The Vorticity Equation

Atmos 5110 Synoptic–Dynamic Meteorology I

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Suggested reading: Lackmann (2011), section 1.5

Holton and Hakim (2013), section 4.3

The vorticity equation allows us to understand the development and decay of circulation systems on all scales, including the large scale.

Derivation

- Take ∂/∂x of the meridional (v) momentum equation and ∂/∂y of the zonal (u) momentum equation (and ignore friction)

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]$$
$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right]$$

- Differentiate, subtract the bottom from the top and substitute $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}$ to obtain

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}$$
$$=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

- But

$$\frac{D\zeta}{Dt}=\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}$$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} = v\frac{\partial f}{\partial y}$$

- Therefore

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + \ v\frac{\partial f}{\partial y} = \frac{D(\zeta + f)}{DT}$$

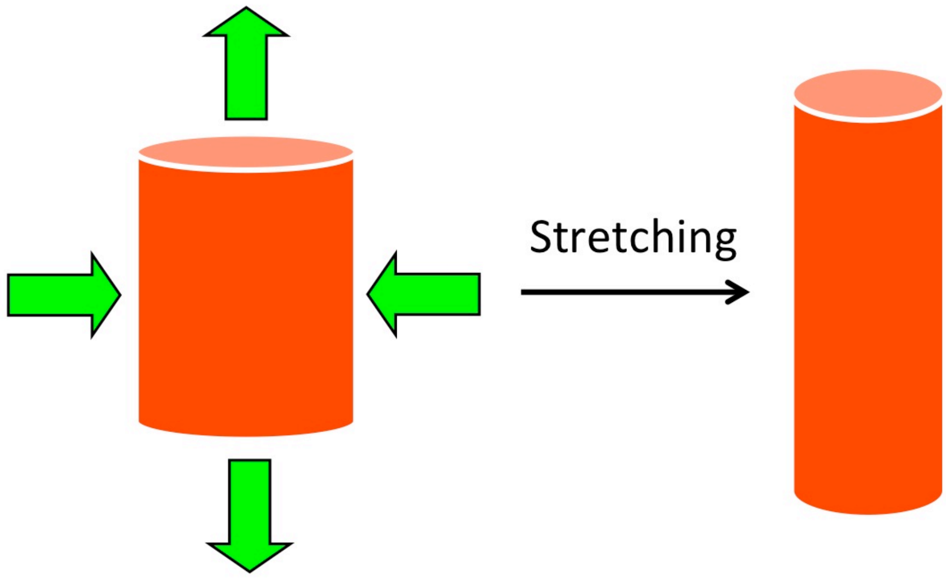
- Substituting and rearranging yields the vorticity equation

$$\underbrace{\frac{D(\zeta + f)}{Dt}}_{\text{Absolute vorticity tendency}} = \underbrace{-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\text{Divergence Term}} - \underbrace{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{\text{Tilting Term}} + \underbrace{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}_{\text{Solenoidal Term}}$$

Absolute vorticity tendency
Rate of change of the *vertical* component of absolute vorticity following the flow

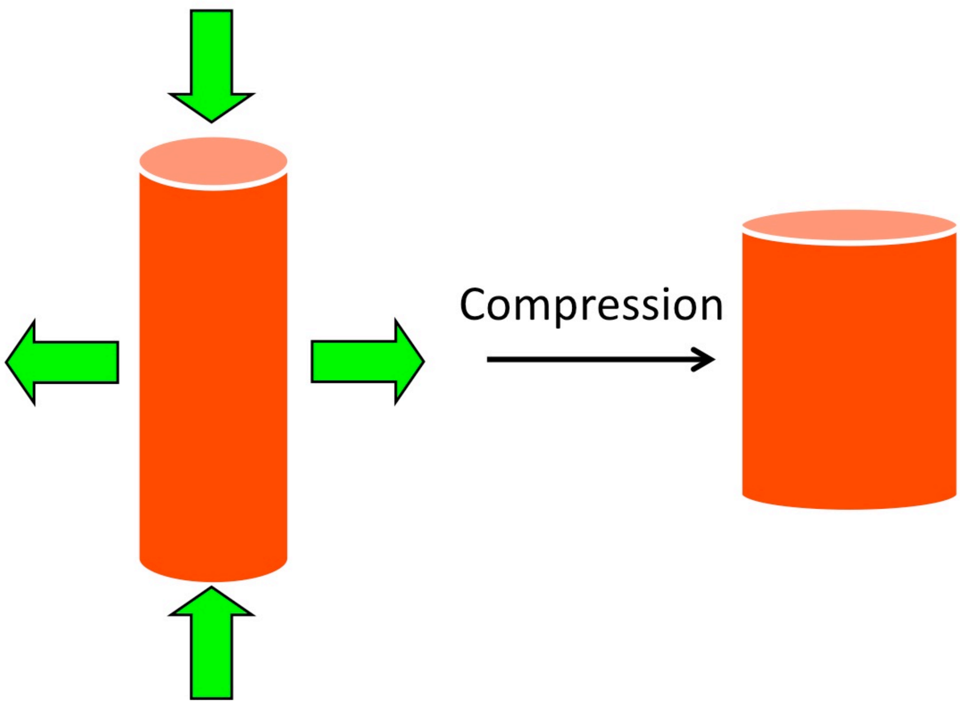
Physical interpretation: Divergence term

- Sometimes called the stretching term since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$ by continuity
- Represents the effects of divergence (compression) and convergence (stretching) on the absolute vorticity (analogy: the ice skater effect)
- Convergence acts to increase the magnitude of the absolute vorticity
    - If anticyclonic, the anticyclonic circulation strengthens
    - If cyclonic, the cyclonic circulation strengthens
    - Applies on smaller scales where $|\zeta| > |f|$ and the sign of the absolute vorticity is determined by ζ.
- Assuming $\zeta + f > 0$ (good assumption on large scale in Northern Hemisphere)
    - Divergence (vertical compression) decreases the absolute vorticity
    - Convergence (vertical stretching) increases the absolute vorticity



$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)<0 \Rightarrow -(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)>0 \Rightarrow \frac{D(\zeta+f)}{Dt}>0$$

$$\Rightarrow \zeta + f \ must\ increase$$



$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)>0 \Rightarrow -(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)<0 \Rightarrow \frac{D(\zeta+f)}{Dt}<0$$

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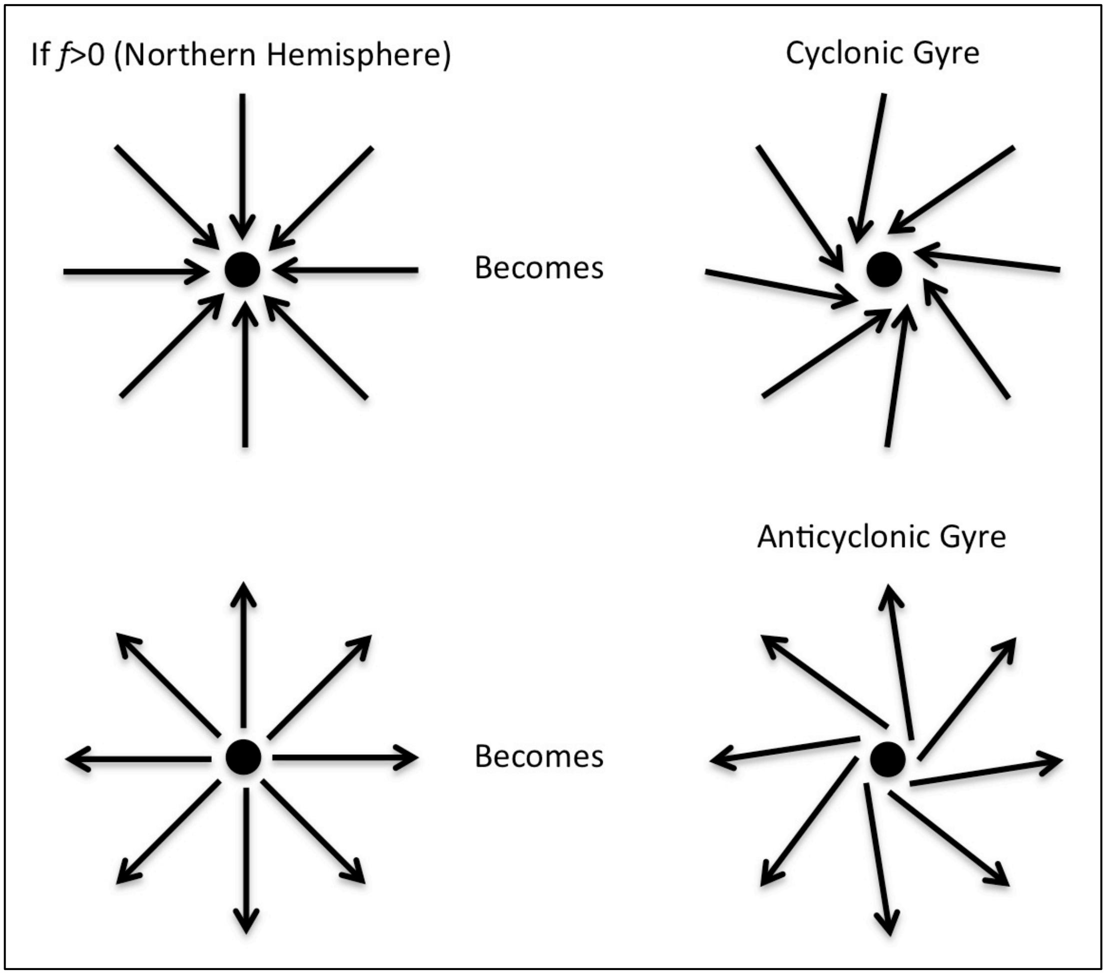
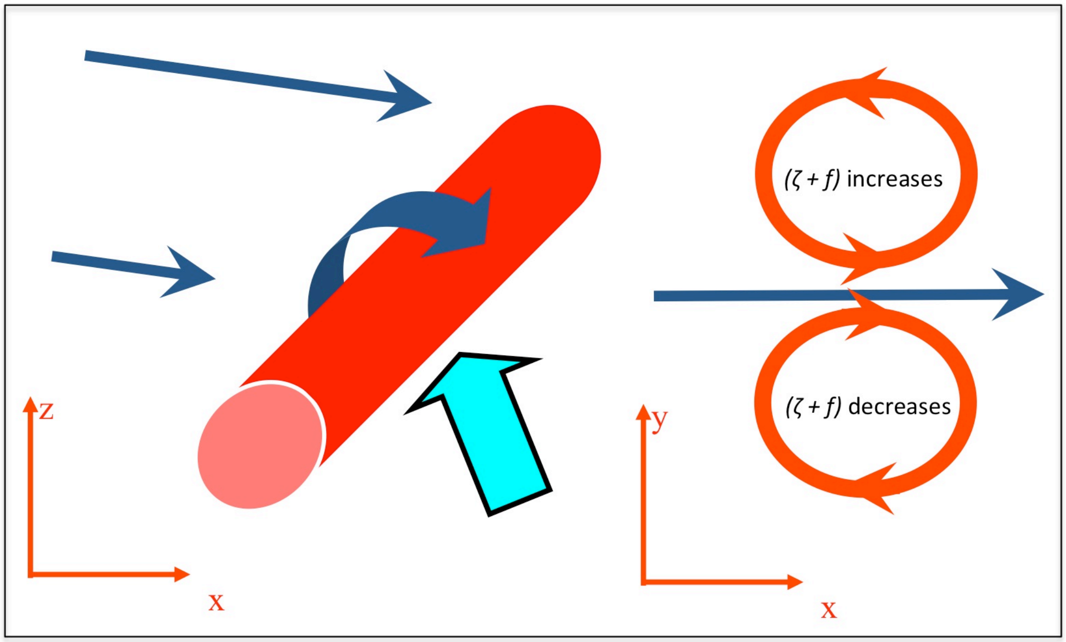


If f>0 (Northern Hemisphere)
Cyclonic Gyre
Becomes
Anticyclonic Gyre
Becomes

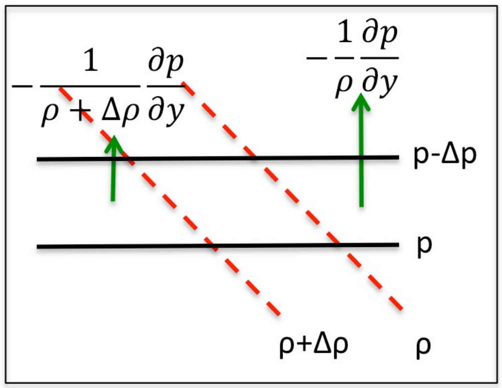
Physical interpretation: Tilting term

- Represents the tilting or twisting of horizontal vorticity into the vertical (or vertical vorticity into the horizontal)
- Important for mesoscale storm dynamics and tornadoes, but not typically a major player on the large scale



Physical interpretation: Solenoidal term

- Only plays a role when the atmosphere is baroclinic [i.e., pressure and density (i.e., temperature) contours cross]
- Vorticity changes result from differential horizontal accelerations imposed by horizontal variations in the pressure gradient force



- Although $\frac{\partial p}{\partial y}$ is constant in the illustration above, the pressure gradient acceleration is larger on the right because the density is lower
- As a result, the relative vorticity gradually increases (i.e., becomes cyclonic if starting from rest)
- Mathematically the above becomes

$$\frac{D(\zeta+f)}{DT} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}\right) > 0$$

since $\rho > 0$, $\frac{\partial \rho}{\partial x} < 0$ and $\frac{\partial p}{\partial y} < 0$

Review

See classquestion.com questions.