveh & Greg, 2019). The ANS theory also influenced the way mathematics learning disabilities (i.e., dyscalculia) are perceived and diagnosed. Some view dyscalculia, or at least sub-types of dyscalculia as a specific problem with the number sense (e.g., Wilson & Dehaene, 2007).

However, not everyone agrees with the ANS theory. Already in 2002 (Mix et al., 2002) and maybe even earlier, some doubts started to arise regarding the interpretation of evidence coming from studies supporting the ANS theory. The main concern was that when comparing two groups of items, the number is not the only thing that differs: the total area of the items, their density, their volume, etc., all change too. I will refer to these as continuous magnitudes, as opposed to discrete numbers. It is impossible to create two sets with a different number of items and identical continuous magnitudes (for a review see Leibovich et al., 2017). Therefore, the question is: what do we process when we process such sets? How do we extract the number of items?

From such questions, my colleagues and I devised a theoretical model called the Approximate Magnitude System (AMS) theory (Leibovich et al., 2017). The main difference between the ANS and the AMS theories is that in the AMS theory, continuous magnitudes are more basic and automatic than the perception of numbers (see also Gebuis et al., 2016). The AMS model assumes that we are born with the ability to compare and manipulate continuous magnitudes. Then, with the development of language, and experience in our environment, we start to notice the correlation between continuous magnitudes and discrete numbers, e.g., usually, when the toy box is fuller, it has more toys than when it's less full. Only later, when cognitive abilities such as the ability to ignore irrelevant information are developed, a child can understand the exceptions to the rule: e.g., maybe there are only two toys in the toy box, but they are big and take lots of space. At that point when the correlation can be used but also ignored, we can say that the child understands the discrete quality of a natural number.

There is empirical evidence for parts of the AMS theory. For example, in a neuroimaging study, adult participants saw two groups of dots and chose the group containing more dots. The number of continuous magnitudes positively correlated with the number of dots ranged from 1-4 continuous magnitudes. The data revealed an area

in the brain that was more active when there were more continuous magnitudes that positively correlated with number. This area was previously found to integrate information from different sources (Leibovich & Ansari, 2017).

Another demonstration of the AMS is Piaget's number conservation task (Piaget, 1952). In this task, children see two rows of the same number of coins, evenly spaced at first, and asked whether they contain the same number of coins or not. Then, the experimenter space out the coins in one of the rows in front of the child and repeats the question. This manipulation usually makes young children (at about the age of 5) claim that the more spaced-out row has more coins. An fMRI study repeated a similar task with adults and demonstrated that although adults responded correctly, in the spaced-out condition, there was more activity in brain areas related to ignoring irrelevant information (Leroux et al., 2009). These results suggest that even adults are initially misled by continuous magnitudes but know to inhibit the irrelevant information and deliver the correct response.

Importantly, studies investigating the role of continuous magnitudes in processing groups of items and extracting numbers are usually done in highly controlled conditions that are far from the learning environment of children. It also rarely considers the way children manipulate the magnitudes around them. Therefore, in the next section, I will bring some examples from educational approaches, that, in my opinion, manifest the spirit of AMS from an educational perspective.

NUMBERS AND CONTINUOUS MAGNITUDES IN EARLY MATHEMATICS EDUCATION

To demonstrate the usefulness of combining knowledge from mathematics education and cognitive psychology in general, I chose to focus on the AMS theory and the teaching of measurement as a case study. For this purpose, I will use mainly the study of Mellone, Baccaglini-Frank and Martignone (2020).

This topic of measurements is of interest here because it goes to the question of how we teach numbers, and to what aim. Numbers are a tool for counting discrete countable objects, but they are also a tool for measurements. In western culture, in most cases, natural numbers are introduced to children as a tool for counting discrete objects before their role in measurement is introduced. For example, Sfard (1991) suggested that the mechanism of counting is the starting point for understanding the concept of a natural number.

In contrast is the point-of-view depicted by the El'konin-Davydov curriculum (Venenciano et al., 2021). The El'konin-Davydov curriculum emphasizes that the learner should develop a general theoretical understanding of a phenomenon before she can master specific cases of the phenomenon. And to develop such an understanding, a learning activity is required. Accordingly, this approach suggests that the child must develop an understanding of numbers by experiencing and measuring continuous magnitudes. From there the child will later develop an understanding of numbers as a tool of measurement.

In the study of Mellone et al (2020), the authors seek to learn how children at the beginning of first-grade deal with rice as a substrate that can be treated as both countable and continuous. For that aim, an interviewer poured about 200 grains of rice into a pile in front of the child and asked him/her to do the same. Then, they asked whether their pile has ‘as much rice as’ the other pile. After the initial response, the children were offered some tools to use: a spoon, clear plastic glasses, a ruler, etc. The aim was to see which strategies children will use to solve the task.

From a cognitive perspective, the ANS theory will predict that children will rely mostly on quantity and only a little on continuous magnitudes (depending on the ANS sub-theory). On the other hand, the AMS theory will predict relying mainly on continuous magnitudes under these conditions.

The results revealed that only 4/14 participants attempted to count the rice to complete the task. Out of them, only two were able to count technically (knew the number-words). The others relied on different continuous magnitudes, like participants in laboratory cognitive experiments. For example, one strategy was to shape the piles into two, two-dimensional shapes and compare their area. Another way was to grasp each pile in one fist and compare how dense it feels. Another was to compare the heights of the piles.

Using the tools, children used plastic cups to compare the height of the two piles. Some used a ruler, but as a divide between the piles and to compare the length but did not use the numbers on the ruler. When children used the word ‘measure’ they meant comparing.

The study of Mellone et al (2020) is a great example of the added value of qualitative research to quantitative research. First, it demonstrated the same phenomenon predicted by the AMS theories, but in a more natural setup and completely different methodology, offering converging evidence from two different fields toward the role continuous magnitudes play in comparing numbers. Second, it elaborated beyond what studies in cognitive psychology have done so far, by an analysis of the interviews under a specific framework (i.e., theory of semiotic mediation). This resulted in identifying potential pivot signs – gestures or words children use that have the potential to be used to introduce new mathematical concepts in an activity organized by the teacher.

A collaboration of cognitive research within the setup of the rice experiment will allow for example, to evaluate if factors such as the ability to ignore irrelevant information (i.e., inhibition), or how much information a child can hold in her mind at once to complete a task (i.e., working memory), are also a contributing factor to the strategy children chose. Another interesting avenue of research is to try and check for a relationship between future math achievements at school and the chosen strategy, or to try the same study with children (or adults) diagnosed with learning disabilities in mathematics. These require the collaboration of educators, math education and numerical cognition researchers.

COMBINING INSIGHTS FROM MATHEMATICS EDUCATION AND COGNITIVE PSYCHOLOGY – THE THE EL’KONIN-DAVYDOV CURRICULUM

Earlier I introduced one way that the ANS theory influences early mathematics curriculum. The AMS theory, and more generally, the approach that continuous magnitudes and not numbers are the building blocks of mathematics, have developed independently from educational theories. However, I believe that the two share some commonalities that can be the source of mutual benefit to mathematics education and cognitive psychology.

The El’konin-Davydov curriculum and the AMS share some similarities that are worth further exploration. For example, the importance of the child’s experience with the environment to develop their own general theories and understanding, corresponds to the AMS theory, suggesting that learning the correlation between discrete numbers and continuous magnitudes requires experiencing the environment. The current AMS theory is relevant to very young ages (from birth), and the El’konin-Davydov curriculum to older ages (from the age of about 4 years). Therefore, one interesting avenue of research is to see whether the AMS theory and the curriculum can be adjusted to discuss similar age groups. I will elaborate more about it in the last part.

Another example for a potential cooperation between cognitive psychology and mathematics education is a curriculum that is based on the El’konin-Davydov curriculum, is the ‘Measure Up!’ curriculum, designed for elementary grades (from age 4):

“Where most contemporary curricula assume number as the most basic concept, MU uses children’s everyday knowledge of measurement as the context for exploring and developing mathematics. Concepts such as unit, iteration, equality, commutativity, and transitivity are first introduced through work with non-numeric quantities of length, area, volume, and mass. This leads to work with the number line, regrouping, place value, and rational numbers” (<https://manoa.hawaii.edu/crdg>)

The El’konin-Davydov curriculum and the MU approach maybe even expand the AMS model. Recently, one study by Schenke et al (2020) reported that a computer game based on the ‘Measure Up!’ curriculum with 4-5 year-olds found gains for children playing with this particular game relative to a control group. Namely, this curriculum can work also as an app with the guidance of a parent, since, as the paper claims, many teachers do not feel comfortable teaching the topic of measurements.

A collaboration with a whole curriculum such as the ‘Measure Up!’ will allow us to get a more in-depth picture of other cognitive factors, that may be non-specific to mathematics that benefit from the curriculum and refine the MU activities even more or make them more ‘Taylor-made’ for every child.

LOOKING TO THE FUTURE

The curriculum that I discussed here, as well as the ones I found, focuses on ages 4 and above. This is despite the perception that is taken almost as an axiom that humans are hard-wired to deal with mathematics. Although some suggestions are referring to earlier ages (see Pound, 2006), to the best of my knowledge, there isn't any formal state-level curriculum for earlier ages. Even in places such as Israel, where a large percentage of infants are in daycare facilities from the age of 3-4 months, no such curriculum exists.

Therefore, my suggestion is to build on the El'konin-Davydov curriculum, and by an interdisciplinary effort and collaboration, to adapt it for younger ages from birth to the age of 3-4 years. These ages can be critical to the development of mathematical abilities. And yet, they are left to chance.

We know from prior studies that the home numeracy environment highly influences children's mathematics abilities (e.g., Mutaf-Yıldız et al., 2020). Creating a curriculum has the potential to benefit children that goes to daycares but also stay at home since the curriculum could be modified to fit the home environment and, as the Measure Up! curriculum demonstrated the elements of the curriculum can be used in an app or a computer game and be used as a source for parents.

Starting to develop mathematical abilities as early as possible also fulfills an important social role in closing gaps between children from different SES. Many studies highlighted the importance of early mathematics curriculum, and demonstrated the success of curriculum interventions at an early age, especially in closing the gaps between children coming from low SES families and those from medium and high SES (e.g., Starkey et al., 2022). By starting early, we can give every child a fair chance in school, and maybe even in life.

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ENHANCING STUDENTS' CONCEPTUAL KNOWLEDGE OF FRACTIONS THROUGH LANGUAGE-RESPONSIVE INSTRUCTION. A FIELD TRIAL

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Language-responsive instruction is thought to enhance mathematical learning, especially for students with low language proficiency. However, empirical evidence for the effectiveness of such kind of instruction in regular classrooms is scarce. We conducted an experimental intervention study with a pretest and a posttest in grade 7 (N = 212). Students were randomly assigned to one of three instructional groups: fraction instruction with or without additional linguistic support, or a waiting control group. Results showed that both intervention groups had higher learning gains than the control group. In addition, language proficiency, fraction pre-knowledge, basic arithmetic competence, general cognitive abilities, and mathematical anxiety were found to predict fraction learning.

THEORETICAL AND EMPIRICAL BACKGROUND

Research shows that students with limited language aptitude benefit less from mathematics instruction (Barwell et al., 2016) and that especially the development of conceptual knowledge is related to learners' language aptitude. In particular, previous research showed that students with lower language proficiency performed more poorly on tasks that required conceptual knowledge than on language-free computational tasks (Prediger et al., 2018). Given that mathematical discourse practices such as arguing and explaining meanings are fundamental for developing conceptual knowledge, this is not surprising: Learners with high language proficiency are able to verbalize their ideas and concepts more effectively to acquire conceptual knowledge. In contrast, learners with low language proficiency may lack the linguistic prerequisites to do so. Thus, instructional approaches are needed that aim to overcome language deficits especially with respect to develop conceptual knowledge (Erath et al., 2021).

Previous research on language-responsive learning in mathematics

Previous research on supporting students with low language proficiency in their mathematical learning has focused on the development and evaluation of language-responsive instructional approaches, both in qualitative and quantitative research. Based on the analysis of the mathematical and language learning processes in qualitative case studies design principles for designing language-responsive instructional approaches were derived (Erath et al., 2021). Complementing these qualitative analyses of mathematics and language learning processes, there are also studies under laboratory conditions that provide empirical evidence for the effectiveness of language-responsive instructional approaches. For example, Prediger

and Wessel (2013) conducted a quasi-experimental intervention study in a laboratory small-group setting in the context of remedial instruction. A language-responsive intervention program fostering students' conceptual fraction knowledge was developed for six lessons of 90 minutes in 2-to-1 sessions (two students with one teacher). The results of a pre-post intervention study demonstrated higher learning gains for the language-responsive instructional approach.

So far, there are only few studies that have investigated language-responsive learning approaches in regular classrooms. Therefore, investigating the effectiveness of language-responsive instructional approaches in differentiated teaching contexts, where students work with different learning materials within the same classroom seems to be an appropriate step toward ecological validity. Moreover, given the wide range of heterogeneity in regular classrooms (Nusser & Gehrler, 2020), such an approach allows to take into account different aspects of heterogeneity with regard to learning outcomes (e.g., different levels of mathematical competence, language proficiency, general cognitive ability, mathematics anxiety).

Fractions

Fraction knowledge is a predictor of academic, occupational, and financial success and is key for the acquisition of later mathematical skills (Siegler et al, 2012). Despite their importance, fractions are notoriously difficult (Siegler & Lortie-Forgues, 2015). Thus, there is a need to improve understanding of the predictors of fraction learning, with the goal of providing theoretical insight into and practical guidance for remedying these problems.

Research suggests that various individual factors predict the acquisition of fraction knowledge. In particular, previous studies have shown that fraction knowledge is predicted by a combination of domain-general competencies (e.g. general cognitive abilities, language proficiency) and specific numerical competencies (e.g. whole number computation) (Vukovic et al., 2014). In particular, mediation analyses showed that numerical competencies were direct predictors of fraction outcomes whereas domain-general competencies predict fraction knowledge directly and indirectly through related mathematical competencies (Hecht et al, 2003; Vukovic et al., 2014). Across several studies, language proficiency emerged as an important predictor of fraction knowledge (e. g. Hecht et al, 2003; Seethaler et al., 2011; Vukovic et al., 2014). Besides these cognitive factors, research also showed that fraction knowledge depends on affective individual characteristics such as mathematical anxiety (Starling-Alves et al., 2022).

Moreover, studies suggest that conceptual knowledge of fractions play a key role in overall fraction competence (Siegler & Pyke, 2013; Siegler & Lortie-Forgues, 2015). While procedural knowledge refers to the ability of performing fraction operations, conceptual knowledge comprises knowledge of concepts and principles, as well as knowledge of why a mathematical procedure works (Crooks & Alibali, 2014; Kilpatrick et al., 2001). Conceptual knowledge of fractions involves knowing special

characteristics of fractions (e.g., relation of numerator and denominator, density of fractions) and different interpretation of fractions (e.g. part of a whole, quotient, ratio) (Behr et al., 1983). If students integrate conceptual knowledge about these interpretations when solving calculations with fractions, it seems plausible that they may recognize incorrect procedures. Therefore, conceptual knowledge is thought to guide correct algorithmic procedures (Siegler & Pyke, 2013). For example, students interpreting fractions as parts of a whole may better understand that adding fractions on a symbolic level requires that the denominators are equal as only parts of the same size can be combined. Therefore, it seems to be essential to enhance students' conceptual fraction knowledge.

RESEARCH AIM

We evaluate the efficacy of a language-responsive instructional approach to conceptual knowledge of fractions with additional linguistic support in a pre-post-intervention study that is situated in regular classrooms. Furthermore, we are interested in how learning effects depend on individual learning prerequisites. In particular, we investigate the relations between individual variables (language proficiency, fraction pre-knowledge, basic arithmetic competence, general cognitive ability, mathematics anxiety) and fraction outcomes.

In this paper, the two research questions are:

(RQ1) Does an intervention on fractions with linguistic support lead to higher learning gains than an intervention without additional linguistic support?

(RQ2) Which individual variables predict learning of fractions?

RESEARCH DESIGN, SAMPLE AND METHODS

Research design. The data was collected in an experimental field trial with a pre-post design and an additional waiting control group. Students in each class were randomly assigned to one of two intervention groups. The waiting control group comprises two classes that participated in the intervention after the posttest. The intervention included a repetition unit on fractions which focuses specifically on conceptual knowledge. It comprised three lessons on (1) the concept of fractions, (2) the equivalence of fractions, (3) the addition and subtraction of fractions. The unit was implemented using one of two intervention materials: The first material (M1) contained additional linguistic support while the second intervention material (M2) did not contain any additional linguistic support.

Intervention material. The intervention materials were based on tasks which have been developed and evaluated in previous research (Prediger et al., 2014; Wessel et al., 2018). Both treatments contained the same tasks, thus all students were taught with the same learning objectives. To specifically support oral and written language production one intervention material (M1) contained additional linguistic support while the second intervention material (M2) did. In particular, M1 included formulation aids and phrases (e.g., sentence starters, word lists, sentence patterns) which were offered in the form of

sentence modules. These sentence modules make required vocabulary available for explaining explorations and provide sentence structures without impairing the conceptual explorations relevant to the acquisition of conceptual knowledge.

Sample. A total of $N = 212$ students in grade 7 participated in the study (M1: $n = 79$; M2: $n = 87$, control group: $n = 45$). These students were from 11 classes and either visited the middle or the lowest track of the three-tracked secondary school system in Germany.

Test instruments

- *Fraction knowledge* was assessed using a test instrument on fractions (Lenz et al., 2020). The test instrument had a satisfactory internal consistency, with Cronbach's $\alpha = .92$ for the pretest and $\alpha = .93$ for the posttest (50 items, $N = 212$).
- *German language proficiency* was assessed using a C-Test, offering economical and highly reliable measures, with Cronbach's $\alpha = .76$ ($N = 212$).
- *Basic arithmetic competence* was measured before the intervention by three subscales of the standardized test "HRT 1-4" (Haffner et al., 2005) with Cronbach's $\alpha = .80$ (3 scale, $N = 212$).
- *General cognitive abilities* was assessed using the "Figural Analogies" subscale of the "KFT 4-12 + R" (Heller & Perleth, 2000), with Cronbach's $\alpha = .88$ (25 items, $N = 212$).
- *Math anxiety* was measured using the items of the Programme for International Student Assessment (Mang et al., 2018), with Cronbach's $\alpha = .80$ (5 items, $N = 212$).
- *Age, gender, socioeconomic status, and multilingual background* were surveyed by a self-report questionnaire.

Data analysis. Repeated measures of analyses of variance (ANOVA) were used to test for the effects of the intervention. This allows comparisons of the significant differences in the mean scores between the intervention group with additional linguistic support, the intervention group without additional linguistic support and the waiting control group. Linear regression analysis was conducted to analyse potential predictor variables for learning outcomes in fraction knowledge.

RESULTS

At pretest there were no significant differences between the two intervention groups and the control group for fraction knowledge and the individual variables language proficiency, general cognitive abilities, basic arithmetic competence, and mathematical anxiety. There were also no significant differences in the distribution of gender, multilingual background, and socioeconomic status ($p > .05$ for all variables). Accordingly, we can assume that the groups were comparable before the intervention.

The first research question asks for differences in the learning outcomes between the three groups (M1, M2, control group). The results of the ANOVA show a main effect of time, suggesting that regardless of group, response accuracy increased significantly from pretest to posttest ($F_{\text{time}}(1, 211) = 168.97, p < .001, \eta^2 = .06$). No main effect of group was found ($F_{\text{group}}(2, 211) = 1,576, p = .209$). However, the interaction effect of time \times group was significant ($F_{\text{time} \times \text{group}}(2, 211) = 14.48, p < .001, \eta^2 = .01$) with a small effect, suggesting that the learning gains in fraction knowledge from pretest to posttest differed to some extent between the three groups. The descriptive data is presented in Table 1. As the mean differences show, both intervention groups achieved comparable learning gains, whereas there is no significant change in the mean difference from pretest to posttest in the control group. Thus, it seems that additional language support did not have a general effect on students' fraction learning. However, this effect may depend on students' individual variables.

Table 1: Learning gains in fraction knowledge

	Pretest <i>M</i> (<i>SD</i>)	Posttest <i>M</i> (<i>SD</i>)	mean differences
Intervention group M1	17.94 (9.91)	24.89 (11.28)	6.95***
Intervention group M2	15.37 (9.17)	22.22 (10.01)	6.85***
Control group	18.87 (8.97)	20.71 (9.40)	1.84

The second research question asks for predictors of learning outcomes in fraction knowledge. Table 2 presents the results of the linear regression analysis.

Table 2: Results of the regression analysis

	<i>B</i>	<i>Std. Errors</i>	<i>p</i>
Language proficiency	0.13	0.04	<.001
Fraction pre-knowledge	0.71	0.05	<.001
Basic arithmetic competence	0.09	0.03	.003
General cognitive abilities	0.17	0.06	.004
Mathematical anxiety	-1.15	0.57	0.045
Treatment group	-1.86	0.47	<.001

The R-squared value for the model was good 1 (adjusted R^2 of 0.77). The regression analysis shows the anticipated result that language proficiency, fraction pre-knowledge, basic arithmetic competence, and general cognitive abilities are significant predictors of fraction outcomes. Moreover, there was a main effect of the non-cognitive variable mathematical anxiety. When controlling for language proficiency, fraction pre-knowledge, basic arithmetic competence, mathematical anxiety and general cognitive abilities, group was a significant predictor.

DISCUSSION

In this study, we investigated the effectiveness of language-responsive instruction of fractions in an experimental design under field conditions.

The results of the ANOVA show significant learning gains in fraction knowledge in both intervention groups (with and without additional linguistic support) whereas no significant learning gains were found in the control group. This result confirms the quality of both intervention materials. That the effect sizes were small is not unexpected, given that (1) the intervention was a short-term intervention of three lessons, (2) the intervention was a repetition unit, hence the students had some pre-knowledge of fractions, and (3) the intervention was conducted under field conditions in regular classrooms, which is why the prior knowledge was heterogeneous unlike in special courses for students with little prior knowledge.

Moreover, the results provide further evidence for the effectiveness of language-responsive instruction. We found that a language-responsive instructional approach with additional linguistic support fostering students' fraction knowledge can be effectively implemented in regular mathematics classrooms. Furthermore, the research design used, extends previous research conducted under highly controlled laboratory conditions with specific subgroups of students (Prediger & Wessel, 2013). In this respect, the present study investigates language-responsive instructional approaches on fractions in a more ecologically valid setting than previous studies. Furthermore, our findings replicate the results of the field study by Prediger and Neugebauer (2021) showing the effectiveness of a language-responsive instructional approach to percentages, a closely related domain.

Our analysis of individual predictors provides insights into the variables that affect learning outcomes in fraction knowledge. We found that language proficiency, fraction pre-knowledge, basic arithmetic competence, general cognitive abilities, and mathematical anxiety predict learning of fractions. These results confirm previous research on predictors of fraction knowledge, which suggest that fraction learning depends on domain-general, specific numerical competencies as well as non-cognitive variables (e.g. Vukovic et al., 2014; Starling-Alves et al., 2022). A notable contribution of this study is the finding that language proficiency emerged as a unique predictor of fraction learning, even in a short term intervention and when controlling for general cognitive abilities and prior mathematical knowledge. This result emphasizes the need for language-responsive instructional approaches to support learners with low language proficiency.

Overall, the results of this study contribute to a deeper understanding of the sources of individual differences in fraction knowledge, which is key given the increasing heterogeneity of students in regular classrooms. Future research is needed to further specify the relations of these individual variables and learning fractions in order to derive corresponding instructional interventions, that take into account potentially different learning gains for learners with different learning abilities.

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Additional information

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ADULTS' AWARENESS OF CHILDREN'S ENGAGEMENT WITH GEOMETRICAL ACTIVITIES

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This study focuses on parents' awareness of geometrical notions that can be promoted among young children before entering first grade. Fifty-two adults responded to an open questionnaire asking parents to list geometrical concepts they believed could be developed during early childhood and to describe situations where children raised geometrical ideas. Findings indicated that two-dimensional shapes were the most frequently cited concepts. Playtime was the most frequent context and composition was the most frequent activity. Workshops for parents may build on studies such as this to increase parents' awareness to other geometrical concepts that can be promoted at a young age and to additional opportunities for developing geometrical knowledge.

INTRODUCTION

For the last 25 years, there has been growing emphasis on early childhood mathematics education and the promotion of children's numerical and geometrical knowledge (Lewis Presser et al., 2015). Acknowledging the importance of these studies, mathematics educators (e.g., Cannon & Ginsburg et al., 2008) have become increasingly interested in how to foster mathematics knowledge during the preschool years. Towards this aim, several countries have instated mandatory mathematics curricula for preschools. However, many young children spend a considerable amount of time at home. Furthermore, studies suggest that for children to take advantage of the academic opportunities provided at preschool, some level of support from the home environment, such as toys that stimulate learning number and shapes, is necessary (Anders et al., 2012). Thus, if we aim to promote young children's mathematical knowledge, the home environment should also be considered. This study is part of a larger project that focuses on adults' knowledge and beliefs regarding the teaching of number and geometry concepts during the early years, and ways of supporting adults' interactions with children and mathematics (e.g., Barkai et al., 2022). In this paper, we focus specifically on adults' awareness of geometrical concepts that can be promoted in young children and their awareness of situations where young children engage with geometrical concepts.

THEORETICAL BACKGROUND

Recognizing the importance of supporting children's geometrical knowledge and reasoning, several curricula have set standards for learning geometry in preschool. For example, state standards in the United States (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), suggest that kindergarten children identify and describe shapes regardless of their orientation and size and to analyse, compare, and compose both two-dimensional (2D) and three-

dimensional (3D) shapes. In Israel, the mandatory national mathematics preschool curriculum (INMPC, 2010) lists specific shapes that preschool children should be able to identify, such as triangles, circles, squares, cubes, cylinders, and pyramids. These curricula take into consideration that the acquisition of geometrical concepts includes both visual and attributional reasoning. Early on, children use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components (van Hiele & van Hiele, 1958). Children at this level can name shapes and distinguish between similar looking shapes. Next, children begin to notice that different shapes have different attributes, but the attributes are not perceived as being related. Later on, relationships between attributes are perceived and definitions are meaningful. Attributes may be critical or not-critical (Tsamir et al., 2008). For a triangle, for example, having three straight sides is critical, but the lengths of those sides is not critical, nor is the orientation of the triangle.

Several researchers and educators have recommended various activities that could promote children's geometrical reasoning. Sorting shapes, for example, has been suggested by several researchers (Levenson et al., 2011; Clements & Sarama, 2022) as a way of helping children differentiate between critical and non-critical attributes of shapes. Those researchers also suggested composing pictures from various shapes, and composing shapes from other shapes. As children must turn and flip shapes to compose their picture, they become familiar with shapes oriented in an atypical way. With regard to 3D figures, Clements and Sarama (2022) recommended building with blocks as a way to foster children's understanding of hierarchical relationships between the parts, and fostering their spatial skills. Educators also stress the importance of language in helping children express their geometrical reasoning.

Recognizing that young children spend a significant amount of time at home, several studies have investigated parents' beliefs and attitudes regarding the importance of promoting children's number and geometry knowledge. For example, Missall, et al. (2015) found that most parents agree that mathematics should be and can be promoted in the years before first grade, while Sonnenschein, et al. (2020) found that most parents believe that enhancing early reading skills is more important than promoting early mathematics skills. Adults also believe that their involvement and intervention with children is important for both numerical and geometrical growth (Levenson et al., 2021), although significantly less participants agreed that they needed to receive guidance to support their involvement.

While holding positive attitudes is important, it is equally important that parents are knowledgeable in the ways of promoting early mathematics. In one study (Cannon & Ginsburg, 2008), parents claimed that they lacked knowledge regarding the goals of learning mathematics at a young age. Sonnenschein et al. (2020) found that many parents wish to receive information from their children's preschool teachers regarding how to support their children's mathematics, including ideas for carrying out fun mathematics activities at home with their children. Focusing on early numerical competencies, a recent study (Barkai et al., 2022) found that adults are mostly aware

of counting, but are not aware of specific skills such as counting backwards, and skip counting. Furthermore, when describing object counting activities with children, few adults described the ways objects were set up, or different sub-competencies, such as one-to-one correspondence and cardinality. In other words, there was a general awareness of numerical activities, but still a lack of knowledge regarding specifics.

Regarding geometry, curricula and research have pointed out the importance of engaging young children with geometrical activities. Yet, studies have shown that parents are more likely to engage their children with numerical activities, than with geometrical activities (e.g., Zippert et al., 2020). Furthermore, it was found that when parents engage their children with block play, they offer little mathematical support, especially in comparison to the numerical support given when playing with card games (Zippert et al., 2020). When shown a list of geometrical activities and asked to report on the frequency of carrying out those activities at home, parents cited naming simple shapes and using position words as the most frequent activities (Missal et al., 2015). Copying shapes and using shapes to make a picture were among the least frequent. In the above studies, parents were given a list of mathematical activities to consider. In the current study, we phrased our questions in an open manner, to investigate what comes to the mind when parents are requested to think about geometrical concepts that can be promoted at a young age, and are asked to describe situations where children engage with geometrical concepts.

The aim of the current study is to investigate parents' awareness of geometrical concepts developed during early childhood. We adopt the following definition: To be aware is to have or show "realization, perception, or knowledge" (Merriam-Webster, <https://www.merriam-webster.com/dictionary/aware>). Specifically, we ask: Are adults aware of geometrical concepts that can be promoted among young children? Are adults aware of situations whereby children raise geometrical ideas on their own or with an adult's intervention, and if so, what are the contexts, activities, and geometrical concepts detected in those situations?

METHOD

The current study was conducted in Israel with a convenience sample of 52 adults (labelled A1-A52), recruited by three researchers from acquaintances in their areas of residence (middle to high socio-economic neighbourhoods). Ethical approval was given by the Institutional Review Board of the researchers' university and informed written consent to participate in the research was collected from all participants. Participants were between the ages of 20 and 40, 94% had an academic degree, and none were preschool teachers. All had children between the ages of three and six years. A researcher met with each participant personally and handed them the questionnaire.

The questionnaire consisted of three open questions:

Q1. In your opinion, which geometrical concepts can be promoted among young children (aged 3-6), before they begin first grade?

Q2. Describe a situation, in which you observed young children (aged 3-6) raising geometrical ideas without adult involvement.

Q3. Describe a situation, whereby during an interaction between a young child (aged 3-6) and an adult, geometrical ideas were raised.

The reason we requested participants to specify situations with and without adult interaction was to offer participants two different contexts for recollecting situations where children engaged in geometrical activities.

Data analysis of responses began with directed content analysis of geometrical concepts mentioned in the mandatory Israel National Preschool Mathematics Curriculum (INMPC, 2010). These concepts included: 2D and 3D shapes, spatial orientation, and symmetry. The situations described by parents in response to the second and third questions were analysed inductively in terms of the context of the situation and the type of activity mentioned by the parent. For example, Nan (all names are pseudonyms) wrote, “The children looked at things around them and compared them to shapes that they recognized such as triangles and squares.” The context was coded as an everyday situation and the activity was coded as comparing shapes. Moran wrote, “Children play with magnets and make shapes.” The context was coded as play and the activity was coded as composing shapes. Additional examples are given in the findings (Tables 2 and 3). Two researchers independently coded all data, comparing codes and reaching full agreement.

RESULTS

We begin by reporting on geometrical concepts mentioned in response to all three questions. All participants responded to the first question by mentioning a least one geometrical concept, with some mentioning three concepts. For example, Sherry wrote: “triangles, squares, circles, different types of lines.” Thus, Sherry related to 2D figures and components of shapes. Table 1 summarizes the geometrical concepts raised by adults for each of the three questions, and the number of adults who related to each one. As can be seen, nearly all parents mentioned 2D shapes in response to the first question. The most frequently mentioned were triangles, circles, and squares with frequencies of 69%, 58%, and 58% respectively. Other 2D shapes mentioned (to a much lesser extent) were: rectangles, rhombuses, hexagons, trapezoids, ellipses, octagons, and parallelograms.

Recall that for Q2 and Q3, participants were requested to describe situations where children, without and then with an adult, raised geometrical concepts. First, note that 11 adults did not recall situations where children raised geometrical concepts on their own, and four did not recall a situation that included an adult and a child interacting with geometry (see last row of Table 1).

Table 1: Geometrical concepts mentioned in response to the three questions (N=52)

Geometrical concept	Examples / descriptions	Q1	Q2	Q3
2D figures	Triangles, squares, circles	50 (96%)	19 (37%)	22 (42%)
Shapes (non-specific)	The term shape was used without specifying a particular shape.	-	10 (19%)	13 (25%)
Shape components	Sides, lines, angles	12 (23%)	2 (4%)	2 (4%)
“Fun” shapes	Stars, hearts	7 (13%)	-	-
3D figures	Pyramids, cubes	4 (8%)	3 (6%)	1 (2%)
Spatial relationships	Behind, near, far	1 (2%)	1 (2%)	-
Other	Length, parallelism	9 (17%)	-	-
No concepts mentioned	-	-	9 (17%)	10 (19%)
No answer	-	-	11 (21%)	4 (8%)

Table 2 summarizes the contexts of the situations described by participants. The most frequent context was play, including doll play and building with magnets.

Table 2: Contexts of situations that involved geometric ideas (N=52)

Context	Example	Without an adult	With an adult
Playing	Playing with playdough.	22 (42%)	24 (46%)
Everyday context	When we eat, we make all kinds of shapes.	10 (19%)	11 (21%)
Drawing	I sat with my son, and we drew lots of circles.	6 (12%)	5 (10%)
Reading	In the book I read to my daughter there were all sorts of shapes.	1 (2%)	4 (8%)
Unclear		2 (4%)	4 (8%)
No response		11 (21%)	4 (8%)

In addition, some parents described activities without mentioning a specific geometrical concept. For example, Natali wrote: “When eating pretzels with their father, the children described the shape of the pretzel.” This was coded as “shapes (non-specific).” There were also parents who just related to the activity, such as “drawing with my child,” without relating to any geometrical concept. It might be that those parents were aware that children were engaging with geometry, but took less notice of the specific geometrical content.

Table 3 describes the frequencies of activities mentioned in parents' descriptions of situations where geometrical concepts were raised, without and then with an adult. In general, there was little difference between the contexts (Table 2) and the types of activities (Table 3) parents described in both situations. The exception, perhaps, is for the activity of discussion, which was only noted in situations where an adult was involved. While this might seem reasonable, children do discuss geometrical ideas amongst themselves, without an adult present (Gejard & Melander, 2018).

Table 3: Activities mentioned in situations that involved geometric ideas (N=52)

Activity	Examples	Without an adult	With an adult
Composing	We made an ice-cream cone from a triangle, 3 lines, and the ice-cream was made from circles, touching each other a little bit.	18 (35%)	14 (27%)
Identifying	The children looked at a clock and said it was a circle.	12 (23%)	13 (25%)
Comparing	The children looked around them and compared what they saw to shapes, like a triangle and a square.	3 (6%)	2 (4%)
Drawing	We draw together in the afternoon.	2 (4%)	6 (12%)
Discussing	My son pointed to an octagon and said 8 doesn't have a name. We discussed this.	-	6 (12%)
Bodily movement	Using their bodies.	1 (2%)	-
Unclear		5 (10%)	7 (13%)
No response		11 (21%)	4 (8%)

Looking further into situations described with adult involvement, we examined whether parents took a teaching stance or merely participated with the child's activity. For example, Sarah took a teaching stance and responded to Q3, "When we passed by a stop sign, I asked my daughter what shape it is and she answered a hexagon." Shawn, however, described a more indeterminate role, "We played a matching game with shapes." Approximately one-third of the parents took a teaching role, with the rest taking a more indeterminate role.

DISCUSSION

Our first aim was to investigate parents' awareness of geometrical concepts that can be promoted among young children. Responses to all three questions of the questionnaire

indicated that parents mostly mentioned 2D shapes. While many children play with 3D figures, such as blocks, parents hardly mentioned 3D shapes in their responses, nor the spatial relationships that can be learned when playing with blocks (Clements & Sarama, 2022). Considering that many young children call a cube a square (Nieuwoudt & van Niekerk, 1997), activities such as playing with blocks or making figures out of playdough, could offer opportunities for parents to discuss with children the difference between 2D and 3D shapes. Recall that in response to the second and third questions, some parents did not describe a specific geometrical concept. This might also indicate less of an awareness to the geometry involved in an activity.

Regarding contexts, the second research question investigated adults' awareness of situations whereby children raise geometrical concepts. The most prevalent context mentioned was playing, while only approximately 20% related to everyday contexts, such as walking around the neighbourhood, and setting the table. This is in line with Cannon and Ginsburg (2008) who found that in everyday contexts, parents helped their children learn language more than mathematics. Yet, everyday contexts are also opportunities for enriching children's geometrical knowledge, opportunities that may be missed if adults are unaware of them.

The activities mentioned in participants' descriptions varied, but like adults' descriptions of numerical activities (Barkai et al., 2022), they were lacking in detail of what exactly was being done. In the future, interviews might be conducted to learn more about the details of those activities. Still, the most frequently noted activity was composition, an activity supported by several researchers (Clements & Sarama, 2022). However, few participants specifically mentioned discussion as part of the activity. This is critical. Comparing shapes can lead children to notice and differentiate between critical and non-critical attributes (Tsamir et al., 2008), but without appropriate discussion, comparison may remain at the level of whole shape reasoning (van Hiele & van Hiele, 1958).

To conclude, parents are interested in fostering children's mathematical knowledge, but, as this study suggests, may not be aware of the variety of geometrical concepts that can be developed at this age, or the range of contexts and activities that may support geometrical development. Workshops for parents could raise parents' awareness to additional geometrical concepts other than 2D shapes, and suggest more ways and activities for fostering geometry, both with adult intervention and without.

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A TEACHING INTERVENTION WITH DYNAMIC INTERACTIVE MEDIATORS TO FOSTER AN ALGEBRAIC DISCOURSE

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In this paper we analyze the developing discourse of high school students on the relations between two algebraic expressions. Using a commognitive perspective, we show in fine-grained detail how dynamic interactive mediators (DIMs) can help students with a history of low achievement find protagonists for their stories, fostering a first important step in the construction of an algebraic discourse.

INTRODUCTION AND THEORETICAL BACKGROUND

A great deal of research has documented students' difficulties in solving equations and inequalities, and in giving meaning to algebraic symbols, unknown and variables (e.g., Kieran, 2022). Common difficulties are related to a procedural view of the equal sign and to the transition to the letter-symbolic form of equations (Carpenter et al., 2005; Radford, 2022). Based on this scenario, this study investigates the effects of a didactical approach for introducing students to equations and inequalities that is based on the presentation of multiple artifacts. These are designed for representing the relationships between two expressions depending on the same variable. We used the software GeoGebra to represent expressions as moving arrows and as weights of a balance (see the next section). We investigate the effects of such an approach taking a commognitive perspective (Sfard, 2008) and analyzing students' *discourse* about the proposed *dynamic interactive mediators* (DIMs) (Antonini et al., 2020). Indeed, recent studies showed how learning to talk about well-designed DIMs can provide students with effective entry points into mathematical discourse (Baccaglini-Frank, 2021).

In line with the commognitive lens, we embrace the idea that algebra is a discourse (Caspi & Sfard, 2012) and, as such, it is characterized by its specific *words*, *visual mediators*, *narratives* and *routines* (Sfard, 2008). Learning algebra can be then described as the process of becoming able to access and express such a discourse. The framework provides many tools to capture and describe this process in a fine-grained way. We now present some of them and their use in our specific case of interest.

In this paper we investigate whether and how students accomplish a *saming process* between the two proposed DIMs, that we will call DIM_A (the one with the arrows) and DIM_B (the one with the balance), for short. Indeed, as expert mathematicians we are able to account for the fact that utterances about the reciprocal movements of two arrows realizing, for instance, $2x+1$ and $x+5$, and corresponding utterances about the behavior of a balance with weights $2x+1$ and $x+5$ seem to be saying "the same thing". According to Sfard (2008), we rationalize the "sameness" by conjuring *abstract objects* and speaking about arrows and balance as *realizations* of the same relation between the two expressions. It follows that the equation $2x+1=x+5$ can be described as either

a *concrete object* (e.g., a balanced off balance) or as a signifier of an abstract object. Until this equation has no realizations for a student, it is just a concrete object and can only be manipulated in well-defined ways. When a student begins to speak of abstract objects and their properties, the discourse has become *objectified*. The development of this form of discourse gives us information about the students’ learning process. In particular, the special property of the objectified discourse is that it *subsumes* the former independently existing discourses, in our case about DIM_A and DIM_B , making it possible to express in the new language almost everything that can be said in any of the original discourses with their own special signifiers (Sfard, 2008).

Description of the DIM designed for this study

For this study we designed a DIM realizing the relation between two expressions depending on the same variable, that we call $DIM_{(A,B)}$ because it embeds DIM_A and DIM_B (Fig.1). On the left part it shows a balance, with the plates described by algebraic expressions, dependent on the unknown weight x and written in a blue and a grey box. A label shows the relation between the plates: when their weight is equal, it is green; otherwise, it is yellow. On the right part there are three arrows moving like a dynagraph (Antonini et al., 2020). There are a red tick mark, realizing x , that is directly draggable bound to the positive x-axis, and a blue and a grey tick mark, realizing the two expressions depending on x , that move indirectly. These two tick marks correspond to the two plates of the balance, that also change simultaneously, with the dragging of x .

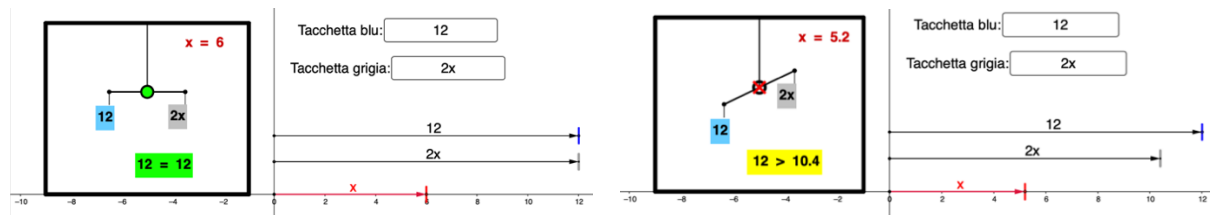


Figure 1: Two screenshots of the $DIM_{(A,B)}$ with the expressions ‘12’ and ‘2x’.

We designed this DIM with the aim of promoting the process of saming between DIM_A and DIM_B . The hypothesis is that the interaction with $DIM_{(A,B)}$ can foster the emergence of new discourses that may lead to the birth of a subsuming discourse. We hypothesized this could happen once students had interacted separately with DIM_A and DIM_B and constructed an “A-discourse” and a “B-discourse”. These are discourses involving the DIMs; their main features are reported in Table 1. In line with the discussion in the previous section, we thought that a possible discourse subsuming A- and B- discourses could involve algebraic symbols (see the fourth column of Table 1). Hence, in the $DIM_{(A,B)}$ we introduced labels with algebraic expressions. The protagonists of this S-discourse are objectified equations and inequalities, while the balance and dynagraph become possible realizations. In our previous examples with the two expressions, the narratives “the balance hangs to the left” in the B-discourse and “the blue arrow is to the right of the grey one” in the A-discourse find a counterpart in the S-discourse with “ $2x+1 > x+5$ ”.

Table 1: Features of discourses about DIM_A , DIM_B and of a possible subsuming.

	A-discourse	B-discourse	S-discourse
Words	Tick mark, arrow, drag, right/left	Balance, weight, plate, up/down	Expression, equal, bigger/lower
Visual Mediators	Red, blue, and grey tick mark, $ax+b$ and $cx+d$ labels, gestures on DIM_A	Red, blue, and grey boxes, $ax+b$ and $cx+d$ labels, gestures on DIM_B	$ax+b$ and $cx+d$
Narratives	“If the red tick mark is at 2, the other tick marks are aligned”	“If the unknown weight is 2, the balance is balanced off”	If $x=2$ then $ax+b=cx+d$
Routines	Drag to the right/left	Put on/take off	Algebraic manipulation

Research questions

This study is part of a greater funded research project exploring the impact of teaching interventions with second year high school students with a history of low achievement in mathematics who are asked to engage in a set of newly designed activities with DIMs in the context of algebra. In this paper, we focus on students' discourse about the relations between two expressions of the same variable. We conjecture that DIMs can play a key in students' development of this discourse, by supporting the construction of abstract objects that may become the protagonists of the subsuming discourse. To investigate such a conjecture we designed the three DIMs above and in this study asked the following specific research questions: *What characteristics (words, visual mediators, narratives and routines) of the discourses about DIM_A and DIM_B does the discourse developed by students in their interaction with the $DIM_{(A,B)}$ have? To what extent does this new emerging discourse subsume the former ones?*

METHOD

Data collection occurred in an out-of-school learning center with twelve 10th grade low achieving students volunteering from three different Italian high schools. Participants attended four 2-hour-long sessions, during which DIMs-based activities were proposed by a researcher. Data were collected in the form of video recordings, students' written productions, and screen recordings of the tablets used for the activities.

This paper focuses on two pairs of students engaged in activities with $DIM_{(A,B)}$, during the 4th session. They are given explorative tasks aimed at fostering a discourse subsuming the previously constructed A- and B- discourses and the researcher's questions seek to promote saming between DIM_A and DIM_B . We analyzed data focusing on three features that we operationalized through guiding questions and communicational indicators. We looked for such indicators to be identified in the transcripts and coded them with the typographic marks shown in Table 2.

Table 2: Analytic scheme.

Feature	Guiding questions	Examples of indicators
Type of discourse	Does it have the characteristics of A-, B- or S- discourse?	See the description of <u>A-discourse</u> , <u>B-discourse</u> and <u>S-discourse</u> in Table 1
Presence of abstract objects	Are there references to an expression as signifier of an abstract object?	“ <i>x+1 is bigger than</i> ”, “ <i>x+1 equals</i> ”, ...
Instances of saming	Are DIM _A and DIM _B described as realizations of the same signifier?	“ is the same ”, “ is equal ”, “ they make the same ”, ...

DATA ANALYSIS

We tell the story of Andrea (A) and Hugo (H) through the analysis of selected short episodes. Then, we present some episodes of Grazia (G) and Lucia (L) interacting with the same DIMs, for discussing similarities and differences in their emerging discourses.

Andrea and Hugo

When Andrea and Hugo first saw the DIM_(A,B) in which the expressions 12 and 2x were defined (Fig. 1), they observed the presence of a “balance that is balanced off” and before interacting with it, they discussed with the researcher (R) as follows:

- 1 H: The two plates are both at the same level
- 2 R: Ok, both the plates are at the same level and what is there on the plates?
- 3 H: Twelve and two x
- 4 R: Mm and why are they at the same level?
- 5 H: Because they both have, we can say, the same weight
- 6 A: Value? The same value, let's say
- 7 H: Because it also tells us that *x equals six* [He points at the red label $x=6$] so *two x equals twelve*
- 8 R: Ok, perfect and instead on this side [She points at DIM_A]?
- 9 H: **It indicates it with the tick marks** [He does not make gestures]

Through a B-discourse, Andrea and Hugo describe the equality between the two expressions for $x=6$. We highlight the first occurrence of the word “value” (turn 6) that can be seen as a seed of subsuming discourse, because it may refer to the “value of the expression”; however, it is used again by Andrea referring to DIM_B as the “value of the weights”. At turn 3, ‘12’ and ‘2x’ are the objects of Hugo’s discourse, to which he seems to refer not yet as abstract, but as concrete objects. Differently, at turn 7, ‘2x’ is used without explicit references to the DIMs. The last sentence expressed by Hugo (turn 9) suggests a partial instance of saming. Indeed, an asymmetric relation emerges between the DIMs: the tick marks indicate what happens on the balance but not vice

versa. The student said that the tick marks indicate “it” referring to what has just been said for the balance, but what “it” means is missing.

The researcher then asked the students describe what happens when dragging x in $\text{DIM}_{(A,B)}$. While Andrea was dragging Hugo stated:

- 10 H: It depends on the value of how much the red tick mark is [He simulates with the finger the movement of the red tick mark on the axis], because if it goes to the right the balance tends to, tends to hang... to the right, while if it is a number smaller than six, the balance tends to hang to the left

This excerpt is characterized again by B-discourse, since it mainly concerns the behavior of the balance. However, there are also some A-discourse narratives and a seed of S-discourse in the utterance “if it is a number smaller than six” since it can refer to both DIMs because the subject is not made explicit by Hugo. There is another occurrence of the word “value”, used as before but this time referring to a tick mark. Therefore, “value” could constitute a word in these students’ emerging S-discourse. However, as evidenced by their writing at the end of the episode (Fig. 2a), the relations between the quantities are uniquely expressed in terms of balance’s swing, confirming the B-discourse to be the prevailing one so far.

In the next activity, two new expressions were defined in $\text{DIM}_{(A,B)}$: $x+5$ and $2x+1$, and students’ exploration took place, prompted as before. When asked to summarize what they observed, Andrea and Hugo again produced a B-discourse to describe the relationships between the two expressions (e.g., “if x equals four...the plates have the same weight. Then, instead, if x is greater than four it tends to dangle to the right”). This summary is accompanied by an inscription that is very similar to that of the previous episode. But “the scale is even” is replaced by “they have the same value” (Fig. 2). This difference suggests the students’ development of an S-discourse because in their narrative about the equality they lost the reference to the balance model by using the term “value”, which had previously been used within both A- and B-discourses. The same expression could indeed refer to the position of the tick marks.

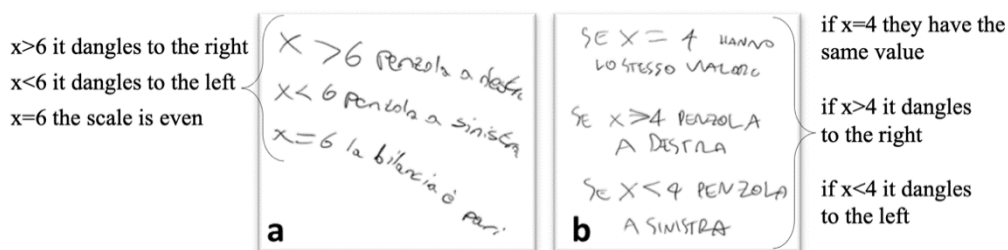


Figure 2: Andrea and Hugo’s written production for the activity on the $\text{DIM}_{(A,B)}$ with the expressions (a) ‘12’ and ‘ $2x$ ’, (b) ‘ $2x+1$ ’ and ‘ $x+5$ ’.

In another similar activity, involving the expressions $2x+4$ and $3x$, Hugo related DIM_A and DIM_B by observing that the behavior of the balance depends on that of the arrows (“The balance tends to dangle to the left ... because the blue tick let's say has a greater value than the grey one”). Then, about the equality, Hugo stated:

- 11 H: All the lines are on the same number, they have the same value.

- 12 R: Ok, right. And the balance?
 13 H: Let's say, it always gives the same value.

The researcher's intervention promoted Hugo's move from an A-discourse (turn 11) to a B-discourse (turn 13). He used "the same value" both referring to the tick marks on the same number and to the balance plates weighing the same. This suggests the emergence of an S-discourse and a process of saming between the DIMs.

We conclude the overview of Andrea and Hugo's story with a short episode, following the previous one, in which an interweaving of A-discourse and S-discourse appeared.

- 14 H: If the tick mark, if we want the blue tick mark greater than the grey one...
 The number, the x has to be smaller than four, while if we want that three x
 that the grey tick mark is greater it has to be a number greater than four
 15 A: When it is at four
 16 H: When it is at four the value is equal

The episode begins with an A-discourse in which Hugo mentions the tick being "greater" (turn 14), probably referring to the value taken on. Interwoven with this discourse, however, there are endorsed narratives both in A- and B- discourses. The A-discourse reappears in the next turns when they said "is at four" instead of "is four" suggesting the focus to be on the position of the red tick. This is accompanied, again, by the subsuming narrative involving "the value". Moreover, we observe that in this episode for the first time there are possible instances of abstract objects (turn 14). For example, when Hugo used "grey tick mark" as a noun for the object "three x".

Grazia and Lucia

In the first activity, after an initial exploration without having yet dragged the tick realizing x , Lucia described $DIM_{(A,B)}$ involving the expressions 12 and $2x$ as follows:

- 1 L: Like the first 12 of the balance is indicated as, as it is the double of this x
 [She does not make gestures], and the same thing for the grey tick mark...
 So they correspond, that is, **it is the representation of the balanced off**
balance made on the line

Lucia initially produced a B-discourse which is then followed by an A-discourse introduced by the expression "the same thing". The student seems to make explicit a saming between the two visual mediators, as also highlighted immediately afterwards by her narrative "it is the representation of the balanced off balance made on the line".

Now we share a short excerpt from the next activity, involving the expressions $x+5$ and $2x+1$, in which we find another instance of saming in Lucia's discourse:

- 2 L: That is, you also notice that the tick marks are not aligned anymore, as
 before, because the balance is not balanced off anymore, but they indicate
 exactly two different values, so the balance is not, is not in balanced off
anymore

Lucia, by mixing A- and B-discourse, refers to a cause-and-effect relationship between the two DIMs, which bound in both directions: "the tick marks (...) because the balance (...)" and then "[the ticks] indicate (...) so the balance (...)". This highlights how DIM_A

and DIM_B played a mutual role in Lucia's discourse, being both signifier and realization one for the other.

Shortly after this excerpt, the description of the behavior of the balance occurred with an interweaving of S-discourse and B-discourse, in which the former was more present:

- 3 L: If x would have been smaller than 4 ehm we expect the opposite situation, so that $2x$ plus one is, we say, it corresponds to a value smaller than x plus 5... And instead the balance will be balanced off when x will be four.

The reference to the DIM_B appears only at the end of the turn. Except for this, the one produced by Lucia is a S-discourse, since it is an endorsed narrative to describe in the same terms the situation that both balance and arrows realize.

We conclude by showing what Grazia and Lucia wrote to summarize their observations about the two pairs of expressions explored in the activities in focus (Fig. 3a-b). Note how this written discourse, unlike Andrea and Hugo's, completely subsumes the A- and B-discourses constructed during the activities with the $\text{DIM}_{(A,B)}$.

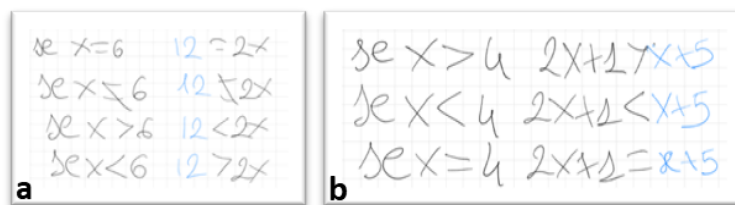


Figure 3: Grazia and Lucia's written production for the activity on the $\text{DIM}_{(A,B)}$ with the expressions (a) '12' and '2x', (b) '2x+1' and 'x+5'.

DISCUSSION AND CONCLUSIONS

The analyses presented above confirm that DIMs can foster low achieving students' participation in mathematical discourse. More specifically, activities with multiple DIMs, designed to be realizations of the same algebraic signifier, can promote the development of algebraic discourse as a form of subsuming discourse. The DIM we designed, embedding a balance (Otten et al., 2019) and a dynagraph (Antonini et al., 2020) as realizations of the relations between two algebraic expressions, accomplishes this by creating the need for a common discourse. However, significant differences between the two pairs of students' discourse emerge. On one hand, Grazia and Lucia accomplish saming between DIM_A and DIM_B , having at their disposal the ingredients for constructing a S-discourse: algebraic expressions, and their relations, as abstract objects of which the two DIMs are realizations. Andrea and Hugo, on the other hand, do not seem to develop this degree of objectification and this results in a lack of protagonists of a possible S-discourse. We see a partially successful attempt to construct a protagonist in their use of the term "value" bridging A- and B-discourses. Even if it is only a first step of the process, this is a success for them, considering the difficulties shown in the previous sessions. A longer intervention might have made a difference, but nevertheless there is a seed of subsuming discourse.

We now discuss limitations and aspects not a priori expected. We thought that fostering saming between DIM_A and DIM_B , through activities asking to speak in different ways

and to look for similarities between the DIMs, could support the construction of a S-discourse. However, we observed how the processes of saming and the process of constructing an S-discourse are closely intertwined, generating a kind of paradox that may be explained in discursive terms as follows: saming needs a subsuming discourse with which to talk about “the same thing” that the two DIMs realize, but, at the same time, the protagonists of subsuming discourses are the products of a saming process. The story of Andrea and Hugo shows how this circularity can be overcome through the use, within the same discourse, of words, visual mediators, narratives, or routines from different discourses; in their case, for example, the word “value”.

In conclusion, in this paper we told the stories of two pairs of students, providing insights for both research and didactic reflections. Especially, the story of Andrea and Hugo can contribute to the literature on difficulties in working with the letter-symbolic form of equations and inequalities (e.g., Carpenter et al., 2005; Kieran, 2022). It also confirms the importance of designing didactical approaches aimed at fostering students’ learning to talk about multiple DIMs realizing the same object, because they can open new doors into mathematical discourse for students like Andrea and Hugo.

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MORE THAN JUST THE BASIC DERIVATION FORMULA: THE IMPACT OF PRIOR KNOWLEDGE ON THE ACQUISITION OF KNOWLEDGE ABOUT THE CONCEPT OF DERIVATIVE

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Based on process-object theories, knowledge about mathematical concepts can be viewed as having an operational and a structural aspect. Ideally, structural knowledge should build on operational knowledge. However, some students are not able to acquire enough operational knowledge and only acquire a pseudostructural conception of a mathematical concept (see Sfard, 1992). In this study with 144 grade 10 students from Germany we examine the effects of different areas of prior knowledge on the acquisition of knowledge about the concept of derivative. We found evidence that substantial knowledge about the concept of function, the concept of slopes of linear functions, and mathematical symbols is necessary to acquire mathematical knowledge about the concept of derivative that goes beyond a pseudostructural conception.

THEORETICAL BACKGROUND

When considering the mental structure of mathematical knowledge, there are multiple theories differentiating between an operational (or process) and a structural (or object) aspect of knowledge about a mathematical concept. These theories include, for example, the *APOS* (action-process-object-schema) theory (Dubinsky, 1991) and the process-object framework by Sfard (1991, 1992). The common core of these *process-object frameworks* is the idea that knowledge about a mathematical concept has an operational and a structural aspect, and a person should ideally acquire knowledge about both to understand said mathematical concept fully. The ideal-typical acquisition of mathematical knowledge would entail that a person first acquires knowledge about the operational aspect of a mathematical concept before proceeding to mentally form a new mathematical object that is structural in nature. Sfard (1991) calls this emergence of a new mental mathematical object *reification*. Once a person has acquired structural knowledge about a mathematical concept, it enables them to perform new processes with this mathematical object. This, in turn, can lead to operational knowledge about a new mathematical concept, eventually resulting in a new mathematical object.

The common core of the process-object frameworks can, thus, be summarized into a hierarchy where the acquisition of operational knowledge about a mathematical concept precedes the acquisition of structural knowledge about the same mathematical concept. Furthermore, knowledge about one mathematical concept functions as a prerequisite for knowledge about a new mathematical concept.

Sfard (1992) adds the idea of *pseudostructural knowledge* to this knowledge hierarchy. She proposes that many students are not able to fully integrate the operational and

structural aspects of a mathematical concept. Instead, students often have some understanding of a mathematical object without fully comprehending the underlying process that leads to the formation of this object. Zandieh (2000) characterizes the resulting pseudostructural conception as “an object with no internal structure” (p. 107) or a “whole without parts” (p. 108). Pseudostructural knowledge provides limited opportunities to operate with a mathematical object but enables students – without comprehensive knowledge about a certain mathematical concept – to acquire some (perhaps limited) knowledge about further concepts.

In the present study, we investigate mathematical knowledge about the concept of derivative according to the process-object frameworks. The concept of derivative is typically introduced along three mathematical concepts that build on each other and can each be seen as operational and structural in nature (e.g., Zandieh, 2000): difference quotient, differential quotient, and derivative function.

Therefore, Figure 1 summarizes the proposed model for acquiring knowledge about the concept of derivative according to the process-object frameworks, including Sfard’s (1992) idea of pseudostructural knowledge.

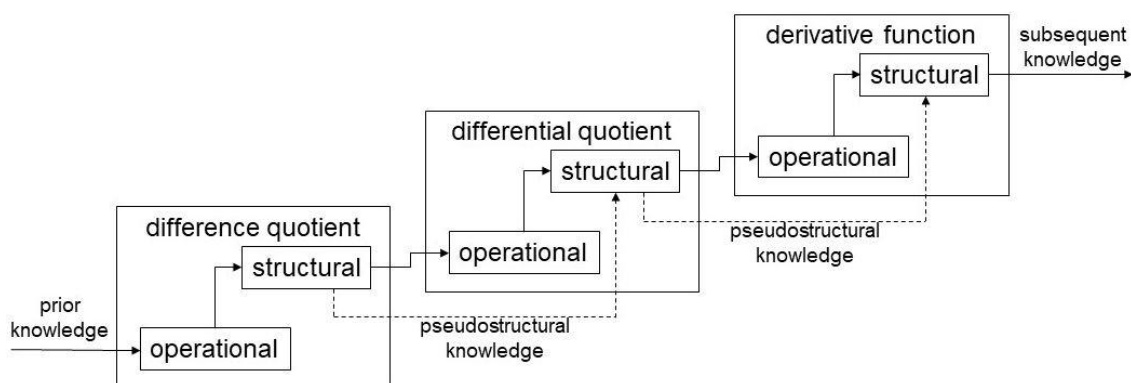


Figure 1: Model for the acquisition of knowledge about the concept of derivative

Understanding the concept of derivative: relevant prior knowledge

Previous studies have shown that students tend to have considerable trouble with understanding the concept of derivative, while solely algebraic differentiation (e.g., derivation formula) seems to be substantially easier (e.g., Orhun, 2013; Orton, 1983). A possible cause of this might be that the concept of derivative builds on many elements of prior knowledge as proposed by the aforementioned knowledge hierarchy.

Following previous research (e.g., Carlson et al., 2015; Thompson & Harel, 2021), we identified six areas of mathematical knowledge that could be considered relevant for acquiring knowledge about derivatives: functions, slopes of linear functions, limits and approximations, manipulation of fractions, algebraic manipulations and equations, and basic knowledge of mathematical symbols. In line with the aforementioned process-object frameworks, there may be operational as well as structural aspects of knowledge about functions, slopes of linear functions, and limits and approximations. For the concept of function specifically, the aspects of knowledge about functions, i.e., mapping, covariation, and function as an object, can be seen as a progression from

operational to structural knowledge (Doorman et al., 2012). For knowledge of slopes of linear functions, an operational perspective is characterized by the use of a gradient triangle, while the structural aspect requires seeing the slope as a whole and, for example, comparing the slopes of different linear functions without having to calculate them specifically. Similarly, knowledge of limits and approximations may also be viewed as operational (dynamic) and structural (static) in nature (see Maharaj, 2010 for a more detailed analysis).

The next two areas of prior knowledge, fractions and algebraic manipulations/equations, relate to students' technical abilities to perform calculations with fractions (adding, subtracting, multiplying and dividing fractions that may include variables), manipulate algebraic expressions (including, e.g., binomial formulas), and transpose equations. The last area of prior knowledge is mathematical symbols, which includes the basic comprehension of mathematical symbols and terminology, such as intervals, sets, and different number ranges.

Research questions

Content from all six areas of prior knowledge could be relevant for learning the three sub-concepts of derivative (difference quotient, differential quotient, derivative function) (Carlson et al., 2015; Orton, 1983; Thompson & Harel, 2021). In our study, we explore whether this prior knowledge can predict the successful acquisition of knowledge about the concept of derivative beyond pseudostructural knowledge.

Using a dedicated study, we aim at investigating the following research questions:

1. To what extent does students' prior knowledge in the aforementioned areas of prior knowledge predict the acquisition of knowledge about the concept of derivative?
2. To what extent does students' prior knowledge predict the acquisition of operational knowledge about the concept of derivative (as an indicator of whether students have acquired more than pseudostructural knowledge about the concept of derivative)?

METHOD

We developed two tests, in order to measure students' knowledge in the six areas of prior mathematical knowledge and about the operational and structural aspects of knowledge about the concept of derivative.

The *prior knowledge test* was broken up into six subtests each containing items on one area of prior knowledge. Each participant received the six subtests in the same order, however there were different versions of each subtest presenting the items in a different order. Each subtest was administered with a separate time limit (5 or 10 minutes). The *knowledge about derivatives test* contained items measuring operational as well as structural aspects of knowledge regarding each of the three sub-concepts difference quotient, differential quotient, and derivative function.

Both tests were administered within 45-minute regular mathematics lessons for the tenth grade in schools in Northern Germany. The data analysis in this research report is based on data of $N = 144$ students (data collection is ongoing). Students received the prior knowledge test before the teachers started the teaching unit on derivatives. Subsequently, teachers designed the lessons on the concept of derivative at their discretion, all teachers followed the same curriculum. After the conclusion of the teaching unit, the knowledge about derivatives test was administered.

Students' knowledge regarding each of the areas of prior knowledge and knowledge about the concept of derivative as a whole (operational and structural aspects included) was measured using a percentage to the available points for each scale. Students' knowledge about the operational aspect of the concept of derivative was measured by using the corresponding items from the knowledge about derivatives test. This subtest indicates whether students have acquired knowledge about the concept of derivative that goes beyond a pseudostructural conception. Due to low reliabilities (Cronbach's α) in two prior knowledge subtests, the subtests on fractions and algebraic expressions/equations were combined to a scale representing technical skills and the items from the limits and approximations scale were reassigned to one of the other scales that fitted the content of each item best. Consequently, all analyses presented in this report are based on four areas of prior knowledge (see Tab. 1).

Multilinear regressions were estimated assessing the predictivity of the four areas of prior knowledge on knowledge about the concept of derivative as a whole and about the operational knowledge about derivatives separately. Subsequently, a cluster analysis was conducted based on the results of the prior knowledge test, grouping students based on their knowledge in the four areas of prior knowledge. For each of the resulting clusters of students, means for the knowledge about derivatives test and its operational subtest were calculated. The results should give an idea which area and which level of prior knowledge is necessary for the acquisition of knowledge about derivatives and especially operational knowledge about the concept of derivative which goes beyond pseudostructural knowledge.

RESULTS

Table 1 shows descriptive data for the administered scales. All scales except one showed an acceptable reliability. The scale for operational knowledge about the concept of derivative showed an almost acceptable reliability ($\alpha = .59$). The mean values revealed that all tests were quite difficult for the students, although the tests addressed standard curriculum content (see below for limitations of the study).

Scale (#items)	Sample item	α	$M (SD)$
Functions (16)	Let $f(x) = 2x + 1$. Calculate $f(x + 1)$.	.62	.34 (.16)

Slopes of linear functions (12)	Indicate the slope of the function $g(x) = -x + 2$.	.64	.33 (.19)
Technical skills (fractions, algebraic expressions, equations) (23)	Simplify as much as possible: $2(x + x^2) - 4x - 5x^2 + x - 5x^2 =$.71	.31 (.14)
Mathematical symbols (12)	Give a number x for which holds: $x \in \mathbb{R}$ and $x \in \mathbb{Z}$ but $x \notin \mathbb{N}$.	.77	.30 (.23)
Knowledge derivative (16)	Let f be a function with $f(x) = \frac{1}{3}x^2$. At what point P is the tangent to the graph of f parallel to the straight line g with $g(x) = 2x + 5$?	.69	.25 (.16)
Operational knowledge derivative (9)	Draw the graph of a function f with the following three properties: slope 0.5 at (0,0), slope 1 at (2,3), slope 0 at (5,1)	.59	.17 (.17)

Table 1: Descriptive statistics and sample items for the administered scales

The results of the multilinear regressions reveal a significant effect of prior knowledge about functions, slopes of linear functions, and mathematical symbols on knowledge about the concept of derivative (Tab. 2). Isolating the operational aspect of knowledge about the concept of derivative, only prior knowledge about functions had an effect.

Predictor	Knowledge derivative	Operational knowledge derivative
(Intercept)	.02	-.04
Functions	.35***	.32***
Slopes of linear functions	.21**	.15#
Technical skills (fractions, alg. expressions, equations)	.01	.08
Mathematical symbols	.12*	.11#
R^2	.36	.27

$p < .10$; * $p < .05$; ** $p < .01$; *** $p < .001$

Table 2: Regression coefficients β and statistical significance

The cluster analysis on the basis of the prior knowledge subtests revealed four clusters of students as the best solution (based on the dendrogram). Table 3 (top part) shows the mean scores on each of the prior knowledge scales for each of the four clusters of

students. It can be observed that, on average, students in Cluster 1 generally performed the best and students in cluster 4 performed the worst on the prior knowledge subtests.

Scale	Cluster 1 (<i>n</i> = 24)	Cluster 2 (<i>n</i> = 30)	Cluster 3 (<i>n</i> = 29)	Cluster 4 (<i>n</i> = 61)
Functions	.56 (.10)	.37 (.12)	.30 (.14)	.25 (.11)
Slopes of linear functions	.58 (.18)	.35 (.13)	.35 (.10)	.21 (.14)
Technical skills (fractions, alg. expressions, equations)	.48 (.11)	.39 (.07)	.35 (.08)	.18 (.08)
Mathematical symbols	.45 (.28)	.54 (.13)	.12 (.10)	.22 (.15)
Knowledge derivative	.44 (.19)	.27 (.13)	.17 (.09)	.20 (.15)
Operational knowledge derivative	.37 (.19)	.19 (.14)	.09 (.10)	.13 (.14)

Table 3: Means (*SD*) for prior knowledge as well as on knowledge on derivative and the subtest operational knowledge on derivative for each cluster

Since the formation of the four clusters of students was solely based on students' prior knowledge, it was of interest how well students in each of these clusters performed on the two scales from the knowledge about derivatives test (Tab. 3 bottom part). The pairwise comparison of the mean scores of the four clusters partly revealed significant differences (see Tab. 4 for the most interesting comparisons). Since we had small cluster sizes, we additionally provide effect sizes (Cohen's *d*).

Scale	C1 vs. C2	C2 vs. C3	C2 vs. C4	C3 vs. C4
Knowledge derivative	1.08 ^{***}	0.58 [*]	0.40 [#]	0.18
Operational knowledge derivat	1.04 ^{***}	0.58 [*]	0.34	0.24

[#] $p < .10$; ^{*} $p < .05$; ^{***} $p < .001$

Table 4: Cohen's $|d|$ for differences of cluster means for the outcome variables

DISCUSSION

The regression results show that knowledge about functions and slopes of linear functions as well as basic knowledge about mathematical symbols predict how much knowledge about the concept of derivative is acquired. As for operational knowledge, knowledge about functions occurs as a significant predictor, while knowledge about slopes of linear functions and basic knowledge about mathematical symbols show a tendency towards statistical significance. Students' technical skills to manipulate fractions, algebraic expression, and equations do not have a significant impact on the outcome scales. These results, which are also reflected in the cluster means in Table 3, emphasize and confirm the importance of specific prior knowledge about functions and slopes of linear functions in acquiring knowledge about the concept of derivative (Carlson et al., 2015; Orton, 1983; Thompson & Harel, 2021).

The effect sizes of the mean differences for the four clusters of students in Table 4 indicate a considerable contrast between Cluster 1 and the three remaining clusters with effect sizes of $|d| > 1$. The students in Cluster 1, who on average performed the best on the prior knowledge test, were able to achieve, on average, by far the best results on both scales from the knowledge about derivatives test. Notably, students in Cluster 1 ($n = 24$, 17 % of all students) are the only ones to achieve a level of operational knowledge about the concept of derivative that goes beyond pseudostructural knowledge (last row in Tab. 3). Following the ideas of Sfard (1992) and Zandieh (2000), this is the only group of students that we can consider having sufficiently integrated operational and structural aspects of knowledge about the concept of derivative. Consequently, all remaining students in Clusters 2-4 ($n = 120$, 83 % of all students) mainly have acquired pseudostructural knowledge about the concept of derivative. When looking back at the results from the prior knowledge subtests (top part of Tab. 3), it is evident that the average scores of students in Cluster 1 for functions and slopes of linear functions are far above the average scores for all students (more than one *SD* above average). The differences between the four clusters in the prior knowledge subtests suggest that substantial knowledge about functions and slopes of linear functions is a necessary precondition for the acquisition of operational knowledge about the concept of derivative. Students from Cluster 1 scored 56% and 58% on these subtests, respectively, while the 37% and 35% of students in Cluster 2 were not sufficient to overcome the level of pseudostructural knowledge.

Limitations

Overall, students' scores on all the administered scales were lower than expected, however, our results are generally in line with previous research in that students' performance is low when measuring knowledge about the concept of derivative (e.g., Orhun, 2013; Orton 1983). It can reasonably be assumed that the main cause for the low prior knowledge are the lasting effects of the Covid pandemic. Nevertheless, we believe that this will have little effect on the findings regarding our research questions since we are interested in the effects of prior knowledge on knowledge about the concept of derivative. Another limitation is the low reliability of some scales which might influence the precision of the detected relations. Since the data collection will be continued, a larger sample might improve the reliability in the final analysis.

CONCLUSION

To summarize, our results indicate that knowledge about the concepts of function and slopes of linear functions as well as knowledge about mathematical symbols predict the acquisition of knowledge about the concept of derivative. Students' technical skills to manipulate fractions, algebraic expressions, and equations seem to have no specific and additional impact. Further, our findings suggest that in order to acquire knowledge about derivatives that goes beyond a pseudostructural conception, students specifically need substantial prior knowledge about functions and slopes of linear functions. Therefore, students' level of knowledge about the concepts of function and slopes of

linear functions might predict whether students can only acquire pseudostructural knowledge or a more comprehensive understanding of the concept of derivative.

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SECONDARY-TERTIARY TRANSITION OF INTERNATIONAL STUDENTS: ONE STUDENT'S EFFORTS TO OVERCOME THE CHALLENGE OF LEARNING MATHEMATICS IN ENGLISH

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The transition from secondary to tertiary mathematics is recognised as being disruptive. Yet, international students enrolling at universities in a foreign social context face different challenges from those of local students, particularly when a change in the language of instruction occurs. Understanding these challenges is important if universities are to support international students in transition. In this study we focus on an international student from China to illustrate how her mathematical identity was shaped by both university structures and her personal experiences of school mathematics in another social context. In particular we demonstrate how her past experiences contributed to the resilience that facilitated a successful transition.

INTRODUCTION

Mathematics is a popular choice for international students attending universities abroad. This may, in part, result from the perception that mathematics is a language-invariant subject (Wood et al., 2007). Yet curricular and other cultural differences between the education systems of different countries means that international students transition to university with different skills, beliefs and understandings about learning mathematics than those of their locally schooled peers. These differences signal the potential for different transitional experiences from those of local students.

Understanding the transition from the perspectives of international students will contribute to knowledge of how different cultural contexts shape student experiences. Di Martino et al. (2022) call attention to a scarcity of research in this area as flagged in their recent systematic review of literature on the transition from secondary to tertiary mathematics (hereafter STT). The review also highlights the need for future research to pay closer attention to affective and sociocultural factors in the STT.

In this paper we use mathematical identity to explore the experiences of an international student from mainland China enrolled in first-year mathematics at a New Zealand university. We aim to illustrate how social structures of tertiary mathematics in one social context and personal experiences of school mathematics in another context can contribute to students' mathematical identity.

RESEARCH BACKGROUND

Transitioning from school to university mathematics as an international student

The STT is widely recognised as a period of disruption. In addition to significant cognitive challenges, students must adapt to a new pedagogical culture (Hernandez-

Martinez & Williams, 2013). Di Martino et al. (2022) suggest that how students experience the challenges of the STT depends on their educational context. They show how students from three different countries in Europe evidenced contrasting experiences of the STT.

Language proficiency also plays a major part in the transitional experiences of students. Barton et al. (2005) showed that first-year students at a New Zealand university suffered at least a 10% disadvantage when their home language was not English. Student such as these may have difficulty following the academic language used in lectures and may thus find themselves heavily reliant on written texts and symbolic modes of working (Wood et al., 2007). Students who are not confident in the language of instruction may also encounter barriers to group work. Hwang et al. (2022) report that the discomfort experienced by a Chinese-speaking student when unable to successfully communicate her ideas to her English-speaking peers inclined her to silence in a groupwork task. While the failure to communicate successfully can be attributed to language difficulties, the discomfort experienced by the student may need further unpacking. Xu and Clarke (2019) suggest that actions disruptive to harmonious relationships may be avoided in Chinese classrooms. A failed communication attempt could be viewed in this light. Similarly, students from cultures with steeply hierarchical student-teacher relationships might perceive questioning the teacher to be disrespectful, despite this being an accepted practice in many Western settings (Lillyman & Bennett, 2014).

How social structures and individual experiences shape mathematical identities

The social structures that exist in educational settings suggest what is valued within the community. Literature shows how structures ranging from instructional practices within specific courses (Voigt et al., 2021) through to wider institutional policies such as assessment practices (Solomon, 2007) can shape the identities of learners in these spaces. The common understanding of success as being related to achievement or examination grades (Gutiérrez, 2013) is often induced by these institutional structures.

On the other hand, literature also shows that experiences unique to the individual can inform mathematical identities. For example, perceptions of fixed ability, which were shown to negatively impact on the mathematical identities of students at a UK university (Ward-Penny et al., 2018), contrasts with the view of many Chinese students that mathematics is a skill that can be improved with practice (Mok, 2020). Research also shows how prior experiences of struggle might shape identities of resilience in some students (Hernandez-Martinez & Williams, 2013) and disaffection in others (Ward-Penny et al., 2018). As international students transition into university mathematics, it is likely that both unique experiences from their home countries, and the structures of their foreign host universities, will shape their identities as mathematics learners.

THEORISING MATHEMATICAL IDENTITY

The concept of positioning plays an important role in the enactment of identity. Positioning theory (Van Langenhove & Harré, 1999) describes positioning as the way rights and duties are ascribed in interactions, offering a way to make sense of the interactional moves through which people clarify who they are and how they see others. Positioning involves inviting people to conform to a *storyline* which offers pre-existing *positions* with incumbent rights and duties. The subject of positioning may accept or comply with the offered storyline, or reject it in favour of their own.

In this paper we conceptualise identity as being enacted, within a specific context, through a multitude of positioning actions. We also adopt the ontological stance of O'Mahoney and Marks (2014) who view the social and subjective elements of identity as being distinct, but related to, one another. O'Mahoney and Marks (2014) characterise social identity as concerning “the actual embodiment of the roles and categories that are generated in social structures” (p. 72) while personal identity “emerges from the embodied, reflexive self, in part forged through the interests and actions of the individual” (p. 71). Consistently, we propose that students enact *social mathematics identities* when positioning themselves in response to recognisable roles offered by social structures in their mathematics learning context. *Personal mathematics identities* materialise when students draw on their individual experiences to position themselves as mathematics learners.

Social and personal mathematics identities interact with one another through *agency*, or the actions undertaken by a student to steer learning towards their goals. Both forms of identity are reproduced and transformed by the student's agentic actions. Accordingly, we define *mathematical identity* as a synergy of social and personal mathematics identities that interact through agency (see Figure 1).

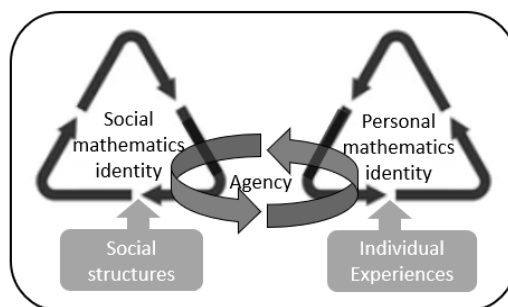


Figure 1: Mathematical Identity (adapted from O'Mahoney & Marks, 2014)

In the context of this paper, social mathematics identity emerges from roles that the university, the mathematics department, lecturers, or tutors offer to mathematics students. Students inhabit these roles in unique ways, enacting recognisable social identities. For example, in describing the ease with which they are able to meet curriculum requirements, a student positions themselves relative to a course structure. Personal mathematics identity is an outward expression of internal processes like preferences or feelings, that are based on individual experiences. To enact their

personal identities, students draw on storylines from their past experiences. For example, a prior experience of productive struggle might enable a student to describe themselves as one who enjoys the challenge of mathematics. Viewing mathematical identity as a synergy of two distinct parts allows us to separately consider the individual's response to opportunities offered by society and their own reflexive understanding of themselves as a mathematics learner.

RESEARCH QUESTION

The overarching aim of our research is to explore the experiences of international students transitioning to university mathematics. Due to space limitations, in this paper we explore the case of one participant from China transitioning to an English-medium university. We ask how the structures of first-year mathematics, and the experience of learning school mathematics in China, shaped the mathematical identity of an international student?

METHODOLOGY

This study took place at a large New Zealand university where around a third of the students in first-year mathematics are international students. This paper presents the case of Sunny (pseudonym), a participant who had completed her schooling in China before enrolling in a Bachelor of Science programme in New Zealand, with majors in computer science and mathematics. Before this, she completed a foundation (or bridging) course in English, which included mathematics and other subjects. Her first university mathematics course covered standard topics in linear algebra and calculus.

Sunny took part in an individual, semi-structured reflective interview with the first author, lasting 23 minutes. It was video-recorded and transcribed. At the time of interview, Sunny was enrolled in a postgraduate level honours programme specialising in computer science. Interview questions encouraged her to summarise her mathematics learning experiences, and to recount how she evaluated progress, and what challenges she faced, both prior to and during the STT.

First-year mathematics incorporates an array of institutional structures which might suggest what is valued in this setting. For example, the language of instruction may suggest that mathematical communication in English is more highly valued than in other languages. Our analysis began by categorising positioning acts described by Sunny as being resourced by university structures or by individual experiences, thus as contributing to social or personal mathematics identities respectively. For example we viewed Sunny's statement that she tried to complete mathematical problems "as fast as possible" as being resourced by the assessment structure of first-year mathematics, which includes time restricted tests and an examination. On the other hand, we categorised her declaration that "it's okay to have some difficulties" as being resourced by individual experiences. We examined the stories Sunny told to understand what challenges she faced in first-year mathematics, and how she responded to these.

FINDINGS

How structures of first-year mathematics shaped social mathematical identity

The storyline within which Sunny viewed success as a mathematics learner was resourced by the structures of her course. She explained how she evaluated her progress in first-year mathematics.

I think for me is like just, when each assignment counts, if I can do all the questions by myself, or maybe I'm not, but that is my goal – I would like to do a lot of questions by myself. And also like, try to be as fast as possible, I mean just take shorter time as that I can easily remember, I have already remember all those things.

Her perception of success as being able to complete tasks both independently and quickly replicated the performance requirements of the mid-semester course test and the final exam.

Sunny went on to describe the challenges standing between herself and success of this nature. While she found that the content in her first-year was “not that difficult,” she believed that “the biggest problem would be just the problem of learning English, and that is the hardest thing.” She explained that she could understand concepts and communicate using written mathematical symbols, but had difficulty communicating her understanding verbally.

I can understand those things, like the symbols. But if you asked me to explain it in English of how those symbols like you can explain to others, it will be a bit hard to do that part.

Understanding course content was not entirely without challenge either. For instance, she shared her struggle to grasp “the definition part, with the, like epsilon all those things.” Sunny asked her course lecturer for help.

I mean I asked them and I still don't understand. They want to help, but you know I just don't understand. I think it's the language along with the concept. The concept is also hard.

In this reflection we see how cultural understandings of student-teacher relationships may have contributed to Sunny's reluctance to persist until concepts were clear. But language also presented a barrier. In these, and other instances, Sunny enacted a social mathematics identity of a student proficient in meeting most mathematical requirements of her first-year course, but struggling to meet communication requirements in the English language.

How individual experiences shaped personal mathematics identity

Sunny also drew on her individual experiences of learning mathematics to enact her personal mathematics identity. She explained that she had completed high school in China before introducing a storyline that positioned “the content here [in New Zealand] is not that difficult comparing to the content learning in China.” Through such storylines we recognise Sunny as a first-year student with advanced prior knowledge and one accustomed to learning difficult mathematical concepts. She later described how “I quite enjoy learning math itself, but there's still some difficulties there. But I

think it's okay to have some difficulties." This relayed her sense of enjoyment when learning first-year mathematics, and strengthened her position as a student who was not only accustomed to challenge, but one who embraced it.

Agentive actions to overcome barriers to success

Sunny's identity as a mathematically competent student who was accustomed to challenge shaped her actions in first-year mathematics. She regarded language as her biggest challenge, indicating that for her the difficulty lay, not with learning mathematics, but with learning it in English. Through her reflections, we see this difficulty manifesting in three aspects: understanding the course lecturers, communicating mathematics verbally, and collaborating with peers.

Sunny found lectures difficult to follow, saying that understanding was easier when verbal explanations were accompanied by symbolic notation. She explained how she would revise the lecture content "again and again by seeing, for me it's more about seeing the lecture notes, and also maybe listening to the recording to help me understand." She also sought external resources that offered visual explanations, such as YouTube videos where "they explain things in some animations, which is better." These actions show how Sunny invested considerable time and effort to overcome the challenge of accessing lecture content delivered in English. Communicating her mathematical understanding to others also presented a challenge. She told how she spent significantly more time completing assignment questions requiring verbal explanations than those requiring calculations.

Collaborative sense-making is an important aspect of first-year mathematics at Sunny's university. Students work together in problem-based tutorials. The mathematics department also offers 'assistance rooms' which are specific spaces where students can gather informally to work on mathematics in the company of others and a tutor. Sunny shared that in both tutorial sessions and the mathematics assistance room, she would "tend to find people speaking Chinese and discuss together about those questions." So she took part in collaborative sense-making activities by seeking out Chinese speaking students with whom she could communicate fluently.

DISCUSSION & IMPLICATIONS

This paper proposes a novel way of conceptualising mathematical identity that proved useful, in Sunny's case, for illuminating the effect of past experiences on more recent actions in a different social context. The same theoretical framework was applied to other participants in the study with similar outcomes. This leads us to suggest that a view of mathematical identity as the synergy of distinct parts may be more generally useful for understanding links between learner engagement across time and contexts.

Sunny's experience highlights some factors that contributed to her successful negotiation of the STT, and illuminates areas where additional support might have been helpful. The agentive actions described by Sunny evidence how her experience of productive struggle with mathematics in China supported practices that enabled her to

overcome the new, language-based challenges of learning mathematics at a foreign, English-medium university (Di Martino et al., 2022; Hernandez-Martinez & Williams, 2013). Her actions reproduced the identity of one who was both accustomed to, and embraced challenge when learning mathematics, and reinforced her identity as a mathematically competent student in the new sociocultural context. Other studies (e.g. Hernandez-Martinez & Williams, 2013) have shown how prior experiences can contribute to resilience in the STT, but studies of international students in the STT are scarce. By exploring how the backgrounds of international students shape transitional experiences, this paper follows Di Martino et al.'s (2022) call for research that contributes to understanding of the STT in different cultural contexts.

We acknowledge the idiosyncrasy of Sunny's circumstances and recognise that not all students have such well-established storylines to draw on, where positions support success in the STT. However Sunny's experience of language difficulties is certainly not unique. Sunny expressed a preference for "seeing" the lecture content by repeatedly re-reading the course notes after lectures. We also note that she did not rely entirely on symbolic representations (Wood et al., 2007). She referenced a range of conceptual representations, seeking animations online and listening to lecture recordings. We call attention to the efforts demanded of international students to overcome language barriers, and encourage host universities to take steps to support them. Providing lecture recordings and written course notes allow international students to access lecture content in ways that, though time consuming, are more suited to their needs. Indexing collections of appropriate resources, such as video animations that aid understanding of course concepts, should similarly be considered.

Sunny's strategy to overcome language barriers when working with other students (Hwang et al., 2022) was to join groups where others spoke her own language, making mathematical communication easier and communal sense-making more productive. Yet communicating with university lecturers remained challenging on two fronts. Both the academic English encountered in these interactions and cultural understandings of the student-teacher relationship (Lillyman & Bennett, 2014) raised barriers to understanding. Mathematics departments might mitigate these challenges by considering how first-year students can be introduced to graduate students, tutors, and lecturers who speak the language of international students in transition.

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SHIFTS IN LOCAL NARRATIVE IDENTITIES: A CASE OF LOW ACHIEVING STUDENTS.

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Narrative identities provide a powerful tool for investigating affective aspects of students learning mathematics. Through the case of Cora, a 10th grade student, I introduce a prototypical local narrative identity of a student with a history of persistent low achievement in mathematics. In this paper I also discuss how a sequence of activities based on the use of digital artifacts, open-ended and unfamiliar tasks, can affect such an identity.

INTRODUCTION

The project within which this study is situated aims at identifying characteristics of low achieving high school students' discourse in the context of algebra, both within a mathematical dimension and from an affective dimension. Using a commognitive approach, we wish to study whether and how such characteristics (mathematizing discourse and identity) change during students' work with appropriately designed activities involving digital artifacts. This paper addresses the case of Cora, focusing on her shift in identity at the end of her participation in the experimentation. In particular, we will analyze and discuss her interviews at the beginning and the end of the two months intervention. Such an intervention made use of digital mediation. Indeed, Baccaglioni-Frank (2021) has shown in fine-grained detail how digital mediation is a way of granting struggling students the possibility of constructing meaningful (both to them and mathematically) narratives. In the research project of which this paper addresses a specific issue, students participated in 5 sessions with a researcher, during which they worked on activities with digital artifacts concerning the notion of equation or function. Cora was assigned to the sequence on functions.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

The commognitive perspective defines *identity* as a “collection of stories”, constructed through the activity of *subjectifying*, that is communicating about the participants of the discourse (Sfard & Prusak, 2005). Utterances of subjectification can be classified into three main levels, based on how general the subjectification is. The first level concerns a specific performance (e.g., She forgot how to solve this equation); the second level a routine performance (e.g., I can't solve equations); and the third level concerns a person's inherent property (e.g., You are terrible at math) (Heyd-Metzuyanin & Sfard 2012). Moreover, a set of subjectifying stories must be *reifying*, *endorsable*, and *significant* to constitute an identity. A story has reifying quality when it attributes stable characteristics to the participants (e.g., I'm good at math), a story is endorsable “if the identity-builder when asked, would say that it faithfully reflects the state of affairs in the world” and a story is significant “if any change in (the narrative)

is likely to affect the story teller’s feelings about the identified person.” (Sfard & Prusak, 2005, pp. 16–17). Hence, when studying identities, it is key to infer whether a subjectifying utterance is reified. The reified utterance can be attained with:

- Third-level subjectification utterances (e.g., He doesn’t understand);
- Recurring first-level and second-level subjectification utterances (e.g., I don’t know how to solve this equation...I can’t do this. Maybe... no, I don’t know it);
- Second-level subjectification utterances followed by adverbs such as “always” and “never” (e.g., She has always had difficulties with equations) (Heyd-Metzuyanin & Sfard, 2012).

In this paper, I will talk about first-person narrative *local* identities, which refer to stories told by students about themselves to the interviewer with whom they interacted for this study, in the specific context of the research center where the project took place. For the sake of brevity, we will speak of these simply as *identities*.

In this frame, Wiener’s attribution theory appears to be a useful tool to gain insight into first-person identity and its possible changes (Wiener, 1985). Wiener classifies the individual’s attributions of failure considering three independent dimensions: the degree of stability (fixed versus variable), the locus of control (internal versus external), and the controllability (controllable by the subject or not). The attributions related to uncontrollable causes, preventing the student from being the protagonist of their learning (Zan & Di Martino, 2009), can affect students’ opportunities to develop richer mathematical discourse. For example, if a student is convinced that they are not capable of solving any mathematical equation because they are bad at math, they are likely to develop a failure identity in mathematics with a possible negative impact on their striving for higher achievement. Thus, we propose to include causal attribution as a characteristic of students that may provide evidence for delineating identities.

In this paper, we explore the following research questions: *What is Cora’s identity built at the beginning of the sessions and how does it change by the end? What aspects of the activities, according to Cora, might have contributed to the shift in her identity?*

METHODOLOGY

The data collection to which Cora took part lasted two months (October-November 2022), and it involved twelve 10th-grade students (ages 14-15) with a persistent history of low achievement in mathematics. All students participated in the project as volunteers, in the afternoons (not school hours), coming to a research center in central Italy. Each participant took an initial interview, then came for five two-hour meetings, and once again for a final interview. Both the interviews and the sequence of activities were conducted by researchers whom the students had never met before.

The goal of the initial interview was to gain insight into students’ mathematizing discourse on the notions of equation and functions, and into the mathematical identities of participants. The interview was organized in two different parts: the first one

consisted of four questions focused on the relationship with mathematics, whereas the second one was a task-based interview.

The final interview was developed for assessing possible variations in the participants' mathematical identity; it consisted of questions on the sequence of activities on functions. During the 5 sessions with the activities with digital artifacts, Cora worked with another student, a classmate from her school (since before the study). All activities were video and audio recorded through multiple cameras and screen captures.

The activities with digital artifacts aimed at fostering students' sense-making processes involving functions as mathematical objects. These activities were mostly open-ended and unfamiliar (different from typical “school tasks” on functions). Functions are usually presented as algebraic formulas at the beginning of high school in Italy, their covariational properties are not usually considered: on the other hand, literature in mathematics education shows the widespread students' difficulties in managing variables and the Cartesian graph (e.g., Thompson & Carlson, 2017, Baccaglini-Frank et al., in press). Baccaglini-Frank (2021) shows how specific activities – where sense-making is strongly stimulated – can contribute to make students' mathematical discourse richer. We conjecture that those activities can also foster changes in students' failure identities.

During the two months of the project, at school the participants were learning how to solve and manipulate fractional equations, so participants were not working on functions at school during the project period.

To gain insight into the participants' identities, we adopted the analytic scheme in Table 1 (Heyd-Metzuyanim & Sfard 2012).

Table 1: Analytic scheme used for the analysis of the data.

Aspects to investigate	Questions guiding the analysis	Indicators
Type of identity being developed (successful or failure).	What direct identifying expressions does the student use in the initial interview? And in the final interview?	Third level subjectifications. Second level subjectifications followed by adverbs such as “always” and “never”.
	What non-direct identifying expressions does the student use in the initial interview? And in the final interview?	Recurrent first and second level subjectifications.
Attribution theory as features of identity.	What elements of unsolicited attributional activity appear in the first interview? And in the final interview?	Explicit reference to cause of success through expressions such as “it is the fault of...”, “maybe because of”.

DATA ANALYSIS

To gain insight into Cora's initial identity, I shall analyze Cora's identifying utterances emerging from the initial interview. For brevity, from the first part of the interview, I included only the expressions where identifying is most present. From the second part of the interview, I included transcriptions from two tasks. I chose one because it concerns the equation Cora studied the previous year and is formulated as a typical "school task". I chose the other task, relating to the functions, because functions are the object of the following sequence of activities.

Initially, Cora seems to have developed a failure identity that is not generalized to all the mathematical activity.

- 1 C let's say that I'm a little savvier with some topics, while other topics I really don't... Maybe I even spend my days trying to understand it but then... it's also the book's fault because basically, it doesn't explain very well, it's the teacher's fault, I just can't do some things...

Such an identity seems to be stable over time because Cora speaks of long-lasting difficulties she has had with mathematics since she was in 8th grade.

While Cora is performing the first task in the interview (solve $13 - a = 13 + 11$), her failure identity appears through a first recurrent level of subjectification.

- 2 C So, here I wouldn't know, I might think of... an equation? ...No, I really don't know. No! Neither. I really wouldn't know. I am thinking that if a is a monomial and the others, well they are integers, and at a first glance I would consider it an equation.

Her identity leads Cora to describe herself as unable to perform the task: she decides to move on to the next question.

- 3 C: No, I prefer to skip this one because I really have no idea, I have no idea how to solve it, I really can't come up with [how to do it].

She makes similar utterances for all the tasks, including the one on functions. [At first, only the Cartesian graph of $y = -2x + 1$ is shown, without any explicit task]

- 4 I: The first thing I ask you is the same as always: have you ever seen something like this before?
- 5 C: Yes, it should be a function. Now, me and functions, they are exactly one of the contents that...with which I have had most difficulties, among them all. Because really, I don't...it's the one I have had most difficulties with. Anyway, this here I think it's a function. That, right, rule about the line in a ...in a space... So, in this case, yes it's a function, it should...

The identifying expressions include a recurring first level of subjectification [5] and a second level ["I really never understood functions, also how ... how we worked on them to find the results, I really never understood them"]. Again, Cora describes herself as unable to perform the proposed task. Therefore, functions are one of the topics through which Cora has built her failure identity.

During the initial interview, Cora repeatedly highlights how the ability to succeed in mathematics is linked to her ability to “use logic” in mathematical activities.

- 8 I: Is there something you like about math?
 9 C: So ehm surely when in the end I succeed in something, I mean the satisfaction in the end, maybe when you have to use logic... in the end, I succeed, and the result is right.

This is also consistent with her rejection of memorized procedures.

- 10 C: I really struggle with all the stuff to memorize, keep in mind, the formulas. So, it gets harder, maybe with logic, it's better. Instead, the things to keep in mind are just not.

So, for Cora to be successful she seems to believe that it is necessary “to use logic” in mathematical discourse [9,10]. Instead, when she tries to rely on her memory, she recognizes that she finds it hard to succeed [10]. Struggling with memorization is not referred to a specific performance but it is described as a stable property of Cora’s, so I consider it part of her identity. Moreover, I expected to find these features in the identities of other low-achieving students.

The attributional theory also provides elements to investigate Cora’s identity. Cora identifies external, uncontrollable causes such as the book and the teacher as the causes of her failure [1]. Such an attribution makes it impossible for Cora to become the protagonist of her own learning; indeed, she cannot be able to have full control over it. Even remembering when, up to 7th grade, she had no problems with math, Cora attributes that success to external and uncontrollable causes, such as good teachers. (“So, let's say until middle school...the first, the second year of middle school was still going well anyway, because the teachers were good, like in elementary school.”).

From the final interview, it appears that Cora's identity has changed during the sequence of activities.

- 11 I: Between these activities, is there anything that particularly affects you?
 12 C: Surely the ones about functions, because when I arrived here and saw that there were activities on functions... I thought that... that I wouldn't have succeeded, in any case, I wouldn't have understood what we were talking about, because [functions] were exactly a topic that I never understood. And instead, seeing that I was also able to reason about it, then being able to answer the questions... That is, I was quite amazed by the fact that I was able to do it, perhaps thinking about it calmly... but that I managed, in the end, to conclude...because I didn't think I could do it.

Cora recognizes herself as “able” and seems "amazed" [12] suggesting how strong her previously constructed identity was. A subjectifying story is defined as *significant* if a change in one of its features affects the feelings of the identity builder; indeed, the change in Cora’s failure identity is not a matter of indifference for her (“I was quite amazed by the fact that I was able to do it”).

Now Cora claims to "reason" about functions and "answer the questions" in the tasks she was assigned [12]. At the beginning of the activities, the failure identity could have

prevented Cora from seizing learning opportunities like the one she was being offered (“I thought that I wouldn’t have succeeded”) but (fortunately) this did not happen. The fact that she was allowed, and, actually, always asked to make sense of what she was saying and doing, is a feature she emphasizes as something she appreciated very much. It is also coherent with her previously described refusal to memorize. Cora not only says that she was able to answer the questions, but that she was able to do it by reasoning [12]. So, being able to make sense of a mathematical content that was so difficult for her seems to have led to a change in her identity.

However, activities based on the use of digital artifacts, the open-ended and unfamiliar tasks, are not the aspects that Cora identifies as having the highest quality.

13 I: In general, concerning this experience, is there anything you liked?

14 C: So, I really enjoyed the experience because, compared to how mathematics is done at school, it is a very different approach... I mean, there is no pressure, there is no ... and I also understood several things that I did not understand before, especially with the work I've done, and ... [...] And so that is, I liked more than anything else the fact that there was no pressure, that it was possible to think calmly about things. [...]

15 I: Do you have any suggestions to improve the sequence of activities?

16 C: I don't have any, because I was comfortable and had a good time. That is, in my opinion, also being able to reason a lot about things really helps.

Reasoning “a lot” [16] and “calmly” [14] about the tasks, the absence of “pressure” [14] suggest that time is the factor that Cora identifies as the determining in her successful experience. This even seems to change her causal attribution of the previous failure.

17 C: Yes, functions in general, for example, I've never understood them well, I've never really had them clear. Partly because we had studied them quickly, partly because... I just didn't understand them

Now Cora points to time as the cause of her previous failure. The cause is still external: the context suggests that Cora means “we studied them too quickly in class” and not “I studied them too quickly”, so whoever decides the time dedicated to functions is the teacher, not Cora. However, time is a much more circumscribed and precise causal attribution than the initial one.

CONCLUDING REMARKS

In light of the above findings, we show that the identity of Cora is a failure identity not generalized to all mathematical activities. Her identity is also characterized by a difficulty in memorizing and by her attribution of the failure to external and uncontrollable causes (such as the schoolbook and the teacher). Such a causal attribution shows how Cora is not able to have full control over her learning.

We believe our data supports the conclusion that, after the sequence of activities, something about Cora's identity changed. Functions, one of the topics Cora believed she could not learn, finally becomes accessible. Believing that she is able to understand

mathematics is the first step in enabling her mathematical discourse; indeed, the literature shows how a failure identity can prevent a student from taking advantage of learning opportunities (Heyd-Metzuyanim & Sfard, 2012).

In the broader project we hope to find shifts in identities during the implementation of the designed activities, like the one described in Cora. Our hypothesis was that such shifts could be favored by the activities proposed, where solution procedures are never given explicitly, digital artifacts are manipulated by the students as they engage in reasoning processes, and their sense-making is fostered through open-ended tasks. Cora, instead, identified having time to reflect as the higher-quality element. This is a relevant finding for two main reasons.

First, thinking that one needs more time to understand math can be read as a deeper change in the student's identity than, for example, believing that one needs digital mediation. Indeed, taking more time to think calmly can be replicated in other learning circumstances.

Secondly, it also affects the previous failure's causal attribution. The previous failure is no longer attributable to the book or, in general, to the teacher but to the fact that the topic was covered in class too quickly. Although it is still an external and uncontrollable attribution, it becomes more specific and influenced: the student cannot change the teacher or the book, but she can certainly ask the teacher to repeat and deal with some topics more slowly.

LIMITATIONS AND IMPLICATIONS FOR FUTURE DIRECTIONS

This study has potential limitations. We have analyzed *local* identities closely linked to the context, but we do not know how this impacts the identities built at school. In future research, we would like to explore the relationship between identities built at the research center and those built at school, and try to create bridges between these contexts. Another limitation is not having analyzed, due to lack of space, the relationship between shifts in identities and shifts in mathematizing discourse. Finally, the importance given to time by Cora provides a significant element for the future redesign of activities.

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ALGEBRAIC STRUCTURE SENSE IN A BLIND SUBJECT

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Literature about the way visually impaired students approach mathematics is still very scarce, especially in the case of algebra, despite the fact that mathematical content is known to be increasingly accessible thanks to assistive technologies. This report presents a case study aimed at describing the process of algebraic symbols manipulation by a blind subject. Results show how the hearing can substitute the eye in the development of structure sense. Data analysis reveals that screen readers and coding languages (as LaTeX) have interesting potentialities in the development of structure sense for blind students.

INTRODUCTION

Several authors in the field of mathematics education have been discussing about the importance of visualization within different mathematical activities (e.g., Presmeg, 2006). The term ‘visualization’ itself refers to the sense of sight that may result as a way of accessing mathematical representations – including geometrical shapes, graphs, and formulas. According to Sfard (2008), the mathematical discourse itself is defined (among other features) by specific *visual mediators*, defined as “visible objects that are operated upon as a part of the process of communication” (p. 133). Radford (2010) describes the process of learning mathematics – particularly in the algebraic context – as a process of domestication of the eye, meaning “a lengthy process in the course of which we come to see and recognize things according to ‘efficient’ cultural means” (p.10). However, for some students, the sense of sight is not, totally or partially, suitable to access mathematical content because of visual impairments or complete blindness.

The literature about the learning of mathematics for blind students is very scarce, but we know that they can access mathematical contents through other senses (e.g., Alajarmeh et al., 2011; Healy & Fernandes, 2011). Indeed, Radford (2010) recognizes that the same lengthy process of domestication could happen for other senses, and Sfard (2008) defines *realizations* of mathematical objects as “perceptually accessible things” (p. 154), without specifying the nature of such perception. Furthermore, she recognizes that gestures can realize mathematical objects (Sfard, 2008). Healy and Fernandes (2011) have, indeed, observed that blind subjects may involve gestures in their appropriation of mathematical meanings; these authors see the gestures used by a blind subject as re-enactions of previously experienced activities. Their arguments are convincing in the case of geometrical figures and solids: the gesture may correspond to past experiences of touching, moving a finger along physical artifacts. Similarly, the description of a graph of a continuous function (or even a discrete graph) can be based on the embodied experience of motion (Núñez et al., 1999). On the contrary, other

mathematical representations – like algebraic symbols – appear much more detached from the sensorial experience and we wonder how blind students can get access to this kind of representations. Assistive technologies and the braille alphabet may be suitable means (Alajarmeh et al., 2011; Armano et al., 2018; Bouck et al., 2016).

To the best of our knowledge, there is no international literature in the field of mathematics education about algebraic symbol manipulation by blind students using screen readers. Hence, this study is a first step forward in filling this research gap by offering a *thick description* (Bell & Kissling, 2019) of the process of algebraic symbols manipulation performed by an experienced blind individual while solving an algebraic task.

THEORETICAL FRAMEWORK

The involvement of digital technologies within the process of teaching/learning mathematics has provided many new opportunities for visually impaired students who can rely on screen readers (and other assistive technologies) to access written text, including algebraic formulas (Alajarmeh et al., 2011; Armano et al., 2018). However, using algebraic symbols for mathematical problems solving does not only require reading the symbols, but being able to act upon them. While studies have focused on how digital textbooks can aid students' algebraic activity (e.g., Bouck et al., 2016), there is a dearth of research about systems for enabling students to act productively on symbols (Alajarmeh et al., 2011).

When we refer to algebraic symbol manipulation, we consider that it encompasses more than just the rote application of transformation rules. It involves a broader competence in using “equivalent structures of an expression flexibly and creatively” (Linchevsky & Livneh, 1999, p. 191), that is briefly named *structure sense*. According to Hoch and Dreyfus (2004), in the context of school algebra, structure sense can be described as composed of six abilities which are: (1) seeing an algebraic expression or sentence as an entity; (2) recognizing an algebraic expression or sentence as a previously met structure; (3) dividing an entity into sub-structures; (4) recognizing mutual connections between structures; (5) recognizing which manipulations it is possible to perform; (6) recognizing which manipulations it is useful to perform.

While describing a specific case study (see Method section), we are here interested in understanding if and how blind subjects can rely on their structure sense while solving an algebraic task, the accessibility of which is provided through digital tools. Our research question is: How can a blind subject rely on his structure sense while solving equations if supported by assistive technology?

METHODS

Due to the nature of our research question and because of the paucity of research literature on the topic, we have chosen to conduct an exploratory case study. Such design is recommended when the aim of the research is “to portray ‘what it is like’ to be in a particular situation, to catch the close up reality and ‘thick description’ [...] of

participants' lived experiences of [...] a situation" (Cohen et al., 2007). Aiming at describing how a subject draws upon his/her structure sense, we interviewed an adult person who has a strong education in mathematics, to whom we refer with the pseudonym of Antonio. Antonio, who has a degree in Physics and has worked as a fellow researcher for 2 years, became blind four years ago due to a degenerative pathology. He learned to use LaTeX with speech synthesis as a visually impaired undergraduate student, 10 years ago.

We opted for a task-based interview: the interviewee was asked to select two equations among those proposed by Hoch and Dreyfus (2004) and to solve them. Because of the space limit, we will present only some excerpts from the solution process of one equation (Figure 1b), about which the solver was particularly talkative. The task was presented through a PDF file (Figure 1) which was implemented with the *Axessibility* package for LaTeX (Armano et al., 2018). By adding a single line of code to the source LaTeX file (line 2, Figure 1a), this package automatically inserts a hidden alternative text in the PDF document at each formula which is then accessible to screen readers (e.g., Jaws, NVDA). On Antonio's computer the screen reader NVDA was installed, allowing him to hear the read-aloud of LaTeX code. The LaTeX code for the equation in focus is shown in the box 'a' of Figure 1. In particular, the command `\frac{}{}` represents a fraction having as numerator the content of the first curly braces and as denominator the content of the following curly braces.

Figure 1 consists of two rectangular boxes, labeled 'a' and 'b'. Box 'a' contains LaTeX source code for a document class, package loading, and a mathematical equation. Box 'b' shows the rendered PDF output of the same code, displaying the equation as a mathematical formula.

```

1 \documentclass{article}
2 \usepackage[accsupp]{axessibility}
3 \begin{document}
4 \[ \frac{1}{4} - \frac{x}{x-1} - x = 5 + (\frac{1}{4} - \frac{x}{x-1}) \]
5 \end{document}

```

$$\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1} \right)$$

Figure 1. Example of LaTeX code (box a) and corresponding compiled PDF (box b).

Aiming at a thick description (Bell & Kissling, 2019) we collected several sources of data including “speech acts; non-verbal communication; descriptions in low-inference vocabulary; [...] recording of the time and timing of events; the observer's comments [...]; detailed contextual data” as prescribed by Cohen et al. (2007, p. 405). This was realized by recording the interview including in the audio- and video-recording of the interviewee (through a webcam) and capturing the interviewee's computer screen. The second author of this report acted as interviewer and took personal notes during the interview. The video was transcribed verbatim by the first author integrating the transcription of ‘what is said’ with descriptions of ‘what is done’ (e.g., Table 1) – as recommended by Sfard (2008) – and with screenshots. Screenshots have been elaborated adding arrows representing the movements of the cursor (Figure 2); the final position of the cursor is represented by a vertical line. The three authors have analyzed this enriched transcript by coding each line with the six components of structure sense

described in previous section. The coding process was discussed and reviewed among the three researchers until consensus was achieved.

RESULTS

The first excerpt refers to the first reading of the proposed equation. Antonio uses the functionalities of the Axessibility package (Armano et al., 2018) to hear the reading of the LaTeX code behind the PDF that we provided to him. Then, he decides to copy/paste the LaTeX code on the Microsoft Notepad application. The NVDA software reads the code out loud while his cursor navigates through the Notepad, as captured in the video.

Table 1. Enriched transcript of the first excerpt.

Line	What is said	What is done
1	Antonio: Let's see the structure. One fourth.	The cursor moves till the denominator of the first fraction and stops right before the closing curly brace.
2	Interviewer: While you are understanding the structure, would you like to tell it?	The cursor moves forward and stops after the minus sign.
3	Antonio: One fourth, yes.	The cursor moves back, before the minus sign.
4	Minus x over... Over x minus one. Minus x equals... Then there is the second member of the equation.	The cursor moves forward and stops right before the equal sign.
5	Five plus, open bracket. Then there is a bracket.	The cursor reaches the opening bracket.
6	One fourth inside the bracket.	The cursor moves back and forth over the <code>\frac</code> command.
7		The cursor reaches the end of the last fraction
8	Minus x over x minus one.	The cursor moves back and forth over the denominator of this fraction, then it goes back to the minus sign.
9	Closed bracket and that's it.	The cursor moves forward till the end of the equation.
10	Here I would start by working on the brackets.	

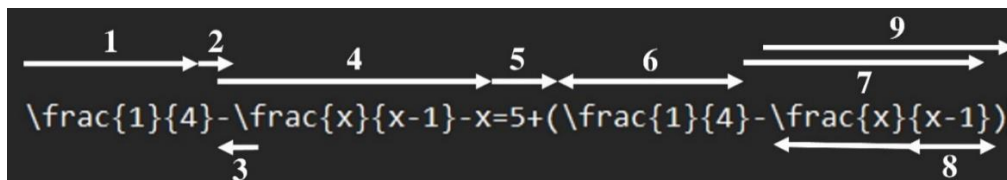


Figure 2. Cursor's movements during the first excerpt. Numbers are keyed to Table 1.

Despite not being familiar with the theoretical framework of this paper, Antonio starts by expressing the intent to understand ‘the structure’ (line 1). Since this strategy corresponds to reading the different sub-structures (the fractions, the parenthesis) of the equation, he is relying on the third component of structure sense, which is “divide an entity into sub-structures”. Indeed, by analyzing the movements of his cursor, we

can see that his attention is initially caught by the first fraction (lines 1-2). Then he analyzes the second fraction and stops when the left side of the equation finishes (line 4). The presence of the bracket is noticed (line 5) and then he goes back and forth over the two fractions within the brackets (lines 6-9). Then, we can notice that – even if the readers would normally force a left to right reading – Antonio analyses the structures of specific parts of the equation by realizing multiple readings of the identified sub-structures. This is particularly true for the fractions; their presence is highlighted by the LaTeX commands and the curly braces identifying the numerator and the denominator.

After this first excerpt, Antonio copies the whole equation on a second line in the Notepad. He recognizes that he can work inside the brackets first (component 5 of structure sense) and prepares the environment for doing so: he creates many blank spaces before the closing bracket (second line in Figure 3), then he states that he can calculate the least common multiple of the denominators (again component 5). He writes the denominator in the obtained blank space within braces (second line in Figure 3) and then adds a couple of braces before (third line) – so preparing the space for the numerator. He performs his calculations for the numerator between these braces (fourth and fifth line in Figure 3) and when he is done, he adds the command `\frac` before the braces (sixth line). Finally, he replaces the content of the brackets with the calculated fraction (last line in Figure 3). Then, he recognizes which manipulation he can realize and uses the braces as containers for organizing the structure of the result of such manipulations. After these manipulations, Antonio decides to work on the fractions on the first side of the equation, as shown in the excerpt in Table 2.

```

\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
)
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
{4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
){4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
{x-1-4x}{4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
{-1-3x}{4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(-\frac{3x+1}{4(x-1)})

```

Figure 3. Different phases of Antonio's manipulations of the fractions in brackets.

Table 2. Enriched transcript of the second excerpt.

Line	What is said	What is done
11	Antonio: Ok. Now let's see what was here.	The cursor moves back to the beginning of the equation.
12	There was one fourth.	The cursor moves till the end of the second fraction
13	And then there was the same thing as in the brackets, but outside.	Then the cursor moves back to the beginning of the equation.
14	Thus... Thus, the result is the same of the other side because it's equivalent.	The cursor moves till the end of the left side of the equation.
15	Then I can copy this.	The cursor moves to the fraction on the right side of the equation, which is then selected.

16		The fraction is copied in a new following line on the Notepad.
17	Then I copy ‘minus x equals’.	He types ‘-x=’.
18	Well, I can copy and paste the second member. Yes, I copy and paste it.	He selects, copies, and pastes the right side of the equation.
19	Thus, since they are... the fractions are equal but opposite in sign... because if... then, I bring at the second member what is in the first member, they are equal and opposite. The result should be...	The cursor goes back to the beginning of the equation and then moves till the end of the right side.
20	x equals minus five.	He moves to a new line and write ‘x=-5’.
21	Let me check...	The cursor goes back to the previous line and moves through the whole line, from left to right.
22	Yes, that should be the result.	

While reading the left side of the equation, Antonio recognizes the same structure of the expression within the brackets (line 13, component 2 of structure sense). Then, he understands that, instead of performing again all the manipulations shown in Figure 3, he can replace the fractions on the left side with the fraction calculated on the right side (lines 15-16, components 4 and 6), which was into the round brackets (Figure 1b). Having two identical sub-structures on the two sides of the equality, he decides to cancel them (lines 19-20, components 4 and 6). Hence, in this short excerpt we can notice many of the components of structure sense intervening and, considering the other parts of the transcript as well, we can observe all the six components of structure sense enacted.

DISCUSSION AND CONCLUSION

We can answer our research question by noticing that all the six components of structure sense have a role in Antonio’s solving of the equation using the Microsoft Notepad and NVDA reader for manipulating the equation represented in LaTeX code. In particular, Antonio uses the reader to read (and re-read) *self-selected* portions of the equation instead of simply reading from left to right. The notepad is used to manipulate the equation both within the same line (differently than what we are used to do with paper and pencil, Figure 3) or connecting different lines (e.g., lines 13-18).

We have noticed that during the first reading (Table 1) the equation is divided into sub-structures (component 3 of structure sense) and then Antonio recognizes which manipulations are possible (component 5) on these sub-structures. As observed for seeing subjects, brackets play a relevant role in structuring the equation (Hoch & Dreyfus, 2004). However, in this specific case, we can notice that also the use of the LaTeX code – allowed by the Accessibility package (Armano et al., 2018) – may play an important role in structuring the equation into sub-structures, since the `\frac` command is often a place where the cursor stops. Furthermore, the LaTeX code becomes not only a tool for reading mathematics, but a tool for doing mathematics as well (in the sense of Alajarmeh et al., 2011). This is visible when the curly braces are

used to organize the space of manipulation, distinguishing the numerator and the denominator of the algebraic fraction (Figure 3).

Antonio provides a telling example of how the LaTeX code can serve as a tool for symbolic manipulation with the interesting ‘side effect’ of being transformable in a PDF file which can be then read both by seeing students and visually impaired ones. As noted by Ahmetovic et al. (2021), LaTeX is a writing system used in all STEM disciplines, then its learning is both useful for academic achievement and for inclusivity. The study presented in this report suggests that the learning and use of LaTeX could promote and support structure sense especially for visually impaired students, but potentially not only.

We have also noticed that Antonio was able to recognize previously met (sub-) structures and use them to shortcut his manipulation (Table 1), so mobilizing many components (2-4-6) of structure sense. This recognition is realized after hearing the reading of the first part of the equation (line 13) by NVDA software; paraphrasing Radford’s (2010) words, Antonio’s ears have gone under a lengthy process of domestication through which they came to hear and recognize things according to an ‘efficient’ cultural mean. The reading of the equation and the memorized ‘sound-track’ acted as realizations of the equation in the sense of Sfard (2008) – being anything but visual. This fact corroborates that other senses than sight can successfully help not only in the rote manipulations of algebraic symbols, but in developing structure sense as well. This suggests that verbalization of the structure of algebraic expression may be an important step in the development of structure sense for blind people, but this is true for all the other students as well (Maffei & Mariotti, 2011). This observation strengthens what has been noted before in the case of LaTeX: adopting inclusive approaches to algebra teaching may be fruitful not only for impaired students, but for the whole class-group as well.

Surly, we must be cautious about the conclusion that we draw from a case study; in particular, we must consider that visual impairments are very different among them. For instance, Antonio was not completely blind during his high school studies and then he might rely on visual memories of algebraic expressions. Different results might be obtained in the case of students born blind and/or that are able to use Braille to read and write mathematical notations. Future developments of our research project will include subjects with different past histories about their disabilities and their learning of mathematics. Nevertheless, we hope that this report could offer a step forward in unveiling the (many) ways in which visual impaired solvers can successfully tackle algebraic equations.

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TEACHERS' LEARNING THROUGH ITERATIVE CONTEXT-BASED MATHEMATICAL PROBLEM POSING

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In a previous study, we used the variation theory of learning to develop a multi-dimensional model (MDM) through which teachers iteratively can modify their context-based mathematical problem-posing (CMPP) products. We now investigate how the MDM can promote teacher learning through iterative task co-design. In a case study, we analyze modifications that a teacher, Adi, embedded in her task during and after tutor-teacher CMPP redesign meetings. We show how tutor MDM-based interventions stimulated teacher learning: some suggestions were not just embraced by the teacher, but also led her to articulate dimensions of possible variation manifested in further modifications she inserted to her task. We suggest implications for future research related to teacher task-designer programs.

INTRODUCTION AND THEORETICAL BACKGROUND

Research has recognized the immense potential of teachers practicing problem posing (PP) and task design for enhancing their mathematical-pedagogical knowledge and their students' performances in problem solving (e.g., Leavy & Hourigan, 2022; Rachamim et al., 2022). That is not to say PP is an easy enterprise for teachers – their initial attempts at PP are seldom of high quality (Crespo & Sinclair, 2008). Then how do teachers, usually novices in task design, “be supposed to formulate good problems?” (Kontorovich, 2020, p.390). Several studies report that tailored professional development can improve teachers' ability to pose valuable problems (e.g., Cai et al., 2020; Grundmeier, 2015). These studies emphasized the need for teachers to receive substantial scaffolding from mathematics education specialists for their PP products to be worthwhile (Rachamim et al., 2022). In this paper, we strive to contribute to our knowledge of teachers learning through PP and task design. A better understanding of what and how teachers learn when engaged in PP and task design can be a basis for developing suitable guiding principles for mathematics education specialists supporting teachers as task designers.

For this purpose, we closely examine the process in which a developing teacher, Adi, modified a context-based mathematical problem that she posed. We focus on the part of her iterative design process that was accompanied by a tutor applying the multi-dimensional model (MDM) – developed for guiding teachers' context-based mathematical problem posing (CMPP) and task design (Marco & Palatnik, 2022). Before introducing the MDM and our research question, we briefly present the *variation theory of learning*, which we used to develop the MDM and in the current paper to conceptualize teacher learning.

Variation theory

The variation theory of learning relies on the premise that learning is always directed at something (phenomenon, skills, or certain aspects of reality) and conceptualized as a qualitative shift in the way of perceiving this “something” (Marton & Booth, 2013). To see or experience an object of learning in a certain way requires the learner to be aware of its specific aspects and discern these aspects simultaneously. Lo and Marton (2012) emphasize that awareness is stimulated by experiencing difference (variation) between two values. When we become aware of a value by contrasting it with another value (e.g., large vs. small), the value is separated from the object of learning, and a dimension of variation is inferred (e.g., size). Then, the object is perceived with both its value and dimension of variation, and the learner can focus on the value alone, naming it and even changing it (Lo & Marton, 2012).

The MDM and research question

Figure 1 presents a visualization of dimensions of variation in teachers’ applied mathematics problem posing, that we developed in a previous study where we investigated modifications teachers incorporated into their tasks during an iterative task design (Marco & Palatnik, 2022). The dimensions we articulated using the variation theory are correctness, authenticity, task assortment (consisting of mathematical diversity, multiple data representations, question-answer format, precision-approximation, and generalization), task flow, and student involvement.

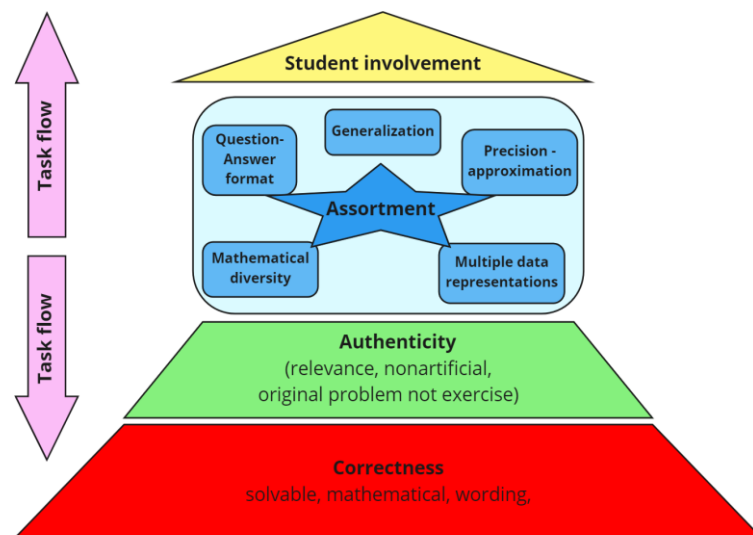


Figure 1: The multi-dimensional model (MDM) of teacher CMPP and task design

Here we elaborate on three main dimensions: Task Assortment – an umbrella dimension that assesses how diverse a task is, containing items that vary along some subdimensions. Specifically, we say that a task has a high assortment if it (i) has items requiring specific case calculations side by side items necessitating generalization; (ii) uses multiple data representations; (iii) blends different mathematical content areas; (iv) has items with diverse question/answer formats; and (v) has both items that require

precise calculations and items that enable approximations. Task Flow – expresses the extent to which items are ordered in a way that evolves the task sensibly towards achieving its pedagogical goals. Student Involvement – the extent to which a task explicitly positions students as active agents in real-life situations (e.g., as consultants in decision-making) and provides human motivation for their mathematical activity. When we developed the MDM, we focused our investigation on teachers' CMPP products, not their PP and task design processes (Marco & Palatnik, 2022). This study addresses the following research question (RQ): What can teachers learn during CMPP iterative task (re)design with a tutor using the MDM?

METHODOLOGY

Research context—PD for CMPP

This paper's data is from the third year of a three-year PD program in which secondary-school teachers designed context-based mathematical tasks. The PD was conducted as a community of practice in which teachers, teacher educators, and researchers collaborate to achieve specific goals (Cooper & Koichu, 2021). We hypothesized that through posing and designing their context-based tasks, teachers would develop their capacity to use mathematics knowledge and skills in real-life challenges, gain ownership over the materials they develop, and have inner motivation to implement them in their classrooms (Koichu, 2020).

This paper focuses on the case study investigating the development of the “Accessibility is not a Privilege” task (see Figure 2) composed by the teacher Adi, during her second year in the PD. We chose this case since Adi made many modifications to the problem she initially posed. The CMPP task design process included the following stages: (i) initial PP design, (ii) receiving feedback from the PD community and redesigning, (iii) first personal meeting with a tutor (the first author, then serving as a community leader), (iv) redesign, (v) second personal meeting with the tutor four weeks after the first meeting, (vi) redesign, (vii) receiving written feedback from the tutor and submitting the final version. From these stages, we report here on stages (iii-vii) focusing on three versions of Adi's task: the first and second versions are the drafts Adi sent for the first and second personal meetings, respectively. The third one is the final version she submitted. Note that the MDM was not yet finalized when the meetings took place – the tutor used it to provide feedback and generate suggestions for task modifications without explicitly presenting its dimensions.

Accessibility is not a Privilege

In Israel, about 18% of the population suffers from physical disability [...] The Equal Rights Law for People with Disabilities (ERL) in Israel states that every person deserves the same rights to access every place and receive any service. The ERL determines all public and residential buildings should be accessible for people with disabilities and in wheelchairs in particular [...] In buildings that are up a flight of stairs, there should be a ramp that allows reaching the entrance to that building.

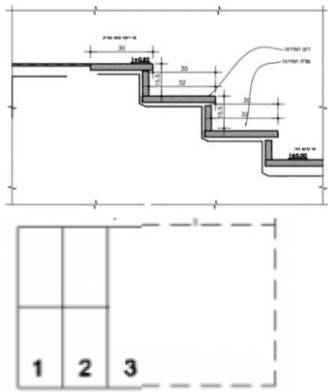
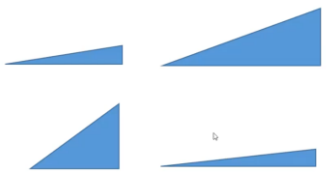
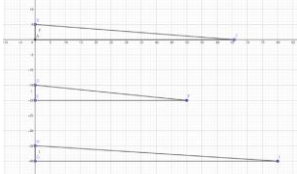
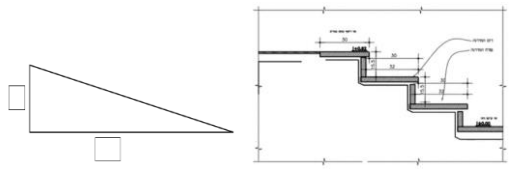
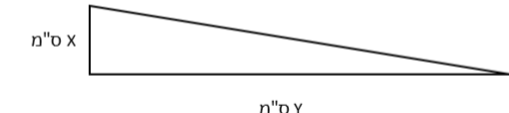
ERL Guidelines for Planning a Ramp:		
<ul style="list-style-type: none"> • Maximum slope for the ramp: 8%. It is advisable to plan a ramp with a moderate slope of 5%. • Ramp minimum width - 1.3 meters. • If the slope exceeds 5%, its length shall not exceed 9 meters. 		
First version (a)	Second version (b)	Third version (c)
<p>(a1) In the figures below, there is a plan for an entrance to a public building and a side section of a stairway. Plan [single masculine form; SMF] a ramp for people with a physical disability that satisfies the ERL guidelines. Pay [SMF] attention to the heights and calculate [SMF] the ramp length according to the regulations.</p>  <p>(a3) Consider [SMF] these right triangles. Data is given in cm. Determine if they satisfy the ERL guidelines.</p> 	<p>(b1) same as a1 (b3) Consider [SMF] these right triangles, each representing a possible ramp for people with a physical disability. Data is given in cm. The upper ramp satisfies the ERL guidelines, and the middle one does not. Determine [SMF] if the ramp in the bottom satisfies the guidelines and justify your answer.</p>  <p>(b4) Does a ramp whose height is 40 cm and length 500 cm satisfy the guidelines? Justify [SMF] your answer.</p> <p>(b5) Why do you think the ERL recommends a 5%-slope, even though the maximal slope permitted is 8%?</p> <p>(b6) According to the guidelines, a mid-plateau that separates between two stairway slopes must be administered after a certain length of the ramp. State your reasons for this regulation.</p>	<p>(c1) Rivka, a regular customer in David's grocery, fell from a ladder when cleaning her house for the Passover Holiday and broke her leg. She will have to be in a wheelchair until her leg recovers. At the entrance to David's grocery, there are a few stairs which makes it difficult for Rivka to shop. Consequently, David decided that it is time to make his grocery accessible to everyone. He needs to plan a ramp at the side of the stairs... The figure below shows a side section of the stairway to David's grocery. Plan [plural genderless form; PGF] a ramp for people with physical disability that satisfies the ERL guidelines. In your plan, pay [PGF] attention to the heights and calculate the length of the ramp according to the regulations. Add the measurement the attached figure below.</p>  <p>(c3) same as b3 (c4) Consider [PGF] a ramp whose height is x cm and length y. Which of the following inequalities is correct according to the ERL guidelines?</p>  <p>(i) $x+y < 0.08$; (ii) $x < 0.08y$; (iii) $xy < 0.08$; (iv) $y < 0.08x$ (c5) and (c6) are the same as (b5) and (b6), respectively.</p>

Figure 2: “The Accessibility Task”, partial comparison between the three versions.

Data collection and data analysis

To assess Adi's learning, we collected data comprising the three versions of her task, video recordings of the two personal meetings, written email exchanges between the tutor and Adi concerning the task, and a video recording of a personal interview with Adi performed by the second author six months after the PD completion.

To analyze the data, we created a three-layered document. The first layer included versions of the task submitted for feedback (i.e., CMPP products, see Figure 2), the second layer consisted of transcriptions of all the video recordings, the third layer contained corresponding screenshots capturing participant's sketches, gestures, and facial expressions during the meetings. Adi's modifications to her task during, between, and after the two personal meetings were categorized according to the MDM dimension(s). We then allocated the parts in the personal meetings where this modification was discussed between Adi and the tutor. Reading the document line-by-line and checking the two other layers, we established who initiated the modification (tutor or teacher) and retrospectively traced verbal utterances, gestures, and sketches justifying the modification. Using the variation theory of learning (Lo & Marton, 2012) as an analytical tool, we discerned cases in which Adi's modifications were accompanied by indicators of her becoming aware of a dimension of possible variation. We then searched for cases of transfer – where Adi used similar MDM consideration for changing other items in the task. Finally, we used the post-PD interview to assess long-term learning, juxtaposing the analysis results with indicators of Adi's awareness of the MDM dimensions.

RESULTS

In this section, we bring empirical evidence to argue that Adi learned in the CMPP iterative design process by developing awareness of and applying task modification corresponding to three dimensions of the MDM: Task Flow, Task Assortment, and Student Involvement.

In the first meeting, before presenting a3 to the tutor, Adi said: “Now, whoever did not apply the Pythagorean theorem yet, will have to use it here.” This utterance shows that Adi did not mind students applying the Pythagorean theorem to two different items of the same task. The tutor implied that a3 was unsolvable since there were no numbers to perform calculations. After Adi said she intended to add them:

Tutor: And what would happen if you won't provide numbers here?

Adi: I will have to make the diagrams accurate (so that) they would have to use a straightedge (to measure). I don't need to give all the sides. Only two numbers for each ramp, in some ramps—a side and a hypotenuse, and in others, two sides.

In her answer, Adi gently rejects the tutor's hinted suggestion not to provide numeric data. First, she doubts the item solvability (i.e., Correctness dimension: “they would have to use a straightedge”) and then specifies which pairs of numbers she will provide

(e.g., “a side and a hypotenuse”). So, at this point, she definitely intended her students to perform numeric calculations using the Pythagorean theorem in item a3.

Tutor: Say, I am thinking aloud here, what would happen if you gave a ‘soft question’ here, without numbers? And you’ll ask if it *satisfies, may satisfy, or doesn’t satisfy* the ERL guidelines. Three options for each ramp.

Adi: Only by way of their appearance?

Tutor: Exactly, through feeling. Intuition...

Adi: You say that when a student approaches this question, he must understand that he himself is going up this ramp sitting in a wheelchair and see if the wheelchair can go through safely.

Here Adi articulates an advantage she sees in the tutor’s suggestion that she previously dismissed. She implies that not providing numbers requires students to “understand” – imagining themselves in the realistic situation as a resource for solving. Adi’s conclusion demonstrates her awareness of the Student Involvement dimension and her acknowledgment of a certain way to apply it in the CMPP task. As the discussion continues, the tutor raises another argument supporting his suggestion:

Tutor: So, we will give them another Pythagorean (theorem)?

Adi: Perhaps they could avoid (using) it (in previous items), but I don’t mind.

Tutor: In a1 they could not avoid it.

Adi: Ah, right, a1 is (with) the Pythagorean (theorem).

Tutor: So OK, they already got it! They did the Pythagorean, they calculated (with) the Pythagorean, OK? We don’t want to exhaust them with seven Pythagoreans. Let’s give them more qualitative understanding now.

Adi: Ok. I love it, actually.

Here the tutor uses questions and exaggeration (“seven Pythagoreans”) to claim that leaving a3 without numbers would increase Task Assortment by increasing the mathematical diversity and thus improve the Task Flow. Adi accepted the tutor’s reasoning (“Ok. I love it”) and later changed the task accordingly (b3, Figure 2). In the second meeting, while debating another modification to the task:

Adi: But they (students) already understand this (the dependence of the slope on the height) in item b2 because they did this calculation, do you understand? I want to do something else that is not based on something they understood in previous items. So that each item will have something special of its own... Because the goal is not to have seven items... Therefore b4, I am doubtful if I should keep it. It feels to me like [in a complaining student voice] “what? another calculation?”

In this excerpt, Adi generalizes the tutor’s argument about the repeated use of the Pythagorean theorem and articulates general considerations that manifest awareness of the Task Assortment and Task Flow dimensions. She expresses it as her desire (“*I want*

to do something else...”) and applies it reflectively to another new item she composed, b4, that was not yet discussed in the second meeting. As the tutor in the first meeting, Adi also uses an exaggeration “the goal is not to have *seven items*” (c.f. “we don’t want to exhaust them with seven Pythagoreans”). Choosing the same number for the exaggeration (“seven”) is a slight linguistic hint connecting the two episodes.

In the post-PD interview, when asked about the final version of her task, Adi reiterated reflective considerations related to Task Flow and Task Assortment, demonstrating that she remained aware of them six months after the PD:

Adi: (At first) I did it very banal: ‘let’s add another item cause it’s nice to have,’ and I made it very computational. And here (version c), I think it opens the mind and simultaneously adds another (mathematical) topic.

We found that Adi applied various design techniques in the modifications accumulated in the final version of the task. Student Involvement was increased through telling a personal story (c1), using plural genderless pronouns (c1-c6) instead of single masculine form (e.g., a1), and explicitly asking for student opinions (“why do you think...” c5). the Task Assortment is enhanced through using graphs in c3 (Multiple Data Representations); adding inequalities in c4 (Mathematical Diversity); asking for verbal explanation in c5 and c6 or turning b4 to a multiple-choice item c4 (Question-Answer Format); and asking for a generalizing relationship between two variables in c4 (Generality). The Task Flow can be seen in Adi’s decision to place c5 and c6 at the end of the task, as she explained in the post-PD interview:

Adi: After you experienced, understood, felt, applied, now, after all these, tell me why you think it is a serious matter (the ERL guidelines). If I would give it (c5 and c6) at the beginning, I’m sure the student would say ‘what does she want from me?’

DISCUSSION

The answer to our RQ is that, during CMPP iterative task design supported by a tutor using the MDM, teachers can (i) become aware of MDM dimensions and (ii) develop various techniques for diversifying task items across these dimensions. In accordance with the variation theory of learning (Lo & Marton, 2012), the MDM was used by the tutor to suggest variation in the task enabling the teacher to experience differences between item versions and become aware of some dimensions of variation (Task Flow, Task Assortment, and Student Involvement). Teacher awareness was manifested by changes she initiated in her task and retention of these dimensions half a year after completing the PD. In this respect, guiding CMPP with the MDM expands the educational toolkit for supporting teachers in PP and task design (Cai et al., 2020; Rachamim et al., 2022).

The current study presents a case of teachers’ learning through a fruitful process resulting in a mathematically and socially sound CMPP product. However, our study also showed that experiencing differences and modifying a task does not always develop teacher’s awareness of MDM dimensions. Comparing multiple cases of other

teachers' CMPP is necessary for understanding the conditions for such teacher learning in this context. Another promising direction for future research is tutor's interventions. Under what conditions tutor's suggestions of modifications will improve the quality of teachers' PP products in the short-run and when significantly and sustainably enhance their task-design capacity.

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TOWARDS THE NOTION OF CONCEPT GESTURE: EXAMINING A LECTURE ON SEQUENCES AND LIMITS

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Gestures, whether derived from teachers/lecturers or students in relation to mathematics education, are becoming more and more present in our field. Our focus here is on concept gesture, which can also be seen as a body metaphor, where (part of) the body attempts to enact or depict the mathematical concept dynamically, both somewhat idiosyncratically yet also, to some extent, generically. In this paper, we illustrate the notion of concept gesture by examining some of a lecturer's pedagogic gestures in a first-year university analysis lecture, focusing specifically on those that relate to or engage with mathematical concepts.

ON CONCEPT IMAGE AND CONCEPT GESTURE

“To understand a gesture is to comprehend a gestalt” (Zwicky, 2019, p. 114)

Forty-plus years ago, Tall and Vinner (1981) published an influential article in which they described the term *concept image* (“the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes”, p. 152), to account for differences between a concept's formal definition and the process and manner in which it is conceived. Their construct emerged in a period dominated by cognitive theories of learning that aimed at explaining individual student mathematical constructions (Bingolbali & Monaghan, 2008). Acknowledging that learning theories have since developed additionally to account for the social, cultural, biological, and embodied dimensions of learning (Radford, 2014), we build on Tall and Vinner's notion and, in this paper, make initial steps towards a conceptualisation of what we here refer to as *concept gesture*.

While a gesture might initially be seen as merely a specific element of a concept image, we note significant distinctions between a concept image and what we call a concept gesture that make it valuable to consider them separately: (a) the notion of concept image is not necessarily visual (as it includes all associated processes and properties), whereas gestures decidedly are; (b) concept images focus on mental cognitive structures, whereas gestures are inherently physical and embodied; (c) the notion of concept image was conceived to examine individuals' mathematical constructions, whereas gestures exist fundamentally in the social (and, frequently, pedagogical) domain; (d) concept image is private while the concept gesture is public, in the sense of customarily being explicitly visible to someone else present; (e) methodologically, concept images are often examined through students' written work (e.g., Biza & Zachariades, 2010), which stays permanently on a page (unless actively erased), whereas gestures occur in real time and then instantly vanish (unless video-recorded).

In this paper, we present initial ideas about the meaning of *concept gestures*, as gestures that convey *conceptual mathematical meaning* (rather than ‘organisational’ gestures, such as a lecturer pointing to a place on the board to attract student attention), and could accordingly be used pedagogically to promote and support students’ learning of mathematical concepts. Similar to Tall and Vinner (1981) who used a single mathematics textbook as a primary data source to exemplify and develop their notion, we shall use a single undergraduate Real Analysis lecture as ours.

ON GESTURES AND EMBODIED COGNITION

Over the past fifteen years, there has been a considerable increase in attention to aspects of human gestures in relation to mathematics education, both by students and by teachers (e.g. see Edwards et al., 2009, for a special issue on gestures). However, the predominant emphasis has been on student gestures and gesturing, and relatively little research has attended to teacher/instructor gestures, particularly at the undergraduate level (Weinberg & Wiesner, 2015). Furthermore, studies that analyse undergraduate mathematics lecturers’ gestures often focus on organisational gestures such as pointing and touching (e.g. Hare & Sinclair, 2015), rather than on what we term *pedagogic conceptual gestures*, namely specific gestures deliberately intended by the teacher to convey some aspect of the mathematical notion under presentation and discussion.

Theoretically, conceptual gestures can be regarded in the wider context of *embodied cognition* (e.g. Núñez et al., 1999). Whereas traditional cognitive learning theories treated gestures, and more generally, the body, as merely potential secondary and transitory steps towards ‘genuine thinking’, more recent education research considers cognition itself to be embodied (Radford, 2014). As argued by Radford (2009, p. 113):

[Thinking] does not occur solely *in* the head but *in* and *through* language, body and tools. As a result and from this perspective, gestures, as a type of bodily action, are not considered as a kind of window that illuminates the events occurring in a “black box” – they are not clues for interpreting mental states. They are rather *genuine constituents* of thinking.

A compelling illustration of the embodied nature of cognition can be found in a study by Healy and Fernandes (2011), which revealed that blind students, who are accordingly not ‘conditioned’ by social conventions of gestural communication, gesture with their body when engaged in mathematical sense-making activities.

In relation to sequences and limits, discussed herein, we draw on Núñez’s (2006) notion of *fictive motion*, which is a “fundamental embodied cognitive mechanism through which we unconsciously (and effortlessly) conceptualize static entities in dynamic terms” (p. 171). In the formal limit definition, there is no motion nor dynamic entities. However, the commonly-heard sentence ‘ a_n gets *closer and closer* to L as n *tends* to infinity’ reveals a metaphorical conceptualisation involving movement and dynamic terms. In this paper, we examine how the fictive motion associated with the sequence limit definition is exhibited and conceptualised in the lecturer’s gestures.

ON METHODOLOGICAL CONSIDERATIONS

The data with which we develop, exemplify, and justify the need for our suggested *concept gesture* construct is a video-recording of a Real Analysis lecture on the topic of the formal definition of a sequence limit. The recording is publicly available on the YouTube channel of the *Technion – Israel Institute of Technology*, and its spoken language is Hebrew. The first author (OM) had originally examined this lecturer's recordings in relation to a wider research project that did not have a specific focus on gesturing. The lecture discussed here was chosen to examine how a lecturer deals with one of the most cognitive-challenging concepts in Calculus / Real Analysis – the formal sequence limit definition. However, it quickly became evident this cognitive-loaded lecture did not contain much written text on the boards, but rather long episodes of him only speaking and prominently gesturing – leading us to inquire into how he was using his body to communicate and conceptualise mathematical ideas.

To be able to analyse the lecturer's gestures meaningfully, we found that we had to 'retune' our senses. Speech is for the ears, while gestures (and writing, albeit differently) are predominantly for the eyes (even though both actions may generate a small amount of sound). But when these communication components occur simultaneously, speech generally receives priority and predominance due to its carried semantic meaning. Accordingly, to attune our senses and perception to the gestures and their meanings, we first examined the entire recording with the sound off, forcing ourselves to engage with the gestures without any associated verbal meaning. We watched it this way separately, making notes individually, and then met together to talk about what we had noticed. Subsequently, we, again separately, engaged with the recording a second time, on this occasion with the video turned off, focusing solely on the acoustic track (and then met again to discuss). Considering that OM is a native Hebrew speaker and the second author (DP) is not, OM could attend directly to the speech semantics and syntax, while DP was able to hear other sounds more easily (e.g. the lecturer's emphasis and cadence), by not being able to be seduced by the meaning. This stage also enabled us to be open to explore and conjecture about potential connections between the gestures observed in the earlier iteration and the speech/sound heard in this, though without being 'locked' on a single connection (which might have occurred had we simply examined the sound and vision together from the outset). Lastly, we both separately attended to the complete recording (vision and sound) and each made a third set of notes, before meeting for a discussion about its entirety (followed, unsurprisingly, by multiple further discussions). When both vision and sound were present, to a certain extent (albeit in different respects) we could encounter much of the experience the students present at the time had had. Also here, the benefit of DP's utter lack of Hebrew comprehension was the marginalisation both of speech and of linguistic text (though not mathematical diagrams or notation), at least semantically. For both of us, the build-up of watching the lecture in three iterations, each time attuning to different senses, allowed us to notice aspects of the bodily mathematical communication, which otherwise we might have not noticed.

EXAMPLES OF CONCEPT GESTURES

In the subsequent two examples of certain gestures, we begin by providing descriptions of the physical bodily movements as observed in the first iteration of viewing with the sound off, and only subsequently explain how we interpreted and inferred their associated mathematical meaning.

Example 1

A kind of gesture that had appeared early on and was then repeated multiple times throughout the lecture was what we here refer to as the ‘arm-sweeping’ gesture (see Figure 1). When enacting this gesture, the lecturer would typically begin with his arm close to his chest (and his hand in a relatively closed position), and then sweep his arm to the side in the direction away from his body (while his hand would open up as well). This gesture was manifested in what we regarded as two versions: a ‘continuous’ version, where the sweeping opening of the arm was done in a smooth movement (see Figure 1); and a ‘discrete’ version, where the opening of the arm involved ‘bouncy’ stops along the movement (no figure is shown due to space limitations, though the general motion could be understood from the following diagram ‘read’ from right to left: $\leftarrow \curvearrowright \curvearrowright \curvearrowright \curvearrowright \curvearrowright \curvearrowright$).



Figure 1: An illustration of the ‘arm-sweeping’ gesture (‘continuous’ version)

During the initial viewings of the video (each of us separately and then together), when the sound was still muted, we were attempting to endow the gesture with mathematical meaning. Though there was no sound, in the meaning-making of the gesture, we could draw on what was written on the board – which was intelligible also to DP, the non-Hebrew speaker, as these were mathematical symbols. For example, in Figure 1 (timestamp 5:50), two sequences had already been written on the board ($a_n = n$ and $a_n = \frac{1}{n}$). The lecturer’s arm movement could then be understood as representing the process of the sequence progressing ‘towards infinity’. This interpretation was here emphasised by the lecture consecutively repeating the gesture four times in a row (5:50-5:55). Moreover, during the enactment of the gesture, not only did his arm move greatly, but also did his entire body, as if he were trying to reach further than what his body physically allowed (see Figure 1), potentially representing the idea of ‘approaching infinity’ with this gesture.

During the first sound-off viewing iteration, we had conjectured that the ‘arm-sweeping’ gesture in its ‘continuous’ version was associated with the idea of a *limit*, whereas its ‘discrete’ version with the terms of the *sequence* itself, or perhaps its indices as they continued towards infinity. While, in reality, an arm’s movement is

always continuous and cannot be discrete, we interpreted the ‘bouncy steps’ in the lecturer’s second version of the gesture as a gestural metaphor for another gesture that could have been made had his body permitted – that of creating discrete points in space that move towards infinity. The later analysis stage that included listening (and understanding the semantic meaning for OM) revealed that there existed some semantic evidence to our original interpretation. For example, in the timeframe 5:50-5:57, the lecturer said, “Regarding their limit behaviour, OK, what does it [$a_n = \frac{1}{n}$] approach?” – where the gesture in its ‘continuous’ version was enacted during the words “limit behaviour”. Immediately afterwards, he continued and said (5:57-6:01): “The more I progress in the indices, the [sequence] elements become closer and closer to zero” – where here the lecturer was gesturing the ‘discrete’ version when saying, “The more I progress in the indices”.

An interesting phenomenon we observed here is that both versions of the ‘arm-sweeping’ gesture did not distinguish different sequences in terms of convergence and divergence (e.g. $a_n = n$ and $a_n = \frac{1}{n}$; see Figure 1). However, upon further examination of the rest of this Real Analysis course recordings, we discovered that in this course, a sequence was considered to have a limit whether this limit was finite or infinite (where a separate formal definition was given in each case). Accordingly, we can interpret the lecturer’s unified gesture for both cases to represent the limit concept as regarded in this course; or in other words, as its concept gesture.

Example 2

Another kind of gesture that repeated multiple times in the lecture could perhaps be referred to as the ‘swinging’ gesture or the ‘oscillating’ gesture. This gesture was typically enacted with the entire right arm creating a ‘trail’ in an arch form, which oscillated back and forth while rhythmically accelerating as it got closer and closer to a single point (see Figure 2a). The first two appearances of the gesture followed the writing of the sequence $a_n = \frac{(-1)^{n+1}}{n}$ on the board, and we had initially interpreted this simply as a gestural ‘mimicking’ of the sequence. However, what became interesting for us is that the meaning of this severally repeated gesture seems to have developed in the continuation of the lecture, and evolved into that which we refer to as a concept gesture in relation to the definition of a limit.

Focusing on analysing the mathematical meaning conveyed beyond the spoken and written words, we noted several episodes in the lecture where no text was added to the board for substantial periods of time. Such episodes significantly drew our attention, particularly in the first round of the data analysis, which involved the viewing of the lecture without any sound. In particular, we identified a 5-minute episode (25:49-30:42) in which the lecturer did not write anything on the board other than drawing three diagrams. What struck us here was that one of these was a *diagrammatic actualisation* of the ‘swinging’ gesture that had been introduced earlier in the lecture, though now realised in written graphic form (see the middle diagram in Figure 2b).

Furthermore, this gesture received further attention in this episode, as the subsequent diagram (bottom diagram in Figure 2b) was an extended variation on the gesture, which was followed by the lecturer enacting the ‘swinging’ gesture once more (29:37-29:38).

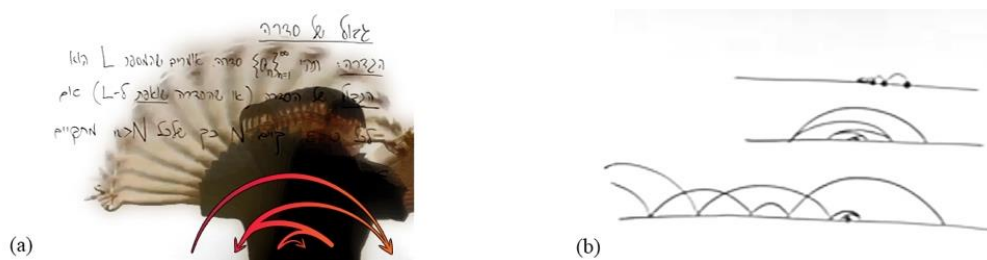


Figure 2: (a) Illustration of the ‘swinging’ gesture; (b) subsequent gestural diagrams

In the initial viewing-without-sound stage, we had attributed potential meaning to the diagrams and associated ‘oscillating’ gesture, which was later confirmed when watching the video with the sound on. The diagrams served as illustrations of different ways in which a sequence may approach its limit – allowing oscillations around a limit in addition to monotonously approaching it (compare middle to top diagram in Figure 2b); as well as permitting ‘random’ values at the beginning of a sequence prior to an oscillation behaviour (bottom diagram in Figure 2b). We further note these diagrams were generic in nature (as gestures generally are), pointing to general ways of approaching a limit, while containing no reference to a specific sequence. Accordingly, we suggest that through the creation of a link between these diagrams and the earlier gesture, the ‘swinging’/‘oscillating’ gesture was endowed with more general meaning; it was no longer a mere ‘mimicking’ of a particular sequence, but rather representative of the conceptual idea of how a sequence may approach its limit from both directions.

Interestingly, the ontological coupling of the gesture and diagram created here was also exhibited through the temporal nature of a gesture being ‘implemented’ on the diagram. Soon after the drawing of all three diagrams (Figure 2b), the lecturer erased them, rather than moving to the next available board. As such, we suggest that the written diagram was not treated as more significant than the fleeting gesture, but rather merely as a temporal transfixing of the gesture, highlighting its associated conveyed ideas.

The deletion of the diagrams was soon followed by that part of the lecture in which the lecturer’s gestures seemed to have played the most significant role. At this stage, the lecturer wrote the formal limit definition on the board – after which he laid his notes, which he had almost continuously been holding in his hands, on the table, and embarked on a 7.5-minute oral and gestural explanation (32:09-39:45) in which only very few clarifying notes were added to the board. During this part, the lecturer created combinations of the various gestures which had been used earlier on – one of which was the ‘swinging’ gesture, which was now enacted frequently and repeatedly. In combination with other gestures depicting a hypothetical limit $L = 1$ and an ϵ -neighbourhood around it, the ‘swinging’ gesture was used to refer to the elements of the sequence a_n being outside or within the ϵ -neighbourhood. Similar to its first appearance (see above), we identified a rhythmical accelerando in the gesture. Initially,

when the sequence elements were still outside the ϵ -neighbourhood, the movement of the right arm was grand and slow. However, when the sequence entered the ϵ -neighbourhood, the gesture became bouncier and quicker as the elements approached the limit. In other words, this gesture, which had initially appeared as a ‘mimicking’ of a particular sequence and later developed into a general idea of how to approach a limit, was here fully realised into a conceptual gesture depicting a central aspect of the limit definition – the relation between the elements of the sequence and the definition statement “for every $n > N$, we have $|a_n - L| < \epsilon$ ”.

DISCUSSION

Reflecting on the lecture exemplified above, the aspect that stood out for us most was how the lecturer was *communicating mathematically* through his gestures (and not just ‘organisationally’). He repeatedly embodied and modified mathematical concepts, where, through a cognitive mechanism of ‘fictive motion’ (Núñez, 2006), he ‘brought concepts to life’ from static objects (e.g. sequence and limit) to dynamic entities (by his body physically ‘sequence-ing’ and ‘limit-ing’). Through repeated and developed use of the same gestures in different contexts – from specific example ‘mimicking’ to representing the general idea of a limit – he created an accumulative mathematical narrative whereby, towards the end of the lecture, different gestures came together and seemed to have formed a combined meaning. Furthermore, his gestures were distinctive in the sense that they involved large-scale movements not only with his fingers and hands, but also using his whole arm and body (see Figures 1 and 2). In relation to potential pedagogical implications, we note Gerofsky’s (2011) findings that students who use whole-body engagement to gesture mathematical concepts seem to be more conceptually flexible and imaginative in their mathematics learning than students whose gestures involve small movements of a finger, hand, or arm.

More generally, we view this lecturer’s pedagogical gestures as illustration of what we refer to as *concept gestures*. We regard concept gestures as gestures that convey conceptual mathematical meaning, and can also be seen as body metaphors, where (a part of) the body attempts to enact or depict the mathematical concept dynamically, both somewhat idiosyncratically yet also, to some extent, generically. Linking this notion back to Tall and Vinner (1981)’s constructs, we suggest that, while concept definition is formal and public, and concept image (as representing mental pictures, associated properties and processes) is private and individual, concept gesture (by a lecturer) is public and individual. In this regard, while ‘concept image’ emerged in a constructivist era in mathematics education, attempting to describe how individual students learn formal concept definitions via mental constructions, ‘concept gesture’ as enacted by a lecturer is befitting the more recent socio-constructivist approaches that examine the social, cultural, and biological roots of learning. This also leads to new questions, such as how do concept gestures feed or shape concept images and concept definitions? Or can some concept gestures support concept definitions better than general concept images? Naturally, future development of this construct would need

to inquire further into the different possible interrelationships among concept gesture, concept image, and concept definition.

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THE ROLE OF TOPOLOGY IN TWO-VARIABLE FUNCTION OPTIMIZATION

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This study uses Action-Process-Object-Schema theory (APOS) to examine students' understanding of optimization of two-variable functions. We propose a genetic decomposition (GD) model of Schema components and relations between components, that students may construct to understand the optimization of these functions. The GD was tested by performing semi-structured interviews with 22 students. The results suggest that students' struggles are mainly related to the topology of domain sets and those relations which help to understand the scope and limitations of some of the theorems commonly applied to solve optimization problems. The study also contributes to a better understanding of the APOS notions of Schema and types of relations between Schema components and their use to model students' mental constructions.

INTRODUCTION

A growing body of research deals with teaching and learning multivariable calculus, including its definition, geometric representation, and differential and integral calculus. Martínez-Planell and Trigueros (2021) discuss this in a survey article. However, very little research deals with the optimization of two-variable functions (López, 2020; Martínez-Planell & Trigueros, 2021). Given the importance of optimization in applications of two-variable functions to science, technology, engineering, mathematics, and economics, this study addresses this significant gap.

The few existing studies consider topics such as Lagrange multipliers (Xhonneux, 2011), the use of digital software for the identification and classification of extrema of two-variable functions (Alves, 2012; Ingar, 2014), the relation between second-order Taylor polynomials, quadratic forms, and the classification of critical points (López, 2020), and two-variable function optimization in contextual situations (Ingar, 2014; Mkhatswa, 2021). Here, we focus on the role of topology in students' understanding of two-variable function optimization.

THEORETICAL FRAMEWORK

We use the notion of Schema in Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014). Since APOS is a well-known theory, we only describe Schemas. A Schema is a coherent collection of mental structures (Actions, Processes, Objects, and other previously constructed Schemas) that are interconnected in a way that allows the individual to recognize when a problem situation falls under the scope of the Schema. Thus, to study Schemas, one may start by specifying what component structures will be considered and what types of relations interconnect these

components. Trigueros (2019) proposed that relations between components can be classified as *correspondence*, *transformation*, and *conservation* relations. *Correspondence* relations are the result of superficial comparison of some Schema components in terms of similarities or differences. They may be constructed through the observation of the repeated use of certain structures, but the individual cannot justify the relation. When an individual explains or justifies a relation between Schema components in terms of other structures, or when the individual can group some components, the relation can be considered as a *transformation relation*. The construction of *conservation* relations is evidenced when one component can be interchanged or fluently used for or with another in different problem situations.

Determining the construction of the different types of relations implies that researchers should consider the student's work in different problem situations related to the Schema studied to distinguish between them.

A Schema is a dynamic structure. It is constantly changing. Its development can be described as a triad consisting of three stages. A Schema is said to be at the *Intra-* stage of development when different components are unrelated or mostly related by correspondence relations. The Schema is at the *Inter-* stage of development when transformation relations have started to form, and most components are interconnected by some type of relation. At the *Trans-* stage, all components are interrelated mostly by conservation relations. When the student is able to determine the particularities of problems related to the Schema, it attains coherence.

GENETIC DECOMPOSITION FOR THE TWO-VARIABLE FUNCTION OPTIMIZATION (TVFO) SCHEMA

An important idea in APOS is that of a genetic decomposition (GD). This is a model of how students may construct a particular mathematical notion. In the case of Schemas, a GD is expressed as a list of component structures and types of relations between them. Due to space constraints, we list the components we focus on without much description, and we only describe the specific relations we will use in this paper.

Components

- Set topology (s) as a Schema including open and closed sets, continuity, and boundary of sets as Processes
- One variable function (1f) as a Schema, including one-variable optimization
- Two-variable function (2f) as a Process or Object
- Partial derivatives (∂f) as a Process
- Critical Points (cp) of two-variable functions and extrema as Processes
- Second partial derivatives ($2\partial f$) for two-variable functions as Processes

Important relations between components

- Partial derivatives and critical points (cp- ∂f) evidenced by identifying points in the domain of the function when it is an open set or at the interior of a closed set, where both partial derivatives are zero; identifying points where one of the

partial derivatives does not exist, and by showing awareness that critical points provide candidates for local extrema.

- Critical points and second partial derivatives (cp-2 ∂ f) evidenced by the application of the second derivatives test (Stewart, 2012) to classify extrema using the Hessian determinant H when $H \neq 0$ and by showing awareness that other methods must be used to classify extrema in case $H = 0$.
- Set, critical point, and two-variable function (s-2f-cp) evidenced by the recognition of the role of continuity and the topology of the domain set in regard to critical points for different domain sets.
- Sets and two-variable function (s-2f) evidenced by the recognition of the topology of the domain of the function.
- Sets, function, and partial derivatives (s-2f- ∂ f) evidenced by the recognition of the topology of the domain and the possibility to compute a partial derivative.
- Sets, two variable functions, and one variable function (s-2f-1f) evidenced by the awareness of the need to separate the domain into its interior and its boundary and the kind of derivative to be used.
- Sets, one-variable function, and critical points (s-1f-cp) evidenced by its use in finding the extrema on the boundary.
- Two-variable function and critical points (cp-2f) evidenced by the use of ad-hoc methods.

METHODOLOGY

Two groups of students from different introductory multivariable calculus sections were chosen to participate. The instructors of both sections were experienced in teaching two-variable function calculus. Eleven students from each section were chosen so that three of them were over-average, five were average, and three were under-average, as determined by their professor. This selection was determined in order to enable us to deduce from observations a wide range of different mental constructions and types of relations. The students participated in semi-structured interviews about one week after the course had ended; each interview lasted approximately one hour. During the interview, the students solved problems, explaining what they were thinking as they went along. The interviewer could ask questions for clarification. The interviews were audio and video recorded, transcribed, translated into English, individually analysed, and discussed as a group by the researchers. Differences were negotiated until a consensus was reached. For the analysis, a matrix was constructed with student names as rows and interview problems as columns. Codes that reflect the proposed relations in the GD were used. Here we only have space to show three of the 17 interview tasks.

1. Let $f(x, y) = x^2 - 2x + y^2 + 2y$, $S_1 = \{(x, y): x^2 + y^2 \leq 9\}$, and $S_2 = \{(x, y): x^2 + y^2 < 0\}$. Discuss the strategy (but do not carry it out!) you would use to find the maximum and minimum values of f on each of these sets.

2f. Let $f(x, y) = x^3 + y^2$ be defined on $S = \{(x, y): x^2 + y^2 \leq 1\}$. How do you find the maximum and minimum values of f on the boundary of S ? Find them.

3a. True or false? The function $f(x, y) = xy$ attains a global maximum, a global minimum, and has a saddle point on the set $S = \{(x, y): 2 \leq x \leq 4 \text{ and } -3 \leq y \leq 1\}$.

RESULTS

We show evidence and discuss a representative sample of students, questions, and the relations they show to have constructed. We organize the information in terms of results from students who showed the construction of the same Schema level.

Intra-TVFO stage of Schema development and the construction of correspondence relations

Twelve of the twenty-two students had constructed mostly correspondence relations between Schema components. We present student C as an example. In question 1:

Student C: I first need to find the critical points of f inside S_1 then I should use the second derivatives test to see if the critical points are max or min for f inside S_1 , then I need to find the critical points $f(x, y)$ on the boundary...

Interviewer: How do you find the critical points of f on the boundary of S_1 ?

Student C: Like the way for the inside of the circle, umm if I solve the equations $f_x = 0$ and $f_y = 0$, then they will be the critical points, now we should check which of them are inside S_1 ... and which of them are on the boundary of S_1 ... now we need to use again the second derivatives test for the critical points which are on the boundary of S_1 ...

Observe that student C relates partial derivatives and critical points (cp- ∂f): “for the inside of the circle, umm if I solve the equations $f_x = 0$ and $f_y = 0$, then they will be the critical points.” He also proposes that after setting both partials equal to zero “now we should check which of them are inside S_1 ... and which of them are on the boundary of S_1 .” Student C considers that there is no difference in the method for finding extrema in the interior and in the boundary of the disk S_1 which suggests that, in this problem, the constructed relation cp- ∂f is a correspondence since the student’s arguments seem to be the result of repeated use. Student C also established a relation between critical points and second partial derivatives (cp- $2\partial f$): “I should use the second derivatives test to see if the critical points are max or min for f inside S_1 ” and later “we need to use again the second derivatives test for the critical points which are on the boundary of S_1 ”. Student C again gives evidence of a correspondence relation. He considers using the second derivative test to classify critical points, independently of the points being on the boundary; that is, he considers similarities in the use of the method even though they do not apply in this case. We note that in Question 3a, he correctly found critical points:

Student C: Let me find the critical points umm finding the partial derivatives and solving them for zero I will have the critical points of f ... so, the only critical point is (0,0).

Also, when set topology did not play a role, he used the second derivative test correctly:

Student C: I think I need to use the second derivative test to see whether the critical point $(0,0)$ is a saddle point or not ..., okay, we have $D(0,0) = -1$ so this point is a saddle point

The student shows the construction of correspondence relations mainly due to repeated use. He seemed to use memorized procedures without considering if they did or not apply. He also confused necessary and sufficient conditions on theorems; when asked if the following assertion is a theorem or not “if f has a maximum or a minimum value at (a, b) , then the partial derivatives of f exist at (a, b) and $f_x(a, b) = f_y(a, b) = 0$ ”, he said:

Student C: This is true, umm because when based on the second derivatives test a function has a min or max at a point like (a, b) so I expect the point (a, b) be a critical point, and so both f_x and f_y are zero at (a, b) .

To summarize, student C showed throughout the interview that all the relations he had constructed were correspondence relations.

Inter-TVFO stage of Schema development and the construction of transformation relations

We classified four students at the Inter-TVFO stage of Schema development. They had interrelated most Schema components and had constructed transformation relations. Student B is an example. In question 1:

Student B: I first solve S_2 , umm we just need to find the critical points $f(x, y)$ which are in the interior of S_2 , then we can compute the values of $D(x, y)$ and f_{xx} for each of the critical points to see if they are min, max or a saddle point.

Interviewer: And S_1 ?

Student B: Since the interior of S_1 and S_2 are the same so we have also the critical points inside the circle for S_1 , then we need to evaluate $f(x, y)$ on the boundary of S_1 using the method of Lagrange multipliers or changing the function $f(x, y)$ to a one-variable function, umm in the next step we can find $D(x, y)$ and f_{xx} for them, but I don't think if we need to use the second derivatives test because now I'm finding the min and max of f on the closed and bounded set S_1 , so I need to just evaluate the values of $f(x, y)$ in all the critical points and candidate points on the boundary, then its biggest value will be max and the smallest value will be min.

Student B's awareness that there is no need to classify critical points on a compact set (“I don't think if we need to use the second derivatives test because now I'm finding the min and max of f on the closed and bounded set S_1 ”) gives evidence of her construction of the relation s-2f-cp. Also, in question 2f:

Student B: The function f is $x^3 + y^2$ which is continuous everywhere, umm when we have limitations on the values of x and y in the form of the ordered points (x, y) umm like $x^2 + y^2 \leq 1$ which is a bounded and closed set then the

values of the function f on S will have its global maximum and also its global minimum.

When asked if a continuous function on a closed set must attain maximum and minimum values:

Student B: I think this is false because the set should be closed and bounded.

Interviewer: Can you give a counterexample?

Student B: If I consider the function $f(x, y) = x + y$ on a closed and unbounded such that the values of x or y can be everything from $-\infty$ to $+\infty$, then the function f doesn't attain its max and min on R^2 .

These examples suggest that student B had constructed a transformation relation for the relation s-2f-cp. However, she did not show the construction of conservation relations. For example, when asked if the function $f(x, y) = |x|$ attains a global maximum and minimum on $S = \{(x, y) | x^2 + y^2 \leq 4\}$, she did not recognize it is a continuous function on a closed and bounded set. A student showing a conservation relation of s-2f-cp would have immediately recognised the existence of extrema in those conditions. She had also constructed some correspondence relations. When asked if the following assertion is a theorem or not, "if f has a maximum or a minimum value at (a, b) , then the partial derivatives of f exist at (a, b) and $f_x(a, b) = f_y(a, b) = 0$ ", she confused necessary and sufficient conditions and said:

Student B: This one is also true because at the max point the tangent lines are horizontal and their slopes are zero, and I know that the values of $f_x(a, b)$ and $f_y(a, b)$ are the values of the slopes of the tangent lines to the function f at the point (a, b) .

Trans-TVFO stage of Schema development

Seven of the twenty-two interviewed students were classified in the Trans-TVFO stage of Schema development. Student A is an example. Due to space constraints, we only show partial results, concentrating on the relation s-2f-cp.

Interviewer: [Question 1] Do you need to use the second derivatives test for S_1 ?

Student A: No, I don't need, umm because we evaluated the min and max of f on the boundary of S_1 and in the interior of S_1 , umm and this question asked us to find only the absolute extremes of f on S_1 which is closed and bounded.

The last citation suggests the construction of the s-2f-cp relation. The construction of this relation was also suggested in every available opportunity, for example:

Student A: [Question 2f] The function f is continuous on S , and the set S is closed and bounded so we can limit both x and y to have their min and max, I mean both x and y are from -1 to 1, umm the expression of f is $x^3 + y^2$ so f has its global max on S for sure.

Student A: [Question 3a] Now I need to look for the global max and global min, since the set of the rectangle S is closed and bounded and the function $f(x, y) =$

xy is continuous on this set [a closed rectangle] therefore it has the global max and global min, umm so the statement is true, umm I can find the value of max and min.

Student A: [When asked if $f(x, y) = |x|$ attains a global maximum and minimum on $S = \{(x, y) | x^2 + y^2 \leq 4\}$] The set S is closed and bounded and the function $f(x, y) = |x|$ is continuous on this set so we can say that the function f attains its global max and its global min on this set. The global min is 0 when we have $x = 0$ and the global max is 2 when $x = -2$ or $x = 2$.

Student A: [When asked if a continuous function defined on a closed set must attain maximum and minimum values] It's false, umm because the theorem should be a continuous function defined on a closed and bounded set.

Interviewer: Can you find a counterexample?

Student A: So, I should design a set which is closed but not bounded, umm like R^2 which is a closed subset of itself but of course unbounded, umm and the function can be for example $f(x, y) = x^2 + y^2$ and in this case we can see f does not attain its global max and min on R^2 .

Student A: [When asked if a function defined on a closed and bounded set must attain maximum and minimum values] The function must be a continuous function. I am thinking for a situation like umm the value of a function like f at the point (a, b) of a closed and bounded set, let say set S , is for example number c I mean $f(a, b) = c$ but the function is not continuous at (a, b) umm and the limit of f at (a, b) be $+\infty$ or $-\infty$, in this case the function f doesn't attain its max and min on the closed and bounded set S , because for some points of S which are very close to (a, b) the value of the function can be as much as big we want.

In all the above examples, student A explained his response with a counterexample, calculation, or argument. The fact that he consistently seemed to think of “continuous on a closed and bounded domain” as interchangeable with “attains its extrema” suggests that s-2f-cp was constructed as a conservation relation. Overall, student A gave evidence of each of the proposed relations in the GD; he frequently explained or justified his reasoning and consistently applied the needed relations throughout the interview. He seemed to interrelate all components with conservation relations.

DISCUSSION AND CONCLUSIONS

We found that when attempting to solve optimization problems, students frequently applied procedures seemingly due to habit or repetition without considering the topology of the domain set. Students' work suggests paying attention during instruction to the topology of domain sets, its relation to extrema, and to necessary and sufficient conditions of theorems. This last suggestion is consistent with the observation of Xhonneux (2011) regarding students' difficulties with the nature of implication and necessary and sufficient conditions and with that of Mkhatswa (2021), who observed that verifying or justifying the extrema was problematic for most students in his study.

On the theoretical side, given that there are few studies about Schema development and even fewer that use the types of relations between Schema components, our study contributes to a better understanding of these notions, particularly in the context of two-variable function optimization.

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REASONING AND LANGUAGE IN RESPONSES TO READING QUESTIONS IN A LINEAR ALGEBRA TEXTBOOK

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We describe the process we used to analyze anonymized responses to “reading questions” embedded in an interactive linear algebra textbook, in a team that involved a textbook author, an experienced instructor, and a researcher. Reading questions are a textbook feature meant to entice students to read the textbook before attending the class where such ideas will be discussed; as students provide responses, the instructors can learn about how students are thinking about the content before a lesson. We generated a categorization of the responses to reading questions on spanning sets and linear independence based on the reasoning and the language in the responses as a first step towards understanding individual student conceptions on span, spanning sets, and linear independence.

There is growing evidence in post-secondary education that user interaction with learning resources (apps, videos, assessment systems) facilitate learning (Heflin et al., 2017). The expansion of the field of interactive textbooks, thanks to the evolution and accessibility of production tools, has significantly facilitated basic research on these resources and as a consequence, the field has been able to advance both theories and methods to investigate the use of these resources using real time data (Trouche et al., 2020). The inclusion of interactive features in textbooks opens a question for researchers about what we gain by having access to the students’ responses as they interact with those features, and for designers, who may want to offer additional instructional supports for teachers and students rooted in student learning needs. In our project (Beezer et al., 2018), the interactive feature, *reading questions*, was added to three undergraduate mathematics textbooks written in PreTeXt (<https://pretextbook.org/>) to entice students into reading the material in the textbooks prior to coming to class. The feature collects students’ responses directly in the textbooks and delivers them to teachers in real time; perusing the responses allow teachers to make decisions to alter plans for their lessons. Our goal is to explore whether de-identified student responses reading questions in these textbooks can be useful to identify students’ conceptions about mathematical notions. As a first step towards this goal, and in collaboration with an experienced instructor (the third author) and a textbook author, who is also an experienced instructor (the second author), we endeavoured to produce a categorization that would go beyond a value judgment of student responses as “right” or “wrong,” and instead push us into unpacking how students were using the ideas to answer the questions. This information will be used in a later analysis to identify the conceptions elicited by individual students. As interactive textbooks become more widely used, finding ways to capitalize on the

information available from student responses could support research efforts to advance understanding of conception development and design processes that capitalize on the responses. As such, this paper presents a proof-of-concept about a process and a tool to undertake this analysis with a larger set of responses and possibly automatizing the categorization.

CONCEPTUAL BACKGROUND

Textbook features fulfill curricular and conceptual functions through organizational devices such as “exposition, explanation, questioning, exercises, examples, and tests... [to] achieve a more active reader of the text and to manifest the aims of the mathematical and cognitive progression” (Love & Pimm, 1996, p. 386). Any type of question (e.g., rhetorical in the text, in exercises) are “the principal means by which the student is encouraged to be an active reader of the text” (Love & Pimm, p. 387). Questions embedded in explanatory text tend to prompt the reader “to complete some task before the narrative can proceed or... inspire the student-reader to explore mathematics further” (p. 386). Using the instrumental approach (Rabardel, 2002) we have documented that students and instructors use this particular feature in more ways than those envisioned by the designers: the students answer the questions after reading the textbook and before class in order to get acquainted with the material and bring questions, but they also use them to verify their own knowledge of the material, or to study for examinations. Teachers, likewise, use the student responses to modify their upcoming lessons but they also use the questions themselves to test their self-knowledge, assign them in class for small and large group discussion, and assign them for a grade after the material has been covered in class. We have also found that students will use the feature, independently of whether their instructors use it, but are likely to use it in consistent ways as their instructors demand it (Mesa et al., 2022; Mesa & The UTMOST 3.0 Team, 2022). This implies that there is a good reason to make use of the responses collected, because it is quite likely that students and instructors will take advantage of those. In the context of interactive textbooks, the availability of such responses suggests that the curricular and pedagogical functions of the textbook can be supplemented through a collective process that involves the practitioners and the textbook authors more intentionally to reflect on how textbook features added can indeed be educational for teachers, students, designers, and authors.

METHODS

We used a grounded approach to analyse 303 responses students provided to five reading questions in two sections of the linear algebra textbook used in the project (Beezer, 2021). The project involved 50 instructors and their students in the United States. We gathered all the responses to reading questions on Spanning Sets (see Figure 8a) and Linear Independence and Spanning Sets from 76 different students across 6 instructors at different institutions. Sample responses to the first reading question in the Spanning Sets section can be seen in

Figure 8b.

Figure 8: Reading questions for (a) Spanning Sets; (b) two student responses.

SS Reading Questions	
<p>1. Let S be the set of three vectors below.</p> $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}.$ <p>Let $W = \langle S \rangle$ be the span of S. Is the vector $\begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix}$ in W? Give an explanation of the reason for your answer.</p> <p style="color: purple;">My answer→</p>	<ul style="list-style-type: none"> When the augmented matrix is put into row-reduced echelon form, we get: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The bottom row of this matrix tells us there is no solution, thus the given vector is not in the span of S.
<p>2. Use S and W from the previous question. Is the vector $\begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$ in W? Give an explanation of the reason for your answer.</p> <p style="color: purple;">My answer→</p>	<ul style="list-style-type: none"> Yes, the vector given is in W. Finding the rref of S augmented with the given vector results in a consistent system with equations $S_1 = -S_3 + S_2$, $S_2 = -S_3 - 1$, and $S_3 = S_3$ (a free variable). Choosing 1 for S_3 gives $S_1 = 1$ and $S_2 = -2$. We show that the vector is in fact a linear combination of S, and therefore in W, by
<p>3. For the matrix A below, find a set S so that $\langle S \rangle = \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of A. (See Theorem SSNS.)</p> $A = \begin{bmatrix} 1 & 3 & 1 & 9 \\ 2 & 1 & -3 & 8 \\ 1 & 1 & -1 & 5 \end{bmatrix}$ <p style="color: purple;">My answer→</p> <p style="text-align: center;">(a)</p>	$\begin{aligned} n[1] [3] [4] [-1] \\ n[2] -2[-4] + [-2] = [8] \\ n[1] [2] [1] [-4] \end{aligned}$ <p style="text-align: center;">(b)</p>

Analysis

We report here the five steps used to develop a categorization of the responses, what we termed placing the responses into “buckets,” which eventually evolved into “math buckets” for the reasoning elicited and the “language buckets” for various aspects of communication.

Spanning Sets starts by defining the span of a set of vectors and giving examples that highlight the mathematical meaning of belonging or not belonging to a set and illustrate that to determine whether a vector is an element of the span of a set of vectors, one needs to find the solution of a linear system of equations with an augmented matrix that includes the vector in question. If the system is consistent, then it is possible to say that there is a solution; once this is established it will be possible to find an actual set of scalars to write the given vector as a linear combination of the vectors in the set. If the system is inconsistent, there will be no solution and the vector will not belong to the span of a given set of vectors. The discussion of spanning sets of null spaces makes the connection that with a trivial null space, a matrix is non-singular and therefore, such a matrix will create a system with a unique solution. Throughout the presentation, the textbook uses Sage cells² to illustrate the same processes.

² A Sage cell is a box that can execute Sage command. Sage is an open-source mathematics software system. <https://www.sagemath.org/>

Figure 9: Summary of textbook content for Spanning Sets.

First, and to orient our analysis, we studied the textbook content to anticipate the expected answers. Figure 9 shows a summary of our description of the content for the Spanning Sets section.

Third, we randomized the responses within questions to eliminate possible associations to students across questions; in this way each response to each question was our unit of analysis and it was not possible to know whether the first response to the first question was provided by the same student across the set. Fourth, we engaged in cycles of reading a response, describing what was stating, putting it in an existing bucket or create a new one, and discussing the inclusion. This constant comparative method was done for each of the answers provided. The third author, a mathematician familiar with the content and with substantial teaching experience, created an initial categorization using the responses to the first reading question in the Spanning Sets section focusing on the reasoning reported. The three authors met every other week to revise discuss the categorization and analyse responses that did not seem to fit the categories created. As new responses to other questions were added, we realized that it was important to differentiate mathematical reasoning (“math buckets”) from the way in which the answer was communicated (“language buckets”). A response could be assigned to only one math bucket but to more than one language bucket. The system worked for five of the reading questions in the two sets. Fifth, we performed a calibration with two advanced undergraduate students (former graders in a proof-oriented linear algebra course taught at the first author’s institution) who received the set of responses and the categorization system and were asked to independently assign the responses to the first question in Spanning Sets to each and meet to discuss their coding process. Once this was done, we all met to discuss their categorization, resolve discrepancies, and contrast with our own categorization. This process led to further revisions of the categories and their meaning. Once we all agreed with the category system, the two coders recoded the all the responses. The labels for each bucket emerged during the discussions and clarification of the responses. There were nine math buckets, three of which are given in Figure 10 (not shown for space reasons: *Sage*, *Solution with no reasoning*, *Generic-valid*, *A method not in the textbook*, *Argument does not Apply*, and *non-codable*), and six language buckets, three of which are given in Figure 10 (not shown: *All clear*, *LaTeX solution*, *Cites a theorem*).

Math Buckets	
Name	Definition and Example
Detailed valid argument connected to the problem	Valid reasoning that is related to the problem at hand (i.e., the answer has the specific details, numbers, or information provided in the problem) and provides specific details about the process. Response may or may not include checking the work.

<p>A in reduced row echelon form: $\{[[1, 0, -2, 3], [0, 1, 1, 2], [0, 0, 0, 0]]\}$. $N(A)$ is then $\{[[2x_3 - 3x_4], [-x_3 - 2x_4], [x_3], [x_4]] : x_3, x_4 \text{ any scalar}\}$. So $\langle s \rangle$ is $\{x_3[[2], [-1], [1], [0]] + x_4[[-3], [-2], [0], [1]] : x_3, x_4 \text{ are any scalar}\}$.</p>	
“Technical difficulties”	Valid reasoning given and there is an error when students relate the answer to the problem at hand.
<p>$\{[-3, -2, 0, 1]\}$. The set only has one vector since there is only one free variable in the rref form of A.</p>	
Invalid	Invalid reasoning OR uninterpretable and no relation to the problem at hand. $S = \{v_1 = [3, 2, 0], v_2 = [1, -1, 1]\}$
Language Buckets	
“Object naming error”	Answer has language that confounds the objects used—possibly because of naming or misapplication of terminology—but otherwise expressing the right idea.
<p>If we make the set S in RREF while setting it equal to the vector, we can come to the conclusion that the vector is in the span because $x = 2$ and $y = -1$ and $z = 0$. So, if we multiply 2 by $[1, 2, -1]$ and -1 by $[3, -4,]$ and 0 by $[4, -2, 1]$ it will become $[1, 8, -4]$ if you add them</p>	
“Object use error”	Answer has language that confounds the objects used: there is an evident confusion in the objects used, incorrect naming, or misapplication of terminology
<p>No, because the spanning set can be reduced in $[2, -1, 0] + z[-1, -1, 1]$ for any z. To get -1, z has to be 3, but the other components will not match those in the vector.</p>	
Unclear	Portions of the text suggest more than one possible way the student is thinking about the problem; the meaning is ambiguous
<p>Yes it is. The three vectors in S and the fourth vector represent a linear combination. This last vector is a solution to the three.</p>	

Figure 10: Final coding system

FINDINGS

The frequency and percent of the 303 responses coded from the 76 students are given in Table 2. We found that the responses were almost equally divided into those that included reasoning leading to a valid argument and those that did not. As the questions were posed with the goal of having students attempt them after reading the material before the ideas are discussed in class, this proportion would suggest that in any given classroom about half of the students would be using the concepts they read in the section reasonably well to answer the proposed questions. For about a quarter of the responses (23%) it will be difficult to identify the reasoning, as the solution is not accompanied by details. In about one fifth of the responses there are issues with the reasoning that could be traced to typos or to inappropriate reasoning. Less than half of the responses (45%) were categorized as exhibiting problems naming or using the objects or had language that was ambiguous.

Table 2: Frequency and percent of the responses coded for the reasoning and language categories.

Math Bucket	<i>n</i>	% ^a	Language Bucket	<i>n</i> ^b	% ^a
Detailed valid argument connected to the problem	96	32%	All clear	169	57%
Solution with no reasoning	69	23%	Cites a theorem	19	6%
Sage	3	1%	LaTeX solution	27	9%
Other valid methods	2	1%	Object naming errors	85	29%
Generic valid	63	21%	Object use errors	10	3%
Technical difficulties	33	11%	Unclear	35	12%
Invalid	30	10%			
Argument does not apply	1	1%			
Non-codable	6	2%			
Total	303	100%			

Notes: a. Excludes non-codable responses.

b. A response could be in more than one language bucket.

Note that a very small number of responses are non-codable (these corresponded to submissions that were written out on paper and handed out to teachers; possibly because of internet connections) or took advantage of Sage. That Sage is not frequently used is not surprising, as we know that only a handful of instructors used it. It is interesting that students of those teachers chose not to write Sage code. We think that perhaps the presentation is such that students feel compelled to replicate the processes outlined—using Sage might be considered cheating. A cross analysis of the two dimensions, shows that most of the answers had both reasoning supporting a valid argument and using appropriate language (62%). This suggests to us, that in general, the questions are at a good level for the students, after reading the text on their own, they can answer the questions give. But we also found a sizable proportion of responses for which this was not the case, and of these, about half were categorized as having language problems with a valid argument. Consider for example the following response coded as “generic valid” and “object naming errors:”

The vector $[-1, 8, -4]$ is in W with the x_1 and x_2 being equations of the first two vectors. (#16, SSRQ1)

The response might be alluding that the third coefficient is a free variable, or it could be that x_1 and x_2 are the names of the scalars, rather than equations. The reference to “two vectors” is unclear as well. The following response illustrates a case categorized as “technical difficulties:”

the vector is not in W because if we take the RREF of the S , then plug our findings into a linear combination the values don't (*sic*) equal the vector $[-1, 8, -4]$ (#29, SSRQ1)

The student seems to be following the process described in the textbook, but we surmise that something might have gone wrong with the calculations that led them to say that the vector was not in W , the span of S . However, the reference to “RREF of the S ,” signals Object use error, as the RREF is not applied to a set but to a matrix.

DISCUSSION AND NEXT STEPS

The impetus of this work was to work with practitioners to investigate what could be said about the responses to reading questions in a specific section of a linear algebra textbook that would be more nuanced than being a “good” or a “bad” answer. The reading questions are given as an incentive for students to read, but they are also there as an incentive for teachers to reflect on what could be done differently in the classrooms. A “good/bad” categorization is not very productive, as it leads to an assumption that it is important to fix a problem (or not) without a clear focus for the fix. With this work we demonstrate that much more can be said about these responses, in particular the process involving the practitioner and the author generated two dimensions, one related to the reasoning, and the other to the language used, and within each of those several components were distinguished. As the collaboration advanced, our discussions led to greater understanding of the complexity of the ideas embedded in the processes. A system that can take the responses and categorize them could be useful for making decisions about what other questions or activities might be productive to use in the classroom. We believe that the responses that suggest object misuse or mislabelling are ripe for proposing activities that could help clarify the nature of the objects. At this point in the work, we know that the categorization taking each response as a unit of analysis is useful in thinking about classroom work and that it could be automatized when a large number of responses are obtained, for example in a large subscription course. But an analysis of the set of responses per student can build better models of how students are thinking about the ideas. This is our next step and we are exploring frameworks that allow us to investigate conceptions and their development (e.g., cKc, Balacheff & Gaudin, 2010; APOS, Dubinsky & McDonald, 2001; commognition, Sfard, 2020) and capitalize on the extensive literature on student thinking about these ideas. A second exploration, that we have started already, relates to finding out how instructors using the textbooks might use the responses and their categorization in teaching.

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DEVELOPING AN INTERNATIONAL LEXICON OF CLASSROOM INTERACTION

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The lexicons documented for the International Classroom Lexicon Project captured pedagogical practices of middle school mathematics classrooms in the original language of ten communities. To better facilitate mutual understanding of local professional languages we sought to develop an international lexicon. We began this research study by focussing on the clusters of words related to the language of assessment and the language of mathematics. In this paper we extend the international work by focussing on the language of interaction present in the Australian, Chilean, Czech and French Lexicons. Studies and theories related to networking strategies and dialogic interactions were used to support a comprehensive comparison of the lexical items related to interaction, and to reveal connections amongst the four lexicons.

BACKGROUND AND RATIONALE

The International Classroom Lexicon Project

In the field of language and mathematics education, research has tended to focus on the language of the student, the teacher language used in the classroom and language appearing in the practice of mathematics (Austin & Howson, 1979; Planas et al., 2018). These areas of research appear to exclude studies of the professional language of teachers, that is, the language *about* the phenomena of the mathematics classroom. In response, the central goal of the International Classroom Lexicon Project was to document the professional vocabulary that teachers use to describe the objects and events that constitute their professional activity in middle school mathematics classrooms (Mesiti et al., 2021). At the core of the research project is the importance of teacher knowledge, and a commitment to share this knowledge with the mathematics education community to improve the reflective practice of teachers.

The professional vocabulary of teachers

The teaching profession, in most contexts, lacks a technical or professional language normally characteristic of established professions such as medicine and law. For example, with respect to the English language, researchers agree that a language for description and analysis remains underdeveloped (Grossman et al., 2009; Lampert, 2000; Lortie, 1975). Similarly, Chilean participants reported that they were unfamiliar with the notion of creating communities with the explicit purpose of engaging in discussions about the problems of practice (Grau et al., 2017). Likewise, Czech teachers tend to use everyday language rather than pedagogical terminology

corresponding to the Czech education tradition influenced by its rich pedagogical history including the ideas of Comenius (1907). To a lesser extent, this is also the case in France despite the influence of didactic research on the professional language of mathematics teachers visible in the French lexicon. The four lexicons examined for study in this paper reflect differences in pedagogical history, educational culture, context, and community (Mesiti et al., 2021). Three of these communities use the world's most spoken languages. English is spoken by 1.131 billion people, Spanish (Chile's official language) is spoken by 534 million people, and French is spoken by 280 million people (Ang, 2020).

In this paper we progress towards an international lexicon, with a focus on the language of interaction. We examine the Australian, Chilean, Czech and French lexicons, benefitting from the authors' involvement in their documentation. The results that follow are analysed in consideration of the following research question: *Given the national lexicons from Australia, Chile, the Czech Republic, and France what is the nature of the connections amongst the cluster of terms that focus on interaction?*

THEORETICAL FRAMEWORK

Linguistic relativity

Researchers agree that linguistic and semantic differences amongst languages have an impact on our experience of the world (Boroditsky, 2001; Levinson, 2003). This theoretical position, characterised as linguistic relativity, is a weaker interpretation of the much-debated Sapir-Whorf hypothesis that 'language shapes thought' (Sapir, 1949). The theoretical position adopted for the International Classroom Lexicon Project is in line with the notion of linguistic relativity; namely, that the differences in vocabulary, from one community to the next, are meaningful, and may indicate a diversity of teacher's perception of the classroom.

Networking studies

The researchers involved in the International Classroom Lexicon Project consider that diversity, and the reflection it supports, is a source of richness in our field. This position is shared by researchers in the networking of theories, who have developed specific tools to address this diversity, especially the Scale of Networking Strategies (Prediger, Bikner-Ahsbals & Arzarello, 2008). A critical notion from this research is the diversity of networking strategies expressed along a continuum. We adapted the strategies *combining/coordinating*, and *synthesizing/integrating locally* to our specific purpose.

RESEARCH METHODOLOGY

The national lexicons

National research teams enacted a 'negotiative' methodology (Mesiti et al., 2021). This approach prioritised teachers as the authority for identification of language used by teachers when in conversation with their colleagues about the practice and phenomena of the middle school mathematics classroom. All items included in the lexicons were

validated locally and nationally as a reasonable representation of the professional lexicon of middle school mathematics teachers in the respective communities.

The lexicons consist of terms in their original language, a closest English translation, a description from the classroom, illustrated with examples (see Table 1 for a selection from each lexicon).

Table 1: Example of lexical items from the four lexicons (Mesiti et al., 2021)

Lexicon	Term	Description	Example
Australian	<i>reflecting</i>	An activity in which students consider the effectiveness or progress of their learning (i.e., their developing knowledge, skills and understandings).	<i>The teacher asks students to identify and describe three new skills they have learnt during a unit of work.</i> <i>The teacher asks students to identify and describe an aspect of their current study that they do not understand.</i>
Chilean	<i>redondear ideas</i> (<i>rounding up ideas</i>)	The teacher synthesises ideas that emerged from dialogue.	<i>The teacher states, after a question-and-answer sequence: "Some of the methods you have explained that are used to determine the sample space are very similar, because they all involve drawing the possible combinations. These drawings are called diagrams."</i>
Czech	<i>shrnutí</i> (<i>summarisation</i>)	Teacher/pupils recapitulate and sum up verbally.	<i>Teacher recapitulates steps of the solution of the problem.</i> <i>Teacher formulates remarks about metacognition.</i>
French	<i>bilan, synthèse</i> (<i>summary, synthesis</i>)	Phase to identify the important points to remember from the mathematical activity carried out.	<i>Teacher finishes a sequence about the Pythagorean theorem and its applications by an overall summary.</i>

Clusters as a unit of comparison

Earlier comparative studies between lexicons were limited to identifying terms that are similarly expressed, or differently expressed but referring to similar phenomena, and those that are entirely absent (Mesiti et al., 2022). Terms were more likely to be inter-related, that is, belonging to similar categories and holding properties in common but not identical. In response to this limitation and inspired by the approaches outlined by

the *Scale of Networking Strategies* (Prediger, Bikner-Ahsbahr & Arzarello 2008) we decided to work with clusters of neighbouring terms. Our first attempt involved working with clusters related to assessment and mathematics (Mesiti et al., 2022).

The Scheme of Educational Dialogue Analysis (SEDA)

Terms related to interaction were of interest to all the researchers. Recognition that the classroom is built on such exchanges prompted our decision to work with interaction as a cluster in a move towards internationalisation. The Scheme of Educational Dialogue Analysis (SEDA) (Hennessy et al. 2016) is an analytic framework for making sense of form and function of dialogic interactions in educational contexts. It is based on sociocultural theory which supports the notion that learning and development are cultural processes enacted through interactions with others (Fernández et al., 2001; Howe, 2010). SEDA was developed to represent productive forms of educational dialogues. It has been used in different settings, disciplines, and school levels. It is important to highlight that SEDA is a coding scheme for communicative acts from dialogic interactions; its unit of analysis is a turn of speech.

Although we did not have ‘codable’ data, we used SEDA because it is a validated tool to observe productive classroom interactions. There is also data available regarding the types of dialogic teaching that could better influence school-age children (Howe et al. 2019). As we were aiming to compare lexicons in terms of their interactional codes, we started by selecting terms related to interaction. Most of these terms were inspired by classroom activities which meant that almost everything could be interactive in nature implying different types of talk. By utilising SEDA we were able to build on a framework already developed, with clear and observable examples, and investigate the way in which synchronous classroom interactions were covered in each lexicon.

Classifying the terms within the interaction-related clusters

A condensed version of SEDA has eight categories: *invite elaboration or reasoning; positioning and coordination; reflect on dialogue or activity; make reasoning explicit; build on ideas; connect; guide direction of dialogue and activity; and, express or invite ideas*. Each category includes sub-categories that operationalise the grouping. As we could not consider a turn of speech the unit of analysis, we decided to use the main categories more than the specificities of each sub-category. The terms within each cluster were independently classified with SEDA by the author from the respective national team. This was followed by a whole group negotiation involving the four authors whereby each classification was scrutinised.

RESULTS AND FINDINGS

Some lexicons were more easily classified than others. Certainly, there were differences in detail of the interactional terms in the lexicons. However, SEDA worked as a referent to talk about each lexicon, its framework worked as a tool for thinking about our own interaction-related clusters. Figure 1 represents the interaction-related cluster for each lexicon (by column) and each term has been coded with the SEDA categories (by row). The terms in italics appear in more than one category.

	AUSTRALIA	CHILE	CZECH REPUBLIC	FRANCE
INVITE	elicit understanding; questioning; reasoning.	asking for help; student notices teacher's mistake; counter-asking; asking for help; elaboration question; information question.	oral exam; revision and practice of knowledge and skills.	asking mathematical questions; launching an activity; re-asking a question; <i>scientific debate</i> ; stimulating ideas or strategies.
POSITION		facilitating learning; lesson closure; putting in common; rounding up ideas; simplification of a problem; solving a mathematical task together; systematisation.	summarization.	comparing, ranking productions; correction; <i>demonstrating</i> ; summary, synthesis; validating/ invalidating productions.
REFLECT	<i>recapping</i> ; <i>reflecting</i> ; <i>reviewing</i> ; <i>summarising</i> .	metacognitive exercise; metacognitive question.	interactive explanation.	<i>commenting</i> ; drawing the attention on, stressing a mathematical point; <i>institutionalisation</i> .
REASONING	clarifying; defining; elaborating; explaining; justifying; modelling; reasoning.	arguing; providing justifications; teacher makes his/her reasoning explicit.	correction of the didactical test; institutionalisation.	arguing; explaining; justifying; <i>kneading-up</i> ; <i>scientific debate</i> ; situation of validation.
BUILD	<i>elaborating</i> ; <i>peer support</i> ; <i>prompting</i> .	building on a student's contribution; developing contents within a problem; paraphrasing what a student says to clarify; mathematical translation of a student's reasoning.	peer-assessment.	<i>commenting</i> ; <i>kneading-up</i> .
CONNECT	recalling; recapping; rephrasing; re-teaching; summarising.		individual consultation with pupils.	<i>institutionalisation</i> ; recalling.
GUIDE	assigning homework; demonstrating; feedback; guiding; modelling; posing problems; scaffolding; wait time; worked example.	promoting peer support; facilitating learning.	talk of the teacher with pupils.	conceptual help; constructive/procedural help; dealing with errors; <i>demonstrating</i> ; dialogic teaching; help with materials; helping, scaffolding; instrumental help; methodological help; skills help.
EXPRESS	answering questions; group discussion; student responses; whole class discussion.	spontaneous student contribution.	clarification; cooperative teaching.	answering a question; formulating; making explicit; reformulating; situation of formulation.

Figure 1: Interaction-related clusters in each lexicon coded by SEDA categories

	AUSTRALIA	CHILE	CZECH REPUBLIC	FRANCE
<i>appropriate code absent</i>	disciplining; encouraging; giving praise; (use of a) hook; motivating; positive reinforcement; reciting.	reciting; use of humor.		encouraging; engaging; disciplining; valuing.
<i>term too general</i>		question-and-answer sequence.		dealing with an incident (didactical/ non-didactical); asking general questions.
<i>term identifies form only</i>	group work; pair work; raising hands.			whole class work; half-class work; autonomous/ guided group work.

Figure 2: Interaction-related cluster terms unable to be coded with SEDA

	<i>terms in lexicon</i>	<i>terms in interaction cluster</i>	<i>terms that were able to be classified</i>	<i>unable to code</i>		
				<i>appropriate code absent</i>	<i>term too general</i>	<i>term identifies form only</i>
				(no.)	(no.)	(%)
AUSTRALIA	61	41	76%	17%		7%
CHILE	74	27	93%	4%	4%	
CZECH REPUBLIC	57	11	100%			
FRANCE	116	42	79%	10%	5%	7%

Figure 3: Summary - Interaction-related cluster terms

Content of the interaction-related clusters

Thinking about our understanding of interaction, about terms from our lexicon that might belong to such a cluster, alongside the SEDA coding scheme, enabled us to usefully identify a set of terms from each of the lexicons. The number of terms ranged from 11 to 42: a small subset for the Czech cluster (11 out of 57; 19%), a slightly larger Chilean cluster (27 out of 74; 36%), a significant cluster from the Australian lexicon (41 out of 61; 67%) and the largest number from the French lexicon (42 out of 116; 36%). Of these terms most could be classified with the SEDA coding scheme (76% Australia; 79% France, 93% Chile) if not all (100% Czech) (see Figure 3).

A significant number of interaction-related terms for the Australian cluster belonged to the SEDA categories of *Guide* (9), *Reasoning* (7), and *Connect* (9). For the French lexicon a sizable number of terms belonged to *Guide* (10), *Reasoning* (6), *Invite* (5), *Position* (5) and *Express* (5). This contrasted with the terms in the Chilean cluster. There were more terms in the categories of *Invite* (6), *Position* (7), and *Build* (4) than *Reasoning* (3), *Reflect* (2), *Guide* (2) and *Express* (1) (see Figure 1). One explanation is that the Chilean Lexicon relates more to simpler discursive exchanges than complex

interactional sequences; it tends to be less about students' language and participation and more about teachers' language and pedagogical approach.

When the Czech Lexicon was undergoing documentation descriptions of activities were condensed into more general terms. The consequence of this is that one term often covered a set of events that can be classified by different SEDA categories. When classifying them using SEDA, the most preferred meaning was chosen; however, it was felt that all terms could be included in several SEDA categories.

Integration of terms in the cluster

An interesting point is the convergence observed in the terms we were unable to code: they were either too general to be associated with one or two categories, or there was no category for them, such as terms denoting more affective interactions (see Figure 2). These terms in the Australian, Chilean, and French clusters related to building connections with students through praise and encouragement. Worth noting is that the term use of humour relates to the social character of Chilean classrooms in which such expressions are important. On the other hand, convergence is not strong regarding terms coded. The distribution of terms amongst categories shows discrepancies. For instance, *Position* is empty for the Australian Lexicon, and contains only one term in the Czech Lexicon, while it is the largest (7) for the Chilean cluster. A similar discrepancy is observed with *Guide*. Similar terms are not necessarily coded in the same way. For instance, summarising is in *Reflect* (Australian) whilst rounding up ideas (Chilean), summarization (Czech) and summary, synthesis (French) are in *Position*. The Australian term does not explicitly invite alternative or challenging viewpoints which was a central characteristic of the Position category.

CONCLUSION

In a move towards internationalisation, we adopted the notion of clusters. Treating a collection of neighbouring terms as the unit of analysis supported us in the creation of valid and meaningful connections amongst the lexicons with the intention of potential enrichment of local professional terminologies. The research confirmed both the challenge of comparison, namely, that analysis engages more than the label and published description of the term. The findings revealed differences and absences amongst the clusters in relation to particular categories of dialogic interaction. It is not clear whether these lexical differences are attributable to an absence of conventionalised practice, however, the extension of language offered by the comparison could be used to deepen understanding of teacher practice and help build meaningful connections amongst the lexicons.

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MEASURING DATA-BASED MODELING SKILLS IN A COLLABORATIVE SETTING

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Data-based modeling is about representing the relationship between two variables in a data set with a suitable function. It can be assumed to depend on individual knowledge about functions, as well as collaborators' knowledge when engaging in collaborative modeling activities. We investigate upper secondary school students' (N = 369) data-based modeling skills in a collaborative student laboratory. We find that data-based modeling skills can be reliably assessed based on a holistic approach by aggregating a range of skill facets. We also find that differences in these skills can be explained by participants' knowledge about functions as well as their collaborators' knowledge. Beyond individual differences, the scores carry substantial information about the group performance, even when controlling for collaborators' knowledge.

INTRODUCTION

There is a consensus that data science is becoming increasingly important (e.g., Schreiter et al., 2022). Data science includes, on the one hand, drawing conclusions from data, but also using data to solve problems (Carmichael & Marron, 2018; Donoho, 2017). Engel (2017), among others, calls for a stronger focus on data science in school education. However, even methods to describe, for example, the relation of two variables in a dataset with functions are mostly absent in German curricula, even for upper secondary school students. It remains an open question, to which extent students can acquire such skills, and how this relates to their knowledge about functions. Moreover, assessment instruments for data-based modeling skills are required. To investigate these questions, a student laboratory for data-based modeling with functions has been established at the LMU Munich. Its main goal is to convey the basic ideas of least square regression techniques to describe the relationship between two variables using a suitable function. The student laboratory is a collaborative learning setting that encourages direct participation in authentic scientific practice, and in which the students can work in small groups on current research topics using (in this case basic) scientific methods in a self-determined manner.

This contribution has three goals: (1) Investigate how data-based modeling skills can be measured within a student laboratory setting. (2) Describe which facets of data-based modeling skills are more respective less difficult to acquire for students in our learning setting. (3) Investigate to which extent interindividual difference in students' acquired data-based modeling skills can be attributed to their own individual knowledge about functions or their group members' knowledge about functions.

Data-based Modeling with Functions

Data-based modeling with functions means that the relationship between two numerical variables is represented by a model function (Vogel, 2014). The determined function can be used for the purpose of prediction (interpolation or extrapolation) and allows to visualize structural characteristic of the context (Engel, 2018). The choice of potential function types (linear, exponential, ...) is usually based on knowledge about the relation between the variables in the given context. Parameter estimation yields an optimal function to describe this relationship numerically. Methods to estimate a suitable function may range from informal processes (such as visually fitting a graph to the data) to more sophisticated modeling techniques (such as least square techniques). From a didactical perspective, limits are set by students' prior knowledge. Least square techniques arrive at an optimal function by choosing parameters by minimizing the sum of squared differences between values predicted by the function and the values in the data (Engel, 2018). Since upper secondary school students have scarce knowledge of programming or statistics, a conveying focus on the basic idea of least-square techniques may be considered an appropriate goal.

Data-based modeling processes

The modeling cycle represents a framework to modeling processes (Blum & Leiß, 2007; Greefrath, 2020). Eichler and Vogel (2013) transferred this cycle to data-based modeling, describing the following facets: Clarifying the phenomenological background, considering the data in different representations, choosing a function type, and estimating parameters for a specific function, as well as validating and interpreting the model by checking its predictions against prior assumptions, and drawing conclusions.

In their framework based on a substantive survey from a sample of data scientists, Lee et al. (2022) describe facets of data-based modeling: This includes framing a real-world problem, considering or gathering data, structuring and visualizing data, identifying and analyzing models that address the initial problem, and finally communicating results and proposing actions.

Both frameworks highlight the search a pattern in a dataset that is to be represented by a model function. This starts by selecting a specific function as a model for the context (*model selection*). In many situations, a structural analysis of the context, physical, biological, economic laws may suggest or help to exclude a specific function type. This requires substantial knowledge either about characteristic properties of a function type (e.g., that constant absolute changes of the independent variable go along with constant relative changes of the dependent variable for exponential functions) or at least about prototypical contexts, that could be described with a function type (e.g., biological growth for exponential functions). Fitting a function graph to a data set and understanding how changing a parameter of the function relates to a change in its graph, and thus to its (e.g., sum-of-squares) deviation from the data, is most likely also dependent on students' knowledge about functions (*model fitting*). Finally, both

frameworks focus on applying the model to make statements about the specific problem or context (*model application*). Depending on the problem and question at hand, this will most likely also require knowledge about the chosen function type, for example the meaning of the parameters in the chosen function (e.g., a slope in a linear function may relate to a velocity), or the global behavior of the function type (e.g., monotonicity, positivity, behavior for very high values).

Measuring data-based modeling skills

There are hardly any established instruments to measure data-based modeling skills. In their review, Cevikbas et al. (2022) describe that covering the various facets in the modeling cycle is decisive to assess mathematical modeling skills. Usually, a distinction is made here as to whether the complete modeling cycle (holistic approach) is assessed as a whole process, or if only facets (atomistic approach) are evaluated in separate parts of the instrument (Blomhøj & Jensen, 2003). Assessing the whole process in one task is considered more valid than the atomistic approach, since it also digs into the coordination of the different facets (e.g., Schukajlow et al., 2015). The atomistic approach, on the other hand, provides independent observations for different facets, and allows to study them separately from each other. Hankeln et al. (2019), for example, constructed a measuring instrument for different facets in the modeling cycle. They concluded that these skill facets measured can be understood as different components of a (holistic) modeling competence. However, this approach may lead to overloading participants with too many tasks on the instrument. In qualitative works, often one complex task is used to study the different skill facets as well as their coordination (e.g., Borromeo Ferri, 2006). In this sense, a third measurement approach is to ask students to work on one complex modeling task, that requires facets, but to code the quality of the different skill facets with independent indicators. This approach requires the coordination of all facets, but also yields separate data on each skill facet.

A further complexity arises, when data-based modeling skills are assessed in collaborative settings: The resulting scores of each participant may partly reflect individual modeling skills, partly skills to collaborate in data-based modeling, and partly the collaborators' skills. The first two may be seen as valid parts of data-based modeling skills, and the second can only be assessed in collaborative settings. However, this raises the question how much the collaborators' skills contribute to the individual scores in data-based modeling, and how this effect can be controlled.

Knowledge about functions as prerequisite of data-based modeling

As argued above, knowledge about functions may play a central role in the acquisition of data-based modeling, including model development, selection, and application. Current works distinguish knowledge about functions into the assignment, covariance, and object aspect (e.g., Weigand & Günster, 2022). Using the model to make specific predictions, e.g. about specific times in the future, requires to determine the value of a function, which falls under the assignment aspect. Prior research has pointed to considerable deficits in students' understanding of functions, especially regarding the

covariance aspect (e.g., Nemirovsky, 1996). This aspect refers to describing how a change in the independent variable relates to a corresponding change in the dependent variable. For example, some function types can be excluded in the model selection, if the covariance properties of a function type do not fit the context (cf. above for exponential functions). Model fitting and application require knowledge about the functions' parameters how they relate to the form of the graph and further function properties. This touches on the object aspect of functions. In summary, it can be assumed that good prior knowledge about functions, and in particular of the covariance aspect, supports the development of skills in data-based modeling.

QUESTIONS

We addressed the following questions in the context of a student laboratory for data-based modeling:

1. Does the combined approach of measuring facets of data-based modeling within one complex task provide an objective and reliable measurement of data-based modeling skills?
2. To which extent do students acquire the different skill facets of data-based modeling in the student laboratory?
3. To which extent can the individual knowledge about functions, or the collaborators' knowledge explain interindividual differences in the development of data-based modeling skills?

METHOD

As part of a student laboratory, which comprised three two-hour online sessions within one week, $N = 369$ secondary school students (17 classes from 11 schools; $M_{age} = 16.4$; $SD = 0.89$; 180 male; 185 female; 4 diverse) worked in small groups on learning tasks conveying the principles of data-based modeling in the context of authentic problems. We implemented a simple model fitting process using the least squares method interactively via sliders with dynamic geometry software. Each small group was supervised by one pre-service teacher, who participated for course credit. In a first phase, students worked on existing data sets throughout a sequence of activities that presented the main principles of data-based modeling with least-square methods. The students did not have to collect, process, or visualize the data sets themselves. Students were asked to select, fit, and apply different models for the provided datasets. In a second phase, students applied these principles regarding model selection, fitting, and application in a new rich data-based modeling task. This task was also solved in small groups, without support of the pre-service teachers. Each student wrote an individual solution to this last task.

The solution from this second phase was used to measure students' *data-based modeling skills*. We coded a total of 13 indicators at different levels for the skill facets (Eichler & Vogel, 2013) of data-based modeling from each student solution. For example, for *model selection* (two indicators), we asked students to argue why certain models are better or less appropriate for the context at hand. For *model fitting* (two

indicators), we evaluated the fit of the students' model to the dataset. We also considered how students use the least squares method to compare different models. For *model application* (four indicators), students were asked to answer two contextual questions based on their model. Finally, the students were asked to describe the least squares method in their own words (five indicators).

To measure students' *knowledge about functions*, we adapted items from existing instruments (eight closed answer items; Klinger, 2020; Lichti & Roth, 2018).

Scores for each measure were aggregated using a partial credit Item Response Theory model. Linear mixed models were estimated to investigate the relation of group composition and knowledge about functions with data-based modeling skills. In the null model, only the (random) effect of the collaboration group was examined. Afterwards, the individual scores for knowledge about functions and the average scores from the rest of the group were added as predictors.

RESULTS

The partial credit scales showed acceptable item fit indices. The WLE reliabilities (data-based modeling: .65; prior knowledge: .67) were acceptable. The interrater reliability for data-based modeling was satisfactory (Cohen's $\kappa=.78$).

In *model selection*, 42 % of the students reasonably argued why the selected function type can map the context, while 48 % can exclude other models based on context. 25 % provided both types of argument.

During *model fitting* 14 % of the students could not show a strategic approach using the least squares method in the second task, after the initial learning phase. 41 % of the students adjusted their function visually to find a model, but did not find the least possible sum of squares. The remaining 45 % indeed identified a model that realizes the smallest possible sum of squared distances. 93 % of the students successfully compared models based on least squares.

For *model application*, 72 % of the students used their model to make a prediction for the future. However, only 55 % interpreted their result in a meaningful way. 31 % of the students used their model to correctly comment on a statement about the covariance aspect of the corresponding context.

The null model showed that 47 % of the variance in students' data-based modeling skills is explained by group membership, indicating a substantial role of collaborative processes. The individual score in knowledge about functions significantly related ($\beta = .32, p < .001$) to data-based modeling skills, and also the average score of collaborators ($\beta = .16, p < .05$) explained interindividual differences. The fixed effects explained 17.4 % of the variance. The group-level explained variance dropped substantially, but still 33 % of the variance were explained by group membership.

DISCUSSION

We draw the following main findings from our study: (1) Addressing the facets of data-based modeling represents an objective and reliable measurement of data-based modeling skills. (2) Different skill facets of data-based modeling are acquired to different extent in the student laboratory. (3) Interindividual differences in the demonstrated skills in data-based modeling in a collaborative setting are predicted by individual knowledge about functions and the knowledge of the collaboration partners.

To assess skills in data-based modeling, we implemented a holistic approach, following the results of Cevikbas et al. (2022) by measuring a range of different skill facets of data-based modeling within a single task. This led to satisfactory results in terms of objectivity and reliability. However, future research should investigate to which extent the results are influenced by the specific data-based modeling task in the assessment. Moreover, not all facets of data-based modeling could be implemented in our task. In particular, representing data, specially using digital representations, might be considered as an additional facet in the future (Eichler & Vogel, 2013; Lee et al., 2022). However, this facet could already be introduced in the lower secondary level (Podworny et al., 2022).

Central principles of data-based modeling can be learned by the students within the student laboratory to a certain extent. Especially the model fitting based on the least squares method is a skill that was shown by the students with high solution rates after participating in the laboratory. Providing arguments for or against a function type during the model selection and application, which mainly build on the covariance aspect of functions posed substantial problems to participants. Deficits regarding the covariance aspect, that were shown in prior research, probably have an effect on task processing (Nemirovsky, 1996). On the other hand, parts of the model application, that require the calculation of values of a function to make a prediction for the future, were solved at higher rates. However, not all students can then interpret or validate their mathematical result obtained in the context, which is in line with similar general results on mathematical modeling (Schneider et al., 2021). This indicates that facets, that depend on the covariance aspect of functions, or which touch on interpretation or validation, require specific focus when designing learning opportunities on data-based modeling.

Our results indicate, that differences in individual data-based modeling skills can possibly be explained by varying individual prior knowledge about functions. This underpins the role of knowledge about functions for learning data-based modeling. Another possible factor in our collaborative setting is the knowledge about functions of the collaboration partners. We found that individual scores are substantially influenced by group membership. Controlling for participants' and their collaborators' knowledge about functions showed that both contribute to explaining interindividual differences in students' scores. Therefore, also the knowledge of the collaboration partners in group work processes on data-based modeling seems to contribute to the

development of data-based modeling skills in a collaborative setting. In particular, variance explained by group membership was reduced when including collaborators' knowledge. This indicates that the group effects can be controlled when measuring data-based modeling skills in a collaborative context to a certain extent, by taking collaborators' knowledge into account. However, a substantial contribution of group membership remains. Further studies are necessary to clarify to which extent this stems from collaborative learning in the first phase, or from the collaborative modeling processes in the second phase of the laboratory.

In summary, we provide a proof-of-concept study of introducing and measuring data-based modeling based on a simplified least-squared method in higher secondary school students in a collaborative setting. We provide evidence that the acquired data-based modeling skills are related to participants' as well as the collaborators' knowledge about functions.

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ANALYSIS OF HOW PRE-SERVICE MATHEMATICS TEACHERS INCLUDE SRL IN THEIR TEACHING PROPOSALS

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From their initial training, teachers must acquire a series of competencies, including self-regulation, allowing both the teacher and the student to reflect, control and modify the various aspects involved in the teaching and learning process. This study analyzes the Master's dissertations of a group of future mathematics teachers to describe the self-regulation actions they claim to promote in the classes carried out during their internship. The results show that these teachers promote practices related mainly to the cognitive, interactional, and epistemic suitability and to a lesser extent, aspects related to the affective, ecological, and mediational suitability.

Previous scientific literature shows that there is a relationship between self-regulation and the academic performance. Pintrich (2000) and Zimmerman (2000) have shown evidence suggesting that self-regulated students are good metacognitive strategy users. They know how to plan, set goals, organize, self-evaluate, and monitor their progress. All this has a positive impact on their learning. The more autonomous and independent they are in self-regulating, the better academic and learning results they obtain. Various authors have highlighted the relevant role of the "development of self-regulation competence" as one of the competencies that future teachers must learn in teacher training programs (Kramarski & Revach, 2009). In the case of the teaching of mathematics, there is also the debate on how to guide future teachers so that their students can develop their self-regulation learning (SRL) competencies. Some studies indicate that future teachers lack knowledge and skills regarding what strategies to use, when to use them, and how, to support their students in developing SRL (Putnam & Borko, 2000). Pintrich has proposed a list of indicators that we have already used to analyze how pre-service mathematics teachers include SRL in their teaching programming (Hidalgo-Moncada et al., 2021). This paper aims to analyze the actions that future mathematics teachers design in their teaching proposals carried out during their internship in a Master's degree program addressed to secondary teachers in Catalonia (Spain), in order to promote students' self-regulation.

THEORETICAL FRAMEWORK

Many authors have highlighted the importance of teachers knowing effective strategies so that they know how to promote SRL in their students. According to Timperley (2008), "reflection on one's own teaching practice" is one of the critical competencies that contributes to pre-service teachers being able to identify whether they can (or cannot) promote SRL among their students. Different authors have highlighted the importance of teacher reflection (Godino et al., 2016; Perrenoud, 2007), claiming that it is a tool to assess the efficiency of the teaching processes carried out (Font et al.,

2018). A reflective nature in the design of the teaching work makes it possible to analyze the quality of the mathematics developed in the classroom. It can also help the teacher to promote self-regulated learning more often (Hidalgo-Moncada et al., 2021).

According to Pintrich (2000), SRL is an active process in which students monitor and regulate their own learning, considering cognitive aspects related to their own learning process and aspects related to motivation and behavior according to a series of set goals in their context. De Corte et al. (2000) suggest that self-regulation positively impacts on students' academic performance. However, there is evidence that many teachers need to promote SRL (Darmawan et al., 2020). Previous studies have identified teaching practices that successfully promote SRL among students. In this paper, we start from this evidence to analyze what kind of practices pre-service teachers use to promote SRL among their students.

METODOLOGY

We analyzed and described the self-regulation actions 20 pre-service mathematics teachers claim to promote in their classes taught during their internship. These teachers participated in a Master's degree program addressed to mathematics teachers (middle and high school) in a Catalan university (2021-2022). The internship lasted two months (approx.), during which the teachers observed the mathematics sessions in a classroom (secondary education level). They had to design and implement a didactic unit for their assignments. At the end of the internship, the teachers carried out a Final Master's Project (dissertation), in which they presented the designed and implemented didactic unit, reflecting on how they had developed it and those aspects they missed. This reflection was carried out based on the Didactic Suitability Criteria (DSC) of the Onto-semiotic Approach (Godino et al., 2016). In this study, a content analysis of the Master dissertations is carried out. For this analysis, a previous instrument was used, consisting of 23 actions that promote self-regulated learning in the mathematics lesson, which are classified according to the six didactic suitability criteria (Hidalgo-Moncada et al., 2021). An example of this characterization is shown in Table 1.

Table 1. Characterization of self-regulation practices according to the criteria of didactic suitability

Suitability Criteria	Self-regulation actions
Epistemic suitability	Promote argumentation and explanation of procedures used.
Cognitive suitability	Promote processes of high cognitive demand, such as generalization, intra-mathematical connections, representation, conjectures, etc.
Interactional suitability	Organize forms of cooperative work.
Mediational suitability	Implement different teaching means that promote the search, processing, and obtaining of information that the student must assimilate.

Affective suitability	Consider the interests of the students, their family and social context, to generate activities for purposes with their interests.
Ecological suitability	Implement different forms of evaluation for the same content.

To analyze the Master dissertations in-depth, *Atlas.ti* has been used, which has made it possible to identify the self-regulation actions that the teachers claim to promote and those that they have not promoted during the internship.

RESULTS

Analysis of SRL Practices Promoted by Pre-Service Mathematics Teachers

According to the data collected, the pre-service mathematics teachers who participated in the Master's Program for Secondary Teachers of some of the Catalan universities during the 2021-2022 academic year, especially promoted practices related to cognitive (22.3 %), interactional (20.1%) and epistemic (19.3%) suitability. These practices include aspects such as teaching strategies that allow students to: solve problems, promote reflection strategies among students, teach them to check their answers, generalize, compare different solutions for the same activity, argue and explain how they solved an exercise, etc. Table 2 presents the results obtained in the analysis of the didactic proposals of each of the twenty pre-service teachers who were part of the sample of the Master Dissertations (MDs) analyzed.

Table 2. Analysis of the Master Dissertations (MDs) by Didactic Suitability Criteria (DSC) (percent).

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
A. suitability	0,0	10,7	7,1	0,0	5,9	7,7	7,1	7,1	5,3	0,0
C. suitability	22,2	17,9	14,3	24,1	17,6	38,5	7,1	35,7	31,6	12,5
E. suitability	5,6	21,4	21,4	20,7	23,5	15,4	35,7	0,0	10,5	12,5
Ep. suitability	44,4	14,3	14,3	17,2	11,8	0,0	0,0	35,7	15,8	25,0
I. suitability	16,7	17,9	25,0	13,8	23,5	23,1	21,4	14,3	21,1	37,5
M. suitability	11,1	17,9	17,9	24,1	17,6	15,4	28,6	7,1	15,8	12,5
Total	100	100	100	100	100	100	100	100	100	100

	P11	P12	P13	P14	P15	P16	P17	P18	P19	P20	Total
A. suitability	5,3	10,5	0,0	0,0	0,0	0,0	3,4	0,0	4,3	5,9	4,5
C. suitability	21,1	21,1	33,3	31,3	33,3	16,7	27,6	11,1	21,7	23,5	23,3
E. suitability	10,5	10,5	13,3	6,3	13,3	16,7	13,8	33,3	8,7	11,8	15,2

	P11	P12	P13	P14	P15	P16	P17	P18	P19	P20	Total
A. suitability	5,3	10,5	0,0	0,0	0,0	0,0	3,4	0,0	4,3	5,9	4,5
Ep. suitability	26,3	31,6	33,3	25,0	13,3	33,3	20,7	11,1	26,1	11,8	20,2
I. suitability	21,1	21,1	20,0	18,8	13,3	16,7	20,7	44,4	21,7	29,4	21,1
M. suitability	15,8	5,3	0,0	18,8	26,7	16,7	13,8	0,0	17,4	17,6	15,7
Total	100	100	100	100	100	100	100	100	100	100	100

* A=Affective, C=Cognitive, E=Ecological, Ep=Epistemic, I=Interactional, M=mediational; P*n* means the person from the sample

On the contrary, if we examine the practices that teachers do not include in their teaching proposals, we find that they tend to be either affective or ecological. Specifically, 23.5% affirm that they do not consider the interests of the students, nor their family or social context, to generate activities related to their interests, promoting a better emotional, motivational, and attitudinal feeling. Something similar occurs in the case of ecological suitability: 17.6% did not include any practice where the study of mathematical content can be linked with the environment and daily life. Intra-disciplinary connections are rarely included in the practices promoted in the didactic proposals elaborated by the pre-service teachers of the selected sample. On the other hand, it is rarer that teachers do not consider practices from other facets, such as cognitive, such as "promoting argumentation and explanation of the procedures used in the proposed activities" in the didactic proposals; only in the case of 11.3% did this happen. This occurs similarly with other "cognitive" practices (such as teaching strategies to solve problems or generalization). Table 3 shows that the percentage of teachers who do not include this type of practice is negligible (5.9 %).

Table 3. Analysis of the Master Dissertations (MDs) by type of practice (percent).

NA20	NA21	NC5	NC9	NC10	NEC22	NEP2	NEP3	NI12	NI14	NI15	Total
5.9	23.5	5.9	5.9	5.9	17.6	5.9	11.8	5.9	5.9	5.9	100

Often, pre-service teachers propose teaching activities integrating different SLR practices. That is the six domains to which the DSCs refer usually appear in groups (with each other). Table 4 shows us that these groupings are not "random" but that we can observe some trends. For example, in the most usual practices (cognitive ones, which appear 83%, that is, 23.3% of the time), on 14 occasions, they were raised in activities where practices related to the epistemic criterion were also proposed, and on ten occasions appear together with the interactional ones. In the case of practices where the interactional aspect is considered (n=75, representing 21.1% of the total), they appear together with the epistemic ones on 20 occasions and with the cognitive ones in 10 cases. That is, the data suggest a tendency to approach SLR through activities that include cognitive, interactional, and epistemic practices. On the other hand, aspects

related to the affective criterion, for example, are hardly considered (at least, from the SLR point of view).

Table 4. Co-occurrence of practices among the Masters’ dissertations (n=20)

	A (n=16)	C (n=83)	E (n=54)	Ep (n=72)	I (n=75)	M (n=56)
A (n=16)	0	0	4	0	1	1
C (n=83)	0	0	6	14	10	6
E (n=54)	4	6	0	1	2	6
Ep (n=72)	0	14	1	0	20	1
I (n=75)	1	10	2	20	0	7
M (n=56)	1	6	6	1	7	0

* A=Affective, C=Cognitive, E=Ecological, Ep=Epistemic, I=Interactional, M=mediational; Pn means the person from the sample; n is the number of quotes

Cognitive, Interactional, and Epistemic Suitability

These three elements are the ones that appear the most in the way in which the teachers in this study propose to work the SLR. However, what exactly do they do? How do they do that? What are the reasons why they propose the practices they propose in their teaching proposals? The following table (Table 5) groups examples of the type of proposals and justifications provided by the teachers who participated in this study in their Master dissertations.

Table 5. Examples of activities or strategies that teachers use to promote SLR in their didactic proposals.

What they do	<p>“When I defined the unit, I took into account that in the activities the processes of communication, experimentation, contextualization, formulation, argumentation, conceptualization, algorithmizing, problem-solving, representation, generalization, and modeling were worked on.” (P1, Cognitive & Epistemic suitability)</p> <p>“Promote inclusive and participatory ways of working, such as working in pairs or small (heterogeneous) groups so that it was the same colleagues who helped each other.” (P14, Interactional suitability)</p>
How do they do that	<p>“... when solving equations is considered, I ask them what they can do to solve the system without doing mentally, if they could find a method. Then they themselves give ideas, explain their conclusions, debate how to make a colleague's proposal and finally reach a conclusion.” (P3, Cognitive, Epistemic & Interactional suitability)</p> <p>“When two students had opposing ideas or when a student explained his own reasoning to his classmates, the students argued their conclusions, results or reflections in order to give them validity and credibility in front of their peers.” (P14, Interactional & Epistemic suitability).</p>

“Usually, when faced with an answer from a student, I would ask the rest of the class what they thought. If they agreed or disagreed. In many cases, when a dynamic debate was generated, some student answered, and we could jointly find a solution to the question or doubt that arose.” (P20, Interactional suitability).

“It was tried that once one or two members of the group had understood in depth the concept that was asked of them, they tried to explain it to their group mates.” (P8, Interactional & epistemic suitability)

Why do they do it “These questions force students to first conjecture hypotheses and possible outcomes and then prove or disprove them, demonstrating the use of exploration.” (P8, Cognitive suitability)

“Among the more theoretical or "file" activities, I have interspersed others with games from different learning platforms to continue to maintain the students' interest in the didactic unit.” (P15 Interactional & Affective suitability)

“Cooperative work is highly recommended when learning, even more so than individual work. It is for this reason that healthy and consensual group work was always advocated.” (P17, Interactional suitability).

The arguments used by teachers regarding how they promote SLR are grouped into three types. On the one hand, those aspects in which teachers point out what they do in relation to practices related to cognitive, epistemic, or interactional suitability. Generally, it is usual to focus on promoting practices such as problem-solving, reflection on the strategies used, generalization, and promoting inclusive and participatory ways of working, in heterogeneous groups.

On the other hand, teachers also explain in their teaching proposals how they intend to achieve everything they say. They propose strategies such as promoting teamwork in a collaborative way (interactional suitability), encouraging practices such as argumentation, explanation, sharing the reasoning elaborated, seeking to give credibility to the results obtained in front of the rest of the classmates, etc. The crucial role played by (self)-assessment is also emphasized: both, the evaluation of one's own work by oneself and the scrutiny of what one has done by the rest of the group. In this sense, the proposals made by teachers usually combine the cognitive part of learning with an emphasis on interaction (collaboration) between students. In some specific cases, reference is also made to the role of the teacher as the "initiator" of the learning process, proposing questions that can lead to connecting mathematical content as the starting point of the didactic sequence.

Finally, we find a third type of practices related to the reasons teachers say to justify their decisions about the practices they propose to use to promote SRL among their students. They highlight aspects such as that the proposed practices "force" students to develop (use, implement) cognitive processes such as conjecturing hypotheses, testing, checking, and different potential responses to the proposed activities. Another frequently cited reason refers to the aspect of interaction: the teachers in the sample for this study generally agree that collaboration between students (sharing, explaining to each other, etc.) greatly contributes to students developing these SRL competencies.

DISCUSSION AND CONCLUSIONS

The change towards student-centered mathematics teaching models has meant that aspects such as the SRL occupy a place of interest in the research agenda in professional development programs (Kramarski & Revach, 2009). Previous research has already indicated that self-regulation is a key competence for learning (De Corte et al., 2000). Studies by authors such as those reported by Robson, Allen, and Howard (2020) show that students who use more self-regulation strategies tend to obtain better scores in their grades. In the same way, Guo, Lau, and Wei (2019) state that self-regulation is related both to improving learning and to obtaining better learning results (more robust learning with understanding). However, despite all this plethora of evidence on the positive impact of using self-regulation strategies, there needs to be more research in the field of mathematics education that explores what practices (how) teachers use to integrate (encourage) self-regulation in their teaching proposals. In the study we report here, some regularities suggest a "perception" shared by the teachers in the sample who participated in the study on how to include self-regulation in their teaching proposals and to encourage what things can be used in self-regulation practices. Above all, according to the results of the analysis of the data collected in the fieldwork, what we see is that these teachers relate self-regulation practices with learning strategies (cognitive suitability criteria) with the forms of interaction in the classroom (interactional suitability criteria), and with the mathematical content to be learned (epistemic suitability criteria). Teachers include practices related to aspects such as argumentation, the formulation of conjectures, the exchange of explanations, and the verification of results, as habitual practices that help their students develop their self-regulation competence to acquire autonomy in their learning process. An aspect that also appears in some of the Master dissertations is the importance that these teachers give to the heterogeneous student work groups. The heterogeneity in terms of ability level in the composition of a workgroup has been highlighted as a key aspect of learning (Díez-Palomar et al., 2020). Suppose there are no different ability levels when a classroom activity is shared. In that case, students cannot help each other (there are no scaffolding processes nor Proximal Zones of Development as Bruner and Vygotsky claimed, respectively). The use of the DSC has allowed us to focus our analysis on highlighting behaviors, and regularities, in the didactic proposals made by the teachers in the sample, using those six suitability criteria that arise from consensus with the research and practitioner community and which have been validated by previous research (Breda et al., 2018; Font et al., 2018).

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LESSON STUDY AND IMPROVISATION: CAN TWO WALK TOGETHER, EXCEPT THEY BE AGREED? ^[1]

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This study reports on one main challenge faced when implementing Lesson Study (LS) in a cultural context different from its Japanese origin: the detailed planning of a lesson in opposition to improvisation during its teaching. We present different ways in which LS and improvisation coexisted for different teacher communities and individuals, and draw some conclusions and implications.

INTRODUCTION

A main component of Lesson Study (LS) is the collective research and planning of a lesson. This includes scrutinizing mathematical ideas, establishing mathematical and pedagogical goals for the lesson, selecting appropriate tasks and problems to realize these goals, deciding on and specifying the teaching moves, thoroughly considering the anticipated responses, proposals and difficulties of students and planning how these may be addressed (e.g., Fujii, 2018). Resources such as the participants' previous teaching experiences, available documented instructional practices and others, nurture the detailed planning process, which aims at leaving almost no room for unexpected events in such a way that there is little need for the teacher to make unforeseen decisions. In contrast, in other instructional scenarios, for example as described by Chazan and Ball (1999), teachers may want to employ "teacher moves [that are] the product of subtle improvisation in response to the dynamics and substance of student discussion" (p. 7), which implies a deliberate action that goes beyond planning. For some teachers, independently of their teaching traditions, improvisation may be considered as an essential part of the spirit of their profession, especially if they are experts (e.g., Borko & Livingston, 1989; Pinto, 2017). Thus, detailed planning and improvisations can be seen as two opposing attitudes to teaching, a contrast that may become a serious hurdle when trying to adapt LS outside of its culture of origin. This was one of the challenges we faced in implementing LS within school-based PLCs (Professional Learning Communities) in Israel, and which is the subject of this report.

ON IMPROVISATION - A BRIEF THEORETICAL BACKGROUND

Crossan and Sorrenti (2002) define improvisation as "intuition guiding action in a spontaneous way" (p. 27). Some of the dictionary definitions of improvisation are "the art or act of composing, uttering, executing, or arranging anything without previous preparation" [2]; and "the act of making or doing something with whatever is available at the time" [3]. The words "intuition", "art", and "with whatever is available" imply that improvisation may not necessarily be a random or arbitrary action. Moreover, in the literature on domains such as jazz or theatre, in which improvisation is pervasive, we even find established "techniques" of improvisation (e.g., Coker, 1980).

In the context of teaching, improvisation is triggered by unplanned classroom situations that require in-the-moment decisions and actions. Improvised moves may be rooted in previous experiences, which even when unconscious or implicit, emerge to shape and drive action. Drawing on Fischbein's (1987) seminal work on primary and secondary intuitions, we propose to distinguish between *primary and secondary experiences*: whereas primary experiences accumulate unconsciously during the natural development of a person from childhood to adulthood, or from a novice to an established professional, secondary experiences accrue as a result of explicit reflection and its subsequent internalized "packaged" morals, as may be the case for experienced teachers. Spontaneous actions are driven by both types of experiences, but while actions based on primary experiences resemble a knee-jerk reflex, constructed from lived-in 'natural' events, actions propelled by secondary experiences are formed in more systematic ways, after these experiences were reflected upon, and 'stored' in a repertoire of automatic or semi-automatic reactions to unexpected situations.

Improvisation in teaching was described by Borko and Livingston (1989) with the metaphor of improvisational performance, where rather than working with a detailed written script, an actor enters the stage with a general description of a situation and a set of performing guidelines, and from that point onwards is continually responsive to new situations. When improvising scenes, performers draw upon extensive repertoires of routines or patterns of action that may be based, as we suggest, on primary and secondary experiences. Similarly, Borko and Livingston suggest that expert teachers may have mental scripts that consist of general outlines of their lessons, and that they fill in the outlines during the lesson according to students' reactions.

In this paper, we report on preliminary results from a study on the Israeli LS project named Math-VALUE (Video Analysis and Lesson-study to Upgrade Expertise). In this project, school-based PLCs were established, with academic support and guidance, to learn about and to locally enact the stages of LS. We focused on teachers' perceptions of improvisation during lesson planning, posing the following research question: In what ways do teachers enact the delicate balance between implementing a lesson planned through LS and the need, or inclination, to improvise?

METHODOLOGY

Settings. As of October 2020, eight Math-VALUE school-based PLCs have been operating across Israel, each including 8-12 middle and high school mathematics teachers (76 in total), with an average teaching experience of 14 years. Each PLC met for a total of 60 hours per school year, either face-to-face, virtually or in a hybrid mode. The schools in which these PLCs were formed represent different sectors of the Israeli society: Jewish and Arab, rural and urban, religious and secular. The facilitators of these PLCs are school-based leaders, each selected from among the local mathematics team. Prior to facilitation, they underwent a year-long preparation course, set as a PLC for leaders in an academic institution, during which they participated in a full cycle of LS and became acquainted with leadership practices and tools to run their PLCs.

Data collection. This study is a small part of a larger research project to investigate the implementation of LS in school-based communities in Israel. Data for the study was gathered from semi-structured individual interviews with 13 randomly selected teachers (1-2 teachers from each PLC), and with three PLC leaders. The interviews were conducted online (about one hour each, on average) and focused on personal and group experiences. All interviews were recorded and transcribed. Due to space limitations, in this report the data is reduced to 4 teachers from 4 different PLCs: Ori, Lili, Naor and Sigal (pseudonyms), with a teaching experience of 20, 8, 4 and 7 years respectively, and Niki, a leader of another PLC with 35 years of teaching experience, who participated in the leaders' LS cycle.

Data analysis. The thematic analysis included reading all transcripts and selecting utterances regarding each of the lesson study stages (planning, implementing and post-hoc reflection). The utterances were classified into emerging themes in order to identify perspectives and phenomena related to improvisation. In this report, we present only selected quotes to illustrate the points made.

FINDINGS

We identified four different ways by which teachers choose to balance (or not) between a detailed lesson plan resulting from LS, and an inclination to improvise, if such exists. In the following, we describe and demonstrate the four types of improvisation found.

Type 1: Planning alongside purposeful built-in improvisation

Ori is an experienced and highly-regarded teacher in a well-known and reputed school located in a large Israeli city. The PLC he participated in worked for several intensive sessions on a collaborative planning of a lesson, which Ori eventually taught in his tenth-grade advanced-level class. The group decided to choose the topic of *introduction to extreme value problems*. Much of the lesson planning time was devoted to designing the task and preparing a student worksheet around it, a stage which involved several rounds of thorough revisions. The task consisted of exploring areas of triangles with a given perimeter, in an attempt to find the one with the largest area. The final product of this LS group was a worksheet consisting of six items. Despite the explicit agreement among group members that the lesson is to be highly interactive, the lesson plan did not include foreseen student questions, proposals or difficulties, nor did it specify ways for the teacher to address students' reactions.

Ori's teaching of the lesson was considered exemplary by both his peers and several mathematics educators who watched the videotaped lesson. The students' responses, including unanticipated reactions, were all encouraged by the teacher and incorporated into the classroom discussions, while linking them to different topics and terms learned throughout the year. Overall, it seemed that the students have led the flow of the lesson.

In the interview, Ori described this practice as typical of his teaching and expressed his belief that students should set the pace of the lesson and lead the discussion:

“I am very associative, and it takes me to all sorts of different and weird places, apropos a lesson plan [...]. The lessons and the discussions are built mostly according to where the students lead them [...]. The tasks often develop from of the questions the students ask.”

Ori stated that this central role of students, as well as the quality of the discussion, are more important to him than the number of exercises solved during the lesson. When relating specifically to the lesson planned in the LS activity, he said:

“The only thing that was special about this lesson was that I had this basic structure in mind. I mean I had a problem with clear sub-problems that I planned to do with them in a particular order. I knew at what point I wanted to start and where I wanted to end, but I did not know how the lesson would develop in the middle. [...] I do not have such a set of lesson plans and exercises. I mostly build my lessons based on my experience and from what comes up in class [...]. This lesson was relatively meticulous in the way it was planned because that was the goal of the LS project, but I didn't know what the students would bring to my doorstep. So in that sense there was a very neat and organized frame, and I had the goals I wanted to achieve, but I was hoping, and I know it works for me, to achieve these points [...]. In retrospect, planning the lesson in more detail would have been meaningless. I think the lesson is built on trusting my intuition and hoping the students will not disappoint and get to very interesting points. Of course, I could not anticipate everything that happened there. But because the students are so good and curious, I could count on them to take the lesson in interesting directions.”

Ori demonstrates a possible way to compatibilize planning and improvisation: the LS activity provided, on the one hand, a structured plan resulting from careful successive revisions of the tasks, as well as a clear vision that unpacked the lesson's goals. Yet, on the other hand, it left the flow of student reactions, questions and proposals open and unanticipated. Ori attributed the lesson's success to the coexistence of these two aspects. In addition, it appears that his profile as an expert teacher, skillful in handling students' reactions and in steering successful discussions, was part of this success.

Type 2: Tight adherence to the planning

Niki is an experienced teacher and head of the mathematics department in a secondary orthodox Jewish school for girls. She participated in the leaders' PLC, and her LS group meticulously planned a geometry lesson, which Niki taught in her ninth-grade advanced-level class. In the lesson, students were given “physical” line segments drawn on transparencies, to be treated as diagonals of quadrilaterals, and were asked to position the segments in various ways to construct different quadrilaterals, then sort them into groups. The detailed lesson plan included anticipated students' responses and explicit suggestions for ensuing teacher reactions. Niki followed the original plan, but despite the careful pre-consideration of possible answers, she faced some surprises. For example, students sorted the quadrilaterals into familiar vs. nameless shapes, instead of focusing on the diagonals' geometrical properties (e.g., equal, perpendicular, bisecting each other). Niki chose to merely re-direct these students' attention to the diagonals, yet in her post-hoc reflection she pointed to an alternative which she would

have preferred: stating specific classification criteria. In the interview, she connected this decision to her commitment to the joint lesson plan:

“It's very different from a lesson I plan by myself, where I feel completely free to make changes while teaching wherever I find it appropriate, and here I had to be pretty close to what we had planned, so that it would really be a ‘lesson study’ and not my specific lesson. Things that had not been planned or anticipated suddenly arose. Of course, I had to apply my own judgment about what I thought my [LS] team would want me to do. [...] I think what I did in the lesson was pretty much what I was expected to do.”

Another dilemma Niki faced was when students were asked to construct quadrilaterals with equal diagonals, and all of them drew only rectangles. She decided not to intervene by asking about other quadrilaterals with that property, and to bring this up only in the later plenary discussion. In retrospect, Niki said that, although she favored a teaching move that would result in everyone constructing other quadrilaterals with equal diagonals, she had no regrets for not following this option:

“I still think there will be other opportunities to do this [...] I would not change the lesson. Its structure was natural and right. We cannot always predict students’ reactions.”

Niki’s case demonstrates a teacher’s choice to set aside her habitual flexibility, and stick to the lesson plan as a deference to the LS group work, which is considered as overriding in-the-moment unplanned action. Thus the extent to which she improvised was minimal. However, minimal improvisation may result from a less cognizant choice, as demonstrated in Lili’s case below.

Lili teaches in an urban school serving mostly students of low socioeconomic status. She taught a lesson on target functions in extrema problems, tightly planned by her LS group. Though a lot of original thought was put into the planning, the main task created was a worksheet for students to tackle individually. Lili described the lesson as follows:

The lesson was so [firmly] planned that there was nothing for me to do but to teach it as planned. There was not much room for digressions here [...]. The students did not need me, and most of the time I just walked around [in class]. I was bored. They did not need me.

In both Niki’s and Lili’s cases, the detailed lesson plan did not leave much room for improvisation. However, there is a difference between these cases: whereas Niki intentionally decided not to deviate from the plan (even if she would have liked to), Lili followed a plan that purposefully minimized teacher-student interactions, thus exempting the teacher from unexpected situations and diminishing opportunities for improvising. Therefore, reducing the improvisation space may be ascribed either to a conscious teacher choice or to a built-in feature of the lesson.

Type 3: Modifying the original plan by adding or omitting parts

Naor teaches in a large rural school, and was part of a group planning a lesson on *graphic and algebraic representations of extremum points*, built around two problems.

When teaching the lesson, he decided to add a third and more advanced problem that he pre-prepared and was not part of the original plan. In retrospect, he said:

“The lesson was fine and overall went as planned. There were almost no situations where I had to improvise, only minor things here and there. [...] But with the third question, I went over the top, jumped straight to some story of graphs and here I realized I had failed. I should have stopped after the two simple problems and give them time to practice. I went beyond the [students’] ‘focus’ time, and instead of slowing down pace and intensity, I accelerated. The truth is that it wasn’t according to the original plan, but I told myself I would do what we planned and if there was more time left, I’ll do another problem, I’ll improvise. [...] Overall it was very interesting to see what happened in the class. [...] one group lost touch, while another group was very much intrigued. During the break, about 15 students stayed and insisted on finishing the third question [...]. But during the lesson I felt I had overdone it.”

Although Naor called his deviation from the collective plan “an improvisation”, it was a pre-prepared alternative that he purposefully designed as a more challenging item. He started by following the collective plan, then presented his alternative. Naor did not regard this deviation from the original plan as unfaithfulness to the group but rather as an expression of autonomy, that allows him to add to the plan. For him, improvisation was not necessarily an in-the-moment spontaneous decision but a modification to the collective plan decided individually, and perceived as a teacher’s legitimate right, even when in retrospect he admitted that his decision was misguided.

Type 4: Improvisation that reverses the entire lesson plan

Sigal is a mathematics and biology teacher in a highly-esteemed and selective school in a large city. She participated in an intensive collaborative planning of a lesson on *arithmetic series*, that she was meant to teach in her tenth-grade advanced-level class. Her LS group worked hard to plan an innovative lesson and a lot of thought was put into the details. However, the video of the lesson Sigal eventually taught showed that it was hardly related to the plan. Tali, the facilitator of this PLC, described it as follows:

“[Sigal] did what she happened to feel like doing, as if there was no lesson plan. That’s not improvisation, it’s not right [...]. She did agree to the lesson plan [...] she had a [printed] lesson plan and she kept looking at it. One of the comments she received from Avi [another PLC member], was ‘what’s this thing of looking at the prints all the time? Like, what, you haven’t studied the lesson?’ Now, not only she kept looking at these prints, but she also did not do what was written there! She did not prepare, did not invest any thought. Avi said to her bluntly, ‘listen, before I go to class, the night before, I study the lesson, I memorize it [...] Maybe you should do the same?’”

We lack data to explain why Sigal decided to ignore the original plan in which she was involved. We speculate that this extreme case, whatever its underlying reasons may be, exemplifies a case of improvisation that completely overturns any planning, and relies solely on in-the-moment alternative decisions, even while keeping an eye on the original plan - perhaps in order to maintain some resemblance to it.

DISCUSSION

Several educational systems worked and still work to “import” LS into their teacher professional development programs (e.g., Quaresma et al., 2018). Many challenges were documented when adopting and adapting LS into a culture different from the one in which it has originated and flourished (e.g., Stigler & Hiebert, 2016; Skott & Møller, 2020). Israel is no exemption. In this report, we focused on one implementation hurdle: planning vs. improvisation. Sharkansky and Zalmanovitch (2000) see improvisation as a cultural artifact, noting that “people and cultures differ in the relative values they place on planning and improvisation” (p. 322). In the case of Israel, they maintain that the intractability of the unique problems faced by the country imposed a flexible way of coping, as opposed to enacting clear-cut, planned solutions. Thus, in the Israeli culture improvisation is pervasive in all facets of life, and education does not escape it. Our findings show that, although the LS PLCs were involved in meticulous planning, in most cases improvisation did take place. Perhaps this is inevitable not only because of the local culture, but since improvisation is deeply rooted in the profession of teaching: it is “what teachers do, regularly and routinely, differently every time, as they improvise, adjust and adapt the routine” (Mathewson & Reid, 2016, p. 43).

However, we have also shown that balancing improvisation and planning may take different forms. Of the four types of improvisation found in this study, two align with dimensions already identified by Mælanda and Espeland (2017): the *communication and dialogues dimension*, referring to how teachers conduct learning-focused dialogues with students based on spontaneous input, was reflected in Ori’s purposeful built-in improvisation (Type 1); the *structure and design dimension*, referring to teachers’ altering the content of the lesson based on reassessing progress, and often relying on a preprepared ‘plan B’, was illustrated in Naor’s modification of the original LS plan (Type 3). The other two types found mark the extreme ends of the improvisation spectrum: tight adherence to the plan while avoiding any improvising (Type 2) at the one end, and complete abandoning of the plan at the other end (Type 4). In between these extremes, it appears that the LS plan served as a boundary that sets the limits within which a certain degree of improvisation can take place.

As possible implications from this study, we suggest that (1) the repertoire of secondary experiences should be increased by explicit reflection in all phases of teachers’ learning, including within LS communities, as the basis for informed improvisation; (2) improvisation does not necessarily need to be an in-the-moment decision, and can be incorporated as a preplanned set of alternative tasks, serving a cognizant choice to improvise, if needed; and (3) teachers’ inclination to improvise should be taken into account by the LS facilitator in monitoring the PLC work.

In conclusion, our local adaptation of LS showed possible ways of coexistence between LS and improvisation. In further studies, we intend to explore in depth why different teachers enact different ways of improvisation, and whether there is a link between such enactments and teachers’ experience and level of expertise.

NOTES

[1] Book of Amos (3:3) – King James English version from the original Hebrew Bible.

[2] <https://www.dictionary.com/browse/improvisation>.

[3] <https://dictionary.cambridge.org/us/dictionary/english/improvisation>

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ATTENDING TO ARGUMENTATION: EXPLORING SIMILARITIES AND DIFFERENCES BETWEEN MATHEMATICS PRE-SERVICE AND IN-SERVICE SECONDARY TEACHERS

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This study comprises a part of a larger study focusing on secondary school pre-service mathematics teachers' (PSMTs) and secondary school in-service mathematics teachers' (ISMTs) noticing of argumentation. Thirty-five PSMTs and 32 ISMTs engaged in analyzing argumentation classroom situations (ACSs) using an ACS-report format. In this paper, we focus on PSMTs' and ISMTs' attention to argumentation. Analysis of the participants' ACS reports revealed similarities and differences between the two research groups. The findings are interpreted in light of both theory and practice.

INTRODUCTION

Students' engagement in argumentation in the mathematics classroom has been widely acknowledged as being important. Participating in argumentation requires students to explore alternative positions while confronting, and evaluating them; to articulate their support or opposition to such positions; and to defend different hypotheses and ideas, thus fostering meaningful insights and deep thinking (e.g., Asterhan & Schwarz, 2016). Previous research findings have shown that teachers tend to encounter difficulties in incorporating argumentation into their classroom practice (Ayalon & Even., 2016) and that argumentation in the mathematics classroom has not yet become commonplace (Umland & Sriraman, 2020).

Teachers' noticing skills – i.e., attending, interpreting, and responding (Jacobs et al., 2010) – are deemed crucial in determining their proficiency and are central to the development of quality teaching (Jacobs et al., 2010). We speculate that teachers who are better able to notice argumentation will possess better skills required to promote argumentation in the mathematics classroom. A number of studies have focused on comparisons between the noticing skills of pre-service mathematics teachers (PSMTs) and in-service mathematics teachers (ISMTs), related, for example, to student's mathematical thinking (e.g., Jacobs et al., 2010). These studies' findings indicate certain differences. For example, PSMTs, in contrast to ISMTs, tend to struggle with attending to, interpreting, and responding to certain facets of student's mathematical thinking (Ibid.). According to the literature, classroom experience influences how and to what extent teachers notice classroom events (e.g., Yang et al., 2021). Yet, no research to date has focused on comparisons between PSMTs' and ISMTs' noticing of argumentation. Such knowledge would enable us to draw appropriate conclusions for professional training programs, for PSMTs and ISMTs alike, aimed at developing their noticing of argumentation and, thus making argumentation in the mathematics

classroom a more common practice. This study seeks to address this lacuna. In this paper, we focus on PSMTs' and ISMTs' similarities and differences specifically in one key component of noticing, namely attending to argumentation.

THEORETICAL PERSPECTIVE

A commonly accepted definition of argumentation is that of van Eemeren and Grootendorst (2004), who maintained that argumentation is “a verbal, social, and rational activity aimed at convincing a reasonable critic of the acceptability of a standpoint by putting forward a constellation of propositions justifying or refuting the proposition expressed in the standpoint” (p. 1). Following this definition, our study regards argumentation as having two important and interrelated aspects – structural and dialogic (González-Howard & McNeill, 2020). *The structural aspect* focuses on discourse in which a claim is supported by an appropriate justification, whereas *the dialogic aspect* focuses on the interactions between students, including co-constructing of arguments, critiquing arguments, mutual respect, and working toward consensus-building (Ibid.). Following Jacobs et al. (2010), and based on the educational literature on argumentation, we conceptualize the noticing of argumentation as a set of three interrelated skills: attending, interpreting, and deciding how to respond. This paper focuses on *attending* skills. *Attending* relates to identifying salient characteristics, structural and dialogic, of the argumentation in a classroom situation (González-Howard & McNeill, 2020). We ask: *What are the similarities and differences between PSMTs' and ISMTs' attending to argumentation?*

RESEARCH CONTEXT AND PARTICIPANTS

A cohort of 35 secondary PSMTs and 32 secondary ISMTs participated in this study, which was conducted in Israel in two separate courses. The PSMTs' course focused on argumentation in mathematics teaching as part of their teacher training. The ISMTs' course focused on argumentation in mathematics teaching as part of their fulfilment of a Master's degree in mathematics education. All participants of both groups possessed either a B.Ed. in math education or a B.Sc. degree with a major in math or in a mathematics-related subject. The ISMTs' years of experience ranged from 1–27 years, averaging 6 years. In both courses, the study was conducted during the fourth session of the course. The earlier sessions focused on discussing theoretical issues related to structural and dialogic aspects of argumentation. Both groups had not been exposed to argumentation formally or explicitly in their academic education.

RESEARCH TOOLS

The research tools included (a) a written Argumentation Classroom Situation (ACS) focusing on the issue of “Abbreviated multiplication formulas” in a 9th grade class; (b) an ACS report format (adapted from Jacobs et al., 2010) that includes prompts related to the three noticing-of-argumentation skills. In this paper, we focus on the skills associated with attending to argumentation. The participants were asked: Describe in detail those ACS sections which you deem important for argumentation, with specific reference to (1) structural aspects, i.e., the arguments (the claim, and justification for

the claim, if such exists) that emerged in the ACS, and the type of justification, and (2) dialogic aspects, i.e., the interactions between participants relating to collaborating on constructing arguments, critiquing arguments, mutual respect, and working toward consensus building. Provide a detailed description of how each aspect is manifested in the ACS, cite examples, and note the specific line numbers.

The ACS: “Abbreviated multiplication formulas”

The following situation took place in a 9th grade mathematics classroom with 20 students. The lesson topic was “abbreviated multiplication formulas.” This was the third lesson on this topic. After the students had learned and practiced the first two abbreviated multiplication formulas: $(x+y)^2 = x^2 + 2xy + y^2$; $(x-y)^2 = x^2 - 2xy + y^2$, they were asked to write "correct" / "incorrect" for the claim: $(a - b)^2 = (b - a)^2$ and explain their answer. Note: The participants received the ACS without the right-hand [Coding] column.

Teacher:	Solve the task and explain your answer.	Coding
Students:	Those two expressions are clearly different; it doesn't take a lot of thought!	Argument 1 Claim: Expressions are different; Justification: counterexample
Teacher	Who wants to explain to us why these two expressions aren't equal?	
Ahmed:	Because if we substitute 10 into a and 5 into b, we get $10-5=5$, $5-10=-5$. 5 is not equal to -5. So, there's no need to keep checking.	
Sareen:	Not true, Ahmed. If we substitute the same value into a and b, then we get two equal expressions, because $0=0$. So then it is possible for the two to be equal!	Argument 2 Claim: Expressions can be both equal or different; Justification: counterexample (Critiquing argument)
Zayid:	So, can we sum up by saying that sometimes they are equal, and sometimes they are not?	(Attempt to reach consensus)
Teacher:	Students, you are saying a lot of nice things. Take a few minutes. Try to substitute different numbers in the two expressions, and see what you get. ...	(Mutual respect)
Teacher:	... Who wants to share their work?	
Ahmed:	I substitute 5 into a and 3 into b and got 4 on both sides. That means that the two expressions are equal.	Argument 3 Claim: Expressions are equal; Justification: empirical

- Teacher: Nice! Ahmed, did you manage to figure out the mistake you made earlier? (Mutual respect)
- Ahmed: Yes, yes! ... I went back over the substitution I used earlier... (Concession)
- Teacher: Nice work, Ahmed! Now I want you to use the abbreviated multiplication formulas that we learned and try to prove the claim. Take another five minutes and try to prove it. (Mutual respect)
- Teacher: Who wants to share their justification?
- May: Can I write my solution on the board? **Argument 4**
- Teacher: Yes, May, please do! Claim: Expressions are equal; Justification: algebraic
- May: May copied her work from her notebook onto the board.
$$\begin{aligned} (a-b)^2 &= a^2 - 2ab + b^2 \\ (b-a)^2 &= b^2 - 2ab + a^2 \end{aligned}$$
- And then she said: I see that it has identical elements, but I can't figure out how to prove that the expressions are identical. (Co-constructing arguments and mutual respect)
- Teacher: Very nice May, you did very good work! There is one more small thing left to do...
- Teacher: Who wants to help May?
- Samir: I did exactly what May did, and then I found that the places of a squared and b squared can be switched because it's addition. [He approaches the board:] $a^2 + b^2 = b^2 + a^2$
- May: Ooof, why didn't I manage to do it myself? I get it now ...
- Teacher: May and Samir, good job! Together you reached the right solution.

DATA COLLECTION AND ANALYSIS

The data included 35 PSMTs' and 32 ISMTs' ACS-reports (total of 77) focused on analysis of the ACS. The reports served as the main data source for characterizing the similarities and differences between PSMTs' and ISMTs' attending to argumentation. The analysis comprised three main stages. In **Stage 1**, for each ACS-report, we classified the participant's responses according to the two aspects of argumentation: structural and dialogic. Responses classified as attention to structural aspects included attending to the elements of arguments which contain claims, justifications and

identifying the types of the justifications. Responses classified as attention to dialogic aspects included attending to the co-constructing of arguments, critiquing arguments, mutual respect, and working toward consensus-building. In **Stage 2**, we coded the participants' attention to structural aspects as follows: A participant who attended adequately, i.e., identified the claim and the justification and its type, to four, three, two, or one out of the four arguments raised in the ACS, received a score of 4, 3, 2, or 1, respectively. We then counted how many PSMTs and how many ISMTs received Score 1, 2, 3, and 4. To compare the scores of attending to the structural aspect between the two groups, a Wilcoxon Signed-Rank test was performed. In **Stage 3**, for each participant, for each of the four dialogic aspects argumentation, we coded the participants' attention as Score 1 in case we found evidence for attending to the specific aspect; otherwise, Score 0. We then counted how many PSMTs (out of 35) and how many ISMTs (out of 32) received Score 0 and Score 1 regarding each of the four dialogic aspects. To compare the scores of attending to dialogic aspects between the two groups, a Chi-square test was applied.

FINDINGS

PSMTs' and ISMTs' attending to structural aspects of argumentation

We used a Wilcoxon Signed-Rank test to compare the PSMTs' and ISMTs' scores of attending to structural aspects of argumentation. The results, displayed in Table 1, indicate no statistically significant difference between the two groups regarding the structural aspect of argumentation ($Z = 0.42, P = .68$).

Table 1: Distribution of scoring of PSMTs' and ISMTs' attending to the *structural aspects of argumentation* (scale 1-4), and Wilcoxon Signed-Rank test.

Group	N	Score of Attending to structural aspects of argumentation				Wilcoxon test	
		1	2	3	4	Z	P
PSMTs	35	3% (1)	14% (5)	31% (11)	51% (18)	0.42	.68
ISMTs	32	0	9% (3)	38% (12)	53% (17)		

As shown in Table 1, roughly half of the PSMTs (51%) and ISMTs (53%) received scores of 4 (i.e., they identified the claim and the justification and its type, for all four arguments raised in the ACS), while most of the remaining PSMTs (31%) and ISMTs (38%) received Score 3 (i.e., they identified the claim and the justification and its type, for three arguments raised in the ACS).

PSMTs' and ISMTs' attending to dialogic aspects of argumentation

A Chi-square test was applied to compare the PSMTs' and ISMTs' scores for attending to dialogic aspects of argumentation.

Table 2: ISMTs' and PSMTs' attending to the *dialogic aspects of argumentation*.

Dialogic aspects	PSMTs % (count) N=35	ISMTs % (count) N=32	Chi-square test	
			$\chi^2(1)$	<i>P</i>
Co-constructing of arguments	71% (25)	88% (28)	2.61	.106
Critiquing of arguments	40% (14)	100% (32)	27.97	< .001
Mutual respect	91% (32)	91% (29)	0.01	.908
Working toward consensus building	34% (12)	84% (27)	17.24	< .001

As shown in Table 2, a significant difference between the two groups was found regarding attending to two dialogic aspects: ‘critiquing of arguments’ and ‘working toward consensus building’. No significant difference between the two groups was found regarding attending to ‘mutual respect’ and ‘co-constructing of arguments.’ Focusing on each group, it is notable that among ISMTs, a high level of attention was paid to all the four aspects. Among the PSMTs, a high level of attention was paid particularly to ‘mutual respect’ and ‘co-constructing of arguments’. ‘Critiquing of arguments’ and ‘working toward consensus-building’ received less attention.

DISCUSSION

Previous studies, not focusing on argumentation, identified differences between pre-service and in-service teachers’ attention to noteworthy features of instruction (e.g., Sherin & van Es, 2005). To some extent, our findings are consistent with these studies in terms of differences between the ISMTs’ and PSMTs’ attending to dialogic aspects of argumentation: The ISMTs statistically significantly outperform PSMTs in terms of attending to critiquing of arguments and working toward consensus building.

Critique and consensus building are two important components of argumentation that distinguish it from *consensual co-construction* dialogue (Henderson et al., 2015), where ideas may be elaborated upon but are not challenged or criticized. Critique in the classroom enables differences to be made explicit and is essential for assessing which idea is correct using mathematical criteria accepted by the teacher and the students (e.g., Staples, 2007). Therefore, teachers’ attention (or lack thereof in the case of the PSMTs) to critique and consensus is important for avoiding missed opportunities to capitalize on classroom situations for engaging students in argumentation. In our case, for example, the teacher’s recognizing the interaction between Sareen and Ahmed allowed students to discuss the correctness of their individual claims, to come up with ideas on how to evaluate each of the arguments, and to build on these ideas in an effort to reach the correct solution. Teachers’ attending to critique is also important because of the emotional aspects involved. Questioning each other’s ideas, being critiqued, and modifying their own claims on the basis of peers’ suggestions, are familiar challenges to many students (Kuhn et al., 2011). Thus, teachers’ attentiveness to these aspects is vital for conveying to their students that critique is an acceptable and expected action.

Therefore, the differences found in attention to argumentation between the two groups are notable.

Still, the results of our study show some similarities in the PSMTs' and ISMTs' attention to argumentation. In the *structural* aspects of argumentation, both groups were able to identify all, or almost all, of the arguments raised in the ACS. In the *dialogic* aspects, participants of both groups paid relatively high attention to aspects of co-constructing arguments and mutual respect. Existing research that focused on PSMTs' perspectives of argumentation found that they tend to discuss argumentation from the structural perspective only (Park & Magiera, 2019). A possible explanation for the PSMTs in our case attending also to some dialogic aspects could be that they had had some prior experience in discussing issues related to both structural and dialogic aspects of argumentation. Moreover, the ACS report format used in the study included explicit prompts related to attending dialogic aspects. Whatever the reason, the fact that certain dialogic aspects were addressed by the PSMTs is encouraging.

Taking into account the small sample of teachers and the specific research conditions in the present study, the findings allow us to cautiously raise a number of questions. One question is: why did the PSMTs attend only to certain aspects and not to others, and why did the teachers refer to all aspects? Researchers point out the possibility that humans' cognition is structured in such a way that critique may be more difficult than construction and that people are biased toward confirmation rather than falsification (Henderson et al., 2015). Is this bias reflected in the case of the ACS analysis by the PSMTs? Another, related question concerns the possibility of a *development* in teachers' noticing of argumentation, reflected in the expansion of attention to dialogic aspects over the years of teaching. Further exploration of these questions may allow us as teacher educators to plan our teaching accordingly.

Another question is related to possible connections between the participants' attention to argumentation and their additional noticing skills, i.e., interpreting and responding to argumentation. Several studies have shown that interpreting and responding are dependent on attending (e.g., Jacobs et al., 2010). An important question, then, is what characterize the PSMTs' interpreting and responding to the argumentation included in the ACS. For example, does their interpretation refer to the interaction between Sareen and Ahmed? Do the alternatives they offer to the teaching strategies of the teacher in the ACS refer to handling critique? We are currently working on analyzing these additional aspects of the participants' noticing of argumentation to address these questions. An additional question relates to possible connections between teachers' attention to argumentation and their actual teaching practice. Will different attention to argumentation be reflected in different instruction related to argumentation in the classroom? Further research is needed to clarify the links between teachers' noticing of argumentation and their teaching for argumentation in practice. To summarize, the research findings contribute to the literature on professional learning, specifically on developing PSMTs' and ISMTs' noticing of argumentation, by providing a snapshot

of their attending to argumentation. Learning more about the similarities and differences between PSMTs' and ISMTs' noticing of argumentation may contribute to teacher educators creating dedicated activities that gradually engage teachers in more complex and challenging processes of noticing that are aligned with a pre-determined sequence (e.g., Santagata et al., 2021). It is our opinion that it may be worthwhile in future research, and in varied contexts, to invite PSMTs and ISMTs to analyze additional ACSs.

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A CARTESIAN GRAPH IS “A THING OF MOVEMENT”

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Adopting a multimodal semiotic perspective, we present the case study of Bea, a 10th grade low-achieving student with a story of difficulties with functions. We focus on a task designed within a Dynamic Geometry Environment (DGE) artefact involving graphs of functions. The analysis of her speech, inscriptions, gestures, and actions on the artefact highlights her productive struggle in coping with the dynamism of the proposed representation, which resulted in an evolution of her semiotic production culminating in a drawing of the Cartesian graph. Results highlight the potentials of DGE activities, fostering the use of multimodal resources for low-achieving students.

INTRODUCTION

Research shows that the study of signs can highlight crucial processes in the teaching-learning of mathematics. In the last decades different semiotic perspectives opened to a *multimodal* approach, focusing on a wide spectrum of signs, as gestures, sketches, manipulations of artefacts, etc. (Presmeg et al., 2016). This approach has enriched research in many areas of mathematics education, including the studies on teaching-learning of calculus with digital artefacts (Arzarello et al., 2009; Ng, 2016). The study reported in this paper aims to contribute to this stream of research, by adopting a multimodal perspective to analyse students' processes during DGE-designed activities involving the construction of the Cartesian graphs of functions. The study is part of a wider research project investigating the impact of digital-integrated artefacts on the learning of high school students with a story of persistent difficulties in mathematics.

Expert mathematicians are able to interpret the Cartesian graph of a function as incorporating the functional relation between the two variables. They are also able to construct them starting from local and global properties of the function. Differently, for many students a Cartesian graph is a mute mark of ink on the paper, being them not able to recognise the two covarying variables and the functional relationship between them. Wide literature highlights the students' need of opportunities to engage in activities on functions that emphasise the covariational aspects and give meaning to the graphical representation (e.g., Antonini et al., 2020; Thompson & Carlson, 2017).

THEORETICAL FRAMEWORK

This study is framed in a multimodal semiotic perspective. We use the notion of 'semiotic bundle' theorised by Arzarello (2006) to analyse a wide spectrum of signs simultaneously involved in thinking and learning processes. Arzarello (2006) firstly defines a *semiotic set* as triplet composed by (a) a set of signs which can be produced by intentional actions (speaking, drawing, gesturing, handling an artefact, etc.), (b) a set of modes for producing and transforming signs, and (c) a set of relationships

between these signs and their meanings. A *semiotic bundle* is defined as a couple made of a collection of semiotic sets and of relationships between them. A semiotic bundle is a dynamic structure since, during the semiotic activity of a subject, new signs can arise, and new relationships between semiotic sets can be created. Arzarello observes that semiotic sets may transform each other, when new signs in a semiotic set are formed as *genetic conversions* of signs belonging to other sets. A semiotic bundle can be analysed *synchronically*, focusing on the simultaneous relation between the semiotic sets, and *diachronically*, focusing on the development of the bundle over time.

In this study we will consider the semiotic bundle composed by four semiotic sets: speech, written inscriptions, gestures, and what we will call *DGE signs*, i.e. those signs produced by intentional actions on the artefact, as the construction or modification of a geometrical object, the dragging of a point, the zooming actions, etc. Our choice to introduce a specific semiotic set for DGE signs requires some remarks. Firstly, we observe that this set of signs fits in the definition of semiotic set, encompassing the (a-c) features presented above. For example, we can consider the dragging of a base point that causes the indirect motion of other objects, thus, modifying the system of signs already present on the screen and producing the dynamic sign corresponding to the observed movements. The activation of the *trace tool* allows to highlight these movements, however, as gestures are signs even without leaving a mark in the air, the dragging produces signs even if the trace is not activated. The choice of including the set of DGE signs allows us to consider dragging actions within the semiotic bundle, in addition to speaking, gesturing, and drawing. Research shows how dragging plays key roles in teaching-learning functions as covariation between two variables (e.g., Falcade et al., 2007). Ng, in a discursive perspective, defines the term *dragsturing* to refer to an “action subsuming both dragging and gesturing characteristics” (Ng, 2016, p. 130), showing how it can effectively foster an evolution of students’ discourse on functions. In our perspective, we can interpret dragsturing as an action allowing the subject to produce a gesture (the finger moving on the screen) and DGE signs (the direct/indirect movements caused by dragging). Moreover, the introduction of DGE signs is prompted by one of the objectives of the research project, in which this study is embedded, that is to provide insights into the impact of digital-integrated activities on students’ learning. By including this set we can contribute to the aim from a semiotic perspective, focusing on the roles of DGE signs for the evolution of students’ semiotic bundle.

In this paper our goal is to investigate whether, and how, the semiotic bundle evolves in the students of the project when coping with a particular digital-integrated activity involving the construction of the Cartesian graph of a function.

METHODS AND TASK DESIGN

Data for this study were collected during an educational path focusing on functions and involving 12 students of 10th grade from three Italian high schools. Five sessions, lasting 120’ each, took place in an out-of-school learning centre and they were

conducted by a researcher. Students worked with touch-screen tablets. Data consist of audio-video recordings of the sessions and screen recordings of the tablets.

In this paper we focus on an activity with a DGE artefact, called *dynagraph*, involving a dynamic representation of a function (for more detail, see Lisarelli, 2023). In this dynagraph two tick marks, labelled A and B, are bound to move on the Cartesian axes. A, representing the independent variable, can be directly dragged, whereas B, representing the dependent variable, moves only indirectly under the dragging of A in accordance with the involved function. The Cartesian graph of the function can be obtained by constructing the point (A,B), activating its *trace*, and dragging A (Fig. 1b). However, this feature of the dynagraph was hidden to students who could only see the tick marks A, B (Fig. 1a). The given task was: “While dragging the tick mark A, can you imagine the trajectory of the point (A,B) on the Cartesian plane? Try to draw it”.

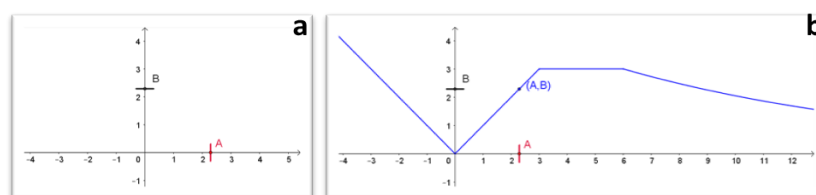


Figure 1: a) The given dynagraph; b) The trajectory of the point (A,B) obtained dragging A – not visible on the screen for students.

The task design involved also an a priori analysis, that allowed us to identify at least two aspects in this task that could generate complexities from cognitive and didactic points of view. The first one concerns the term ‘trajectory’ that has a twofold meaning (Falcade et al., 2007): either as a set of positions reached by the point (A,B), or as a static object that is the trace of the entire path taken by (A,B). The second one concerns the presence of different levels of dependency involving the elements of the dynagraph, and their movements. Indeed, since B is uniquely determined by A, the point (A,B) can be seen as uniquely dependent on A. However, the movement of (A,B) on the plane depends on both the direct movement of A, and the indirect movement of B.

CASE ANALYSIS

We present the case of Bea. In a preliminary interview, in front of the Cartesian graph of $y = -2x + 1$, she stated to have difficulties with functions (“To me they are a bit like the meanest topic”), without succeeding in working on the given graph. In this section we analyse four episodes from the fourth session. In the transcripts, ‘I’ stands for ‘interviewer’, and words in *(italics)* describe actions in the moment they are made.

Episode I.

In the first ten minutes, Bea explored the file (Fig. 1a) by dragging A, initially, on positive numbers and then focusing on the position of B when A is between 0 and 6, which she described by writing “ $0 < A < 6 \rightarrow 0 < B < 3$ ”. When asked by the interviewer, Bea has correctly constructed a blue point corresponding to (A,B) for $A=6$ and $B=3$. Then Bea is asked to describe “where this point is when A varies between 0 and 6”.

- 1 Bea: The blue point follows the direction of A. (*With the right hand she drags A from 6 to 3, and, at the same time, she moves her left index finger horizontally on the screen from the blue point toward left*). However, it always remains at the same height. Except, indeed, until it arrives to the point three. (*She stops dragging and remains still*). Where not only... I mean the point does not only follow the trajectory of A, so it is not only parallel to A, but also to B, so it goes down, it goes down. (*She drags A from 3 to 0, consequently B moves from 3 to 0. At the same time, she moves her left index finger on the screen diagonally toward the origin, Fig.2*).

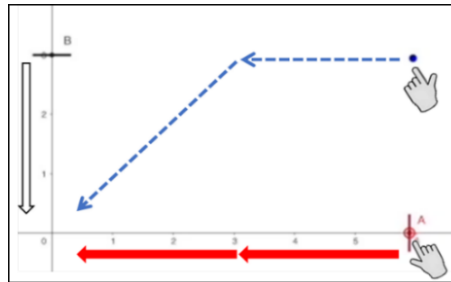


Figure 2: DGE signs and gesture of turn 1. A is dragged from 6 to 3 and then from 3 to 0 (bold arrows). B moves from 3 to 0 (outlined arrow). The blue point remains still. The dotted arrows show the path of Bea's left index finger on the screen.

In this short excerpt, Bea describes the movement of the point (A,B) for the first time, having until now only referred to the movements of A and B. Bea's description employs a series of multimodal semiotic resources. With her right hand she drags A, while B moves indirectly. With her left index finger she indicates the trajectory of the blue point corresponding to (A,B). Note that this point does not move while A is dragged, however Bea manages to represent this movement in a gesture embedded in the screen and co-timed with the movements of A and B. This dragsturing action allows Bea to represent the trajectory dynamically, as a series of positions reached by (A,B). Finally, in her speech, Bea refers to the dependency of the movement of (A,B) on the ones of A ("[it] follows the direction of A") and of B ("it is not only parallel to A, but also to B"), the latter when B is not constant. However, in her gesture the dependency on B is always present. In other terms, dragsturing adds semantic elements to Bea's speech. The bundle of speech, dragging, and gesturing allows Bea to represent the trajectory of (A,B) and the dependencies between the movements of A, B, and (A,B).

Episode II.

Right after the previous episode, the interviewer asks Bea to "try to draw it".

- 2 Bea: Eh... [with concerned tone]
 3 I: You already did it!
 4 Bea: Yes, but the problem is to put it here (*she points to the white page of the tablet*), how do I do? Because it is a thing of movement.

Encouraged by the interviewer, Bea draws a Cartesian plane on which she marks the values 3 and 6 on the x-axis and the value 3 on the y-axis (Fig. 3a). Then she continues:

- 5 Bea: Like...from here to here (*she draws an arrow between 3 and 6 on the x-axis, Fig. 3b*), the point... it remains unchanged and so it remains always

here. (*She indicates over the just drawn arrow*). Wait I'll try to draw it parallel. Here it is. (*She draws a red point over the arrow at the same height of the line '3' of the y-axis, Fig.3*).

At this point, Bea drags A between 3 and 0 in the two verses, then she adds:

- 6 Bea: From three to zero, instead... that is, from zero until three (*she draws an arrow from 0 to 3 on the x-axis, Fig. 3d*), the point... it moves. So, it can go or towards down or towards up, but always on three. (*She draws two red vertical arrows over the x-axis, and then a horizontal line, Fig. 3e*).

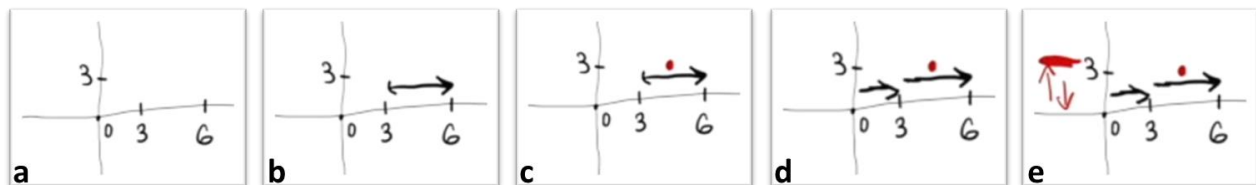


Figure 3: Inscriptions of turns 4-6.

This episode shows Bea's attempt to make a drawing of what she described in episode I. Initially, Bea makes explicit her difficulties in coping with this task (turns 2-4), which requires to represent with a written sign the "thing of movement" that she has previously described with other semiotic resources (the dragsturing of turn 1, Fig. 2). A genetic conversion to enrich the semiotic set of inscriptions is necessary and Bea obtains this by drawing a series of arrows. First, she focuses on $3 < A < 6$, drawing an arrow representing the movement of A and a red point at height 3 (Fig. 3c). Then, focusing on $0 < A < 3$, she draws one horizontal and two vertical arrows corresponding, respectively, to the movements of A and B (Fig. 3d-e). The bundle made by Bea's speech and inscriptions, with the temporal order of the drawings (the arrows for A's movement are followed by those for B's movement), allows her to represent the dynamism experienced and described in episode I. However, the arrows still refer to A and B, whereas the trajectory of (A,B) is not represented with a written inscription yet.

Episode III.

Bea explores the dynagraph for $A < 0$ and observes that A and B "are opposite". When asked, she constructs (A,B) as a blue point $(-6,6)$. Then the interviewer asks:

- 7 I: What trajectory did the blue point do to go from zero?
 8 Bea: It enlarged, like it started from here and then it did like this. (*Gesture of Fig. 4*). I mean, it made a diagonal.

The interviewer invites Bea to "make a drawing of this". Bea draws a Cartesian plane, writing 'A', 'B', and 'A,B' all in correspondence of the origin. Then she continues:

- 9 Bea: Then, if I enlarge A B, I mean, if B goes here and A goes up (*she draws a horizontal arrow from B and a vertical arrow from A, Fig. 5a*) [omissis: Bea observes to have inverted A and B]. Basically, this moves in diagonal (*she draws an arrow in diagonal starting from the origin, Fig. 5b*).
 10 Bea: If instead these lower (*she draws a vertical arrow and a horizontal one, opposite to the ones of turn 9*), this does like this (*she draws a diagonal arrow toward the origin, Fig. 5c*).



Figure 4: Gesture of turn 8.

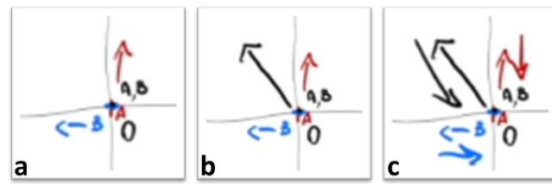


Figure 5: Inscriptions of turns 9-10.

This episode shows the presence of new signs which enrich, firstly, the semiotic set of gestures and then that of inscriptions. The first one is Bea’s gesture of turn 8 representing the trajectory of (A,B). This is the second time that she performs a gesture referring to it – the first one occurred in episode I (Fig. 2) – however now it is not embedded in the screen with a dragging action as before, but it is performed in the air, detached from the tablet. It is an iconic gesture that represents, by itself, the movement of (A,B). Then (turns 9-10) we can observe the presence of a new genetic conversion in the semiotic set of inscriptions. Bea initially draws the same arrows of episode II to refer to A’s and B’s movement, but then she enriches the drawing with two diagonal arrows corresponding to the trajectory of (A,B) (Fig. 5b-c). This is a crucial point, since it is the first inscription made by Bea referring to the movement of (A,B). These arrows, despite being the same written mark than the others, have a very different genesis. They do not represent the movement of something visibly present on the screen, rather they are the conversion into written signs of the gestures previously used by Bea.

Episode IV.

The interviewer constructs the point (A,B) as the intersection of the lines perpendicular to the axes and passing by A and B, so that it moves in dependence of A and B. Then she asks Bea to drag A and to describe what she sees. Bea does not seem surprised by the movement of (A,B) and, referring to (A,B) for positive A-values, says:

- 11 Bea: It goes as a stair... I mean the shape is like this (*she rapidly makes a trace on the screen, without marking, close to the point (A,B)*). Wait I’ll try to... (*Bea takes the other tablet and draws the inscription shown in Fig. 6a*).

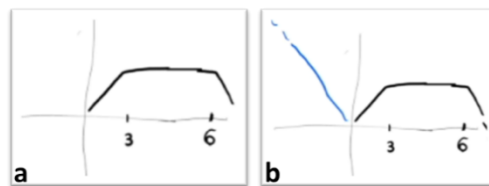


Figure 6: (a) Bea’s inscription of turn 11 and (b) after turn 12.

Bea is then asked to consider negative A-values but before dragging A she says:

- 12 Bea: The point does...it follows the diagonal (*similar gesture of Fig.4*).

After dragging A on negative values, she adds a new line to her drawing (Fig. 6b).

This episode shows how, with the mediation of the point (A,B), Bea manages to produce an inscription that an expert would recognise as the Cartesian graph of the function defined in the dynagraph (Fig. 6b). In her drawing, “the shape” (turn 11) of the trajectory of (A,B) is represented by a series of lines without explicit reference to

the movements of A and B and to the verse of movement of (A,B), as previously happened. Bea's speech also mirrors this aspect (turn 12), because the verb 'to follow' that in episode I was used to refer the dependency of (A,B) on A ("[it] follows the direction of A"), is instead used here to refer to the shape of the movement itself.

DISCUSSION

The synchronic analysis highlighted the fundamental role played by the bundle of different semiotic sets in Bea's activity becoming a crucial resource for her to cope with the task. In all the four episodes, by using different signs, mostly personal and not mathematically coded, she effectively represented the trajectory of (A,B).

The diachronic analysis highlights an evolution of the semiotic bundle in response to the task of drawing the trajectory of (A,B), characterised by an enrichment of signs in the semiotic set of inscriptions. Initially, no inscription is made by Bea who represents the movement of (A,B) with bundles of speech, gestures, and DGE signs (episode I). Then, drawings of arrows, coloured tick marks, and dots appear (episode II). This is Bea's initial attempt to represent in the semiotic set of inscriptions the dynamism of the dynagraph. Then this semiotic set further evolves (episode III) when Bea adds to her drawing the arrows representing the movement of (A,B) which, until now, she has only referred to with gestures on the tablet (turn 1) or in the air (turn 8). This is a crucial genetic conversion which introduces the trajectory of (A,B) as a new character within the inscriptions. The evolution culminates when Bea's drawing corresponds, for an expert, to the Cartesian graph (episode IV). In this inscription, mediated by the new constructed point (A,B), some elements of the previous ones are lost (the arrows for the movement of A and B), and others are transformed (the arrows for (A,B) become a straight line). We interpret this as a form of *semiotic contraction*, "the mechanism for reducing attention to those aspects that appear to be relevant [...] We need to forget to be able to focus" (Radford, 2008, p. 94). Bea, in drawing the final inscription, 'focuses' on "the shape" of the trajectory of (A,B), 'forgetting' the directions of movements of A, B, and (A,B) and the role of time characterising the dragging actions.

CONCLUDING REMARKS

The multimodal approach and the theoretical lens of the semiotic bundle allowed us to observe, analyse, and describe the story of Bea dealing with a dynagraph. This story is, firstly, a story of struggle. Her question "How do I do it? Because it's a thing of movement", synthetises the core of the problem. The task of drawing the trajectory of (A,B) requires Bea to face a semiotic complexity (to represent with a static inscription the dynamic and co-timed movements of A, B, and (A,B)) and a logical complexity (to take into account the dependencies between these movements). The analysis conducted in this study enriches the literature on functions in relation to students' difficulties with graphs (e.g., Thompson & Carlson, 2017), by allowing these complexities to come into focus and thus providing insights for further research. On the other side, this is also a story of productive struggle. At the end, after a rich intertwining of signs, genetic conversions and external mediations (as the interviewer's requests to focus on (A,B),

or the construction of (A,B) as a dynamic point), Bea's semiotic production converged towards the educational aim of the activity: the Cartesian graph. This is highly relevant from a didactic point of view, considering Bea's initial difficulties. Her graph is not a mute trace of ink, as in the preliminary interview, but a sign echoing all her semiotic production and thus rich of personal significance. This result confirms the didactic potentials of dynagraphs, also for low-achieving students (Antonini et al., 2020).

Lastly, our choice of extending the analysis of the semiotic bundle to the set of DGE signs enabled us to observe that many signs produced by Bea developed as genetic conversions of DGE signs. Therefore, our analysis provides an initial contribution to the issue of investigating the impact of digital-integrated activities on students' learning by showing that such activities can have a genetic role, i.e. they foster students' generation of new signs that enrich not only the set of DGE signs, but also other semiotic sets. These signs could be distant from mathematical ones, as in the case of Bea, nevertheless they can be effective didactic resources allowing the teacher to engage a *semiotic game* (Arzarello et al., 2009) bridging them with mathematical signs.

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INSTRUCTIONAL SHORT VIDEOS IN CALCULUS: THE MATHEMATICAL DIDACTICAL STRUCTURES AND WATCHING PATTERNS

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The current study aimed to understand what the structure and design features of an effective mathematical instructional video are. The study focus is the didactical perspective of videos devoted to the explanation of mathematical topics in Calculus. In this paper we present findings based on two case studies taken from a larger collection of 24 mathematical instructional short videos. The analysis of each video is based on a mathematical didactical analysis and on user watching pattern analysis. Our findings suggest four didactical features that might serve as a starting point for explaining the user analytic data: 1) the appearance of an incentive trigger early in the video, 2) examples relevance to the mathematical concept presented, 3) the use of representation, 4) graded progression of the mathematical ideas within the video.

INTRODUCTION

Instructional videos for education are highly utilized and their use is increasing rapidly in the last decade (de Koning, Hoogerheide & Boucheix, 2018). The researchers states that Videos for education are considered as one of the most used content formats for delivering instruction in online courses. In parallel to the increase use of instructional videos in education, the methods in which they are embedded in courses and classes are evolving. Yet, challenges related to video design, production, and the development of effective pedagogical methods for incorporating them within courses are yet unanswered (Hansch et al., 2015). Importantly, there is no consensus as to what makes an educational video effective and how should it be measured (Woolfitt, 2015). Our aim in this study is to increase our understanding on these two issues.

In this paper we present findings based on two case studies taken from a larger collection of 24 mathematical instructional short videos. The analysis of each video is based on a mathematical didactical analysis and on user watching pattern analysis.

THEORETICAL FRAMEWORK

Two main approaches can be found in the literature concerning how to measure the effectiveness of instructional videos. First, an indirect method, which measures the indirect effect on the learner, such as: students' course grades, course dropout-rate and students' feedbacks. Second, a direct method, based on examining the actual way in which the video is used by the learners (Kim et al., 2014). This research uses the second method mentioned, by evaluating the videos' watching patterns.

The context of the study

Previous research focused on tutorial type videos concerning the quadratic function and its different representations, target for 9-th grade students (Netzer, 2019). The current research is an expansion of the previous one in two ways: (1) analysing a wider variety of instructional mathematical videos, and (2) further developing and refining the method used. Therefore, this research is focused on mathematical videos about Calculus, which is one of the main subjects in the Israeli High-school math curriculum for the highest-level students. Also, the Calculus videos are a major part of the instructional videos collection within a set that was developed by the ministry of education, the “Haetgar 5” project (shorturl.at/gksFG).

Research questions

In the context of instructional videos aimed at presenting mathematical ideas on calculus for high-school students, we ask: (a) What is the mathematical-didactical structure of the videos? (b) What can we learn from the user analytics data? (c) Can the identified mathematical-didactical structure explain the user analytics data?

METHODOLOGY

The research is based on Multiple-case study approach (Creswell, 2013). According to Creswell, a subject is selected, and several cases are reviewed in which the subject is investigated or demonstrated. In this research the focus is the didactical perspective of videos devoted to the explanation of mathematical subjects in Calculus. Hence, each video is considered a case study. For each video, we analyse two types of data: (a) the mathematical-didactical review of the video content, and (b) the video’s watching pattern.

The research population are the video watchers. These videos are open and available for all over the internet. The research assumption is that vast majority of the videos watchers are students from 10th to 12th grades who are learning for their high-level mathematics in high school, for matriculation examination. This assumption is based on the fact that the videos are an integrated part of the “Haetgar 5” project, which was targeted for this population.

Following the pre-processing stage of the research, 24 videos were selected. The videos were grouped according to three mathematical content categories: 1) Mathematical-conceptualization, including videos that present a mathematical concept, feature, or definition. 2) Mathematical-procedure, including videos that present a certain mathematical algorithm or process. 3) Unfolding-problem, videos that show an evolving or progressing problem that builds up several mathematical ideas. In this paper we elaborate on the study of two videos. Both videos are under the category of mathematical-conceptualization videos which includes 10 videos overall. The two videos were selected to be presented here due to the variety of the findings in their analysis.

The videos content mathematical-didactical review included several stages. First, the segmentation of each video timeline according to the subject didactical ideas that were presented, such as: the introduction of a new mathematic definition, a demonstration of a mathematical feature, the presentation of a question, an example and its solution. Second, the analysis of each segment, reviewing aspects such as: The method used to present the segment mathematical idea; What type of visuals were used? How both visual and narration supported the communication of the mathematical ideas? Was there a call for action for the watcher? Third, a didactical analysis of how the different video segments interleave and relate to one another including issues such as: What is the method in which the mathematical ideas within the video were built up? How the examples supported the ideas discussed and what was their role in the video? What type of mathematical visuals and representations were used and how do they relate to each other? Was there any reference to known misconceptions concerning the mathematical subject?

Each video's watching pattern was built using google analytics platform. The platform provides several video usages attributes per each specific video. One of them, that was especially used in this study, is named the watching percentage. The watching percentage is a series of 100 data inputs which are calculated by the platform as following: the video timeline is divided to 100 parts (named moments), per each moment a ratio in percentage is provided in the series. The ratio presents the number of watching session per each specific video moment (watching session means the number of times the moment was included in any play session) in respect to the number of watching session of the video opening moment. Drawing the matrix data according to the video timeline provided us with the graph presenting the video's watching pattern (Figure 11). In addition, per each of the three video categories an average watching pattern was calculated based on all watching pattern of the videos forming the category. For example, the black line in Figure 11 represents the average watching pattern of all 10 videos under the mathematical-conceptualization videos category.

FINDINGS

In this chapter we present the findings related to two videos, the Limit video and the Concave video.

The mathematical content of the Limit video is about the limit value of certain functions at given domains. The video can be divided into two main parts, each has its own goal (Figure 1). The goal of the first part (sections B to F) is to demonstrate that different types of functions behave differently while approaching their limits boundaries. Three different functions are reviewed at three different domains. The functions are e^x , x^2 and $\ln(x)$. The three domains are: a) when x increases to infinity, b) x positive and decreases to zero, c) when x decreases to minus infinity. A table format presentation is used to present the value changes of each function in each domain. The first part ends with a summary section in which the behavior of all three functions in each of the different three domains is summed up into a single table.

The second part of the video (section G to J) includes two examples, the goal of both examples is to demonstrate how to solve a conflicting situation when at a certain domain of x the value of a function is determined by the product of two opposing values: one approaching the value of zero while the other reaching infinity. Both examples, in the second part of the video, are utilized to demonstrate interesting function situations. The functions are: $y = (1 - x)e^x$ and $y = x \cdot e^{\frac{1}{x}}$, each feature quite a unique function behavior. The first has a horizontal asymptote only on the left side of the x axis while none on the right. The second includes an undefined domain value in which on one side the function has a vertical asymptote while on the other side the function approaches a hole point.

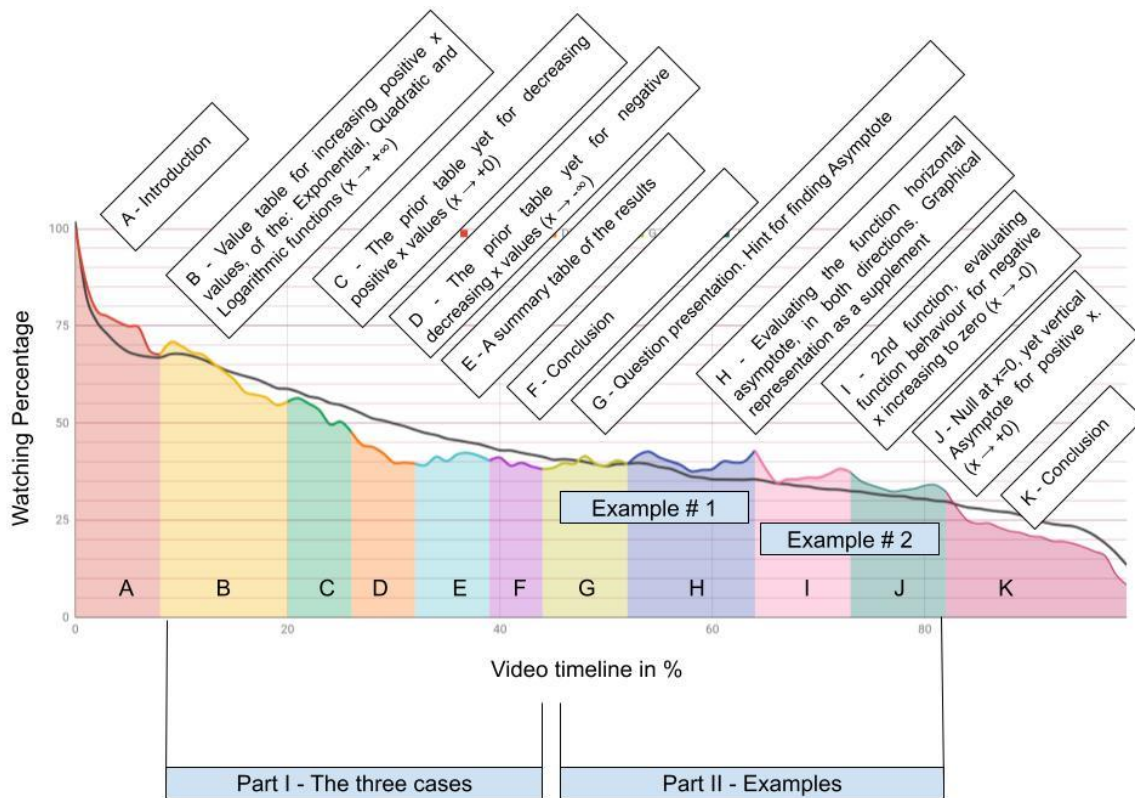


Figure 11: Limit video watching pattern with section breakdown

In Figure 11 we can review both, the video watching pattern and the prior didactical partitioning. We can observe that the first part of the video has a lower-than-average watch percentage, as opposed to the second part that is slightly above the average watch percentage.

As for the first part of video, utilizing the didactical lens, we can suggest three possible didactical factors that might explain the low watch percentage of the first part. First, this part is missing a clear incentive or motivation. There is no clear connection of the discussion during this part to the video’s main goal. Questions such as: why were these three functions specifically selected for discussion? What is the need to display and compare their values in these specific domains? All remain unclear during this part.

Second, the presenter states that the exponential function is the dominant function of all three. The claim is clearly supported by the table presenting the function values at the first domain (positive increasing x values). Yet, it is not clear how this statement can be concluded in the other two domains in which each function approaches a different value in the additional tables. Third, there is extensive use of table form as representation during all this part, while lacking any graphical representation. The lack of utilizing multiple representation supporting the mathematical phenomenon discussed during this part might be one of the causes for the low watch percentage.

In contrary to the first part, reviewing the second part we can suggest three possible didactical factors that might support the above average watch percentage value in the graph (Figure 11, sections G - J). First, both examples are highly relevant. Both clearly show the mathematical situations which are at the heart of the discussion in this video. Specifically, both present conflicting situations where at a certain domain of x the value of a function is ambiguous, because of two opposing values that their product is questionable. Second, the examples are gradual, the second is more challenging than the first, and both complement each other. The second example not only demonstrate an additional case to the first, but a different situation and a more complicated one. Third, use of good mathematical representation. Both examples include at their final stage a graphic representation of the result. Interestingly the peaks in the watch percentage graph at the end of sections H and J correspond to the moments in which the graphs are presented in the video.

The mathematical content of the Concave video is an introduction to an additional function characteristic, concavity, mainly when it is concave up or down. Didactically, the video can be partitioned to five main parts (Figure 12). First, the introduction of two functions and the discussion concerning their similar and different characteristics. Second, the definition of concave up and down of a function. Third, a sample question is presented and explained. Fourth, the definition of the second derivative of a function and its relation to the function concave states. Fifth, a short summary including a presentation of a textual summary table. We focus our discussion here on three out of the five parts: the first, third and fourth part.

The first part (section B and C) presents an interesting call for the viewers to analyze what are the similar and different characteristics of two given graphs. At first the similar function characteristics are reviewed (section B), such as: the positive domain of the function, function increase domain, positive slope, the absence of internal extremum. While the viewers of this video should be familiar with the characteristics suggested in the similar part discussion, the upcoming differences discussion in section C introduce new terminology of function characteristics. Three new definitions are used in the latter part: a) the position of a chord connecting two points on the graph; b) the position of a tangent line to a point on the graph; c) the function rate of change. All three allowing different visual approaches to measure what later will be defined as the function concave state.

The third part (section F and G) includes a short exercise. The question is presented in section F and immediately after, in section G, the solution is presented. The didactical goal of this part is to show an example that correspond to the mathematical ideas presented in the prior definition part (section D and E). The viewer is requested to identify which of the displayed four graphs is concave up? The solution presented utilizes the two visual reasoning methods that were presented in the first part: the chord and the tangent line positioning in respect to the graph, both are shown graphically.

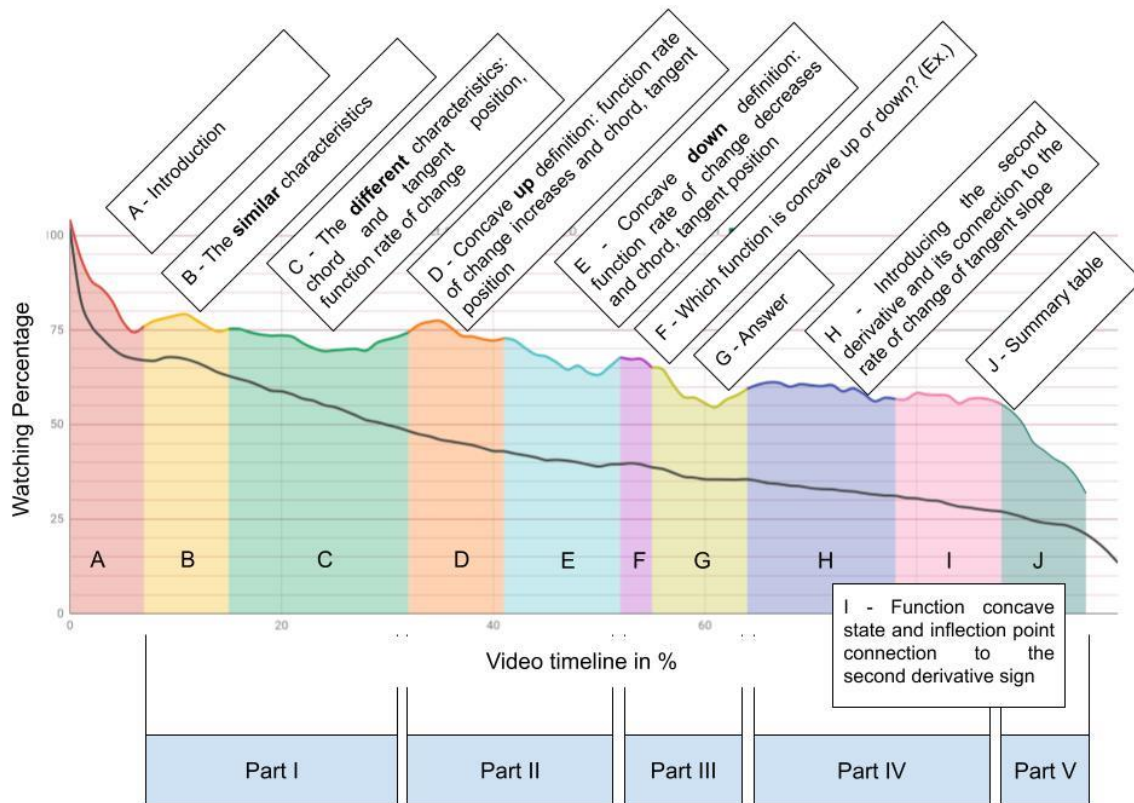


Figure 12: Concave video watching pattern with section breakdown

In the fourth part two new mathematical definitions are introduced: The second derivative in section H and the inflection point in section I. A graphical representation of a function is presented demonstrating both concave states and an inflection point in between. We can suggest two didactical benefits in the decision to introduce both definitions at this stage: First, the second derivative builds up on the prior discussion concerning the rate of change of the function slope, which was presented in the first part of the video. Second, after introducing previously the two visual methods for defining the graph concave state, there is a need to introduce an accurate analytic method.

Reviewing the video's watching pattern (Figure 12) we can clearly note the above average watching percentage during all video duration. Next, we suggest six possible didactical features within the video parts that might explain the high watching percentage. First, the open question presented at the beginning. Second, the function

characteristic review build-up – familiar features first, while new and interesting features later. Third, the introduction of a variety of methods for identifying function concavity supporting a wide overview of the subject, as opposed to presenting a single method. Fourth, a good didactical timing and presentation of the short tutorial example, in the third part, allowing further elaboration on prior mentioned mathematical definitions. Fifth, building up new mathematical terms while basing them on prior presented mathematical ideas. Sixth, expanding the reasoning from visual to analytic.

DISCUSSION

Following the above findings, we would like to suggest four didactical features that might serve as a starting point for explaining the user analytic data. First, the appearance of an *incentive* trigger early in the video, while the Limit video was missing a clear motivation in its first part, the Concave video utilized the open question at the beginning as an incentive for watching the video. Second, the examples *relevance* to the mathematical concept presented in the video, the examples in the Limit video are highly relevant both clearly show the mathematical situations which are at the heart of the discussion in the video. The example used in the Concave video also supports the prior mathematical ideas. Third, *representation*, the Limit video utilized a single representation during the whole first part, yet later in its second part an additional graphical representation of the result is included. The Concave video added the analytic reasoning to both visual methods that were presented earlier which can allow a broader understanding of the subject. Fourth, *graded progression*, the mathematical ideas within the two examples in the Limit video as elaborated previously as well as their order of appearance is a clear example of this factor. In the Concave video it can be viewed in the function characteristic build-up, which demonstrates well this feature, as well as the progress of the mathematical terms introduce during this video.

Kay, in his study on developing a framework for creating effective instructional video podcasts (Kay, 2014), named four categories for design characteristics: establishing context, providing effective explanations, minimizing cognitive load and engaging students. Mapping the four didactical features mentioned in the previous paragraph to Kay's categorization, we can conclude that the first two didactical features, incentive and relevance, can be related to the Kay's first category, establishing context. While the last two didactical features, representation and graded progression, can be related to Kay's second category, providing effective explanation.

The didactical features raised in the findings of the current study, although yet to be further evaluated in the overall research, can contribute to the design and development of mathematical video learning pedagogy.

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CONSTRUCTING A PROOF AFTER COMPREHENDING A SIMILAR PROOF – RELATION AND EXAMPLES

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A common assumption is that proof construction can be learned through proof comprehension. To contribute to this idea, we conduct three different studies with over 296 students. In each study, we ask students to comprehend a given, acceptable proof and afterwards to construct a similar proof. Based on the data, we analyse relations between proof comprehension and proof construction regarding proofs in Number Theory and Analysis. As some students comprehend a proof but are not able to construct a similar proof, we analyse inappropriate proof constructions. Possibly consequences for learning interventions are discussed.

INTRODUCTION

Undergraduate students often read written mathematical proofs, e.g., given by their lecturers or in textbooks (Mejía-Ramos et al., 2012). At least some of those proofs are given to the students as learning opportunities because proofs are “bearers of mathematical knowledge” (e. g. Hanna & Barbeau, 2010, p. 85; Rav, 1999, p. 20) and students should not only be able to comprehend proofs but learn new methods, tools, strategies, or concepts from them (Hanna & Barbeau, 2010). One underlying assumption is that students learn how to construct proofs by studying given (sometimes quite similar) proofs. Thus, if students are able to comprehend a mathematical proof, they should also be able to construct a similar mathematical proof on their own. This assumption is analysed regarding proofs in Number Theory and Analysis.

THEORETICAL BACKGROUND

Proofs are “the heart of mathematics” (Rav, 1999, p. 6) and thus learning how to prove is one of the most important goals when studying mathematics. The concept of proving can be divided in three broad activities (Mejía-Ramos & Inglis, 2008, 2009): proof reading, proof construction, and proof presentation.

Reading a proof can be distinguished in proof evaluation and proof comprehension. Proof evaluation means that a given proof is evaluated by different criteria like correctness, comprehensibility, or beauty. In contrast to this, *proof comprehension* refers to reading a given, correct (acceptable) proof. The reader does not have to judge if the proof is wrong but can focus on comprehending the used methods and arguments. A model of how to assess proof comprehension was introduced by Mejía-Ramos et al. (2012). They divided proof comprehension in aspects that refer to local parts of the proof (like the meaning of terms and statements) and aspects where one must consider the whole proof (like giving a summary or getting to know how different parts of the

proof are connected to each other). Neuhaus-Eckhardt (2022) extended this model in combination with theories about general text comprehension and proposed three dimensions. In addition to local and global aspects of a proof, she explicitly separated the dimension “transferring the general ideas or methods” that refers to aspects beyond the given proof. This third dimension may be of particular importance if students learn from given proofs methods or ideas how to construct similar proofs.

Proof construction refers to developing and writing one’s own proof. Three types of activities can be divided (Mejía-Ramos & Inglis 2008, 2009): exploration of a problem, estimation of truth of a conjecture, and the justification of a statement estimated to be true. In their assignments, students often have to prove a given statement and they can assume that this statement is true. As this refers to the third proof construction activity, we focus on this activity in our studies.

It is well known that many students struggle when justifying a statement, especially at the beginning of university (e.g., Moore, 1994, Weber, 2001). Research put forward ideas to help students to overcome those difficulties, for example transition-to-proof courses (e.g., Kempen & Biehler, 2019), or provide supporting strategies to them (e.g., Weber, 2001). Another, quite common idea is that in many lectures very detailed proofs which use typical methods (e.g., proof by contradiction) and ideas how to prove certain statements (e.g., convergence of sequences) are presented to help students to get to know these methods and ideas and finally to construct proofs on their own (Liebendörfer, 2018). Some lecturers even show proofs that are very similar to the proofs that the students have to do themselves in their assignments. Therefore, an underlying assumption is that students are able to learn how to construct proofs by reading and comprehending given, acceptable proofs (Mejía-Ramos et al., 2012).

Till now only a few studies give hints for this assumption and explicitly analyse the relation between proof comprehension and proof construction. To our knowledge, studies show small to medium correlations between proof comprehension and proof construction (e.g., Waluyo, 2021, $r = .265$). Reasons for these small relations could be that the two proving activities differ more than assumed so that it is not so easy for students to transfer methods or ideas of one proof to construct another similar proof. When comprehending a given, acceptable proof, students can focus on understanding the text, symbols, and given information, whereas proof construction is likely to require different problem-solving strategies than proof comprehension, e. g., testing different proof methods. Other variables that are important for mathematical learning processes, such as prior knowledge (e.g., Ufer et al., 2008) or self-concept (e.g., Rach et al., 2019), could also influence both activities to different extent.

Due to the small number of studies, it is yet not possible to evaluate how helpful proof comprehension is to construct proof. Especially when considering rather similar proofs, for example first reading a proof and then immediately constructing a proof using the same ideas as in the given proof, one would expect rather high relations between these two activities. Analysing exemplary solutions will give an additional

insight what could be problems for students when trying to adapt the ideas of one proof to construct another. Therefore, our research questions are:

RQ 1: To what extent is proof comprehension of one proof connected to proof construction of another similar proof?

RQ 2: What are typical inappropriate solutions to the proof construction task and which reasons may lead to these solutions?

SAMPLES AND METHODS

We analysed the relation between proof construction and proof comprehension in three different studies with undergraduate students. In two studies, we used proofs from Number Theory and students in preparatory courses before the first semester ($N_1 = 113$, $N_2 = 119$). In the third study, we analysed data from students at the beginning of their second semester ($N_3 = 64$) using proofs in Analysis. For answering RQ 2, we also include data of a pilot study of the third study with students at the end of their first Analysis course ($N_p = 58$) to enlarge the body of students' proof constructions.

Most of the students were enrolled in mathematics, economics mathematics, or teacher education programs. Each of the study took place in a lecture so that the students had a time restriction of 30 minutes to complete the proof comprehension test and to deal with the proof construction item.

Proof comprehension was measured using a statement and proof in Number Theory (suppose that p and $p + 2$ are both primes with $p > 3$. Then their sum $2p + 2$ is divisible by 12, Adler & Coury 1995) resp. Analysis (mean-value theorem). To each proof a test with multiple-choice and open items based on the three dimensions of proof comprehension was constructed (Number Theory: 10 items, Analysis: 9 items). Due to reliability analyses two items of the Number Theory test and one item of the Analysis test were not included when calculating the sum of the tests. The reliabilities of the final tests were acceptable (table 1).

We measured the performance of proof construction using a similar statement in Number Theory (suppose that p is a prime with $p > 3$. Then $p^2 + 1$ is divided by 12) resp. Analysis (Rolle's theorem). Similar methods as in the given proofs can be used to prove those two statements. In Number Theory the students only had to construct a small part of the proof because they just started university. In the proof of the first statement is shown that $p + 1$ is divided by 2 and we asked the students to show, that $p^2 + 1 = (p - 1) \cdot (p + 1)$ is divided by 4, so they only had to show that $p - 1$ is also divided by 2. In Analysis the students had to construct a full proof of Rolle's theorem using similar ideas as in the given proof of the mean-value theorem. The students already knew the statements of the two theorems from their lectures in the first semester, but with slightly different proofs. In the given proof of the mean-value theorem an auxiliary function h was used and two cases have been distinguished: h is

constant and h is not constant. The proof of Rolle’s theorem can be carried out similar by using an adapted auxiliary function h .

Students’ proof constructions for each study were coded by two different persons with two points if the proof was acceptable, one point when there were only minor mistakes, and zero points when the proof was wrong. When students wrote nothing at all and they worked on the items beforehand, we coded zero points instead of a missing value because we assume that the students had enough time to work on all items. The interrater reliability was satisfactory for each study (table 1).

Table 1: Descriptive Analysis of proof comprehension and proof construction scales.

Variable	$M (SD)$	Min	Max	Reliability	N
Proof comprehension				Cronbach’s α	
Number theory 1	3.91 (1.94)	0	8	.60	113
Number theory 2	4.67 (1.95)	1	8	.61	119
Analysis	3.12 (1.65)	0	8	.52	64
Proof construction				Cohen’s κ	
Number theory 1	0.73 (0.83)	0	2	0.77	113
Number theory 2	1.24 (0.82)	0	2	1.00	119
Analysis	0.16 (0.48)	0	2	0.99	64

RESULTS

RQ 1: To what extent is proof comprehension of one proof connected to proof construction of another similar proof?

To answer this question, we measured Pearson’s correlation coefficient for each study using R and the package lavaan (Rosseel, 2012). Missing values were estimated using the Full-Information-Maximum-Likelihood method.

In both studies referring to Number Theory the correlation was medium to high (study 1: $r = .51, p < .01$; study 2: $r = .49, p < .01$). In Analysis the correlation was medium ($r = .38, p < .01$), but only 11 people wrote something on the proof construction item (table 2). Noticeable is that 51 students (21 in Number Theory 1, 13 in Number Theory 2, 17 in Analysis) who did not construct an acceptable proof (0 points) show medium to good results (4-8 points) in the proof comprehension test. In the following, we identify typical inappropriate solutions and discuss reasons for this.

Table 2: Frequencies of proof construction.

Study	Acceptable (2 points)	Minor mistakes (1 point)	Wrong (0 points)	Nothing written (0 points)	Total
Number Theory 1	27	28	18	40	113
Number Theory 2	58	32	12	17	119
Analysis	3	4	4	53	64
Pilot study Analysis	12	7	13	26	58
Total	100	71	47	136	354

RQ2: What are typical inappropriate solutions to the proof construction task and which reasons may lead to these solutions?

Typical problems in inappropriate solutions were that students conducted only the first steps of a proof and did not finalise it or that the ideas of the given proof were only copied not adapted. To illustrate these students' problems, we show two examples.

In Number Theory, many students only showed, that $p + 1$ is even like in the given proof, ignoring the fact that they had to prove that $(p + 1) \cdot (p - 1)$ is divided by 4. One student even copied the given proof and did not adapt it to the new statement at all (fig. 1).

Fig. 1: Translated proof construction item in Number Theory with a solution (in italics).

Theorem:

Suppose that p and $p + 2$ are both primes with $p > 3$. Then their sum $2p + 2$ is divisible by 12.

This statement can be proved using similar methods as in the first statement. The start of the proof is shown below. Complete line 2 of the proof in the same way as above.

Proof:

Since $p^2 - 1 = (p + 1)(p - 1)$, one shows that $(p + 1)(p - 1)$ is divided by 3 and 4. Because p is odd...

$p^2 - 1$ is even and therefore divided by 2.

Let $k \in \mathbb{N}$. Then p is of the form $3k + 1$ or $3k + 2$

..and therefore divisible by 4.

$p^2 - 1$ is also ...[do not add proof here].

This student did not explain why $p^2 - 1$ is even which could be done by adapting an idea from the given proof. He continued the proof by starting to show that $p^2 - 1$ is divided by 3 as in the given proof which is not required for this part of the proof. It

seems that this student only copied the given proof and did not adapt it to the new situation. This is not surprising because the student got only 3 points in proof comprehension. But students with a good proof comprehension also show similar problems.

In Analysis most of the students showed only the first case (h is constant) of the proof which can be proven straight forward, whereas case two (h is not constant) is more complicated. An example is given in fig. 2.

Fig 2: Translated proof construction item in Analysis with a solution (in italics).

The following statement can be proved like the mean-value theorem above if the ideas are adapted. Prove the following statement, by (for example) first computing the auxiliary function h for g as above and then considering the cases “ h constant” and “ h not constant”. (You are not allowed to use the mean value theorem.)

Rolle’s theorem

Let the function $g: [a, b] \rightarrow \mathbb{R}$, $a < b$ be continuous and differentiable in (a, b) and $g(a) = g(b)$. Then there exists $\xi \in (a, b)$ with $g'(\xi) = 0$.

Proof:

The function h is obviously continuous on the interval $[a, b]$ and differentiable in (a, b) . In addition, it holds – also obviously – that $h(a) = h(b)$ since $g(a) = g(b)$. If h is constant, then $h'(x) = 0$ holds for all $x \in (a, b)$. The assertion follows from this.

This student had recognized beforehand that the auxiliary function h for this statement equals g . The student then follows the given proof and even adapt it to the function g in Rolle’s theorem but he/she did not give a complete proof. The student only presented the first case (h constant) even though both cases are mentioned in the item assignment. This student got 5 points in proof comprehension and zero points in proof construction.

DISCUSSION

Learning how to prove is an important learning outcome in mathematics university courses. However, many students have problems constructing a proof on their own (e.g., Moore, 1994, Weber, 2001). Lecturers present proofs to their students with the idea that students comprehend the proofs and learn how to construct proofs on their own. With this contribution, we analyse this idea.

The results show that there are medium correlations between proof comprehension and proof construction of similar proofs, replicating already existing studies (e.g., Waluyo, 2021). Still the results are somehow unexpected because we used proofs for comprehension and construction which are very similar to each other and thus, we had assumed that if students comprehend one proof quite well, they will also be able to prove the second similar statement. However, the results suggest a more complicated relationship between proof comprehension and construction. One reason could be that the processes of proof comprehension and proof construction differ too much and more

than comprehending the given proof is needed when using it to construct another similar proof. Thus, it is necessary in mathematics courses not only to present proofs but also to explicate strategies to students how to transfer ideas of one proof to another proof.

The descriptive analysis indicates that only a few students constructed an acceptable proof even if they read a similar proof before and “only” had to adapt the used methods to the new statement. Especially in Analysis (study 3), many students were not able to construct a proof at all although they already knew the statements of the two theorems from their lectures. This could be surely due to time restriction in the study. In addition, a low self-concept regarding proof (Rach et al., 2019) could hinder proof constructions because students assume that they can not construct a proof on their own and thus won’t even start to try. In addition, the exemplary inappropriate proof constructions show that some students only copied the given proof and even students with a good proof comprehension did not or only slightly adapt the given ideas to the new statement. Missing prior knowledge or a lack of specific strategies (Weber, 2001) could be reasons for this so more research is needed.

Limitation of our studies are the small sample sizes as well as the instruments which show only acceptable quality. Research concerning other proofs (of further mathematical fields) is needed to replicate the findings and identify more reasons why students could (not) transfer an idea from one proof to another. In addition, it could be analysed which strategies are helpful to teach students how to use given proofs to construct their own proofs because to comprehend proofs is a useful skill when studying mathematics but comprehending a proof does not mean to be able to construct a similar proof automatically.

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THE ROLE OF TEACHERS' PERSON CHARACTERISTICS FOR ASSESSING STUDENTS' PROOF SKILLS

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Mathematical content knowledge, methodological knowledge, and problem-solving strategies are important prerequisites for constructing mathematical proofs. To optimally support students in proving, teachers need to assess their students' proof skills. Teachers' person characteristics, such as professional knowledge and motivation, are expected to influence their assessment quality, also regarding mathematical proof skills, which we empirically investigated with 150 future teachers. Results indicate that teachers' professional knowledge supports the assessment of methodological knowledge. However, the assessment of problem-solving strategies was mostly independent of teachers' characteristics and motivational characteristics only tentatively affected the assessment of mathematical content knowledge.

INTRODUCTION

Secondary school students are often overwhelmed by the complexity and variety of mathematical proofs (Stylianides, 2019). To tackle this issue, students' learning of mathematical proof skills needs to be better supported. To do so, teachers need to know the prerequisites an individual student brings for successfully constructing mathematical proofs. In other words, teachers must be able to adequately assess relevant facets of mathematical proof skills.

Mathematical Proof Skills

The construction of a proof as a coherent and comprehensive line of arguments to validate a mathematical assumption requires various (sub-)skills. Prior research has already investigated possible facets of mathematical proof skills (Chinnappan et al., 2012). Three facets can be seen as particularly important prerequisites of a student to construct a mathematical proof: mathematical content knowledge, methodological knowledge, and problem-solving strategies (Sommerhoff, 2017). Mathematical content knowledge refers to knowledge of the mathematical subject area of the proof, such as the definition of a parallelogram and congruence theorems (Chinnappan et al., 2012). Methodological knowledge refers to knowledge about the concept of mathematical proof in general. It comprises knowing that mathematical proofs start with certain premises, followed by a rigid line of arguments to culminate in a conclusion. Methodological knowledge also comprises knowledge about what types of arguments are legitimate within a proof in a given context (Heinze & Reiss, 2003). Lastly, problem-solving strategies refer to heuristic strategies, which can be related to the mathematical subject area of the proof, but also refer to metacognitive strategies such as monitoring the proving process and adapting the proving strategy accordingly

(Chinnappan et al., 2012). Even though prior research provided evidence for the role of these facets for students' proof construction, research focusing on the assessment of these facets from a teacher-centered perspective is scarce.

Teachers' Assessment Skills

Generally, teachers' assessment skills are seen as skills that teachers need to cope with various assessment situations and to assess student characteristics accurately (Urhahne & Wijnia, 2021). The most frequently used measure of teachers' assessment skills is judgment accuracy. Unfortunately, according to prior research, teachers' judgment accuracy leaves room for improvement. This led to a call to facilitate future teachers' assessment skills already in university teacher education. To explore reasons for improvable judgment accuracy, the assessment process has been increasingly focused recently (Herppich et al., 2018). For example, to explain varying judgment accuracies regarding the assessment of students' mathematical proof skills, teachers' written notes taken during the assessment may provide insights whether and to what extent the above-mentioned three facets of mathematical proof skills have been considered.

During the assessment process, the assessed students' mathematical proof skills become evident in more or less salient cues regarding mathematical content knowledge, methodological knowledge, and problem-solving strategies. Teachers are then required to use their knowledge about the relation of these cues to the skills to be assessed to infer a final judgment (Brunswik, 1955). Typically, the more relevant cues are perceived and the more meaningfully they are interpreted, the more accurate is the final judgment. However, depending on the teacher, not all cues may be noticed. Moreover, cues can vary in the degree of inference, as some classroom events require more inferences to be identified as a relevant cue than other classroom events. For example, the student's statement "I checked the book and found that I used the definition of a parallelogram incorrectly" is a salient and thus low inferential cue regarding the student's mathematical content knowledge. However, it may also be a cue that the student has good monitoring strategies while working on a task. This, in turn, is less obvious and therefore a rather high inferential cue regarding the students' problem-solving strategies. This also points towards different degrees of inference of cues, depending on the variable that is assessed. Regarding mathematical proof skills, Codreanu et al. (2021) investigated the degree of inference of cues regarding mathematical content knowledge, methodological knowledge, and problem-solving strategies. They hypothesized that cues for mathematical content knowledge are rather low inferential, whereas cues for methodological knowledge and problem-solving strategies are higher inferential. In their study, these different degrees of inference appeared to have affected their participants' judgment accuracy, as mathematical content knowledge was assessed more accurately than methodological knowledge, which, in turn, was assessed more accurately than problem-solving strategies.

Role of Teachers' Person Characteristics for Assessment Skills

This research of Codreanu et al. (2021) suggests that the degrees of inference of cues can influence teachers' judgment accuracy. To explain this, teachers' person characteristics, such as motivation and knowledge, may come into play. A teacher with high motivation and high knowledge may more easily notice and interpret a high inferential cue than a teacher with low motivation and lacking knowledge, who may not even notice this cue. This underpins that teachers' characteristics may influence their judgment accuracy. Recent theoretical conceptualizations of assessment skills highlight the role of the teachers' characteristics for assessing students. Motivational characteristics and knowledge are seen as important characteristics of teachers for adequate assessment processes and high judgment accuracy (Herppich et al., 2018). Findings regarding the role of these teachers' characteristics on the assessment of the different facets of mathematical proof skills enable teacher educators to adapt their support for the assessment of mathematical proof skills based on the teachers' characteristics. Although the role of teachers' characteristics for the assessment of mathematical proof skills has not been specifically investigated so far, there is some research on the role of teacher characteristics for teachers' assessment skills in general. Regarding the teachers' knowledge, content knowledge (CK) and pedagogical content knowledge (PCK) are seen as important. CK refers to mathematical knowledge about the mathematical subject itself. Regarding the assessment of mathematical proof skills, the teacher, for example, needs to be able to solve the proof task by himself/ herself to assess the students' solution. Teachers' PCK that is relevant for assessments refers, for instance, to knowledge about typical student misconceptions. For example, Ostermann et al. (2018) designed an intervention fostering future teachers' PCK in the context of the assessment of functional thinking. Their intervention led to higher judgment accuracies of the future teachers. Whereas research has mostly focused on the role of teachers' knowledge for student assessment, the role of motivational characteristics has been investigated to a smaller extent. For example, teachers' self-efficacy regarding student assessment and motivational aspects of teachers' self-regulation in general have been discussed (Holzberger et al., 2013). Furthermore, Kron et al. (2022) recently empirically found teachers' interest in assessment skills to be relevant for the accuracy of their judgments in the context of decimal fractions.

So far, prior research has investigated the role of teachers' person characteristics for student assessment mainly from a variable-centered perspective. However, person-centered approaches, such as a latent profile analyses (LPA), allow a particular focus on individuals and can pave the way for possible personalized support of individual subgroups (Tetzlaff et al., 2021). A LPA aims at identifying subgroups of participants with similar scores ("profiles") across pre-specified indicators. Nickl et al. (2022) used a person-centered approach for investigating the role of teachers' person characteristics for assessment skills. However, they focused on learning of assessment skills, used rather general measures regarding the assessment process (e.g., word count), and did not specifically consider the above-mentioned three facets of mathematical proof skills.

THE PRESENT STUDY

The present study aims at investigating the role of knowledge (CK, PCK) and motivational characteristics of future teachers (interest, self-efficacy, and self-regulation) for the assessment of mathematical proof skills from a person-centered perspective. To do so, we use profiles of teachers' person characteristics identified by Nickl et al. (2022). To measure assessment skills, we draw on data from a video-based simulation as an approximation of practice that allows for a reliable and scalable measurement of assessment skills. Particularly, we ask the following RQs:

RQ1: Does judgment accuracy regarding the three facets of mathematical proof skills (mathematical content knowledge, methodological knowledge, and problem-solving strategies) differ between the identified profiles?

RQ2: Do the identified profiles consider the three facets of mathematical proof skills to a different extent during the assessment process?

METHODS

Sampling and Study Design

$N = 150$ future teachers (f: 101, m: 47, NA: 2) participated in the study, which was conducted in teacher education seminars at different German universities in 2020 and 2021. Participation was promoted as optional course content and was remunerated.

The study was conducted in an online environment and lasted approximately 90 minutes in total. Future teachers were first asked to self-report their interest for diagnosing, their self-efficacy, and their self-regulation. They were then asked to complete a test measuring their CK and PCK regarding proof in geometry (see Nickl et al., 2022, for details). Afterwards, we asked participants to complete a validated video-based simulation, in which we measured their assessment skills (Codreanu et al., 2021). In the video-based simulation, participants were asked to assess two simulated 7th grade students' mathematical proof skills. Participants were first familiarized with the assessment situation and the facets of mathematical proof skills. Then, they gathered information by watching up to ten videos showing student-teacher interactions. In these videos, the two simulated students worked on the same proof task ("Prove that the opposite sides of a parallelogram are equal in length!"). The videos depicted different stages of the working process of the simulated students. During the videos, participants were asked to take notes for their assessment. When participants felt that they had gathered enough information, they rated both students' mathematical proof skills on a four-point Likert scale (eight items per student; three items regarding the students' mathematical content knowledge, three items regarding methodological knowledge, and two items regarding the problem-solving strategies). The participants' Likert ratings were compared to an expert rating. A match with the expert solution was scored with one point; else, zero points were given. This measure of judgment accuracy was then standardized for better interpretability to the range from 0: no match with the expert solution to 1: all ratings matched.

Data Analysis

To answer the research questions, we draw on teachers' characteristics profiles from a LPA that have been identified in a prior publication using the same data set (Nickl et al., 2022). This LPA used CK, PCK, interest, self-efficacy, self-regulation as indicators and identified three profiles: One profile (called *knowledgeable*, containing $N = 61$) had above average CK and PCK scores and reported average scores on motivational characteristics (interest, self-efficacy, and self-regulation). The second profile (*motivated*, $N = 37$) had slightly below average CK scores and average PCK scores, but reported above average motivational characteristics. The third profile (*potentially struggling*, $N = 52$) was below average across all indicators. Furthermore, in preparation of answering RQ2, we coded future teachers' notes from the video-based simulation. In the coding process, we separated notes into single text segments (e.g., sentences) and identified how many of these single segments referred to mathematical content knowledge, methodological knowledge, and problem-solving strategies, respectively. Interrater reliability was substantial (Cohen's $\kappa = .75$). Finally, to answer both research questions, we used the BCH method and Wald's χ^2 test to calculate and compare means of the three profiles regarding judgment accuracy (RQ1) and the number of notes regarding each of the facets of mathematical proof skills (RQ2).

RESULTS

Descriptive results regarding RQ1 and RQ2 can be found in Table 1.

	Knowledgeable <i>M</i> [<i>SD</i>]	Motivated <i>M</i> [<i>SD</i>]	Potentially struggling <i>M</i> [<i>SD</i>]
<i>Judgment accuracy (0-1)</i>			
M. content knowledge	.423 [.035]	.364 [.051]	.327 [.039]
Methodological knowledge	.423 [.029] ^{a,b}	.293 [.039] ^a	.307 [.031] ^b
Problem-solving strategies	.217 [.029]	.328 [.046]	.263 [.035]
<i>Number of text segments (in total)</i>			
M. content knowledge	3.69 [0.32] ^c	3.67 [0.57]	2.32 [0.35] ^c
Methodological knowledge	4.64 [0.40] ^{d,e}	2.81 [0.39] ^d	3.32 [0.43] ^e
Problem-solving strategies	5.73 [0.13]	4.57 [0.21]	5.33 [0.18]

Table 1: Descriptive results regarding both RQs. Variables, in which the profiles differed significantly in bold. Significant post-hoc tests marked with matching letters.

Regarding RQ1, descriptive differences of the profiles regarding judgment accuracies of mathematical content knowledge ($\chi^2 = 3.61$, $p = .165$) and problem-solving strategies ($\chi^2 = 4.01$, $p = .135$) did not reach statistical significance. Differences regarding judgment accuracy of methodological knowledge were significant ($\chi^2 = 10.65$, $p = .005$). Post-hoc tests showed that the *knowledgeable* profile had a

significant higher judgment accuracy than the *motivated* profile ($\chi^2 = 6.23, p = .013$) and the *potentially struggling* profile ($\chi^2 = 7.73, p = .005$), whereas the *motivated* and the *potentially struggling* profile did not differ significantly ($\chi^2 = 0.06, p = .805$).

Regarding RQ2, the profiles did not differ significantly in the number of text segments in their notes regarding problem-solving strategies ($\chi^2 = 1.19, p = .552$). However, the profiles differed significantly in the number of text segments regarding mathematical content knowledge ($\chi^2 = 9.00, p = .011$) and methodological knowledge ($\chi^2 = 10.44, p = .005$). For both facets, the *knowledgeable* profile wrote more segments than the *potentially struggling* profile (each $\chi^2 > 4.97, p < .026$). The *knowledgeable* profile also wrote significantly more segments regarding methodological knowledge than the *motivated* profile ($\chi^2 = 9.22, p = .002$), but differences were not significant regarding mathematical content knowledge ($\chi^2 < 0.01, p = .980$). The *motivated* profile and the *potentially struggling* one did not differ significantly regarding mathematical content knowledge ($\chi^2 = 3.27, p = .070$) and methodological knowledge ($\chi^2 = 0.63, p = .429$).

DISCUSSION

This study investigated the role of (future) teachers' characteristics for their assessment of mathematical proof skills. For this, three different teacher profiles were used in a person-centered approach. Our results show significantly higher judgment accuracies of the *knowledgeable* profile regarding the assessment of methodological knowledge compared to both other profiles. One reason for this may be that the cues regarding methodological knowledge were higher inferential and the more advanced knowledge of the *knowledgeable* future teachers allowed them to better notice and faster process relevant cues (Brunswik, 1955; Codreanu et al., 2021). That the *knowledgeable* profile writes significantly more text segments than the other profiles about methodological knowledge in the assessment process substantiates this interpretation. This underpins the role of future teachers' knowledge for the assessment of methodological knowledge and suggests that possible support for the *motivated* and the *potentially struggling* profile may consist in providing additional knowledge. To do so, prompts could be an effective means in the context of assessment skills (Amador et al., 2022).

Surprisingly, results are different for problem-solving strategies. Regarding judgment accuracy, no profile has significant advantages. Descriptively, the accuracy of two of the profiles is even close to guessing probability, indicating low judgment quality, independently of teachers' characteristics. Against this backdrop, the focus on problem-solving strategies in the assessment process is interesting (see Table 1): Across all profiles, participants write descriptively more text segments regarding problem-solving strategies compared to both other facets of mathematical proof skills. This may show that future teachers notice various cues connected to problem-solving strategies, but they fail to derive accurate judgments. This may have two reasons (Brunswik, 1955): It could be that they do not notice the *relevant* cues. This may reflect that there are low inferential cues pointing to problem-solving strategies (e.g., the student draws a sketch), which are noticed by future teachers, but that high inferential

cues are missed (e.g., student notices that he/ she used a wrong definition; see example in the introduction), which however might be relevant. It could also be that future teachers notice relevant cues but cannot process them sufficiently for an accurate judgment. This would be in line with prior research reporting problems of novices in interpreting noticed cues (Wolff et al., 2017). In that regard, further research is needed clarifying reasons for these deficits in the assessment of problem-solving strategies.

Regarding mathematical content knowledge, profiles did not differ significantly in their judgment accuracy. However, to examine the role of motivation regarding the assessment of mathematical content knowledge some tentative findings are of interest: On the one hand, the *motivated* profile assesses mathematical content knowledge descriptively more accurate than both other facets of mathematical proof skills and descriptively more accurate than the *potentially struggling* profile. In addition, the *motivated* profile's number of notes corresponding to mathematical content knowledge is descriptively on a par with the *knowledgeable* profile, which is not the case regarding both other two facets of mathematical proof skills. This might indicate that motivation plays a role for the assessment of mathematical content knowledge, which is assumed to require less prior knowledge than the other two facets due to lower inferential cues (Codreanu et al., 2021). Thus, the *motivated* profile may partially be able to compensate its lack of knowledge. Again, further research is necessary to substantiate this tentative finding regarding the assessment of mathematical content knowledge.

Altogether, the results indicate that the extent to which teachers' person characteristics can influence student assessment of mathematical proof skills varies over the three facets of mathematical proof skills. Thus, teacher educators may support teachers' skills to assess mathematical proof skills by different approaches. They may provide cognitive (e.g., prompts for knowledge activation for supporting the assessment of methodological knowledge) or motivational support (e.g., utility value interventions for supporting the assessment of mathematical content knowledge) – depending on the specific facet they want to support. How effective ways of supporting teachers' knowledge or motivation can be designed in particular is currently being investigated.

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ZPD NOTICING – A VIGNETTE-BASED STUDY INTO PRE-SERVICE TEACHERS’ ANALYSIS OF AN ALGEBRA CLASSROOM SITUATION

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Identifying the zone of proximal development (ZPD) for individual learners has shown to be a key for providing students with adaptive help in their mathematical learning. For its high relevance for classroom practice, fostering ZPD-related noticing in pre-service teachers (PSTs) appears as primordial. As empirical research on this aspect of teacher expertise is still scarce, this study examines PSTs’ ZPD noticing in an algebra context. The vignette-based design uses a specifically designed classroom situation in which a teacher reaction is not consistent with the ZPD in the learning of a group of students. The empirical findings from a sample of 26 PSTs indicate difficulties in ZPD noticing of the PSTs, among other insights. We conclude that specific vignette-based learning opportunities might be a pathway for strengthening PSTs’ ZPD noticing.

INTRODUCTION

The notion of *zone of proximal development* (ZPD) was suggested by Vygotsky (1978) and defined as “The distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (p.86). The ZPD can be considered as a theoretical tool to identify content-related learning steps which are in reach of the learner, given her/his prior knowledge. According to the students’ identified ZPD, teachers should adapt the needed guidance and help, to support the learning processes of the students.

Combining these considerations, which are mostly closely linked to specific mathematical content, with approaches to teachers’ *noticing* abilities promises a high added value for enabling pre-service teachers (PSTs) to react to learners’ needs in the classroom. Moreover, this combined perspective enables assessing whether teachers are able to detect potential mismatches between the learners’ needs and teacher reactions which do not lie within the student’s ZPD and hence are hardly useful for encouraging further learning. Despite these potentials, empirical research into *ZPD noticing* (i.e., noticing whether reactions to learners fall into their ZPD) is still scarce. Corresponding to this research need and using a vignette-based research design, we designed a vignette showing a situation in an algebra lesson which aligns with our goal to examine teachers’ ZPD noticing. The answers of 26 PSTs to the vignette provide insight into their ZPD noticing through the PSTs’ analyses (1) of the student’s prior thinking, (2) of the ZPD-inconsistent teacher’s reaction and (3) through whether the PSTs suggest alternative reactions which can be seen as consistent with the ZPD.

THEORETICAL BACKGROUND

Noticing in the sense of knowledge-based reasoning on classroom situations (Sherin et al., 2011) is considered as a major aspect of mathematics teacher expertise, for its focus on connecting observations with relevant criteria rooted in the professional knowledge of mathematics teachers (e.g., Shulman, 1986). Kersting et al. (2012) emphasise the term of *usable knowledge* for expressing how aspects of classroom situations are interpreted against specific knowledge-based criteria – in contrast with rather inert professional knowledge, which is not used for situation interpretation. Making connections between situation aspects and professional knowledge is a common feature to a broad variety of approaches to teacher noticing in mathematics education (e.g., Fernández et al., 2018; Fernández, & Choy, 2020). A model for describing such noticing mechanisms and related processes uses the *awareness* construct: awareness of specific criteria drives a criterion-based analysis cycle between observations in the classroom situation and their knowledge-based interpretation (Kuntze & Friesen, 2018). If, for example, a mathematics teacher is aware of the importance of the zone of proximal development (ZPD), s/he will be likely to interpret observations against the background of related professional knowledge. In this context, a teacher might notice more successfully the learners' current development and possibilities of fostering their further learning, than it would be the case for a teacher who is not aware of ZPD-related criteria.

This example already highlights the high practical relevance of the ZPD concept together with the focus on teachers' noticing. By the short term *ZPD noticing*, we will in the following understand any components of teachers' noticing related to the zone of proximal development (cf. Vygotsky, 1978) – among these,

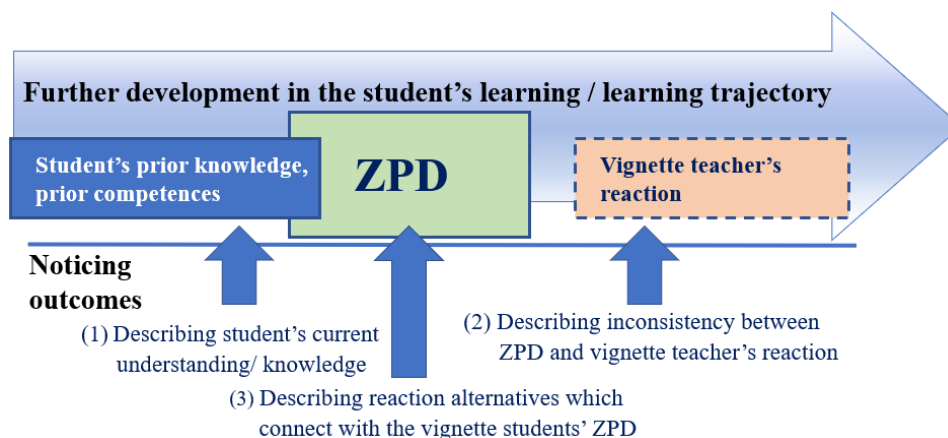
- noticing related to the students' current state of development (i.e., their current knowledge, understanding and learning needs),
- noticing whether teacher reactions to learners fall into the students' ZPD, and
- noticing of possible reaction alternatives adapted to the learners' ZPD

can be expected to play prominent roles (see also Fig. 1). Consequently, ZPD noticing emphasises an adaptive analysis of the learners' needs and possibilities to promote learning steps accessible to the students (Hardy et al., 2019; cf. Kuntze et al., 2021).

This study is focused on ZPD in the context of algebra learning. An extensive literature exists on learners' development (e.g., Demonty, Vlassis & Fagnant, 2018; Kieran, 1992), including the description of learners' development in so-called *learning trajectories* (e.g., Gurbuz & Özdemir, 2020), which describe possible sequences of learning steps also in a long-term perspective. From early algebra learning in primary school grades up to more abstract and less number-based approaches in the secondary school grades (Demonty, Vlassis & Fagnant, 2018), the learning of algebra is described as a process consisting of a long chain of insights and challenges for the learners. While in this process, students are learning how to deal with patterns in number sequences, how to use the concept of variable and how to build up and use symbolic algebraic

expressions. These – also empirically well-grounded – theory elements can be used to identify the ZPD of students, also in ZPD noticing. A teacher who is aware of the learners’ ZPD in algebra learning can be expected to analyse where the students are in their learning related to dealing with variables, for instance. Also, they can analyse whether a possible teacher reaction could help the learners to progress on a learning trajectory in relation to their ZPD towards e.g., a flexible use of variables in algebraic terms. Three example outcomes of ZPD Noticing in this context are shown in Fig. 1.

Figure 1: Model-like overview of key components of ZPD noticing in view of research needs and the corresponding purposefully designed vignette



For investigating ZPD noticing, *vignettes* offer unique methodological possibilities: We understand vignettes as *representations of practice* (Buchbinder & Kuntze, 2018) which can purposefully be designed and used for eliciting noticing (e.g. Kuntze et al, 2022; Skilling & Stylianides, 2020). In particular, ZPD noticing can be examined by designing vignettes representing situations in which the ZPD plays a key role.

Using such a vignette-based approach, the study presented here aims to examine whether PSTs notice students’ current state of development, whether teacher reactions to learners fall into the students’ ZPD, and whether PSTs suggest alternative reactions which connect with the students’ ZPD. Accordingly, our research questions are:

- (1) Do the PSTs describe the ZPD as far as the side of the students’ understanding is concerned?
- (2) Do the PSTs notice the inconsistency between the vignette teacher’s reaction and the ZPD of the vignette students?
- (3) Do the PSTs suggest reaction alternatives which connect with the vignette students’ ZPD?

DESIGN AND METHODS

The study is based on the vignette shown in Figure 2, developed and designed by the third and the second authors of this paper and inspired by Palatnik & Koichu (2015).

Figure 2: Vignette (inspired by Palatnik & Koichu, 2015) in English translation

Algebra class:
 Class grade – 7th grade. The class has already studied and experienced pattern generalizations problems, and this is their last lesson on that topic. So far, the students have successfully dealt with the problem of the type: $n + a$ and an (where n is the variable and a is a parameter).

Today we are going to work on a problem, which I am going now to write on the board

Every straight line divides a pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts?
 A. Solve for $n = 1, 2, 3, 4, 5, 6$.
 B. Find a recursive formula for the case of n straight cuts.
 C. Find an explicit formula for n straight cuts.

So, what does it mean to have one piece and two pieces? I will show this with an illustration on the board

No lines – one piece One line – two pieces

Now, you can work on this task with your seat partners

After about 10 minutes of circulating the class

How is it going?
 Good, I think
 Can I see what you are doing?

Three lines: Four lines:

You can see in the notebook, we found that 1, 2, 3, and 4 straight lines will lead to 2, 4, 7, and 11 pieces

I think that for the following straight lines, it should be 16 and 22 because it goes by one more each time. When going from 2 to 4 it is 2, when going from 4 to 7 it is 3, going from 7 to 11 it is four so going from 11 will be five, so we will get 16. It is like $n+1$, right?

So, if I have seven lines, it will be $7+1=8$? Or is it $22+1=23$?

No! It is $22+7=29$. It is one more than before! Before we added 6 to the 16, so now we are adding 7, one more, to the 22!

The vignette presents a situation from an algebra classroom in a 7th grade class, where the students are asked to find a formula for the maximal number of pieces which can be obtained by cutting a pizza with n straight cuts. A vignette student explains his reasoning in an example-based numerical way. His difficulties in analysing a general, variable-based algebraic formula provide insight into his ZPD in algebra learning. The

teacher’s response is focused on the student’s “ $n+1$ ” utterance. She relates to this as a suggestion of a solution formula and tries to disprove it by providing two substitutions. However, as the teacher’s reaction is outside of the students’ ZPD, and the student grasps the teacher’s answer as a misinterpretation of his answer and offers more explanation. One cannot conclude from the vignette that the teacher’s conjecture-falsification approach is useful for the learners, in relationship to their thinking.

26 pre-service teachers from a South-German university of education participated in this study, and were asked the following, regarding to the abovementioned vignette: (a) explain what the student(s) meant mathematically by their statements; (b) interpret the teacher’s intention behind her response; (c) judge on the helpfulness of the teachers’ answers for the students’ learning; and (d) describe how the PSTs would have reacted in the classroom situation. In all these sub-tasks, the PSTs were explicitly encouraged to justify their answers with their observations from the classroom dialogue presented in the vignette. All written answers were collected.

Data analysis was done according to a criteria-based interpretive analysis (Mayring, 2015). First, we analysed the vignette and created a basis for our coding. To answer the first research question, coding was focused first on the PSTs’ analysis of the students’ mathematical reasoning and whether they noticed that the students struggle to move from example-based numerical thinking to the use of variables and a more abstract algebraic thinking. For the second research question, we categorised the PSTs reasoning on the teacher’s reaction and coded the answers to see whether the PSTs identified her reaction’s inconsistency with the students’ ZPD. Regarding the third research question, the offered alternative strategies were categorised in accordance with their potential to connect with the vignette students’ ZPD. This analysis was of a bottom-up nature, combined with our proposed base coding.

RESULTS

Analysing the students’ thinking and noticing the ZPD

Almost all the PSTs (24 out of 26) offered a mathematical analysis of the vignette students’ answer, based on the notebook drawing (see Figure 2 above) and the discussion with the vignette teacher. They explained what the students mean by “ $n+1$ ” and indicated that the students have found a pattern for the number of pizza pieces. The PSTs used different representations in their analysis, such as verbal or algebraic expressions, tables, or sketches. One example is presented in Figure 3.

Number of straight lines n	1	2	3	4	5	6
Number of pieces a	2	4	7	11	16	22
		+2	+3	+4	+5	+6

Figure 3: Example of a mathematical analysis of the students’ answer, in a table

13 PSTs noticed that the students might have been on the right pathway and described their algebraic ZPD in the way that the vignette students still think of the pattern in a number-based recursive way and do not use variables in a correct way in the sense of a more general algebraic thinking. For example, one of these PSTs wrote:

[...] The students do not realize that n should be added to the number of pizza pieces. [...] They do not succeed in changing their representation to the recursive formula. The students have recognized that the previous step plays a role in the calculation of the maximum number in subsequent steps, [...] but they have not yet found a recursive formula.

We should mention that often, these PSTs made the mathematical analysis in their answer to question (a), and only later, usually in their answer to (c), they indicated that the students have not correctly completed the variable-based step of generalisation.

Identifying the inconsistency

10 of the PSTs interpreted the teacher's response as an encouragement of the students to explain and reason their answer, and to further think of the expression " $n+1$ " and whether it is the correct solution (sample PST answer: "I think the teacher intends for the student to explain his calculation and his chain of thought even more precisely"). In contrast, 12 PSTs indicated that in her response, the teacher tried to disprove the students' formula by inserting numbers in the expression " $n+1$ ":

The teacher probably knows what the students mean by their results. The teacher takes up the formula and substitutes the numbers [...]. With these statements she wants to convey to the students that it is important to clearly define what the variable n stands for. [...] Using concrete numerical examples, she makes it clear to the two students that the formula is incorrect [...] and needs to be revised. This means that the teacher wants to use concrete numerical examples to point out that the formula is wrong.

4 PSTs gave other answers, like that the teacher did not understand the students' idea. Regarding the helpfulness of the teacher's response, 13 PSTs found the teacher's response helpful, especially when considering the goal of reasoning the students' answer. The other 13 PSTs were more critical and indicated the response did not lead the student into the generalisation of the recursive formula. In regard to the inconsistency between the teacher's reaction and the students' ZPD, 13 PSTs identified this inconsistency, such as in the following example:

In her response, the teacher indirectly points out to the students that something in their formula is not right. [...] However, the student does not seem to understand this hint. He seems to believe that the teacher has not yet fully understood their good suggestion and responds to the teacher's response by repeating his thoughts. In this respect, it seems the teacher's supposed goal is not fulfilled, since the response did not help the students to recognize their mistake and to think of an alternative solution.

Offering alternative strategies

7 PSTs offered alternatives which were coded as *connected with the vignette students' algebra learning ZPD*. These answers combined offers to add another representation, such as a table or a sketch, from which the students can better understand the pattern

and how it can be generalised. In addition, some of these answers included questions the teacher should ask which can support this trajectory. Hereinafter, one PST wrote:

[...] I would have encouraged the students to enter the relevant values in a table and examine them. To think about the formula, I would first ask what n is and ask them to use their $[n+1]$ formula to calculate the number of parts in each column, so they will realize the formula is wrong and needs to be revised. Since the students have already understood the relevant information but cannot express their ideas symbolically, I would ask them to describe verbally how to determine the number of parts. [...] I would clarify to the students that the description is correct, but they should now translate it into a formula.

12 answers were coded as *partially connected* with the vignette students' ZPD. In these answers, the PSTs offered that the teacher should ask the students to clarify what the variable n means or offered that the students would add a sketch or a table so they can conclude the formula, but with no further guiding questions. 5 PSTs offered general pedagogical strategies, such as that the students should work in groups so they can reflect on their solutions or that the teacher should clarify the problem further and discuss the solutions with the whole class. 3 PSTs thought that the students' answer was wrong, as the lines should all pass through the centre of the circle. Therefore, their answers offered that the teacher should concentrate on correcting the students' mistake.

DISCUSSION AND CONCLUSIONS

In this study, PSTs analysed a classroom situation, where an inconsistency exists between the students' algebra learning ZPD and the teacher's reaction. The findings suggest that the more than half of the PSTs did not fully achieve the ZPD noticing outcomes presented in Figure 1: (1) noticing the students' ZPD, (2) the inconsistency between this ZPD and the vignette teacher' reaction and (3) offering reaction alternatives which connect with the vignette students' ZPD.

ZPD Noticing in this sense is hence complex. It requires both mathematical content knowledge and mathematical pedagogical content knowledge. Moreover, awareness of ZPD-related criteria (Kuntze & Friesen, 2018) in the context of algebra is needed. As noticing is a knowledge-based process (Sherin, Jacobs, Philipp, 2011) one possible conclusion is that the PSTs' professional knowledge should be fostered, together with ZPD-related awareness. As the questions which were given to the PSTs did not lead them directly to ZPD noticing, it does in particular not appear to be possible to substitute such awareness by providing PSTs with extensive analysis instructions.

Against this background and the importance of ZPD noticing, we suggest that PSTs experience specific vignette-based learning opportunities, accompanied with guidance and instructions which aim to these goals, as well as exposure to diverse teaching strategies and practices which support teachers' ZPD noticing in various contexts.

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ON THE CONNECTION BETWEEN BASIC MENTAL MODELS AND THE UNDERSTANDING OF EQUATIONS

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Basic mental models (BMMs) of equations have been proposed as structures describing conceptual understanding of equations. Two of these BMMs are those of equations as relations and equations as objects. We are interested in the relation between these BMMs and special errors associated with working with equations. In this study we concentrate on very basic equations in the form of $a \cdot x = b$ and $a + x = b$. We are interested in obstacles, errors and misunderstandings concerning these prototypes of equations. An empirical investigation shows that two types of errors, the reversal error and the attribute error, are statistically related to the BMMs students have established.

INTRODUCTION

Equations are basic elements in all fields of mathematics and mathematics education. However, many studies have revealed that students have problems with the understanding of equations, especially with the equals sign, and the solving of equations. The equals sign is seen as an instruction “to work it out now” (Kieran, 1981) or “to do a calculation” (Arcavi et al. 2017, p. 55). While this perspective is important and correct in primary school, it is still present and becomes problematic in lower secondary school (see Borromeo Ferri & Blum 2011).

“A limited conception for what the equals sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equals sign represents a relation.” (Carpenter et al. 2003, p. 22)

Without this “relational view” students will have problems in interpreting expressions correctly and setting up equations properly. This paper concentrates on two particular types of errors, the *reversal error* and the *attribute error*, and investigates how they are linked with each other on the one side and with the two basic mental models of equations as objects and as relations on the other.

The paper first describes the theory of basic mental models (BMMs) and the two error types in detail. Then an empirical investigation is presented which looks for relations between BMMs and these two types of errors.

THEORY

The equals sign

The equals sign has different meanings or perspectives in mathematics and mathematics lessons. It can be seen as an operation sign, e.g., in $3 + 7 = 10$ with the meaning of “results in”, it is a relational sign, e.g., in $29 + 36 = 30 + 35$ or $3 \cdot x + 5 =$

$x - 1$, it can express an identity or an equivalence, e.g., in $a \cdot (b + c) = a \cdot b + a \cdot c$ or it can give a functional relationship, e.g., $V(r) = \frac{4}{3}r^3\pi$ or $f(x) = x^2$. These different meanings result in different conceptions and perceptions connected with equations.

Basic mental models of equation (BMMs)

The concept of BMMs has a long tradition in German didactics where they are called “Grundvorstellungen” (vom Hofe & Blum, 2016). They describe, from a normative point of view, the conceptual mental models students should develop in order to grasp the meaning of concepts and apply them in an adequate and sensible way. Weigand et al. (2022) describe four BMMs of equations, based on mathematical aspects of equations. These are:

- **Operational BMM:** An equation is understood as a calculation or transformation. The equals sign is seen as an operational sign, which indicates a reading direction of the equation in the sense of a “resulting-in” sign.
- **Relational BMM:** An equation is understood as a task to determine numbers or quantities for the expressions on both sides of the equation to get the same value or quantity on both sides. The equals sign is seen as a relational sign. The variable here is understood as an unknown which has to be determined.³
- **Functional BMM:** An equation $T_1(x) = T_2(x)$ is a comparison of two expressions which are understood as functions with $y = T_1(x)$ and $y = T_2(x)$. Here, too, the equals sign is understood relationally, but the the variable is seen as varying over its domain.
- **Object-BMM:** An equation is regarded as a mathematical object that has characteristic properties, such as the number of possible solutions, the definition range or special solution algorithms.

Meanwhile, there are some empirical investigations concerning the structure and the independence of different BMMs of a concept, e.g., of the concepts of function, derivatives and integral (see Greefrath et al., 2021). However, there is a lack of research concerning the relevance of BMMs for solving problems in special fields of mathematics. The BMMs of equations are still a theoretical concept and it is not much known about their effect on solving problems. The present paper is a step in this direction. We especially emphasize the *Relational BMM* and the *Object-BMM* in analysing errors while formulating and interpreting equations.

³ We refer to the three central BMMs for variables without explaining their background (see e.g., MacGregor and Stacey (1997) for details): The variable as a general number, the variable as an unknown number and the variable as changing number or quantity.

Basic difficulties while working with equations

Students have problems while working with equations (see e.g., Arcavi et al., 2017, p. 95 ff.). In this study we concentrate on the very basic equations in the form of $a \cdot x = b$ and $a + x = b$. We are interested in obstacles, errors and misunderstandings concerning these prototypes of equations. Without the competence of interpreting and operating with these kinds of equations, the understanding of more complex equations is not possible. Moreover, we concentrate on two errors in relation to these equations, the *reversal error* and the *attribute error*. In particular, these errors show misunderstandings when dealing with the equals sign as a relational sign.

The prototype of the *reversal error* is provoked by the professor-and-students task (Clement, Lochhead, & Monk, 1981), that reads in its original version:

Write an equation for the following statement: There are six times as many students as professors at this university. Use S for the number of students and P for the number of professors.

While the correct solution is $S = 6 \cdot P$, many students write the reversed relation: $P = 6 \cdot S$. A lot of explanations for this error have been proposed. Already Clement et al. (1981) investigated the possibility of a syntactical transformation of the sentence into an equation. However, MacGregor & Stacey (1993) found that even relations presented in pictures can lead to the error. The error also occurs with additive relations.

We came up with another idea of an explanation of the reversal error within a test on this error. Students should write down an equation that expresses that the river Rhine (length r km) is 200 km longer than the river Elbe (which is e km long). The reversal error, $r + 200 = e$, can be explained if the expression $r + 200$ is not interpreted as a summation, but as “ r is 200 more than another quantity”. Moreover, this view explains a variant of the error that we observed in our studies: There was also the (wrong) answer “ $e - 200 = r + 200$ ”. While this error version resists explanations by syntactical translation or other approaches in the literature, it can be seen as an *attribute error*: The Elbe has the attribute (property) of being 200 km shorter ($-$) while ($=$) the Rhine has the property of being 200 km longer ($+$). This supports the thesis of MacGregor & Stacey (1993), according to which the equals sign is not necessarily understood as a numerical equality, but as a sign for a comparison of a different kind.

Attributes are quite common in mathematics. E.g., arrows are attributes to declare that \vec{v} is a vector, or the plus sign in \mathbb{R}^+ denotes positive numbers. Confusion with operations is likely because some operations look very similar to attributes. E.g., the complex conjugate \bar{z} of a complex number z is an operation that maps one number to another, yet it looks similar to the vector attribute. The absolute value $|x|$ is an operation that maps \mathbb{R} to \mathbb{R}_0^+ , but it may be misunderstood as giving x the attribute of being non-negative. Similarly, in “ $-x$ ” the minus sign should be understood as an operation (namely the opposite of x) but may be misinterpreted as a negative number.

We name this type of error *attribute error*. Students may look at e.g., $x + 5$ as a declaration that x is 5 more than some reference quantity. This attitude is supported by textbooks that contain tasks like this: “Write in symbols: x is increased by 5“. The students are then expected to write $x + 5$ which might be understood as changing the value of x by 5 or as statement that x is larger by 5 than some reference.

This discussion leads to the hypothesis that at least some reversal errors might result from an underlying attribute error and hence there should be a correlation between their occurrences. The attribute error was to the best of our knowledge first discussed in Oldenburg & Henz (2015). This present paper investigates the hypothesis that BMMs of equations, the reversal and the attribute error are correlated. We try to answer the following research questions:

- Is there a relation between the *attribute error* and the *reversal error*?
- Is there a relation between the *Relational BMM* and the *Object-BMM*?
- Are there relations between these error types and the BMMs?

THE TEST

To answer the research question, we use data for a subset of the items of an algebra test. The whole test takes a broader view on algebraic competence and includes e.g., items on substitution and on simplifying expressions. In this subtest we analyse measures of four scales, two on basic mental models and two on the error types described above:

- RevErr: 4 items about the reversal error
- AtrErr: 5 items about the attribute error
- RelBMM: 5 items about the Relational BMM
- ObjBMM: 7 items about the Object-BMM

The *Relational BMM* was measured e.g., by the following items (translated versions):

- It is known that $r = s + t$ and $r + t + s = 30 + 2x$. Determine r .
- In Phantasia you don't measure the temperature in Celsius. Our temperature 0°C corresponds to 10° and 100°C corresponds to 50° . Give a formula for the conversion from Celsius temperature T to fantasy temperature P .

The items for the *Object-BMM* require to look at equations as a whole, e.g.,

- A solution of $(x + 1)^3 + x = 349$ is given by $x = 6$. Use this knowledge to find a solution of $(5x + 1)^3 + 5x = 349$. (from Küchemann, 1979)
- Solve the equation $x^2 + 2x + 1 = 0$.

Most items of these two scales haven been graded on partial credit scale with 0 points for a wrong answer, 1 point for a partially correct answer and 2 points for a fully correct answer. Some easier items have been graded only by 0 (wrong) or 1 point (correct).

To measure the *reversal error* four items have been used. For each item an equation had to be set up. Three of the equations are of additive type (such as the Rhine-Elbe-example above), the last one is multiplicative:

- At a school, there are 20 times as many students as teachers. Let s stand for the number of teachers and S for the number of students. Write this as an equation.

For each of the four tasks a score was given to measure competence in avoiding the error: -1 for explicitly writing the erroneous version of the equation, 0 für writing nothing interpretable, 1 for writing almost the correct equation and 2 für the correct equation. The scale made of these 4 items is called RevErr. Note that the scale is oriented such that high values indicate a high competence in avoiding the error.

The items to measure the competence to avoid the attribute error were all of the following form: An expression was given and students should judge whether a given verbal statement expresses the same information. Some examples:

- x may be any real number. Is it true that $-x$ is negative?
- If x is any real number. Is $|x - 1|$ the same as $+x + 1$?
- Does $|x - 1|$ mean that $x - 1$ is not negative?

As with RevErr the scale AtrErr is oriented so that high values indicate high performance, i.e., few errors of that type.

The test has been completed by 123 teacher students from two second year courses. Participation was not mandatory and no further information (such as age, sex) has been recorded to avoid privacy issues. The students had not had lessons on algebra education before, but they had studied some mathematics at university level. This explains that the solution rate for many items is quite good. For example, only 51 reversal errors were committed (each of 123 students had 4 tasks, i.e., rate 10,3%). However, 42% marked falsely as correct that $|x - 1|$ means that $x - 1$ is nonnegative.

RESULTS AND FIRST INTERPRETATIONS

The written test results were coded and analysed with the R statistical program.

The internal consistency of the scales was assessed by means of the Cronbach alpha coefficient. Results show good values for all scales: RevErr: 0.74, AtrErr: 0.83, RelBMM: 0.79, ObjBMM: 0.86.

The four scales all correlate positively, as shown in table 1. All coefficients are significant (correlation test with Kendall's correlation).

Table 1: Correlation matrix of the four scales.

Correlation	RevErr	AtrErr	RelBMM	ObjBMM
RevErr	1	0.317**	0.213*	0.264**
AtrErr		1	0.525***	0.583***
RelBMM			1	0.686***
ObjBMM				1

First, there is a strong correlation between the two BMM scales. Either of these BMMs correlates strongly and highly significantly with a higher attribute error avoidance competence. On the other hand, the reversal error avoidance competence only correlates moderately, although significantly with the BMMs.

The strong correlations of the AtrErr scale with the other scales can also be confirmed by a linear regression model $AtrErr \sim RelBMM + ObjBMM + RevErr$. The standardized beta coefficients are $AtrErr \sim 0.135RelBMM + 0.153ObjBMM + 0.106RevErr$ and all three are significant, the coefficient of ObjBMM is even highly significant. Of interest is also a model that predicts AtrErr simply from the basic mental models, i.e., $AtrErr \sim RelBMM + ObjBMM$. Here the standardized coefficients are $AtrErr \sim 0.141RelBMM + 0.168ObjBMM$, and both are significant.

However, in the other regression $RevErr \sim RelBMM + ObjBMM + AtrErr$ only the last coefficient is significant: $RevErr \sim 0.002RelBMM + 0.075ObjBMM + 0.386AtrErr$. Similarly, if one just wants to predict RevErr by means of the BMMs: The regression equation $RevErr \sim RelBMM + ObjBMM$ is fitted to $RevErr \sim 0.056RelBMM + 0.140ObjBMM$ and this is not significant ($p \approx 0.06$).

The fact that the two types of errors are connected can also be demonstrated by comparing the group consisting of those students that made no reversal error, and the remaining ones. The mean of AtrErr for the first group is 3.34, while for the second group is only 2.50, which is a significant difference by the Wilcoxon test with $p=0.017$ and an effect size of Cohen $d=0.33$. In a complementary decomposition two groups were defined by scoring in AtrErr below resp. above average. The RevErr score shows a highly significant group difference, with $p=0.001$ and an effect size of $d=0.40$.

The relevance of the acquisition of BMMs for not committing errors can also be seen when looking at the sum scales $BMM := RelBMM + ObjBMM$ and $Err := AtrErr + RevErr$. They correlate with 0.48.

To shed further light on these relations a statistical implicative analysis (Gras et al., 2008) gave the following implications sorted by implicative intensity φ :

- | | |
|--------------------------------------|--------------------------------------|
| 1) ObjBMM \rightarrow AtrErr 0.997 | 2) RelBMM \rightarrow AtrErr 0.996 |
| 3) RelBMM \rightarrow ObjBMM 0.992 | 4) ObjBMM \rightarrow RelBMM 0.98 |
| 5) AtrErr \rightarrow ObjBMM 0.948 | 6) ObjBMM \rightarrow RevErr 0.943 |

- | | |
|---------------------------------------|---------------------------------------|
| 7) AtrErr \rightarrow RevErr 0.941 | 8) RelBMM \rightarrow RevErr 0.909 |
| 9) AtrErr \rightarrow RelBMM 0.902 | 10) RevErr \rightarrow AtrErr 0.744 |
| 11) RevErr \rightarrow ObjBMM 0.651 | 12) RevErr \rightarrow RelBMM 0.609 |

First, look at 7) and 10). The difference of the implication weight shows that 7) is more important, so mastering the attribute obstacle predicts a good performance on reversal errors tasks but not vice versa. This gives support to the hypothesis that attribute error misconceptions may underly many occurrences of the reversal error. Implications 3) and 4) simply reflect the high correlation between ObjBMM and RelBMM and show that there is no particular direction on their mutual relation.

Implications 1) and 2) show that high BMMs predict good performance in attribute error tasks. It is instructive to interpret this from the logical contraposition: The implication $A \Rightarrow B$ is logically equivalent to $\neg B \Rightarrow \neg A$. Hence, one may read 1) and 2) as expressing that mastering attribute error tasks may be a necessary (in the statistical sense) requisite for high BMMs. However, the opposite implications 5) and 9) have high implicative intensities as well so that the directional effect is not very strong.

INTERPRETATIONS

Both basic mental models considered in this paper correlate highly but still can be clearly separated. Concerning the two error types, regressions, correlations, group comparisons and implications indicate that they are related so that the first research question can be answered affirmatively. Moreover, results show that having especially a distinct *Object-BMM* indicates a strong resistance against these errors. Overall, the test results indicate that putting more emphasis in developing BMMs may be beneficial for avoiding the reversal and the attribute errors.

The statistical implicative analysis given above sheds some further light on directional effects between these scales. This may give hints (but not proofs) on possible causal connections. When interpreting the above numbers on the relevance of *Object-BMM* one should have in mind that this BMM is usually considered to be the most advanced form of understanding equations and this may explain its importance.

CONCLUSIONS

The study presented here underpins that BMMs of equations are an important part of understanding equations, which means here the ability of formulating and interpreting equations. They are positively correlated with the avoidance of the reversal error and the attribute error. This especially means that developing the BMMs of equations is a good strategy not only for understanding equations but also for avoiding errors like the reversal and attribute error. The fact that even teacher students at university show considerable difficulties with these tasks further supports the suggestion to address the semantics of equations more deeply by building up these BMMs. A first step is the early development of the relational view already in primary school, e.g., with examples like $39 + 17 = 40 + 16$. Learning steps for this development can be found in Stacey

(2011). In the following years in lower secondary school the *Relational BMM* has to be developed on manifold representations especially on the enactive and iconic level, also integrating environmental situations. Moreover, the meaning of the *Object-BMM* has been underestimated in relation to understanding and interpreting equations already in lower algebra, starting with very basic types of equations. However, a learning strategy for the development of this BMM has still to be constructed.

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MOTIVATIONAL AND EMOTIONAL ENGAGEMENT MEDIATES THE EFFECT OF FEATURES OF EDUCATIONAL TECHNOLOGY IN MATHEMATICS CLASSROOMS

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Potentials of digital tools compared to paper-based learning environments are, e.g., using congruent gestures on realistic models—which can be beneficial for learning outcomes. We argue that this effect is mediated by an increase in motivational-emotional engagement. To test this hypothesis, we conducted a randomized controlled trial with $N=292$ students (6th grade). In an experimental group, students developed the ‘part of many wholes’ concept in a digital learning environment which made use of congruent gestures; in a control group, students worked with the same material paper-based. We assessed students’ knowledge of the part-whole concept before and after the intervention, and asked about their motivational-emotional engagement during the lesson. The results of the mediation analysis are in line with our hypothesis.

INTRODUCTION

The demanding task of learning fraction concepts

As learning fractions is challenging for students (Lortie-Forgues et al., 2015), but essential for mathematical thinking in general (Bailey et al., 2012; Siegler et al., 2012), it is a stated goal of mathematics education research to support students in developing a thorough grounding of fractions by building conceptual knowledge. In this context, the need for conceptual change plays a major role in the expansion of the number domain from natural numbers to rational numbers—here, fractions—as the concepts anchored in learners’ minds are for the most part no longer suitable and in some cases even hindering (Vamvakoussi & Vosniadou, 2004). These necessary conceptual changes can be supported by building new, plausible concepts (Reinhold et al., 2020).

One key concept students have to acquire very early is the part-whole concept (Behr et al., 1983). In this concept, the idea of the ‘part of one whole’ and the ‘part of many wholes’ can be distinguished—where in the latter the new whole is formed from several wholes based on equal sharing (Streefland, 1991). To do this, several individual elements must first be grouped into a whole in order to structure this new whole into equal parts and to consider the required proportion of these parts. This is a relevant step in students’ development of fraction concepts, but one hurdle for learners is the realization that both subconcepts—the part of *many* wholes and the part of *one* whole—do describe an equivalent fraction (e.g., $1/4$ of 3 circles is the same as $3/4$ of one circle). A relevant question is how students can be supported during learning.

Features of educational technology supporting cognition and motivation

One approach that has proven successful to support students' acquisition of mathematical concepts is utilizing specific features of educational technology (Hillmayr et al., 2020)—whereby the specific cause and effect mechanisms have not yet been fully clarified.

On a cognitive side, particularly appropriate gestures, such as self-performed suitable hand movements, can serve to facilitate the learning of complex mathematical content, according to the embodied cognition theory (Wilson, 2002). More specifically, building up the 'part of many wholes' concept may be supported by fitting the gestures used to the to-be-developed mental model (here: equal sharing by cutting and distributing pizzas with fingers). This is one argument why the use of touchscreens in suitable simulation-based learning environments can be beneficial for learning.

However, meta-analyses suggest that the use of digital media may not only positively influence cognitive but also non-cognitive aspects, such as attitudes towards the subject matter, or motivation, which, however, depends on their implementation (Higgins et al., 2019); this is especially true for learning fractions (Reinhold et al., 2021). In particular, a positive and learning-promoting effect of digital simulation-based learning environments through 'representational scaffolding' is assumed (Fischer et al., 2022).

Theoretical modelling and empirical clarification of the relationship between cognitive and motivational-emotional effects of digital learning environments in mathematics is a central research desideratum, which this study aims to contribute to.

The mediating role of student classroom engagement

Based on these considerations and previous empirical results, we propose an explanatory approach in which features of digital learning environments first have a motivating effect in the specific learning situation, and this positively influences students' utilization of the learning opportunities—which results in an indirect learning-promoting effect of digital tools.

The theoretical foundation for this model (see Fig. 1) is a synthesis of (1) the general model for determinants and course of motivated action as product of person and situation (Heckhausen & Heckhausen, 2018), (2) the utilization-of-learning-opportunities model (Seidel, 2014) and (3) the concept of student classroom engagement as a multifaceted process variable (Fredricks et al., 2004).

By motivational-emotional engagement we refer to learners' affective reactions in the classroom, such as interest, boredom, happiness, sadness, and fear (Pekrun, 2006; Fredricks et al., 2004). In the present study, we consider motivational-emotional engagement a latent construct consisting of situational interest (i.e., the interest in the learning object, Hidi & Renninger, 2006), contextualized motivation (i.e., the pure enjoyment of the learning object), and subjective effort (i.e., the extent to which learners consider themselves actively participating in the learning process, Krapp, 2005; Ryan & Deci, 2000).

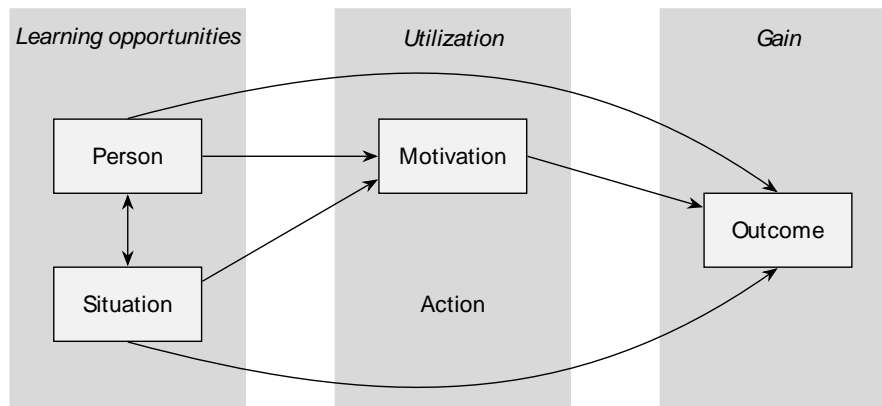


Figure 1. Effect of digitally supported mathematics teaching mediated by a change in motivated action—here with focus on motivational-emotional engagement.

Although learning motivation is usually described as extrinsic, since it aims at specific learning outcomes, it can become intrinsic through learning out of interest, and thus results in higher learning performance, especially with regard to conceptual learning and depth-oriented learning strategies (Krapp, 2005). Moreover, it is assumed that when learners actively engage—i.e., show effort—with learning materials due to situational interest or intrinsic motivation, they are better able to coherently integrate them into mental representations (Moreno & Mayer, 2007), which shows a positive influence on learning outcomes (Pekrun, 2006).

Regarding that, we suggest that higher motivational-emotional engagement (due to features in digital learning environments) can lead to higher behavioral and cognitive engagement (Fredricks et al., 2004), which then can cause higher learning outcomes.

The present study

In the present study, we investigated whether the relationship between the effect of features in digital learning environments on fractions and learning outcomes is mediated by an increase in motivation. We hypothesize that interactive manipulations in digital simulation-based learning environments (using congruent gestures on realistic models) in the exploratory phase of mathematics lessons positively affect contextualized motivation, situational interest, and subjective effort—and that these correlate positively with learning gains. The study design followed the explanatory approach for the effect of digitally supported mathematics teaching through a change in motivated action (Fig. 1), whereby the digital learning environment represents the situation in comparison to a paper-based learning environment, and the learning gain represents the outcome.

METHOD

To answer this, a 90-minute randomized controlled trial was conducted. In the present curriculum, fractions are covered for the first time in the sixth grade, which assumes that learners have not been exposed to fraction learning in school lessons before and have little to no substantial prior knowledge of fraction concepts.

Sample

A total of $N = 292$ sixth graders from German schools in Baden-Württemberg participated in the study. For the intervention, the students from $k = 13$ classrooms were each randomly assigned to either the control condition ($n = 141$, paper-based learning environment) or the experimental condition ($n = 151$, digital simulation-based learning environment).

Material and instruments

As the digital simulation-based learning environment we used an electronic textbook on fractions that focuses on conceptual knowledge building and has already been shown to be effective in authentic learning scenarios in a large cluster randomized controlled trial (Reinhold et al., 2020). The learners approach the ‘part of many wholes’ concept vividly and realistically by means of ‘distributing pizzas’ in an exploration task with specific stimuli (Streefland, 1991). With reference to embodied cognition theory the intuitively designed tasks are to be worked on by means of finger movements: pizzas are cut by a dragging motion of the finger from the edge of the pizza, automatically presenting a knife. The distribution of the individual pizza pieces follows the same principle: a hand appears as soon as the finger is held on the respective piece and disappears as soon as it is moved away from the touch screen (Fig. 2). The students are asked to give a written statement on how they solved the task; they can switch to the next task in the exploration phase in their own pace.



Figure 2. Cutting and distributing pieces of pizza in the digital simulation-based learning environment

Motivational-emotional engagement was operationalized through contextualized motivation, situational interest, and subjective effort via self-reports in a questionnaire with five items each on four-point Likert scales. Actual self-assessment was ensured using inverted control variables:

- Contextualized motivation: I found today's math lesson exciting. Today's math lesson was fun. Time flew by in today's math lesson. I was motivated to learn in today's math lesson. I found today's math lesson boring.
- Situational interest: I found today's math lesson interesting. Today's math lesson made me curious about fractions. Today's math lesson fascinated me. I am not

interested in what I learned in today's math lesson. Now I really want to learn more about fractions.

- Subjective Effort: In today's math lesson I tried very hard. In today's math lesson, I tried to shirk. In today's math lesson, my mind was elsewhere. In today's math lesson I didn't care about anything. In today's math lesson I participated a lot.

To assess learning outcomes, relevant prior knowledge about fraction concepts was assessed before the intervention by means of a pretest, and content-specific fraction knowledge on the 'part of many wholes' concept was measured after the intervention with a posttest. All instruments were piloted in a pilot study with $N = 43$ students.

Procedure

Informed consent was obtained from school administration (Regional council of Freiburg, Department 7 'school and education', reference number 7-6499.2), students, and parents prior to the study.

In each classroom learners were randomly allocated to experimental and control condition through face-down card drawing. The experimental group worked with the digital simulation-based learning environment during the exploration phase and the control group worked with the same material in the form of a paper-based version: Both lessons were based on an identically developed workbook, which only differed in the experimental manipulation, i.e., the digitalization of the individual exploratory phase of the mathematics lesson on the 'part of many wholes' concept. In the control condition, the above-described task is given in paper-based format, requiring students to draw the results of their equal sharing process. In both conditions, (1) researchers conducted the lesson after this exploration phase, (2) results of the exploration phase were written down, and (3) individual practice tasks were administered paper-based.

The time frame of the individual successive instructional phases was determined in advance between the study leaders to create reliability and prevent falsification of the measurement results.

Statistical evaluation

To compare the mean values of the two experimental conditions, we used independent samples t -tests. To test our mediation hypothesis, we estimated a structure equation model both without and with the relevant indirect effect in R with *lavaan* using Bollen-Stine bootstrapping method with a sample of 1,000 and standardized coefficients.

RESULTS

Descriptive Results

The results support our hypotheses. A significant difference in favor of the experimental group was found regarding Contextualized motivation, $t(287) = -3.76$, $p < .001$, Situational interest, $t(286) = -2.75$, $p < 0.01$, and Subjective effort $t(284) = -2.11$, $p < .01$ (Tab. 2).

	α	Control		Experimental		t	p	d
		M	SD	M	SD			
Pretest	0.76	5.73	3.99	6.17	3.79	0.97	0.33	0.11
Contextualized Motivation	0.89	2.91	0.69	3.22	0.72	3.76	< 0.001	0.44
Situational Interest	0.85	2.66	0.68	2.89	0.71	2.75	< 0.01	0.33
Subjective Effort	0.56	3.01	0.55	3.14	0.52	2.11	< 0.01	0.25
Posttest	0.75	5.76	4.75	6.55	4.90	1.40	0.16	0.16

Table 1. Descriptive results and independent sample t -test.

Mediation Analysis

We asked whether the effect between features in digital learning environments on fractions and learning outcomes is mediated by motivational-emotional engagement. Posttest outcomes in the experimental group (working with the digital learning environment on iPads) should be higher than in the control group (working paper based) because of an increase in state motivation. Our mediation analysis supported this hypothesis. The hypothesized structure equation model (given in figure 2) showed a reasonably good fit to the data, $X^2(3) = 3.98$, $p = 0.26$, $CFI = 0.99$, $TLI = 0.99$, $RMSEA = 0.03$, 90 % CI = [0.00, 0.11]. In this model, we estimated the latent variable Motivational-emotional engagement via Contextualized Motivation, Situational interest, $\beta = 0.73$, 95% CI = [0.44, 1.03], and Subjective effort, $\beta = 0.74$, 95% CI = [0.38, 1.14].

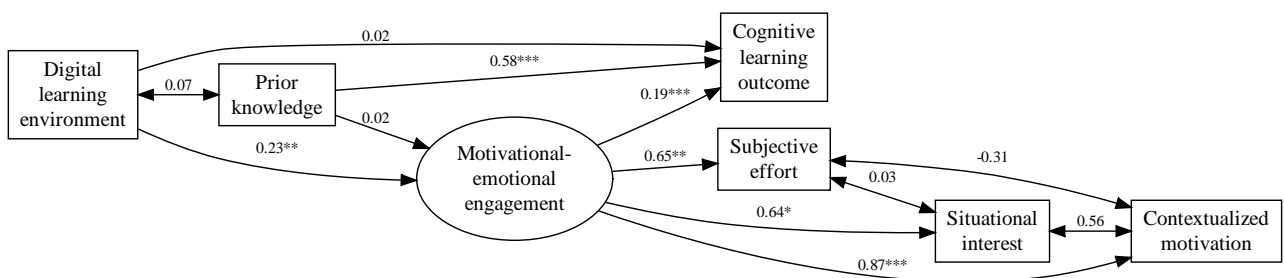


Figure 3. Structure equation model to test the hypothesized mediation of the effect of the digital learning environment on the cognitive learning outcome regarding the part of the whole concept via the latent construct motivational-emotional engagement.

For the posttest outcome on the ‘part of many wholes’ concept of fractions as dependent variable, and including the intervention group as independent variable, the mediator analysis showed no significant direct effect of features in digital learning environments (i.e., the intervention group) on posttest performance, $\beta = 0.17$, 95% CI = [-0.72, 0.98]. The intervention group had a significant positive effect on the mediator motivational-emotional engagement, $\beta = 0.41$, 95% CI = [0.16, 0.61], and

motivational-emotional engagement had a significant positive effect on posttest achievement, $\beta = 1.05$, 95% CI = [0.59, 1.38]. In line with our hypothesis, we found a significant indirect effect, $\beta = 0.43$, 95% CI = [0.15, 0.71]. For all other path estimates, consider Figure 3.

DISCUSSION

The goal of the present study was to find out whether interactive manipulations in digital simulation-based learning environments in the exploratory phase of mathematics lessons positively affect contextualized motivation, situational interest and subjective effort—and that these correlate positively with learning gains.

Motivational-emotional engagement showed significantly better results on average across all scales in the experimental group with the digital simulation-based learning environment. On a descriptive level, the experimental group did show better posttest outcomes—yet, this effect was non-significant, which can be attributed to the fact that only the exploratory phase was instructed digital, and the practice phase was paper-based, as in the control group.

The results indicate that engagement plays a key role in learning performance and thus underline the importance of supportive features in digital learning environments, especially in learning domains that, like fractions, are a hurdle for students.

It remains to be clarified *how* engagement affects learning. To this regard, we made the hypothesis that a higher motivational-emotional engagement leads to a higher cognitive and behavioral engagement (cf., Fig. 1), i.e., more utilized real learning time, and thus to greater learning success. For this purpose, we have already collected process data, such as students' descriptions of their approaches, their problem-solving time, and their word count, which will be evaluated in a next step.

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HOW DOES MATHEMATICAL KNOWLEDGE FOR UNDERGRADUATE TUTORING DEVELOP? ANALYSING WRITTEN REFLECTIONS OF NOVICE TUTORS

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Research has become interested in the mathematical knowledge that undergraduate tutoring involves. This study introduces a mechanism that describes how this knowledge can develop through the work of tutoring. The mechanism emerged from an analysis of 24 reflections written by 10 novice tutors on noticeable incidents that took place in their tutorials. The tutors were undergraduate students at advanced stages of their mathematics degrees, and their teaching unfolded as part of an elective course in mathematics education. The mechanism proposes that tutors can find themselves in contingent situations, where their mathematical knowledge is insufficient. To fulfill the emerging pedagogical need, tutors initiated reflexive actions of mathematics learning to prepare for similar contingent situations in their future tutoring.

RATIONALE AND BACKGROUND

Over the last two decades, research in undergraduate mathematics education has become interested in tutors (also referred to as “teaching assistants” and “graduate student instructors”). In many colleges and universities worldwide, tutors are employed by the mathematics departments to contribute to their instruction. The scope of the tutor roles vary from one tertiary context to another, ranging from working in drop-in mathematics support centers to leading regular problem-solving sessions for smaller groups of students enrolled in a course (e.g., Speer et al., 2005). In many countries, tutor-student interactions play a key role in undergraduate mathematics education, meaning that the way in which the former teach can impact how the latter learn (e.g., Kontorovich & Ovadiya, accepted). This raises questions about what mathematical knowledge tutoring involves and how this knowledge develops.

Research into these questions is in its infancy. The emerging findings indicate that undergraduate tutoring “requires mathematical knowledge beyond content knowledge of the course” (Johns & Burks, 2022, p. 2). John and Burks (2022) show that tutors employ variations of the types of knowledge that are familiar from research in teacher education (e.g., in the terms of Ball et al., 2008, knowledge of content, curriculum, and students). But, unlike school teachers, tutors have rarely completed extended educational programs to prepare for the work of teaching. Indeed, tutor training is typically confined to several workshops that are often independent of the disciplinary subject matter (e.g., Speer et al., 2005). Yet, John and Burks (2022) demonstrate that tutors with only a few years of experience can hold impressive *mathematical knowledge for tutoring* (*MKTut* hereafter). So, how does this knowledge come about?

One possible answer is that MKTut develops through the work of tutoring. Research provides evidence of school teachers and mathematics educators growing new insights and refining their mathematical knowledge via teaching (e.g., Leikin & Zazkis, 2010). Occasional references to learning through teaching feature in the self-reflections of experienced university lecturers (e.g., Kontorovich, 2021). By analogy, it is reasonable to propose that tutors can learn through teaching as well.

We aim to explore how MKTut develops through the work of tutoring. In this study, we focus on a sub-domain of this knowledge: *specialized content knowledge*—not just “common” knowledge of mathematical facts, but knowledge that enables nimbleness of ideas and practices that are distinctive for mathematical teaching (Ball et al., 2008). To address the aim, we scrutinize written reflections composed by novice tutors about their tutoring of first-year mathematics courses. The use of systematic reflection to investigate and promote professional knowledge for teaching is consistent with research on tutors (e.g., Speer et al., 2005) and school teachers (e.g., Mason, 2002).

THEORETICAL FRAMEWORK

John and Burks (2022) argue that only some aspects of existing frameworks of mathematical knowledge for school teaching are relevant to undergraduate tutoring. Yet, given the developmental stage of this area, we build on the existing frameworks.

Mason (2002) maintains that attention and noticing lie at the heart of all professional practice, teaching included. He conceives attention as a complicated human mechanism, in which noticing is responsible for distinguishing some things from their surroundings and getting them through to the level of awareness. Without this awareness, it is impossible to act on these things, that is, to react to them.

Mason (2002) argues that people notice things insofar as they are unexpected, i.e. contingent. Contingency is a dimension in the Knowledge Quartet—a theory concerning the mathematical knowledge that teachers apply in a classroom (Rowland et al., 2015). The dimension concerns situations where a teacher encounters an unanticipated event and is challenged to deviate from their agenda. In Rowland et al.’s (2015) study on elementary classrooms, students’ contributions to the lesson constituted the majority of contingencies. These included instances where students provided surprising answers to a question and spontaneous reactions to an activity. The researchers show that the teacher’s response to the contingency can be of three kinds: to ignore, to acknowledge but put aside, and to acknowledge and incorporate.

According to Leikin and Zazkis (2010), contingent situations in secondary-school classrooms can lead teachers to develop new mathematical ideas. Considering teaching as a partially improvised activity (Rowland et al., 2015), we propose that contingencies can take place in university tutorials as well.

Noticing can stem from a disturbance, for instance, when a teacher experiences a need for a certain piece of professional knowledge. Harel (2008) uses intellectual need to refer to circumstances where “disciplinary knowledge [is] born out of people’s current

knowledge through engagement in problematic situations conceived as such by them” (p. 898). In these cases, one’s existing state of knowledge is insufficient or inadequate, and additional knowledge must be acquired to reach an equilibrium.

Stylianides and Stylianides (2022) reframe the construct of intellectual need for the case of prospective teachers. In the context of proof teaching and learning, the researchers use a pedagogical need to capture teacher’s readiness to develop conceptualizations of proof that are new to them. We see no reason to confine this reframing to a particular mathematical topic. Accordingly, with a pedagogical need, we refer to a teacher’s openness to develop a new piece of mathematical knowledge for teaching. Indeed, Stylianides and Stylianides (2022) stress that pedagogical needs are linked to the teacher’s perceptions of how relevant the new piece of knowledge is to their teaching context.

Noticing can be developed through disciplined reflection on professional experiences (Mason, 2002). Disciplined reflection should not be confused with Schön’s (1987) reflection-on-action—an umbrella term that includes “anything from vaguely thinking back over what happened, to [...] calling upon theories to explain and justify [it]” (Mason, 2002, p. 15). We use the term reflexion to stress the disciplined aspect of one’s reflection. This includes careful documentation of an incident, while aspiring to avoid judgements and implicit assumptions, and successive introspection of the incident with a deep inward gaze. Mason argues that such monitoring of the incidents of the past can prepare teachers to reflect-through-action, that is, to become aware of, and prepared to, modify their practice in the midst of that practice.

METHOD

Our data came from “Mathematics learning through teaching”—a mathematics education course (*MathEd* hereafter) that was offered in the mathematics department at a New Zealand university. The course was not required by any particular program. It mostly attracted undergraduates in the last semesters of their mathematics majors, and who were interested in educational issues.

The central activity of the MathEd course was tutoring in “bridging” (pre-academic) non-credit courses and first-year courses for non-mathematics majors. The MathEd students (*tutors* hereafter) were allocated to groups of up to 25 students, and they led, in pairs, ten one-hour tutorial sessions throughout a semester. The tutors were expected to assist the students with the course content, by supporting their autonomous work on sets of problems. The problems were provided by the course lecturers.

After each tutorial, the tutors were expected to submit a written reflection-on-action, where they accounted for a specific incident that had drawn their attention in the tutorial. This was part of their MathEd coursework. The reflection guidelines asked the tutors to provide a detailed description of an incident they considered significant and encouraged critical questioning of their in-the-moment actions. The tutors were also encouraged to formulate inferences that would be useful for their further teaching. Every week, selected reflections were shared and discussed in the MathEd course.

Over three semesters, we collected hundreds of reflections that focused on myriad of issues. At the first stage, we reviewed each of them to identify those that referred to tutors' learning of mathematics. The process converged to 24 reflections written by ten tutors. These reflections became our data corpus.

The reflections underwent inductive analysis, driven by the question, "how did the incident that the tutors noticed spur the development of their specialized content knowledge for tutoring (*SCKTut*, hereafter)?" We iteratively compared between tutors' reflections, while attending to the differences and similarities between the described incidents and tutors' actions. These comparisons gave rise to initial elements of a mechanism that conceptually connects between the tutorial incidents, the follow-up activity, and tutors' knowledge development. The emerging categories and conceptualizations were applied to the whole data corpus to ensure that they account for the key aspects that the tutors stressed in their reflections.

FINDINGS

We open with the presentation of the mechanism that emerged from the data analysis. Then, we illuminate some of its components with excerpts from a single reflection.

SCKtut development through tutoring: An overview

In accordance with our assumption, many tutors' reflections described contingent incidents that unfolded in tutorial classrooms. Somewhat similarly to Rowland et al. (2015), tutorial contingencies included surprising questions that the students asked and mathematical challenges that they faced when working on the assigned problems. The former pertained to situations where students experienced some intellectual need and turned to tutors with a request to fulfill it. Not all contingencies of the latter type involved the tutors directly. For instance, one of the tutors wrote,

I overheard a discussion in one of my groups where one of the students stated that "a line and a plane can be non-parallel and not intersect in 3 dimensions." This caught my attention because as far as I know [...] this was impossible. I was curious about this student's "non-parallel non-intersecting line and plane" so I asked him if he could elaborate further.

This quote illustrates that the tutors not only coped with contingencies that the students presented to them, but also chose to get involved in the contingencies that they noticed.

All reflections in the data corpus described situations where tutors found themselves in a pedagogical need for a certain piece of content or specialized content knowledge. In these situations, the tutors' state of mathematical knowledge was either insufficient or inadequate to handle the contingency "on their feet". For instance, a tutor could explain the problem solution or justify a particular move, but only with advanced mathematics that went beyond the scope of the particular tutorial. Such situations entailed a particular type of reflection-on-action (or reflection on a struggle to execute an appropriate instructional action, in this case), where the tutors autonomously pursued the development of their mathematical knowledge. Examples of this pursuit

included re-solving the focal problems after the tutorial, consulting with the relevant literature and online resources, and seeking assistance from mathematically versed others (e.g., other tutors, course lecturers). The mathematics in the focus of these actions was the “piece of the puzzle” that the tutors were missing in the contingency. We refer to this activity of the tutors as *reflexive actions of mathematics learning*. In all collected reflections, the tutors maintained that the learning actions resulted in the successful development of the target mathematics.

The learning actions exemplified above took place after the tutorials. However, some reflections referred to reflexive actions that the tutors managed to take “on the fly” to navigate the contingency as it unfolded. For instance, one tutor described his struggle to explain the transition from $\frac{1}{\frac{5x-1}{x}}$ to $\frac{x}{5x-1}$ in a first-semester course. The tutorial was dedicated to inverse functions, and the tutor assumed that the students will be fluent in fraction manipulation. When he “stopped to think”, one of the students suggested to represent the fraction as $1 \div \frac{5x-1}{x}$. The tutor acknowledged the idea and incorporated it in their solution to produce an elaborated explanation (cf. Rowland et al., 2015). In other reflections, the tutors described how they asked the second co-tutor to weigh in. In such cases, the peer tutor took charge and resolved the contingency. These reflections attest to a high level of reflection-through-action that the tutors demonstrated by being aware of classroom resources and using them in-the-moment to address their pedagogical needs. The reflections also depicted these reflexive actions as affording the tutors a chance to advance their mathematical knowledge.

A reflection on one learning journey

We use excerpts from a reflection of Ann (pseudonym), who tutored a “bridging” course. This reflection serves two purposes: (i) to show that a successful resolution of a contingency in a classroom can still entail reflexive development of SCKtut; and (ii) to introduce a new type of reflection that we discerned in tutors’ reflections.

After the first tutorial on the concept of functions, Ann submitted a reflection that revolved around the following problem: “For $f(x) = x^2 - 2$ evaluate: (a) $f(2)$; (b) $f(2 - x)$; (c) $f(x + h) - f(x)$ ”. This is how Annie described the focal incident:

During the tutorial, I had more than three students ask me how to solve (b) and (c). I tried to explain this by telling them that function is like a factory. The variable x is the input, and $x^2 - 2$ is the machine. But they told me they didn’t understand it at all. So I added more content to my explanation and said that this is a factory that makes apple pies, whatever [is] in the brackets is the apple we need to put in the machine to make an apple pie. So to solve (b) we just use $2 - x$ to replace x that in the function.

Ann wrote that “all students got it”, referring to her second explanation. The incident still made her reflect on the two explanations that she provided. In her words,

After the tutorial I was thinking about what’s wrong with my first explanation. [...] The key issue here is their failure to understand that ‘ $2 - x$ ’ here is a variable. Why my second

explanation made them understand it, maybe, because I told them that whatever is in the brackets is an apple. You don't need to think about how to deal with 2, just circle everything in the bracket and put them into formula and replace the x . In fact, I told them ' $2 - x$ ' is the input variable.

Ann wrote that she searched for the notion of variable in the mathematics encyclopedia and found out that,

'In elementary mathematics, a variable is an alphabetic character representing a number, called the value of the variable, which is either arbitrary, not fully specified, or unknown.' It means variable is not referring to x, y, z it just means not fully specified or unknown.

So I think next time if someone ask me about a similar problem. I will ask them to tell me what the meaning of variable. Is that mean x, y, z ? Can a, b, c be variable as well? I will ask them to think about this and refer back to the definition of function. And then I will use my weird apple pie example to help them understand the definition. And also I need to give them chance to tell me what they don't understand about the question and refer their issue back to definition.

In the MathEd lesson, Ann confirmed that the incident was contingent to her. Before the tutorial, she presumed that finding $f(2)$ will prepare the students for the remaining parts of the problem. In spite the contingency, Ann succeeded in presenting a general approach to the problem solution. She even managed to elaborate on it when the students sought additional explanations. In other words, Ann successfully reflected-through-action and satisfied the intellectual need that the students presented her with.

Ann's *resolution* of the contingency engendered a posteriori pedagogical need to understand "what's wrong with my first explanation" and what in the second explanation made students "get it". Drawing on her mathematical knowledge, Ann connected students' intellectual need and her "apple" metaphor to the concept of variable. She followed with further reflexive action of turning to the literature to clarify a formal concept definition. We note a qualitative difference between a more conceptual approach to variables that Ann presented in her reflection-on-action compared to a rather procedural explanation that she described as providing to her students (i.e., "we just use $2 - x$ to replace x "). This change illustrates how reflexive actions can lead a tutor to expand their mathematical knowledge.

In the last part of her reflection, Ann generates a series of questions she may use "if someone asks me a similar problem". Two observations can be made regarding these questions. First, they invite the asker to engage with the notions of variable and function on a conceptual level. Second, they initiate an exchange, providing the asker with opportunities to express, clarify, and reflect their current state of knowledge, while articulating their intellectual needs. The reflection suggested (and Ann later confirmed) that these aspects were not part of the incident that took place in her classroom.

We propose that Ann put her expanded SCKtut in use to *reflect-toward-action*, i.e. to sketch instructional actions that she could undertake when faced with a similar mathematical problem. Broadly speaking, reflection-toward-action prepares tutors for

future contingencies in which they may experience similar pedagogical needs. What is interesting, in Ann’s case, is that there is no evidence to suggest that she experienced a pedagogical need in the classroom. Indeed, she described this need arising “after the tutorial”. This suggests that tutors’ reflection-toward-action can be aimed not “just” at coping with similar contingencies, but at becoming ready to react to them in a more aware manner that is faithful to their newly developed insights. Reflection-toward-action featured in eleven reflections in our data corpus.

SUMMARY AND DISCUSSION

Figure 1 offers a visual summary of the proposed mechanism of SCKtut development through the work of tutoring. The mechanism suggests that contingencies that tutors face in their tutorials can engender a range of pedagogical needs. A sub-set of them may spur tutors to initiate reflexive actions of mathematics learning. Some actions are taken to address the contingency “on-the-fly”, while others unfold outside the tutorial walls. Notably, our data suggest that tutors can pursue learning-oriented actions even when they think that their reflection-through-action addressed the contingency successfully. Reflexive actions can include reflection-toward-action, in which tutors delineate instructional moves that they may take if similar situations arise in the future.

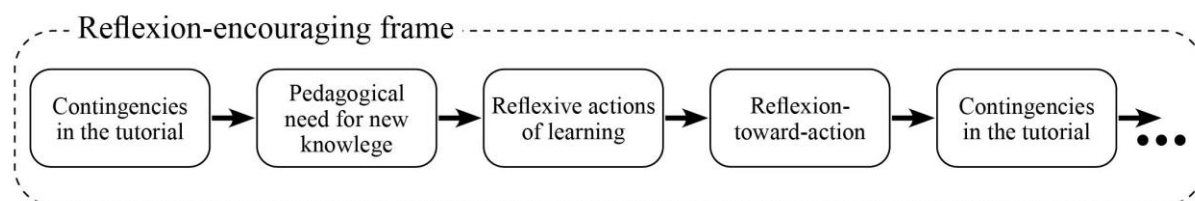


Figure 1: Mechanism of SCKtut development through tutoring

A *reflexion-encouraging frame* is a critical element that contextualizes the proposed mechanism. Indeed, we do not believe that the reflexions that our tutors composed were accidental. They emerged as a response to certain guidelines, and they were part of tutors’ coursework. The course attracted a particular student cohort that was led to conduct critical inquiry into mathematics and its education. Overall, the course posited that teaching is an endeavour through which tutors’ mathematical knowledge can develop. Within this multi-layered frame, tutors were expected to reflect-on-action in a disciplined manner, and deep introspective reflections were encouraged.

The focal mechanism contributes to research on undergraduate tutoring. The study offers evidence to suggest that tutors can advance their mathematics knowledge through tutoring. In this sense, tutors emerge, not unlike school teachers and teacher educators (e.g., Leikin & Zazkis, 2010). Notably, our tutors were taking their first teaching steps. They tutored first-year and “bridging” (pre-academic) courses, which many tutors initially labelled as “basic” and “easy”. This may suggest that a multi-layered reflection frame within which the tutors operated played a key role in their mathematics learning. That said, our findings emerged from self-reflections that the tutors produced as part of coursework. Thus, much more research is needed to understand the complexity of MKTut and its development in different contexts.

Let us consider the MathEd course where soon-to-be mathematics graduates turned into tutors. The course was led by scholars in the didactics of mathematics who were members of the mathematics department. This is not the only department where didacticians and mathematicians work side-by-side. Thus, we propose that MathEd courses offered as part of mathematics programs can provide a promising path for tutor training and for advancing the quality of undergraduate mathematics instruction.

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