The finitary content of sunny nonexpansive retractions

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(joint work with Ulrich Kohlenbach)

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Fix $x \in C$ and put, for all $t \in [0, 1)$, $T_t : C \to C$, defined, for all $y \in C$, by

$$T_t y := tTy + (1 - t)x.$$

The Browder-Halpern result

It is easy to see that for each t, T_t is a t-contraction and thus, by Banach, there is a unique point x_t with $x_t = T_t x_t$ or

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(note that $x_0 = x$).

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Theorem (Browder, 1967; Halpern, 1967)

In the framework above, if X is a **Hilbert** *space, then for all $x \in C$ we have that $\lim_{t \to 1} x_t$ exists and it is a fixed point of T. Moreover, this $p \in Fix(T)$ satisfies $p = P_{Fix(T)} x$ – that is, for all $q \in Fix(T)$,*

$$\|x - p\| \leq \|x - q\|,$$

or, equivalently, for all $q \in Fix(T)$,

$$\langle x - p, q - p \rangle \leq 0$$

(the "variational inequality").

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Let us see what this may mean in our case.

Convergence and metastability

A convergence statement usually looks like

$$\forall \varepsilon \; \exists N \; \forall n \geq N \; \|x_n - x\| \leq \varepsilon.$$

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In turn, this is equivalent to a Herbrandized variant of it, called "metastability" by T. Tao (at the suggestion of J. Chayes), expressed as

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As this is a Π_2 statement (in a generalized sense), by the metatheorems of proof mining one can extract from its proof a *rate of metastability*, i.e. a *bound* $\Theta(\varepsilon, g, \ldots)$ on the N.

Past results - I

In our case, since we deal with an approximating curve $(x_t)_{t\in[0,1)}$, we appropriately modify the metastability statement into

$$\forall(t_n) \nearrow 1\, \forall\varepsilon\, \forall g\, \exists N \leq \Theta(\varepsilon, g, \ldots)\, \forall m, n \in [N, N+g(N)]\, \|x_{t_m} - x_{t_n}\| \leq \varepsilon.$$

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Such a bound has been extracted by Kohlenbach (Adv. Math., 2011) for the Browder-Halpern case, and its additional parameters are

- $\alpha : \mathbb{N} \to \mathbb{N}$ such that for all n and all $m \geq \alpha(n)$, $t_m \geq 1 - \frac{1}{n+1}$;
- $\gamma : \mathbb{N} \to \mathbb{N}^*$ such that for all n, $t_n \leq 1 - \frac{1}{\gamma(n)}$;
- a bound b on the diameter of C.

He extracted bounds out of analyses of both Browder's and Halpern's proofs.

Past results - II

Although Halpern's proof was easier to analyze, the Browder analysis proved crucial in extracting a rate of metastability for another nonlinear analysis result, namely Wittmann's theorem from 1992 – the essential improvement was *the elimination of the use of weak compactness*.

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Later, Saejung in 2010 generalized Wittmann's result to a class of nonlinear spaces called CAT(0) spaces. A rate of metastability was extracted in this case by Kohlenbach and Leuştean (Adv. Math., 2012). Here the novelty was *the elimination of Banach limits*.

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Why do we care about eliminating proof principles?

The inner workings of proof mining

Remember how proof mining works:

$$S \vdash \forall x \exists y \varphi(x, y) \Rightarrow S' \vdash \forall x \varphi(x, tx).$$

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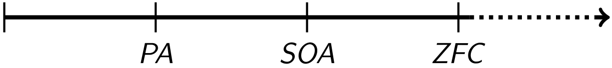

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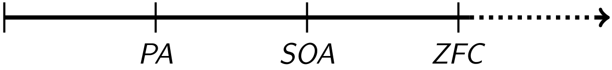

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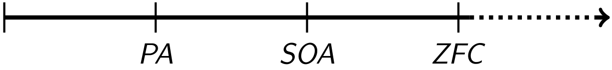

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The point of the simplifications before was to show that the System T functionals are sufficient for expressing the desired rates. Also, see the recent approach of Ferreira/Leuștean/Pinto (Adv. Math., to appear) via the bounded functional interpretation.

Enter Reich

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Theorem (Reich, 1980)

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What property does this $p \in Fix(T)$ satisfy? (We expect it to be relevant, since the corresponding variational inequality turned out to be in the Browder analysis.) To find out, we delve into the theory of uniformly smooth spaces.

Some Banach space theory

If X is a Banach space, one defines the duality mapping $J : X \to 2^{X^*}$, for any $x \in X$, by

$$J(x) := \{x^* \in X^* \mid x^*(x) = \|x\|^2,\ \|x^*\| = \|x\|\}.$$

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Then X is called **smooth** if for any x, y in its unit sphere there exists

$$\lim_{h \to 0} \frac{\|x + hy\| - \|x\|}{h},$$

which can be shown to be equivalent to the fact that for all $x \in X$, $J(x)$ is a singleton.

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In this case, one defines $j : X \to X^*$ to be the unique section of J. Since, when X is a Hilbert space, for all x, $y \in X$, $j(x)(y) = \langle y, x \rangle$, one puts for all $x^* \in X^*$ and $y \in X$, $\langle y, x^* \rangle := x^*(y)$.

Uniformly smooth spaces

A Banach space X is **uniformly smooth** if the limit in the definition of smoothness exists uniformly in the pair (x, y), or, equivalently, if there is a $\tau : (0, \infty) \to (0, \infty)$ such that for all $\varepsilon > 0$, all x in the unit sphere and all y in the closed ball of radius $\tau(\varepsilon)$,

$$\|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\|.$$

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It is a classical result that if X is uniformly smooth, then j is uniformly continuous on bounded subsets. An explicit modulus of uniform continuity $\omega_\tau(b, \cdot)$ was extracted by Kohlenbach and Leuștean in 2012. Also, Körnlein proved in 2015 a strong converse of this property, requiring only the uniform continuity on bounded subsets of an *arbitrary* section of J.

Sunny nonexpansive retractions

Let E be a nonempty subset of C and $Q : C \to E$. We call Q a **retraction** if for all $x \in E$, $Qx = x$. If Q is a retraction, we call it **sunny** if for all $x \in C$ and $t \geq 0$, $Q(Qx + t(x - Qx)) = Qx$.

Proposition (Variational Inequality)

A retraction $Q : C \to E$ is sunny and nonexpansive iff for all $x \in C$ and $y \in E$,

$$\langle x - Qx, j(y - Qx) \rangle \leq 0.$$

As a consequence, there is at most one sunny nonexpansive retraction $Q : C \to E$. We may now say that the point p in Reich's theorem satisfies $p = Q_{Fix(T)}x$, where $Q_{Fix(T)} : C \to Fix(T)$ is the unique sunny nonexpansive retraction.

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Usually, they use Banach limits, in a more intricate way than in the proof of Saejung's result. We found, however, a 1990 proof by Morales, who replaces it by the "ordinary" limit superior, which still lies above the strength of first-order arithmetic, though only barely so. In addition, there is a proof segment which uses some even stronger principles that needs to be dealt first.

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We first put, for any n, $x_n := x_{t_n}$. The asymptotic regularity property is immediate:

$$\begin{aligned} \|x_n - Tx_n\| &= \|t_n Tx_n + (1-t_n)x - Tx_n\| \\ &= \|(1-t_n)(x - Tx_n)\| \leq (1-t_n) \cdot b \to 0. \end{aligned}$$

A use of strong principles

The crucial segment defines a function $f : C \to \mathbb{R}_+$, for all $z \in C$, by $f(z) := \limsup_{n\to\infty} \|x_n - z\|$. Let K be the set of minimizers of f. The claim is that there is a $p \in K \cap Fix(T)$.

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\begin{aligned}
f(Ty) = \limsup_{n\to\infty} \|x_n - Ty\| &\leq \limsup_{n\to\infty} (\|x_n - Tx_n\| + \|Tx_n - Ty\|) \\
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so $Ty \in K$. Now, since K is a closed convex bounded nonempty T-invariant subset of a uniformly smooth space, we have that (!) there is a $p \in K \cap Fix(T)$.

On uniqueness

We try to find an alternative path to the claim. Of course, *a posteriori* the point in $K \cap Fix(T)$ is unique, as it is simply the limit p of the sequence (x_n), characterized by $f(p) = 0$.

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Is there a way of obtaining this uniqueness *a priori*?

Answer: Yes, if we use an appropriate *modulus of uniqueness*, which exists if the space is in addition **uniformly convex** (still covering the L^p case).

Uniformly convex spaces

A Banach space X is **uniformly convex** if there is an $\eta : (0, 2] \to (0, 1]$ such that for all $\varepsilon \in (0, 2]$ and all x, y in the unit sphere with $\|x - y\| \geq \varepsilon$, one has that $\left\| \frac{x+y}{2} \right\| \leq 1 - \eta(\varepsilon)$.

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Proposition (Zălinescu, JMAA 1983)

Let X be uniformly convex with modulus η and $b \geq \frac{1}{2}$. Then there is a $\psi_{b,\eta} : (0,2] \to (0,\infty)$ such that for all $\varepsilon \in (0,2]$ and all x, y in the closed ball of radius b with $\|x - y\| \geq \varepsilon$, one has that

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$$\left\|\frac{x+y}{2}\right\|^2 + \psi_{b,\eta}(\varepsilon) \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2.$$

In 2018, Bačák and Kohlenbach have obtained an *explicit* formula for $\psi_{b,\eta}$.

Reproof of the claim

With this modulus of uniform convexity for the squared norm in mind, we may prove that there is a $p \in K \cap Fix(T)$ by:

- taking a minimizing sequence for f;
- making sure, using the modulus and a variant of the previous T-invariance argument, that it is an approximate fixed point sequence;
- showing, again using the modulus, that any such sequence must be Cauchy;
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Still, some problems remain:

- an amount of (Π^0_1-)comprehension is used to pass to the limit in this argument;
- most importantly, there is this constant use of $\limsup$'s which is also problematic.

Removal of comprehension axioms

The first (tedious) step is to replace the ideal elements (limits, fixed points) by approximate ones. For example, it turns out that in the previous argument, only arbitrarily good minimizers are needed.

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The second step is to replace the $\limsup$'s by *approximate* $\limsup$'s, in a process known as *arithmetization* (this is possible mainly because the $\limsup$'s are used pointwise and not as an operator in itself). Let us see what this means.

Approximate lim sup's

Definition

Let (a_n) be a sequence of reals and $\varepsilon > 0$. A number $a \in \mathbb{R}$ is called an ε-**approximate limsup** (or simply an ε-**limsup**) for (a_n) if:

- for all n there is an m such that $a_{n+m} \geq a - \varepsilon$;
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What makes approximate $\limsup$'s suitable for proof mining is that the following existence proof uses only Π^0_2-induction.

Proposition (Π^0_2-IA)

For all $b, k \in \mathbb{N}$ and for all sequences of reals (a_n) contained in the interval $[0, b]$, there is a $p \in \mathbb{N}$ with $0 \leq p \leq b \cdot (k+1)$ such that $\frac{p}{k+1}$ is a $\frac{1}{k+1}$-limsup of (a_n).

“Where the tyre hits the road”

Of course, it is not enough to show that their existence uses only acceptable principles, one must also make sure that approximate lim sup’s can reliably replace lim sup’s in our arguments. This is made possible by the following.

Lemma

Let $\varepsilon > 0$. Let (a_n), (b_n) and (c_n) be sequences of reals and q, q' and r be $\frac{\varepsilon}{4}$-limsup’s of them, respectively. If $q \leq r + \frac{\varepsilon}{2}$ and $q' \leq r + \frac{\varepsilon}{2}$, then for all N there is a k such that $a_{N+k} \leq c_{N+k} + \varepsilon$ and $b_{N+k} \leq c_{N+k} + \varepsilon$.

Some reverse math

What is even more interesting is that the existence of approximate $\limsup$'s is actually *equivalent* (over PRA) to Π^0_2-IA. This can be proven by adapting an argument from:

U. Kohlenbach, Things that can and things that cannot be done in PRA. *Ann. Pure Appl. Logic* 102, no. 3, 223–245, 2000.

List of simplifications

To sum up, we obtain the following sequence of "intermediate" proofs:

1. the original proof of Morales from 1990
2. the proof using uniform convexity instead of strong set-theoretic principles
3. the removal of ideal elements
4. the pointwise replacement of $\limsup$'s with approximate $\limsup$'s
5. a no-counterexample-style proof of the last segment
6. the fully analyzed proof

Out of those, proofs (3), (5) and (6) may be found in full in our paper. Let us consider now the details of the last step, i.e. the extraction procedure itself.

The analysis of the existence of approximate $\limsup$'s

Out of all steps, the extraction of the functionals for of the existence theorem for approximate $\limsup$'s may be the easiest, as it is almost mechanical. This is so because its proof can be formalized in a standard Hilbert-style proof system using just elementary operations and the axiom of Π^0_2-induction, whose functional ("ND") interpretation (into T_1) may be found in the following paper:

C. Parsons, On *n*-quantifier induction. *J. Symbolic Logic* 37, 466–482, 1972.

Further analysis

- The following steps in the proof are progressively more complex, featuring e.g. Π_3-induction (yielding T_2-level functionals) and a large amount of nestings. In the analysis one thus needs to make use of the *Dialectica* interpretation of

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- The end-product is a *realizer*, i.e. a formula for the N in the metastability statement, which is not fully computable because of the case distinctions introduced by contraction-like behaviour such as the above. To obtain the *rate* of metastability, one majorizes this term, removing in the process the non-computable parts.

On complexity and tameness

The majorization process proceeds smoothly and yields a purely numerical term. A close analysis of the term shows that the functional can actually be defined in T_1, and it is an open question whether it is actually in T_0 or whether some different proof may produce a T_0-definable rate of metastability, similarly to all the rates obtained in proof mining so far. For more information regarding this phenomenon, see:

U. Kohlenbach, Local formalizations in nonlinear analysis and related areas and proof-theoretic tameness. Preprint, 2019.

Applications

- The rate of metastability thus obtained can be used as an input to a previous partial analysis by Kohlenbach/Leuștean of a proof of Shioji/Takahashi (Proc. AMS, 1997) for the convergence in our setting of the Halpern iteration.

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- In addition, a slightly modified argument (using a resolvent construction) works also if one replaces the nonexpansive mapping T with a more general pseudocontraction (required to be uniformly continuous), i.e. one that satisfies, for all x, $y \in C$,
$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2.$$

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- This more general bound completes an analysis of Körnlein/Kohlenbach of a proof of Chidume/Zegeye (Proc. AMS, 2004) for the convergence of the Bruck iteration.

All this can be found in:

U. Kohlenbach, A. Sipoș, The finitary content of sunny nonexpansive retractions. arXiv:1812.04940 [math.FA], 2018.

Thank you for your attention.