Lecture 5: Minimum Cost Flows

- Flows in a network may incur a cost, such as time, fuel and operating fee, on each link or node.
  - ◇ Suppose in addition to a capacity c_{ij}, each arc of a flow network is assigned a cost a_{ij}. The cost of a flow $x = (x_{ij})$ is

$$\sum_{i,j} a_{ij} x_{ij}$$

We now pose the problem of finding a minimum cost flow for a given flow value v.

Min Cost Flow Problem

$$\min \sum_{ij} a_{ij} x_{ij}$$

s. t.

$$\sum_j x_{ji} - \sum_j x_{ij} = \begin{cases} -v, & i = s, \\ 0, & i \neq s, t \\ v, & i = t, \end{cases}$$

$$0 \leq x_{ij} \leq c_{ij}$$

W. S. Jewell, "Optimal Flow through Networks," Interim Technical Report No. 8, Massachusetts Institute of Technology, 1958.

R. G. Busacker and P. J. Gowen, "A Procedure for Determining a Family of Minimal-Cost Network Flow Patterns," O. R. O. Technical Paper 15, 1961.

Examples

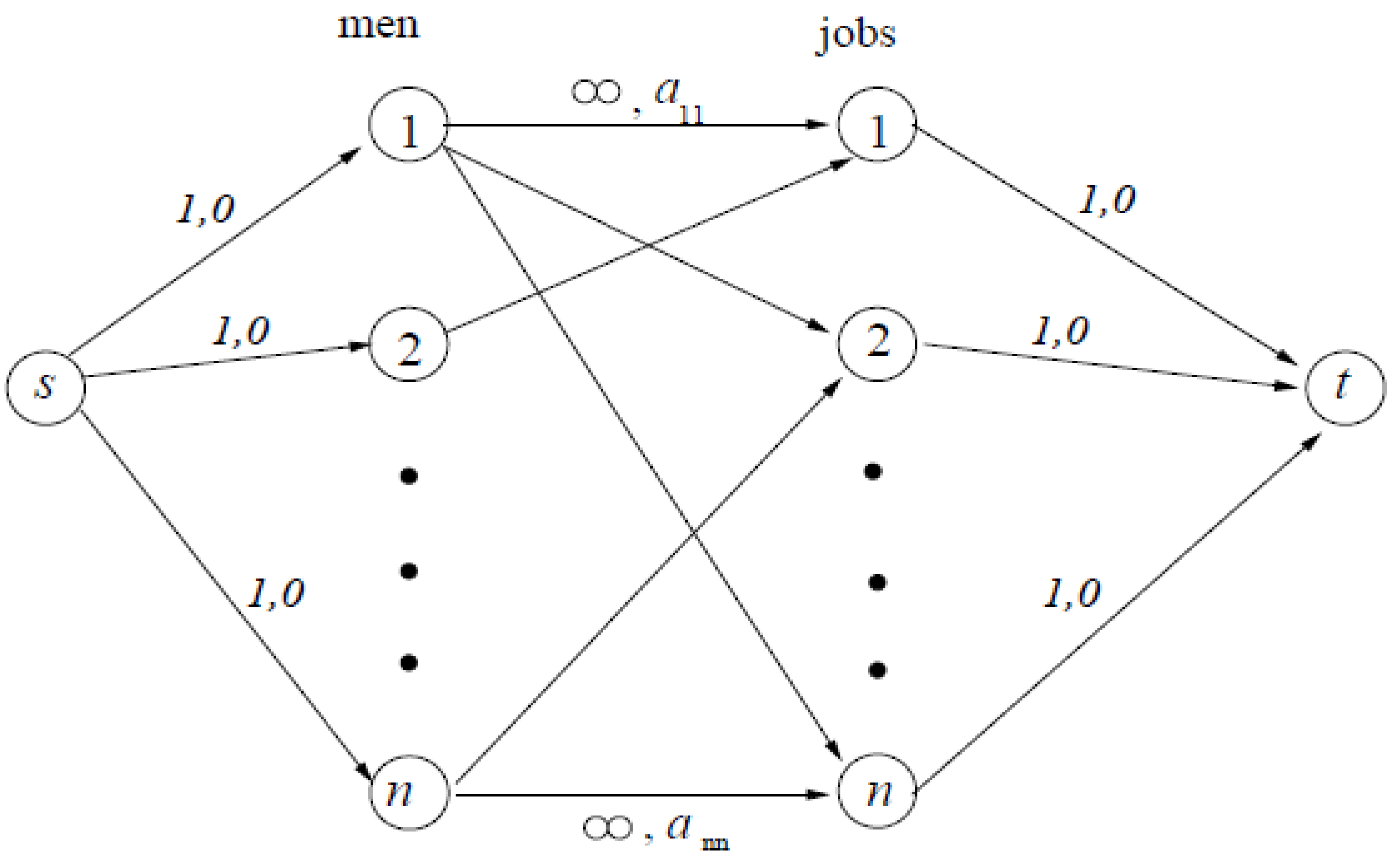
- Supply chain management – deliver goods using *v* trucks from a warehouse *s* to a customer *t* subject to road restrictions and toll fees.

- Telecommunication network planning – send message in *v* packets from node *s* to node *t* subject to bandwidth capacity and transmission loss.

Another Example: Assignment Problem

- There are n men and n jobs. The cost of assignment man i to job j is a_{ij}. For what man-job assignment is the total cost minimized?

Example: Assignment Problem



Questions

- Is the problem feasible?
  - feasibility check
- If a flow is feasible, how to tell it is optimal or not?
  - optimality condition
- How to find an optimal solution?
  - algorithm design
- How much work involved in finding an answer?
  - complexity analysis

Feasibility Check

- No problem

  - just solve a max flow problem and check.

Optimality Condition

- A new concept of "cost of augmenting cycle" is needed.

- Basically, if the current flow allows one more unit to flow in a cyclic manner in part of the nodes with a negative cost incurred, then the current flow is not optimal.

- No augmenting cycle with negative cost !

This concept will be further developed.

Algorithm Design

- Take a "greedy approach"

  - do what you can do best for a current flow value

  - do what you can do best to send more flow

  - stop when you reach the flow required

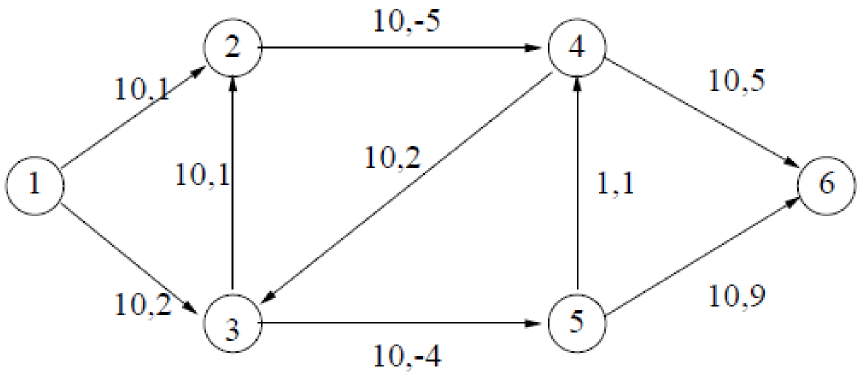
Intuitions

- Start with a flow with value $v' \leq v$.
- Augmenting path will maintain the flow conservation law and increase the value v'.
- When v' reach v, we have a feasible solution.
- Would like to find a minimum cost flow with value v' at each stage.
- Would like to augment the flow value to generate a minimum cost flow with respect to the new flow value.

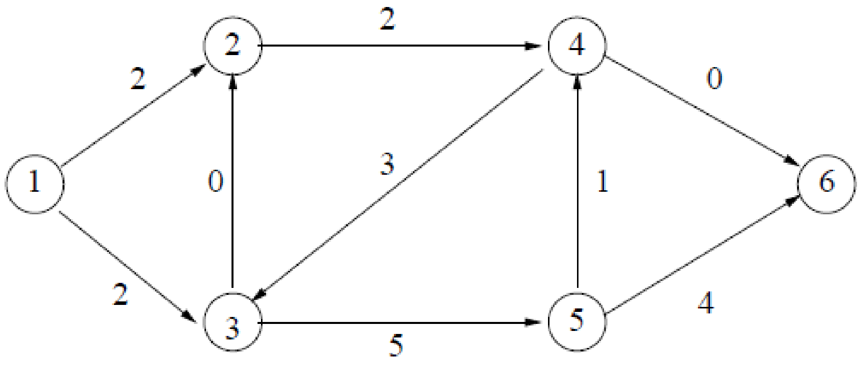
Cost of an augmenting path

- Let us define the *cost of an augmenting path* to be the sum of the costs of forward arcs minus the sum of costs backward arcs. Thus the cost of a path is equal to the net change in the cost of flow for one unit of augmentation along the path. An *augmenting cycle* is a closed augmenting path. The cost of an augmenting cycle is computed in the obvious way, with respect to a given orientation of the cycle.

Cost of augmenting paths and cycles

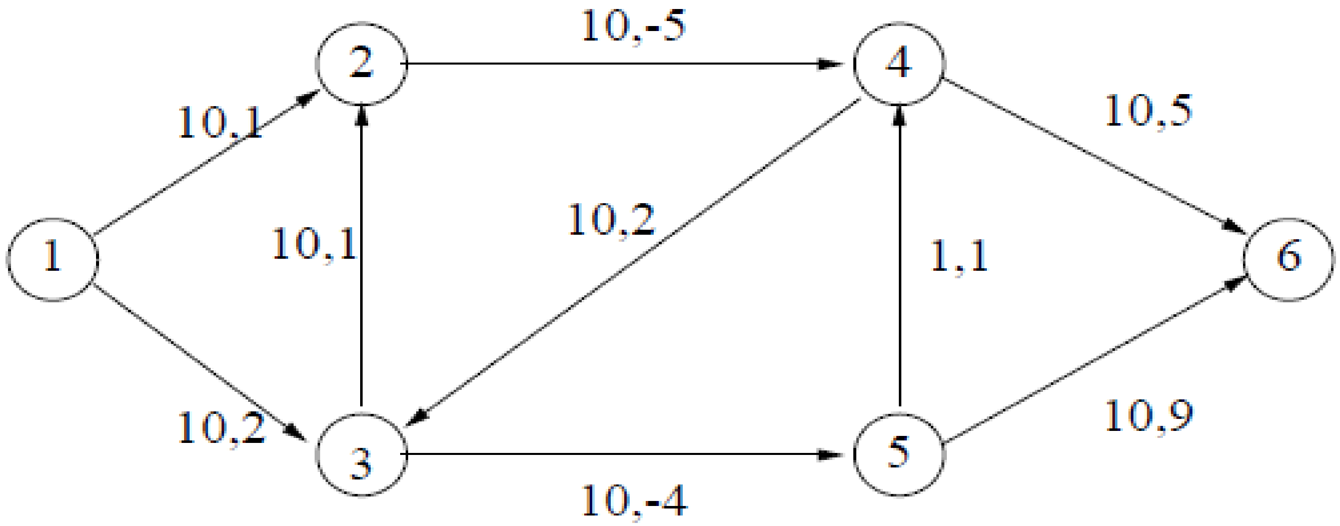


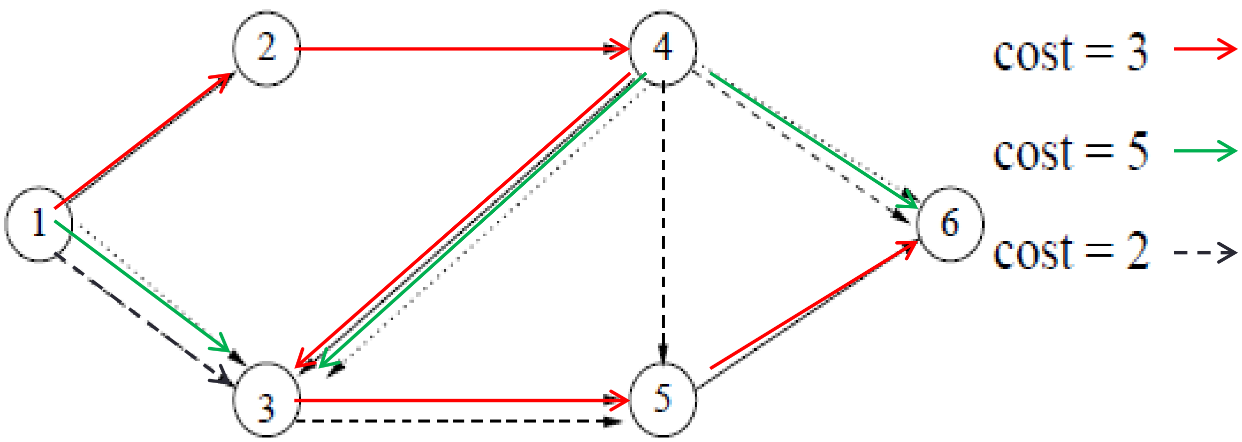
max flow = 20



Flow with v' = 4

Cost of augmenting paths





Observations

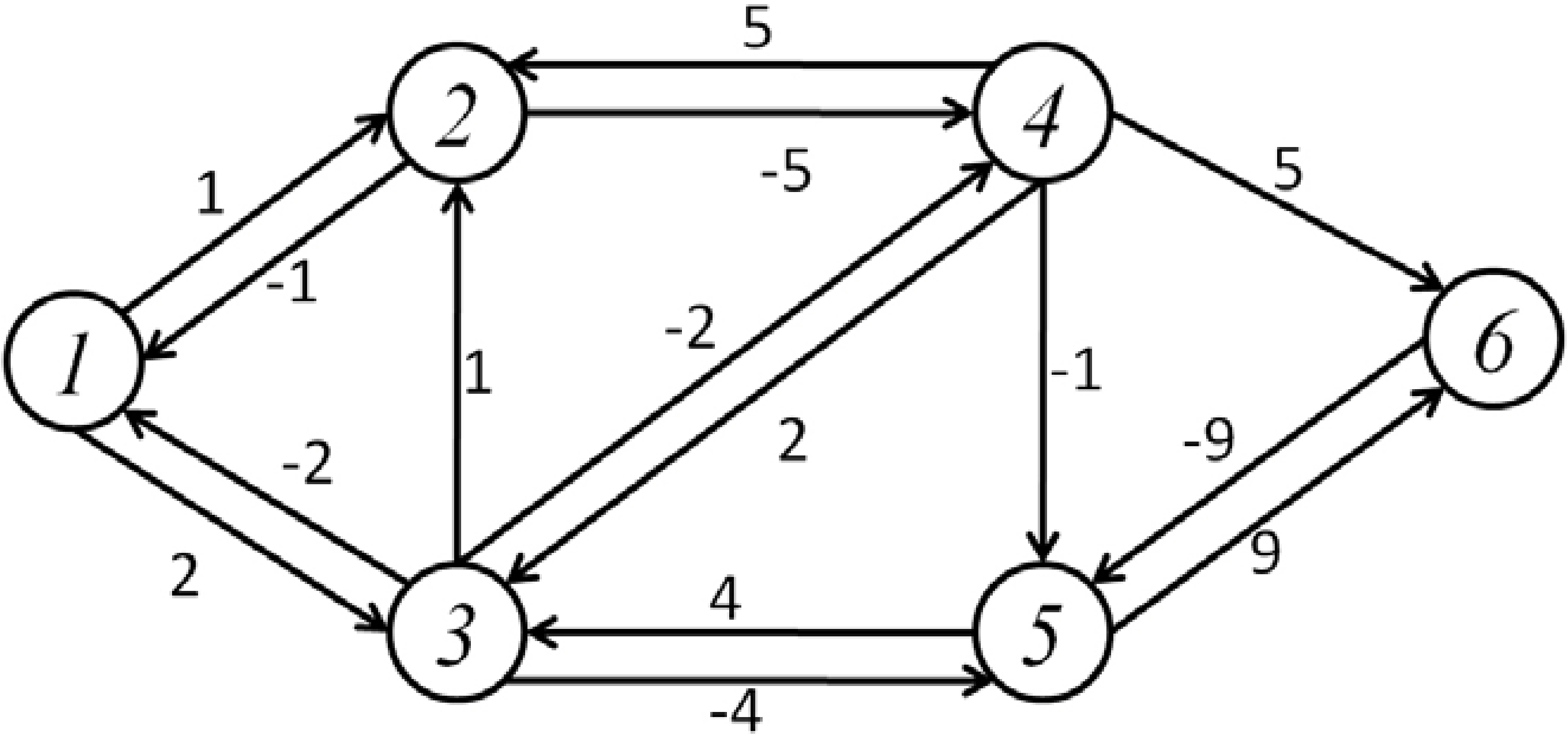
- Cost incurred by pushing one more unit of flow through the network via the augmenting path.

  - Interested in the minimum-cost augmenting path.

- How to find the minimum-cost augmenting path?

  - Think about the shortest path algorithm.

Min-cost augmenting path



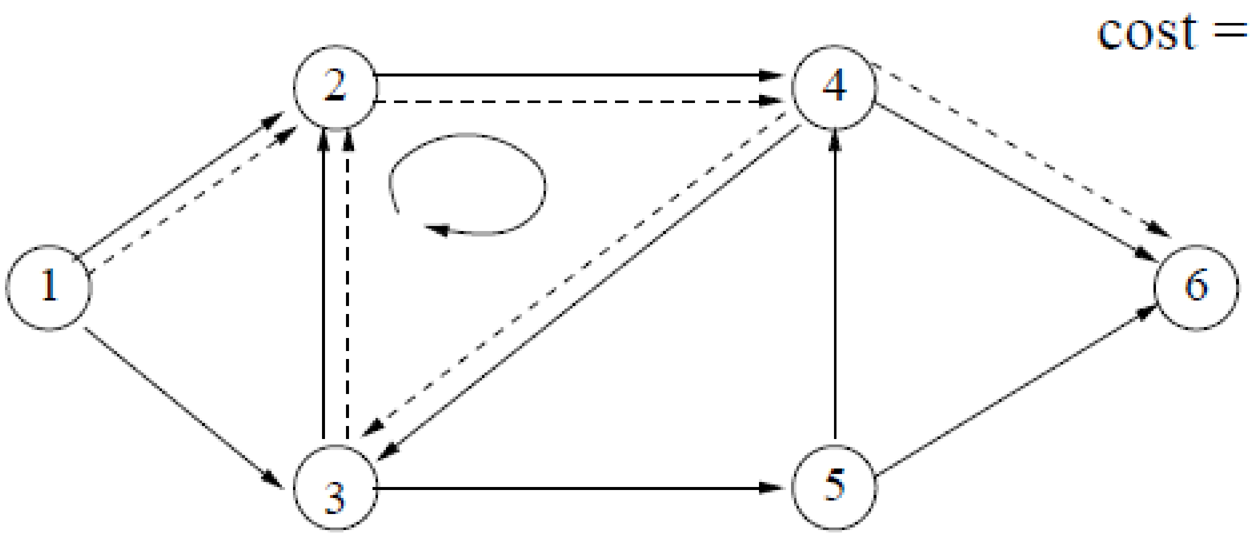
Min-cost augmenting path

- A minimum-cost augmenting path can be found by means of a shortest path computation. Specifically, for a given flow $x = (x_{ij})$ and arc costs a_{ij}, let

$$\bar{a}_{ij} = \begin{cases} a_{ij}, & if\ x_{ij} < c_{ij},\ x_{ji} = 0 \\ min\{a_{ij}, -a_{ji}\}, & if\ x_{ij} < c_{ij},\ x_{ji} > 0 \\ -a_{ji}, & if\ x_{ij} = c_{ij},\ x_{ji} > 0 \\ +\infty, & if\ x_{ij} = c_{ij},\ x_{ji} = 0 \end{cases} \quad (7.1)$$

- Where we understand that $a_{ij} = +\infty$ if (i, j) is not an arc of the flow network. A shortest (s, t) directed path with respect to arc lengths $\bar{a}_{ij}$ corresponds to a minimum cost (s, t) augmenting path. A negative directed cycle corresponds to an augmenting cycle with negative cost.

Augmenting Cycle



augmenting cycle with cost = -2

Property: Increase flow along an augmenting cycle with negative cost results in a lower cost solution with the same flow value v'.

Conjectures

- 1. A flow is of minimum cost if and only if it admits no augmenting cycle with negative cost ?

- 2. (A minimum cost flow of value v') + (augmenting δ along a minimum cost augmenting path) = (a minimum cost flow of value v' + δ) ?

Main Theorems

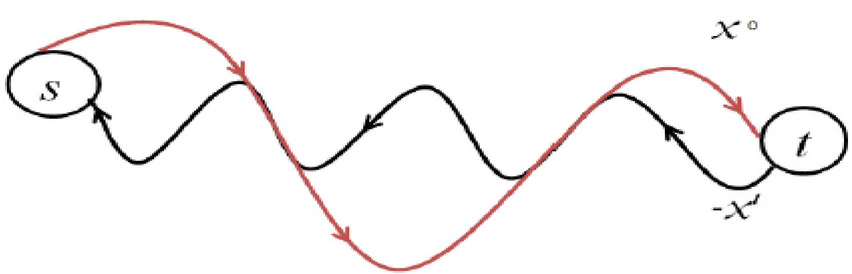
- **Theorem 7.1** A flow of value v is of minimum cost if and only if it admits no flow augmenting cycle with negative cost.

- **Theorem 7.2** *(Jewell, Busacker and Gowan)* The augmentation by δ of a minimum cost flow of value v along a minimum cost flow augmenting path yields a minimum cost flow of value $v + \delta$.

Proof of Theorem 7.1

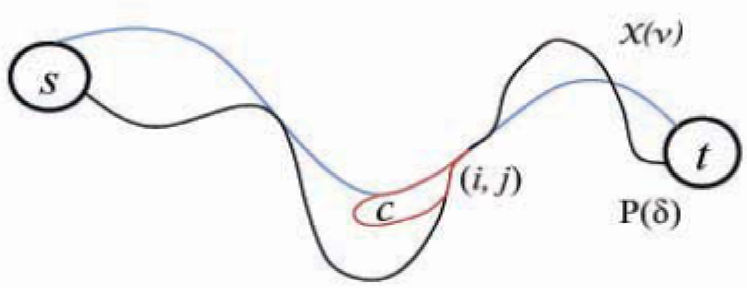
Proof: (i)If the current flow $x' = (x'_{ij})$ admits a flow augmenting cycle with negative cost, then x' is obviously not optimal.

(ii)If the current flow $x' = (x'_{ij})$ admits no flow augmenting cycle with negative cost, then we need to prove that x' is optimal. If not, then $\exists\ x^0 = (x^0_{ij})$ of value v such that $x^0 \neq x^1$ and $c(x^0) < c(x^1)$. Now consider the flow $y \triangleq x^0 - x^1 \neq (0)$. It can be expressed as a sum of augmenting cycles w.r.t. x'. Because $c(x^0) < c(x')$, at least one of these cycles has negative cost w.r.t. x'. This is a contradiction!



Proof of Theorem 7.2

Proof: By Theorem 7.1, it suffices to show that the flow resulting from augmentation along a minimum cost augmenting path does not admit a negative augmenting cycle. Suppose such a cycle C were introduced. Then C must contain at least one arc (i, j) of the minimum cost augmenting path P. But then $(P \cup C) - (i, j)$, or some subset of it, would be an augmenting path with respect to the original flow, and would be less costly than P, contrary to the assumption that P is minimal.



Min-Cost Flow Algorithm

Step 0 (Start) Let $x = (x_{ij})$ be any (s, t) flow with value $v' \leq v$, where v is the desired flow value. This initial flow can be the zero flow, or a flow of value v, perhaps determined by the max-flow algorithm. Or if a flow $x' = (x'_{ij})$ of value $v' > v$ is known, one can let $x = (v/v')x'$.

Step 1 (Elimination of Negative Cycles)
(1.1) Apply a shortest path algorithm with respect to arc lengths $\bar{a}_{ij}$ with the objective of detecting negative cycles. If no negative cycle exists, go to Step 2.
(1.2) Augment the flow around the corresponding augmenting cycle to obtain a less costly flow of the same value v', then return to Step 1.1.

Min-Cost Flow Algorithm

Step 2 (Minimum Cost Augmentation)

(2.0) If the existing flow value $v' = v$, the existing flow is optimal and the computation is completed. Otherwise, proceed to Step 2.1.

(2.1) Apply a shortest path algorithm with respect to arc lengths $(\bar{a}_{ij})$ with the objective of finding shortest path from s to t. If no shortest path exists, there is no flow of value v and the computation is halted.

(2.2) Augment the flow by δ, where $v' + \delta \leq v$, along a minimum cost (s, t) augmenting path as determined by the shortest path computation. Return to Step 2.0.//

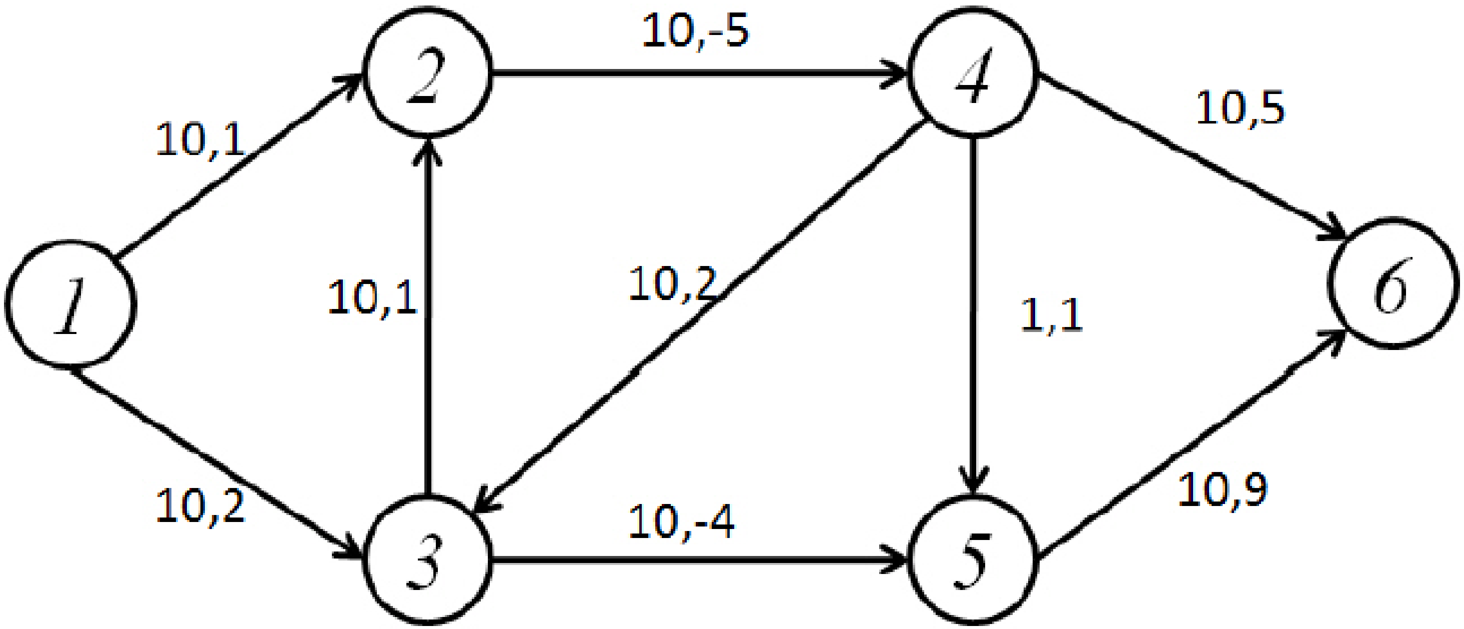
Complexity Analysis

- If one begins with the zero flow and no negative cycles exist with respect to the arc costs a_{ij}, then at most v augmentations are required, provided all capacities are integers. Each augmentation requires a shortest path computation which is $O(n^3)$. Hence in this situation the overall complexity is $O(n^3 v)$.

- Edmonds and Karp have shown that, once negative cycles are eliminated, it can be arranged for the shortest path computation to be carried out over nonnegative arc lengths. Thus, Dijkstra's $O(n^2)$ shortest path algorithm can be applied. The complexity bound of $O(n^3 v)$ is then reduced to $O(n^2 v)$.

- We should note that the minimum cost flow algorithm is well-suited to a parametric ananlysis of minimum flow cost as a function of flow value v.

Example

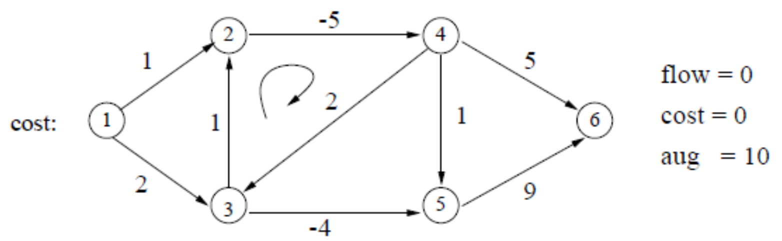


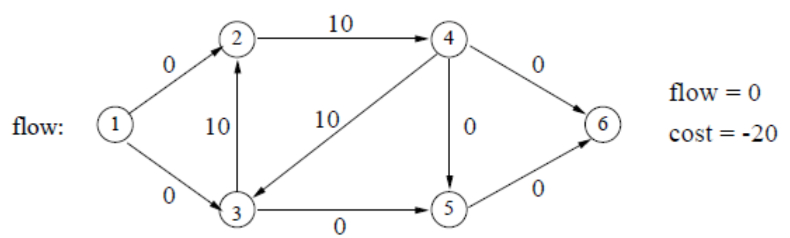
max flow= 20

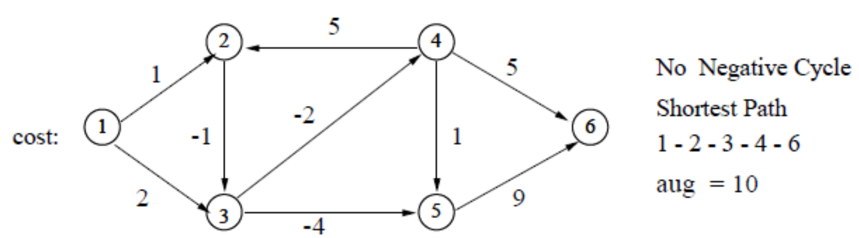
Find a min cost flow $v = 15$.

Example

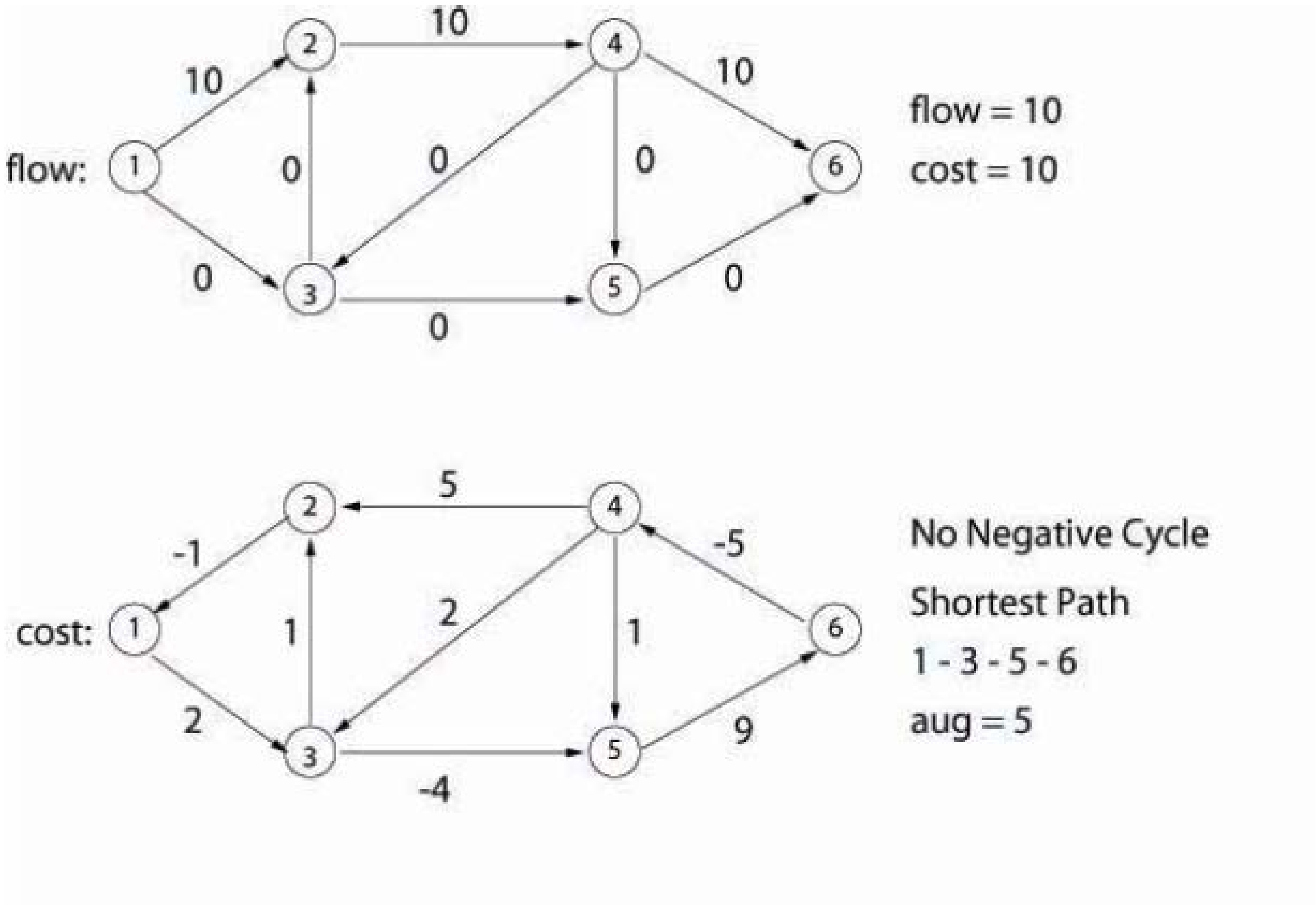
v=15, start with $x = (0)$.





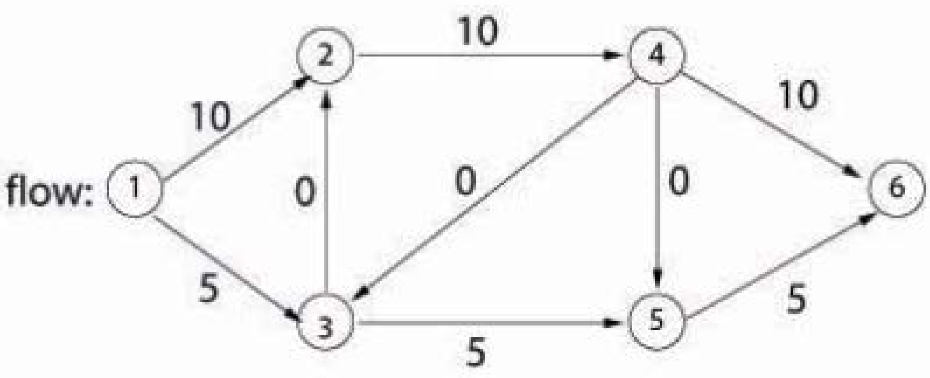


Example



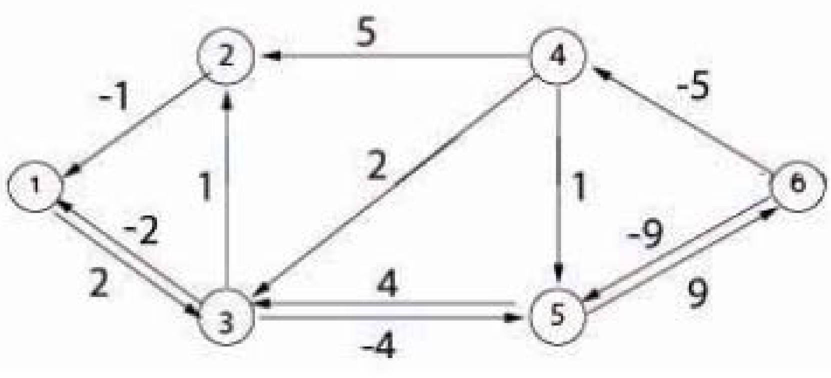
flow:
10
10
10
0
0
0
0
0
0
flow = 10
cost = 10
cost:
5
-1
-5
1
2
1
2
9
-4
No Negative Cycle
Shortest Path
1 - 3 - 5 - 6
aug = 5

Example



flow = 15

cost = 45



No Negative Cycle

flow = 15

Optimal cost = 45

Network with Gains and Losses

- A supply chain of perishable products.
- An international money market with different currencies
- More

Network with Gains and Losses

- Suppose that flow is not necessarily conserved within arcs. If x_{ij} units of flow go into the tail of arc (i, j), then $m_{ij}x_{ij}$ comes out at the head, where m_{ij} is a nonnegative flow *multiplier* associated with that arc. If $0 < m_{ij} < 1$, the arc is *lossy*, and if $1 < m_{ij} < \infty$, the arc is *gainy*.

  Let

$$x_{ij} = \text{the amount of flow } \textit{into} \text{ arc } (i, j).$$

Network with Gains and Losses

- We require the satisfaction of capacity constraints,

$$0 \leq x_{ij} \leq c_{ij}$$

and the satisfaction of conservation conditions at each node other than s or t:

$$\sum_j m_{ji} x_{ji} - \sum_j x_{ij} = \begin{cases} -v_s, & i = s \\ 0, & i \neq s, t \\ +v_t, & i = t. \end{cases} \quad (2.2)$$

Note that v_s is not necessarily equal to v_t, because of flow losses and gains within arcs.

Network with Gains and Losses

- Define $v_s - v_t$ to be the loss of the flow. We shall be concerned with the problem of finding a minimum loss flow for a given flow value v_s or v_t. That is, given v_s maximize v_t, or given v_t minimize v_s.

Currency Conversion

- An exchange rate has been established such that for each unit of currency i one receives m_{ij} units of currency j. There is a bound c_{ij} on the number of units of currency i that one can so convert.

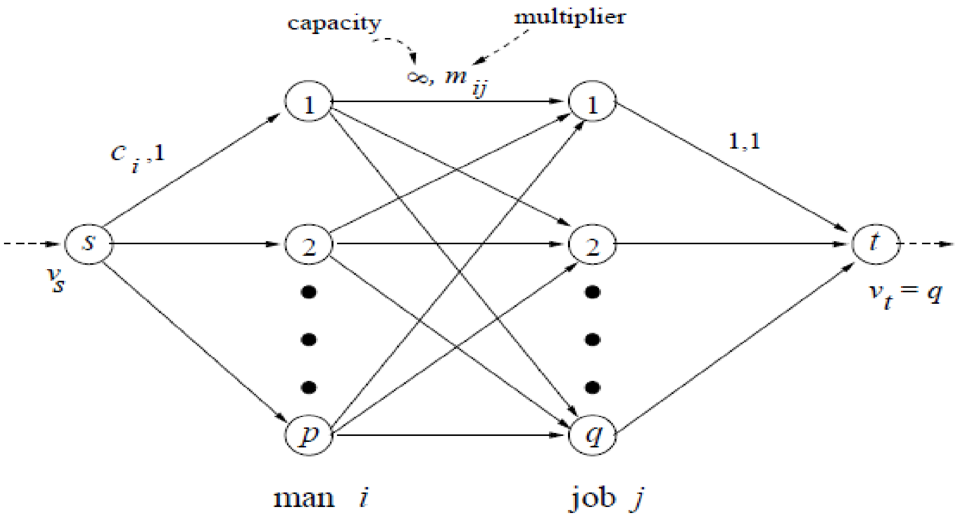
Work Assignment

- There are p men and q jobs. Any man is capable of performing all the work on any given job, or the work can be apportioned among several men. One hour of time by man i is sufficient to complete a fraction m_{ij} of job j. Man i is available to work no more than c_i hours. How should the men be assigned to the jobs so that the jobs can be completed with the smallest possible total number of man-hours of labor?

Work Assignment

Define x_{ij} := # of hours man i works on job j.

$$\begin{array}{ll} \min & \sum_{ij} x_{ij} \\ \text{s.t.} & \\ & \sum_j x_{ij} \le c_i, \quad i = 1, \dots, p, \\ & \sum_i m_{ij} x_{ij} = 1, \quad j = 1, \dots, q, \\ & x_{ij} \ge 0 \end{array}$$



Comments

- ◇ The theory of minimum loss flows if quite parallel to the theory of minimum cost flows, and we can develop a computational procedure parallel to that presented in the previous section.

Major Results

- Theorem 8.1 In a network with losses and gains, a flow of source value v_s is of minimum loss if and only of it admits no endogenous flow augmenting path to *t.* A flow of sink value v_t is of minimum loss if and only of it admits no endogenous flow augmenting path to *s.*

- Theorem 8.2 In a network with losses and gains, the augmentation of a minimum loss flow of sink value v_t along a minimum loss augmenting path of capacity d_t yields a minimum loss flow of value $v_t + d_t$.

Minimum Cost Circulation Problem

- A generalization of the shortest paths problem, maximal flow problem, minimum cost flow problem.

$$\min \quad \sum_{ij} a_{ij} x_{ij}$$

s.t.

$$\sum_j x_{ji} - \sum_j x_{ij} = 0 \quad \forall\, i, \ \text{(circulation)}$$

$$0 \le l_{ij} \le x_{ij} \le c_{ij} \quad \forall\, i, j.$$

$$l_{ij} : \quad \text{lower bound}$$

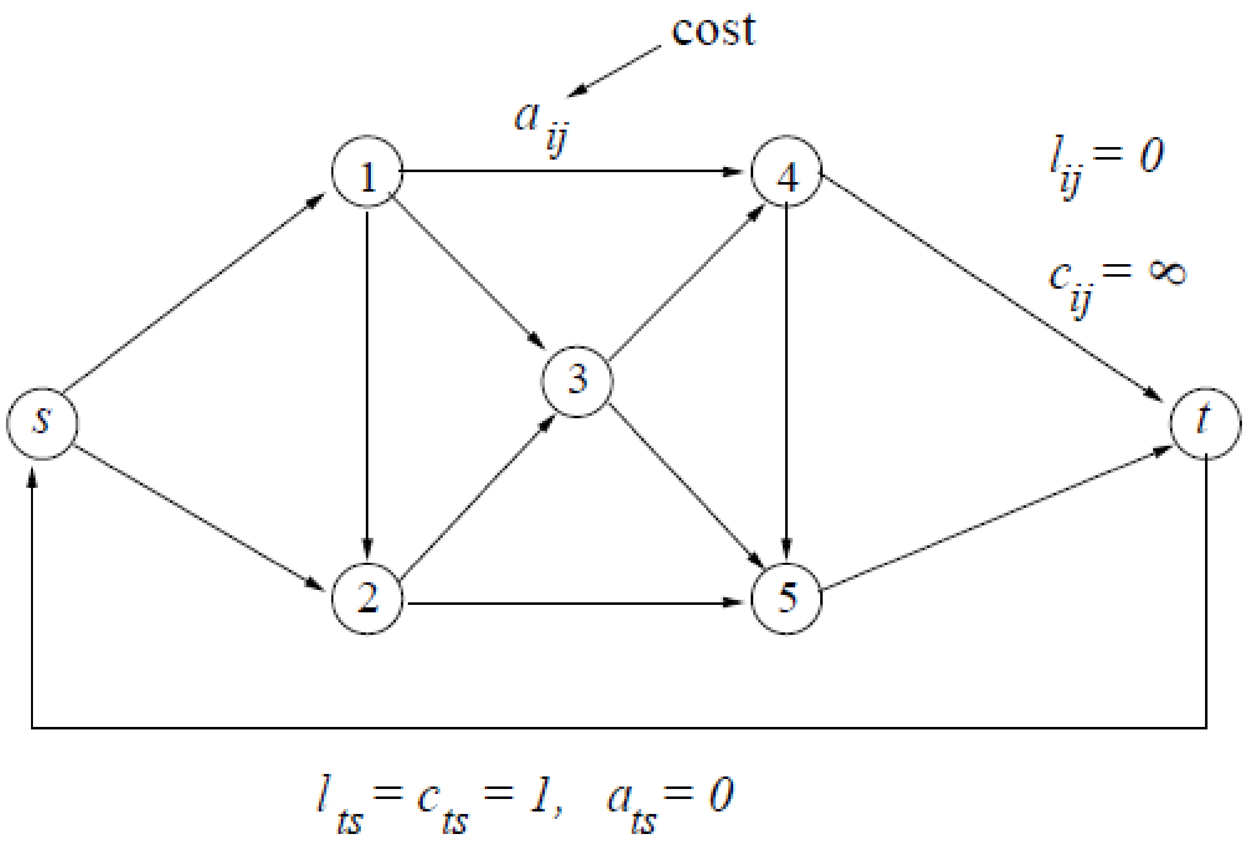
- Flow conservation law holds for all nodes. (no sink, no source)
- $(a_{ij},\ l_{ij},\ c_{ij})$ specifies the problem associated with a given diagraph.

What’s a Circulation?

- In addition to a capacity c_{ij} for each arc (i, j), we may designate a *lower bound* l_{ij} and require that $l_{ij} \leq x_{ij} \leq c_{ij}$.

- A *circulation* is simply a flow in a network in which conservation conditions are observed at all nodes. That is, there is no source or sink.

Example: Shortest Path Problem

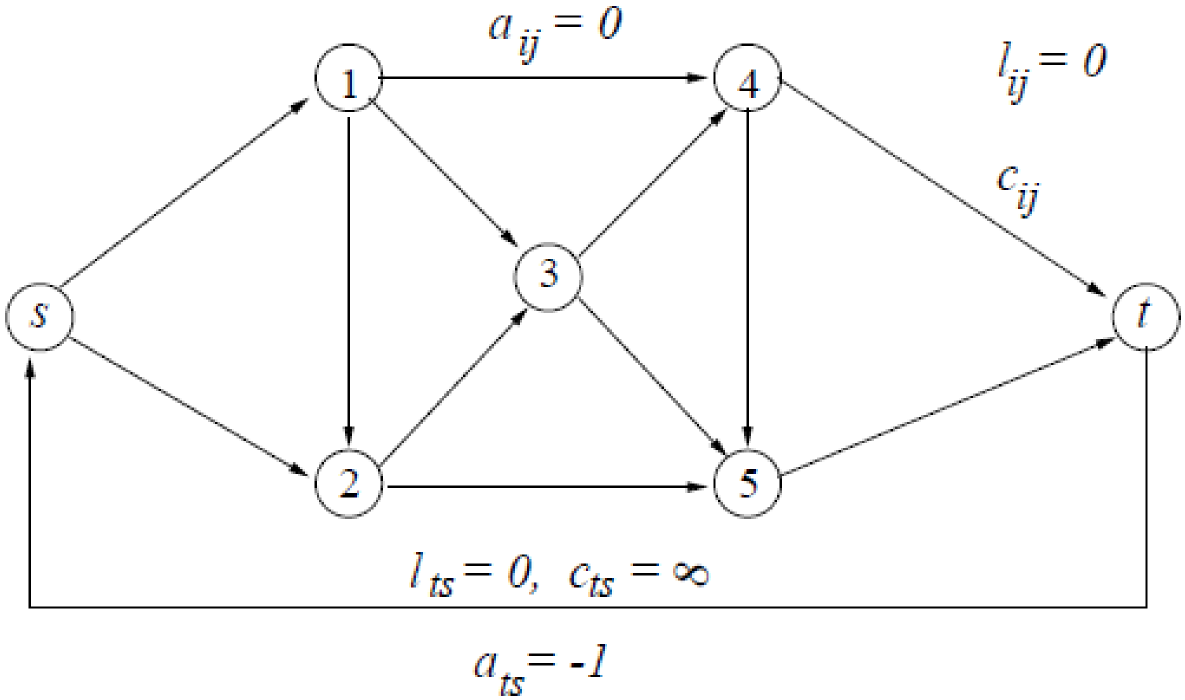


Shortest Path Problem

- To find a shortest path from s to t in a network with arc lengths a_{ij}, add a return arc (t, s) with $l_{ts} = c_{ts} = 1$. For all other arcs $(i, j), l_{ij} = 0, c_{ij} = +\infty$, and a_{ij} is as given.

- To find shortest paths from s to all other nodes, add return arcs (j, s) from all nodes $j \neq s$, with $l_{js} = c_{js} = 1$.

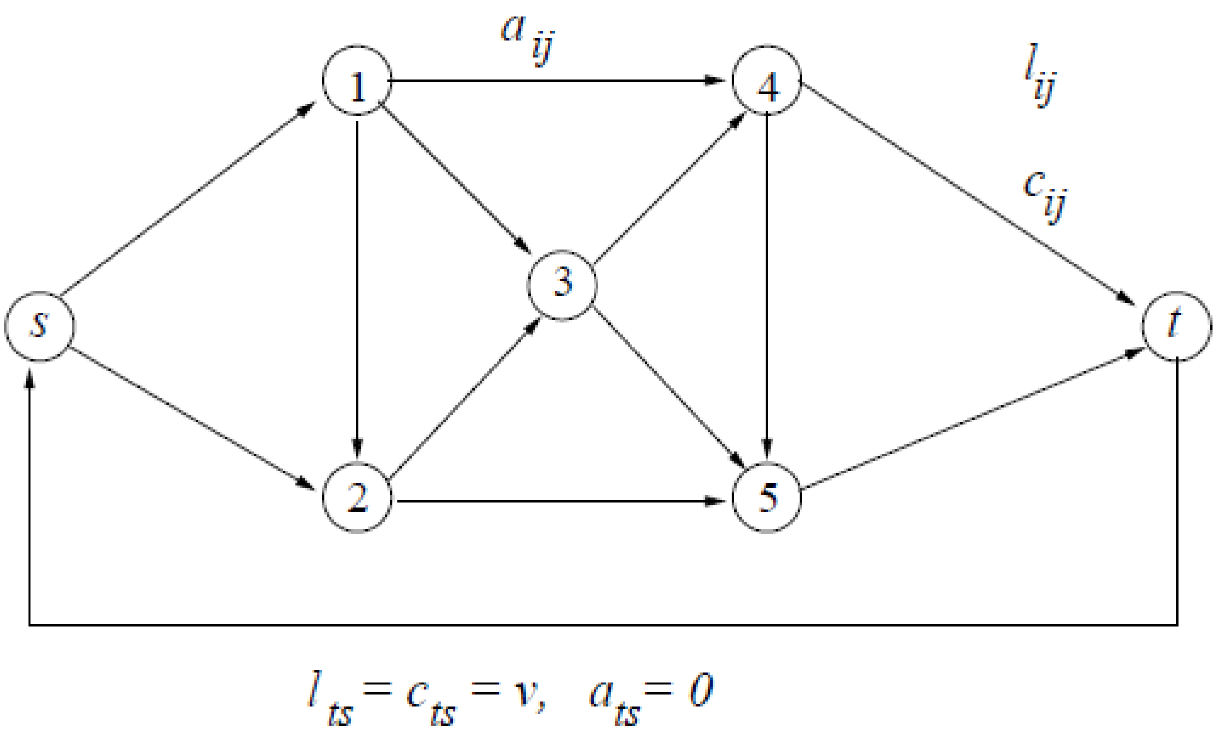
Example: Maximum Flow Problem



Maximum Flow Problem

- To the given flow network with source s and sink t add a return arc (t, s) with $l_{ts} = 0, c_{ts} = +\infty$ and $a_{ts} = -1$. For all other arcs (i, j), the lower bounds (if any) and capacities are as given and $a_{ij} = 0$. (For a minimum flow problem, set $a_{ts} = 1$.)

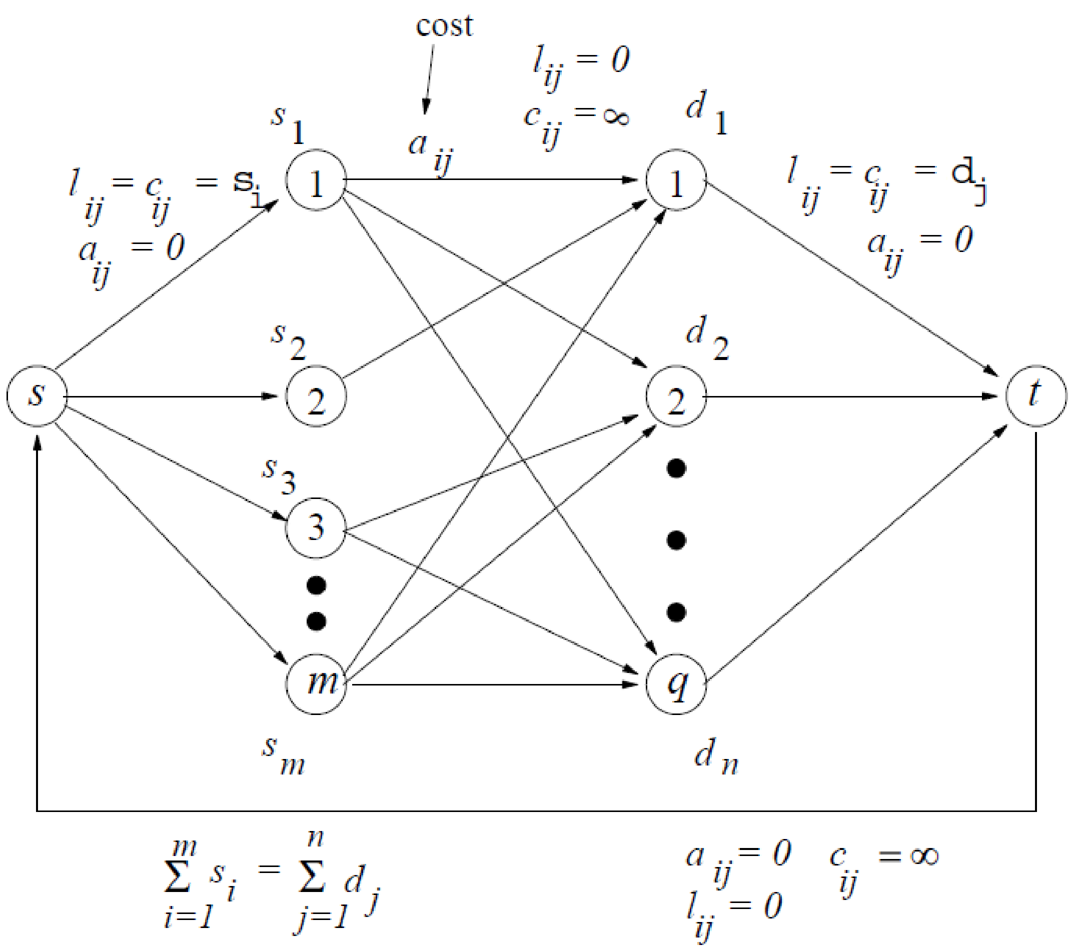
Example: Minimum Cost Flow Problem



Minimum Cost Flow Problem

- Add a return arc (t, s) with $l_{ts} =$ ~~0~~ v, $c_{ts} = v$, and $a_{ts} = 0$. The lower bounds, capacities, and costs of all other arcs are as given.

Example: Transportation Problem



Questions

- 1. Feasibility
  - Is the min cost circulation problem always feasible?
  - How do we know there exists a feasible “circulation” or not?
  - How do we find a feasible circulation, if it exists?
- 2. Optimality
  - How do we know a feasible circulation is of the minimum cost?
- 3. Algorithm
  - How to find an optimal circulation?
  - Complexity?

Feasible Circulation Problem

- Is there a feasible circulation in a given network with lower bounds and capacities?

Set $a_{ij} = 0$, for all arcs (i, j).

Observation 1 - Feasibility

- Think about this simple fact:

  - For a feasible flow, the sum of flows coming into a node at the lowest level required

    should be no more than

    the sum of flows going out of a node at the highest level allowed.

Any Proof?

If there exists a feasible circulation $x = (x_{ij})$, then for any cutset (S, T)

$$\sum_{i \in S} \left(\sum_j x_{ji} - \sum_j x_{ij} \right) = 0.$$

Then

$$\sum_{i \in S} \left[\sum_{j \in S} x_{ji} + \sum_{j \in T} x_{ji} - \sum_{j \in S} x_{ij} - \sum_{j \in T} x_{ij} \right] = 0.$$

Hence

$$\sum_{i \in T} \sum_{j \in S} l_{ij} \leq \sum_{i \in T} \sum_{j \in S} x_{ij} = \sum_{i \in S} \sum_{j \in T} x_{ji} = \sum_{i \in S} \sum_{j \in T} x_{ij} \leq \sum_{i \in S} \sum_{j \in T} c_{ij}.$$

Conjecture for a Stronger Result

- In a network with lower bounds and capacities on the links, there exists a feasible circulation if and only if

$$\sum_{i \in T, j \in S} l_{ij} \leq \sum_{i \in S, j \in T} c_{ij}$$

for all cutsets(S, T).

Proof of the Conjecture

- The “If part” is easy, i.e., we have just showed that “if there exists a feasible circulation, then the inequality holds for any cut set.”
- How about the “only if” part? i.e., “if there exists no feasible circulation, then there is cut set for which the inequality fails.”
- The question is that “do we have a way to find a feasible circulation, when there is one?”

Observation 2 – Find a Feasible Circulation

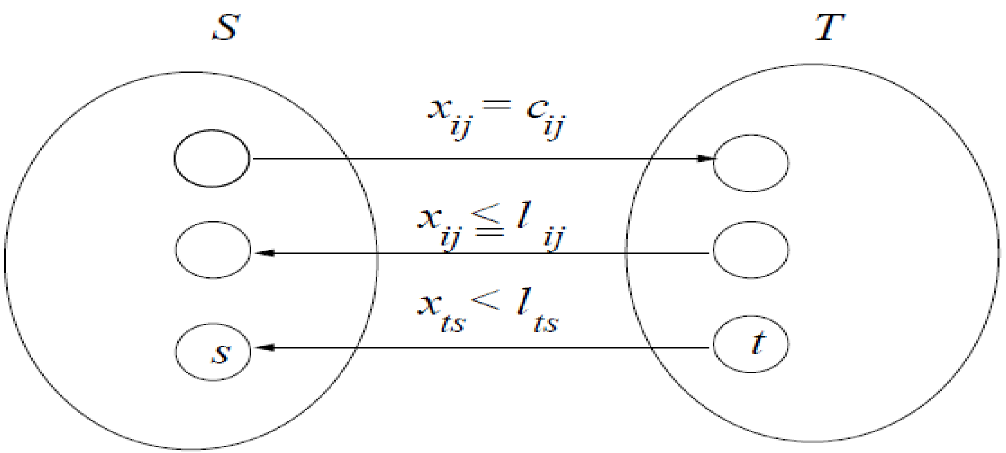
- Here is how to find a feasible circulation in a network with both lower bounds and capacities, if such a circulation exists. Begin with the zero circulation. If all lower bounds are zero, this circulation is feasible. Otherwise, find an arc (p, q) for which $x_{pq} < l_{pq}$. Construct a flow augmenting path from q to p where this path is of the conventional type, except that we require $x_{ij} > l_{ij}$ for each backward arc and δ is chosen such that $\delta \leq x_{ij} - l_{ij}$. Augment the flow from q to p by δ, and repeat until $x_{pq} \geq l_{pq}$. Then feasible circulation is obtained, if the network admits such a circulation.

Find a Feasible Circulation

- For the above mentioned procedure, notice that all links in the augmenting path won’t reduce its link flows under the lower bounds, nor increase its link flows above the capacities along the path. Moreover, it will strictly increase the flow value of x_{pq} by an integer value. Hence we can continue this process until x_{pq} is over l_{pq}, and then work on other links with flow value below its lower bound, if there is indeed a feasible circulation.
- When this process fails, what will happen?

Observation 3

If no feasible circulation exists, then at some point an augmenting path cannot be found for some arc (t, s) with $x_{ts} < l_{ts}$. Let $S = \{\text{nodes can be reached from } s \text{ by an augmenting path}\}$ and $T = S^c$. Then



Since the flow conservation was maintained at each step, we have

$$\sum_{i \in S} \sum_{j \in T} x_{ij} = \sum_{i \in T} \sum_{j \in S} x_{ij} < \sum_{i \in T} \sum_{j \in S} l_{ij}$$

$$\sum_{i \in S} \sum_{j \in T} x_{ij} = \sum_{i \in S} \sum_{j \in T} c_{ij}$$

Major Result

- **Theorem 9.1** *(Hoffman)* In a network with lower bounds and capacities a feasible circulation exists if and only if

$$\sum_{i \in T, j \in S} l_{ij} \leq \sum_{i \in S, j \in T} c_{ij} \quad (9.1)$$

for all cutsets(S, T).

Corollary

- **Corollary 9.2** *(Generalized Max-Flow Min-Cut Theorem)* Let G be a flow network with lower bounds and capacities and which admits a feasible (s, t)-flow. The maximum value of an (s, t)-flow in G is equal to the minimum capacity of an (s, t)-cutset, where the capacity of cutset(S, T) is defined as

$$c(S,T) = \sum_{i \in S, j \in T} c_{ij} - \sum_{i \in T, j \in S} l_{ij}.$$

- Notice that when the lower bounds $l_{ij} = 0$, we have the conventional max-flw min-cut set theorem as before.

Corollary

- **Corollary 9.3** *(Min-Flow Max-Cut Theorem)* Let G be a flow network with lower bounds and capacities and which admits a feasible (s,t)-flow. The minimum value of an (s,t)-flow in G is equal to the maximum of

$$\sum_{i \in S, j \in T} l_{ij} - \sum_{i \in T, j \in S} c_{ij}$$

over all (s,t)-cutsets (S,T), or equivalently, the negative of the minimum capacity of a (t,s)-cutset.

Observation 4 – Optimality Check

- Primal Problem vs. Dual Problem

$$\text{minimize} \sum_{i,j} a_{ij} x_{ij}$$

subject to

$$\sum_j x_{ji} - \sum_j x_{ij} = 0, \quad all\ i \qquad (10.1)$$

$$0 \le l_{ij} \le x_{ij} \le c_{ij}, \quad all\ i,j$$

maximize

$$\sum_{i,j} l_{ij}\lambda_{ij} - \sum_{i,j} c_{ij}\gamma_{ij}$$

subject to

$$u_j - u_i + \lambda_{ij} - \gamma_{ij} \le a_{ij} \qquad (10.2)$$

$$\lambda_{ij}, \gamma_{ij} \ge 0$$

$$u_i \ unrestricted.$$

Optimality Conditions

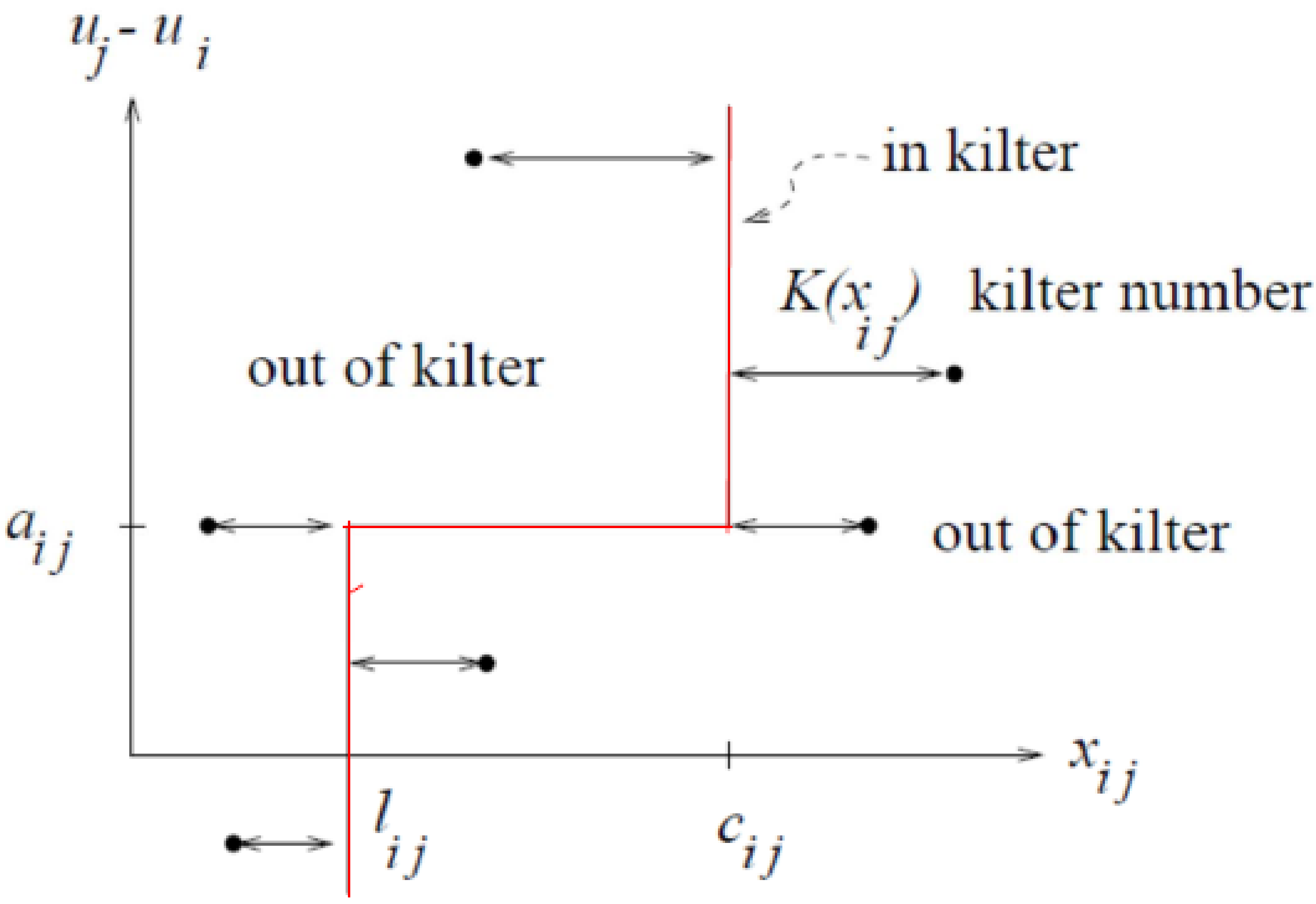
◇ The orthogonality conditions which are necessary and sufficient for optimality of primal and dual solutions:

$$
\begin{aligned}
x_{ij} > 0 \quad &\Rightarrow \quad u_j - u_i + \lambda_{ij} - \gamma_{ij} = a_{ij} \\
\lambda_{ij} > 0 \quad &\Rightarrow \quad x_{ij} = l_{ij} \\
\gamma_{ij} > 0 \quad &\Rightarrow \quad x_{ij} = c_{ij}.
\end{aligned}
$$

Above conditions are equivalent to the following:

$$
\begin{aligned}
x_{ij} = l_{ij} \quad &\Rightarrow \quad u_j - u_i \leq a_{ij} \\
l_{ij} < x_{ij} < c_{ij} \quad &\Rightarrow \quad u_j - u_i = a_{ij} \qquad (10.3) \\
x_{ij} = c_{ij} \quad &\Rightarrow \quad u_j - u_i \geq a_{ij}
\end{aligned}
$$

“Good” Optimality Conditions

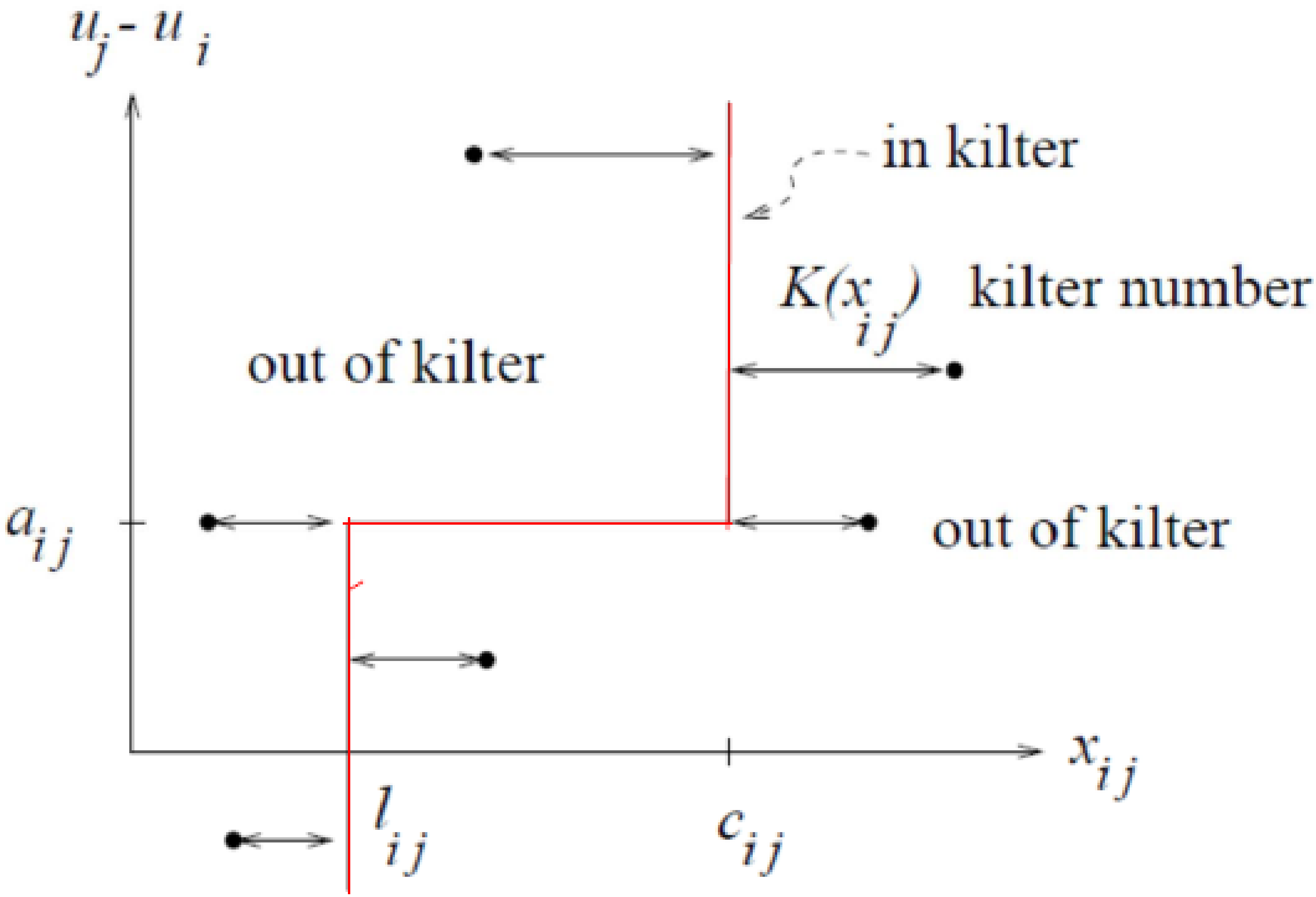


Kilter Conditions

- We refer to conditions(10.3) as *kilter conditions* and represent them by a *kilter diagram* for each arc as shown in Figure 4.14. Points$(x_{ij}, u_j - u_i)$ on the crooked line are *in kilter* and those which are not are *out of kilter*. To each point $(x_{ij}, u_j - u_i)$ we assign a kilter number $K(x_{ij})$ equal to the absolute value of the change in x_{ij} necessary to bring the arc into kilter. Thus,

$$K(x_{ij}) = \begin{cases} |x_{ij} - l_{ij}|, & if\ u_j - u_i < a_{ij} \\ l_{ij} - x_{ij}, & if\ x_{ij} < l_{ij}, u_j - u_i = a_{ij} \\ x_{ij} - c_{ij}, & if\ x_{ij} > c_{ij}, u_j - u_i = a_{ij} \\ 0, & if\ l_{ij} \le x_{ij} \le c_{ij}, u_j - u_i = a_{ij} \\ |x_{ij} - c_{ij}|, & if\ u_j - u_i > a_{ij}. \end{cases}$$

Kilter Diagram



Motivation

- Given a circulation with a zero value of the sum of kilter numbers of each link, then the circulation must represent an optimal solution.

- The question is “how to find such a circulation?”

- Can we design an algorithm to reduce the total kilter number in each iteration until an optimal solution is found?

Observation 5 – Algorithm Design

- The objective of the out-of-kilter method is to obtain a circulation (x_{ij}) and a set of node number (u_i) such that the kilter condition (10.3) are satisfied.
- The sum of kilter numbers $(\sum_{i,j} K(x_{ij}))$ should be reduced in each step by making a change either in the circulation or in the node numbers.
- The change is made by applying Minty's painting theorem (Thm 7.2 of Ch. 2).

Minty's Painting Theorem

For any green-yellow-red colored digraph, and any given yellow arc (t, s), there exists exactly one of two cases:

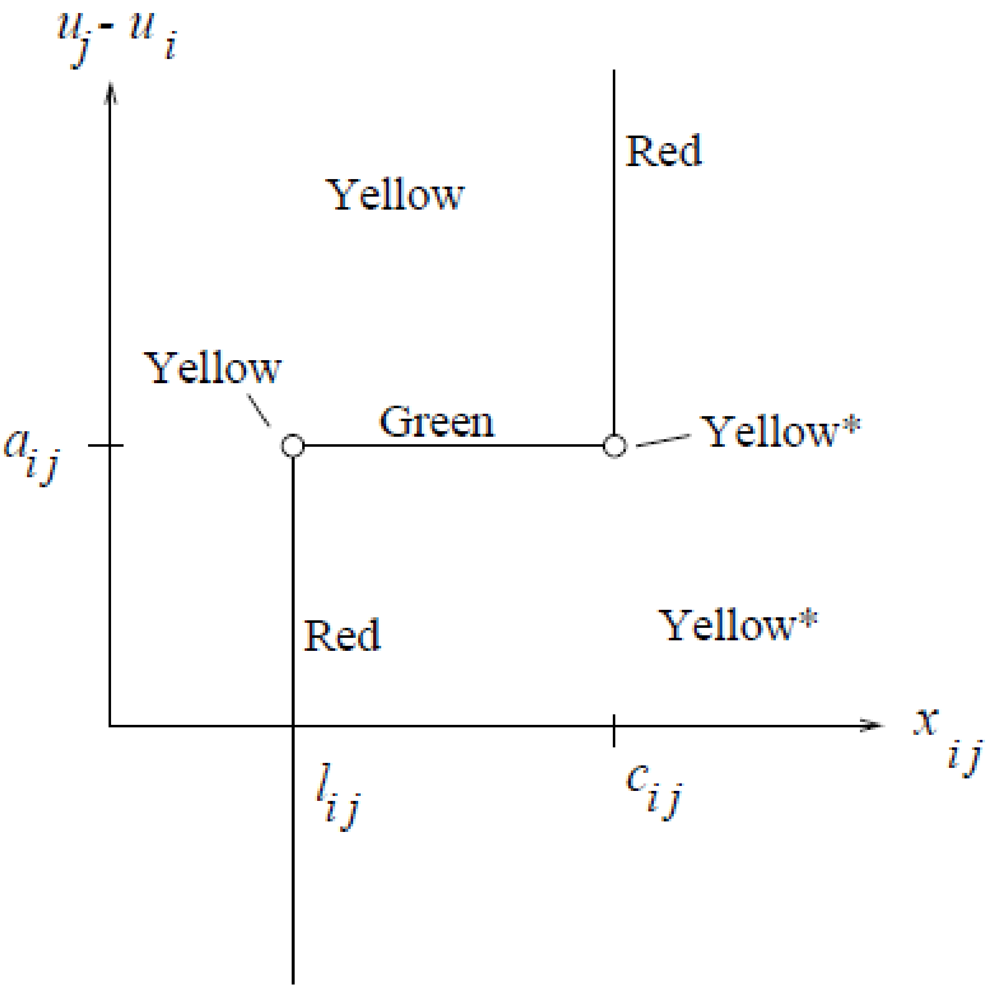
(Case 1): A yellow-green cycle containing (t, s) and all yellow arcs are oriented in the same direction as (t, s).

(Case 2): A yellow-red cocycle containing (t, s) and all yellow arcs are oriented in the same direction as (t, s).

In case 1, we adjust the circulation.

In case 2, we adjust the node numbers.

Observation 6 – Painting the arcs



Rules for Painting Arcs

G: (increase/decrease x_{ij})

$$(10.4) \quad l_{ij} < x_{ij} < c_{ij} \quad \& \quad u_j - u_i = a_{ij}$$

Y: (increase)

$$(10.5) \begin{cases} x_{ij} < c_{ij} & \& \quad u_j - u_i > a_{ij} \\ x_{ij} \leq l_{ij} & \& \quad u_j - u_i = a_{ij} \\ x_{ij} < l_{ij} & \& \quad u_j - u_i < a_{ij} \end{cases}$$

Y*: (decrease) ← reverse direction of (i, j)

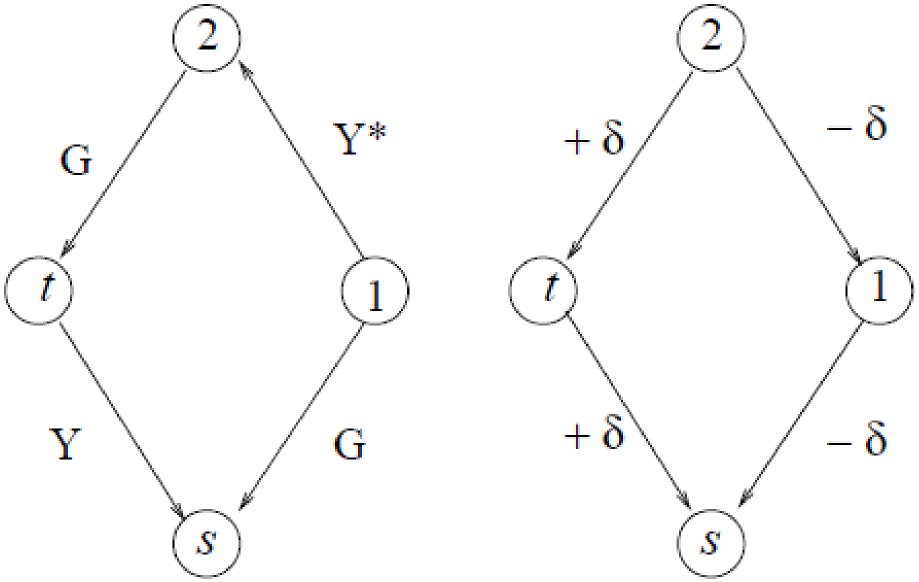
$$(10.6) \begin{cases} x_{ij} > c_{ij} & \& \quad u_j - u_i > a_{ij} \\ x_{ij} \geq c_{ij} & \& \quad u_j - u_i = a_{ij} \\ x_{ij} > l_{ij} & \& \quad u_j - u_i < a_{ij} \end{cases}$$

R: (no change in x_{ij})

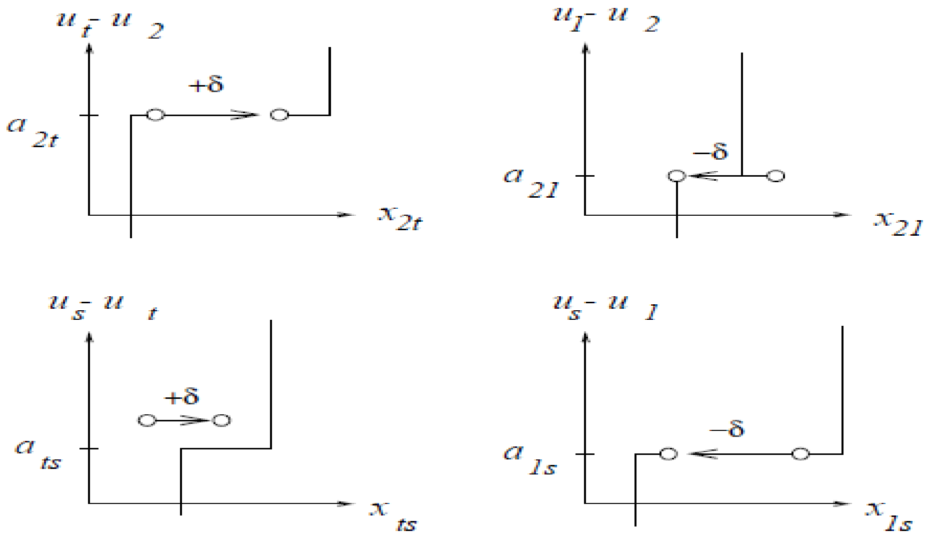
$$(10.7) \begin{cases} x_{ij} = c_{ij} & \& \quad u_j - u_i > a_{ij} \\ x_{ij} = l_{ij} & \& \quad u_j - u_i < a_{ij} \end{cases}$$

Applying Minty's Theorem

(Case 1) There exists a yellow-green cycle C, in which all yellow arcs are oriented as (t, s). An increase by a small amount $\delta > 0$ results in a new circulation with strictly smaller kilter number.



Case 1 (taking Y+ for Y and Y- for Y*)



How large can δ be?

(i) If $u_j - u_i = a_{ij}$, $(i, j) \in Y^+$ or G^+
δ can be no larger than $c_{ij} - x_{ij}$,

(ii) If $u_j - u_i = a_{ij}$, $(i, j) \in Y^-$ or G^-
δ can be no larger than $x_{ij} - l_{ij}$.

(iii) If $u_j - u_i > a_{ij}$, $(i, j) \in Y^+$ or Y^-
δ can be no larger than $|c_{ij} - x_{ij}|$.

(iv) If $u_j - u_i < a_{ij}$, $(i, j) \in Y^+$ or Y^-
δ can be no larger than $|x_{ij} - l_{ij}|$.

Case 1

Let

$$\delta_1 = \min_{(i,j)}\{c_{ij} - x_{ij} \mid (i,j) \in Y^+ \cup G^+,\ u_j - u_i = a_{ij}\}$$

$$\delta_2 = \min_{(i,j)}\{x_{ij} - l_{ij} \mid (i,j) \in Y^- \cup G^-,\ u_j - u_i = a_{ij}\}$$

$$\delta_3 = \min_{(i,j)}\{|c_{ij} - x_{ij}| \mid (i,j) \in Y^+ \cup Y^-,\ u_j - u_i > a_{ij}\}$$

$$\delta_4 = \min_{(i,j)}\{|x_{ij} - l_{ij}| \mid (i,j) \in Y^+ \cup Y^-,\ u_j - u_i < a_{ij}\}$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$.

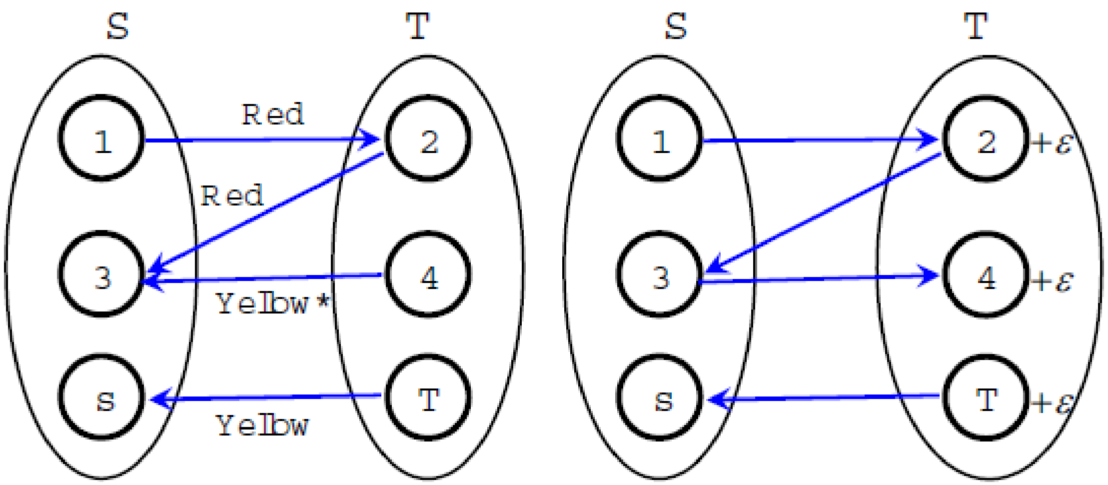
Case 1 – Two possible outcomes

1. If $\delta > 0$ is finite, the kilter number of at least one arc is decreased while other arcs' kilter numbers are not worse than before.

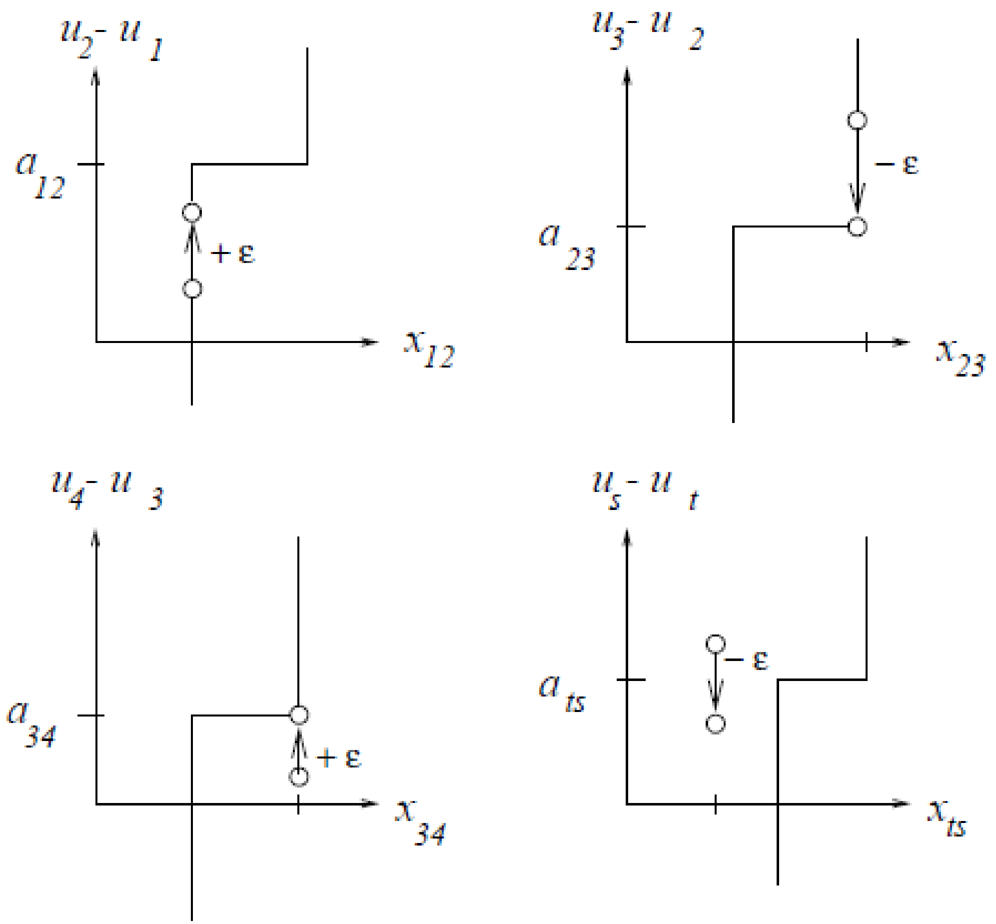
2. If δ is unbounded, there is no finite optimal circulation. This can happen when capacities of arcs in the cycle are infinite and the net cost of circulation around the cycle is negative.

Case 2

(Case 2) There is a yellow-red cocycle (S, T) with $s \in S, t \in T$, in which all yellow arcs are oriented as (t, s). An increase by a small amount $\varepsilon > 0$ in the node numbers of all nodes $i \in T$ will not increase the kilter number of any arc, and may actually decrease some of kilter numbers.



Case 2



Case 2

How large can ε be? Let

$$
\begin{aligned}
\varepsilon_1 &= \min_{(i,j)}\{a_{ij} - (u_j - u_i) \mid (i,j) \in R^+,\ x_{ij} = l_{ij}\} \\
\varepsilon_2 &= \min_{(i,j)}\{(u_j - u_i) - a_{ij} \mid (i,j) \in R^-,\ x_{ij} = c_{ij}\} \\
\varepsilon_3 &= \min_{(i,j)}\{a_{ij} - (u_j - u_i) \mid (i,j) \in Y^+,\ l_{ij} < x_{ij} \le c_{ij}\} \\
\varepsilon_4 &= \min_{(i,j)}\{(u_j - u_i) - a_{ij} \mid (i,j) \in Y^-,\ l_{ij} \le x_{ij} < c_{ij}\}
\end{aligned}
$$

Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$

Case 2 – Three possible outcomes

1. $\varepsilon > 0$ is unbounded, then each set in the definition of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is empty. This can occur only if

$$\begin{cases} x_{ij} \geq c_{ij}, & \forall (i,j) \text{ with } i \in S, j \in T \\ x_{ij} \leq l_{ij} & \forall (i,j) \text{ with } i \in T, j \in S \\ x_{ts} < l_{ts} \end{cases}$$

Since the net flow from S to T is zero, so

$$\sum_{i \in S} \sum_{j \in T} l_{ij} > \sum_{i \in T} \sum_{i \in S} c_{ij}$$

By Theorem 9.1, no feasible circulation exists.

2. ε is finite and equal to ε_3 or ε_4. At least one out-of-kilter arc is brought into kilter, and no kilter number of any arc is increased.

Case 2 – Three possible outcomes

3. ε is finite but less than ε_3 and ε_4.

   No out-of-kilter arc becomes kilter.

   No kilter number is increased.

   At least one red arc will become yellow.

   Some yellow arcs may become red.

   No green arcs are affected.

How to paint the arcs?

Labeling procedure as in the proof of the Minty's Thm in chapter 2.

Green: Two way street

Yellow: One way street

Red: Blocked street

- Start with s (initially labeled),
- All nodes reachable from s are successively labeled,
- If t is reachable from s, then backtracking leads to a yellow-green cycle,
- If t is not reachable, let $S = \{$all nodes reachable from $s\}$, $T = S^c$, then we have a yellow-red cocycle (S, T).

Outline of the Out-of-Kilter Algorithm

Step 0: (Start)

Let $x = (x_{ij})$ be a circulation and $u = (u_i)$ be any set of node numbers.

Say $x = (0), u = (0)$.

Step 1: (Painting and Labeling)

(1.0) Check kilter numbers to stop.

(1.1) Minty's Painting Theorem.

Step 2: (Change in Circulation)

return to step 1.

Step 3: (Change in Node Numbers)

return to step 1.

Out-of-Kilter Algorithm

- *Step 0 (Start)* Let $x = (x_{ij})$ be any circulation, possibly infeasible, but satisfying conservation conditions, and let $u = (u_i)$ be any set of node numbers. It is desirable to start with x, u such that the sum of the kilter numbers is small, but $x = (0), u = (0)$ will do.

- *Step 1 (Painting and Labeling)*
  (1.0) If all arcs are in kilter, halt; the existing circulation is optimal and u is an optimal dual solution. Otherwise paint the arcs green, yellow, and red, in accordance with rules (10.4) through (10.7). Set $\pi_i = +\infty$ for all nodes i. Choose any arc (t, s) which is out of kilter and apply the permanent label "$\emptyset$" to s. No other nodes have labels.

Out-of-Kilter Algorithm

(1.1) If all permanently labeled nodes have been scanned, go to Step 3. Otherwise, find a permanently labeled but unscanned node i and scan it as follows: For each yellow or green arc (i, j) and for each green arc (j, i), if j does not already have a permanent label, give j the permanent label "i" (replacing any existing tentative label). For each red arc (i, j), if $x_{ij} = l_{ij}$ and $u_j - u_i - a_{ij} < \pi_j$ give j the tentative label "i" (replacing any existing label) and set $\pi_j = u_j - u_i - a_{ij}$. For each red arc (j, i), if $x_{ji} = c_{ji}$ and $a_{ji} + u_j - u_i < \pi_j$ give j the tentative label "i" (replacing any existing label) and set $u_j = a_{ji} + u_j - u_i$.

(1.2) If node t has been given a permanent label, go to Step 2; otherwise, go to Step 1.1.

Out-of-Kilter Algorithm

- *Step 2 (Change in Circulation)* Identify a yellow-green cycle C by using the label on t to backtrace to s. Determine δ by (10.8). If δ is unbounded, there is no finite optimal solution and the computation is terminated. Otherwise, increment or decrement the flow in each arc in C by δ. Erase all labels on nodes and go to Step 1.0.

Out-of-Kilter Algorithm

- *Step 3 (Change in Node Numbers)* Let S contain the all permanently labeled nodes and T contain the remaining nodes. (S, T) is a yellow-red cutset. Determine ε by (10.9). If ε is unbounded, no feasible circulation exists and the computation is terminated. Otherwise, add ε to u_i for each node i in T. If Case 2 applies, go to Step 1.0. If Case 3 applies, subtract ε from π_i for each node i in T and make the labels permanent on all nodes for which $\pi_i = 0$. Then go to Step 1.1.

- The out-of-kilter method is easily adapted to handle piecewise linear convex arc costs.

Convergence Proof

- Assumptions
  - all lower bounds and capacities are integers;
  - the initial circulation is integral.
- Implications
  - $K(i, j)$ are non-negative integers in each iteration.

- Let K = sum of the kilter numbers for the initial circulation.

Reasoning

- Each discovery of a yellow-green cycle decreases at least one kilter number by some integer $\delta \geq 1$.

  Therefore, no more than K revisions of the circulation are necessary.

- Each discovery of a yellow-red cocycle results in either (i) an out-of-kilter arc becomes kilter or (ii) at least one red arcs become yellow.

For (i), it reduces at least one positive kilter number to zero. Therefore, no more than $\min(m, K)$ times in all.

For (ii), it means an additional node i in T will become reachable from s the next time the labeling procedure is applied. Also all nodes reachable from s remain reachable. Therefore, (ii) can occur at most $n-1$ times in succession before either a cycle is discovered or an out-of-kilter arc becomes kilter.

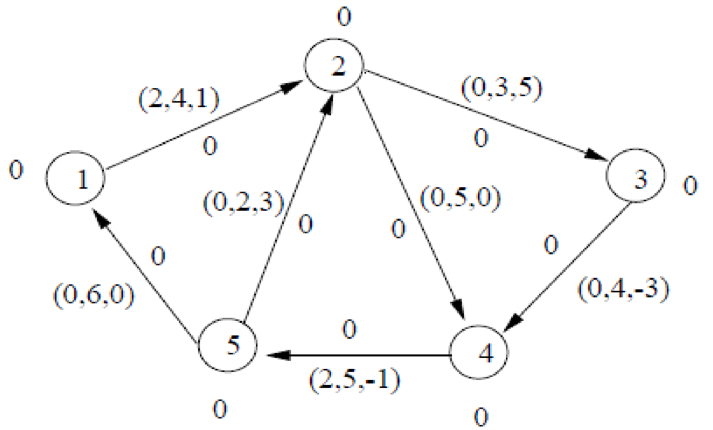
Complexity Analysis

- At most $n - 1$ labeling procedures are needed either to find a yellow-green cycle or to bring an out-of-kilter arc into kilter. In either case, the total kilter number K is reduced at least by one.
- $O(nK)$ labeling procedures required.
- Each labeling procedure requires $O(m)$ elementary operations.
- Total complexity $= O(mnK)$.
- An efficient implementation can preserve the labels after the discovery of a cocycle for which (ii) applies. Hence, the complexity is $O(mK)$.

Complexity Analysis

- We concluded the discussion of the out-of-kilter method by establishing a bound of $O(Km)$ on the number of steps, where K is the sum of the arc kilter numbers for the initial primal and dual solutions. If $x = 0, u = 0$ are taken as initial solutions, then K may be as large as the sum of all arc capacities, which are assumed to be integers.

Example

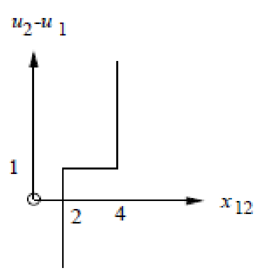


(l_{ij}, c_{ij}, a_{ij})

$x=(0)$

$u=(0)$

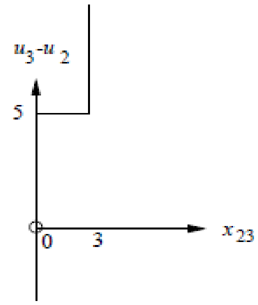
Check Kilter States



out-of-kilter

$k_{12}=2$

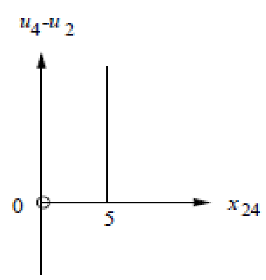
yellow



in-kilter

$k_{23}=0$

red

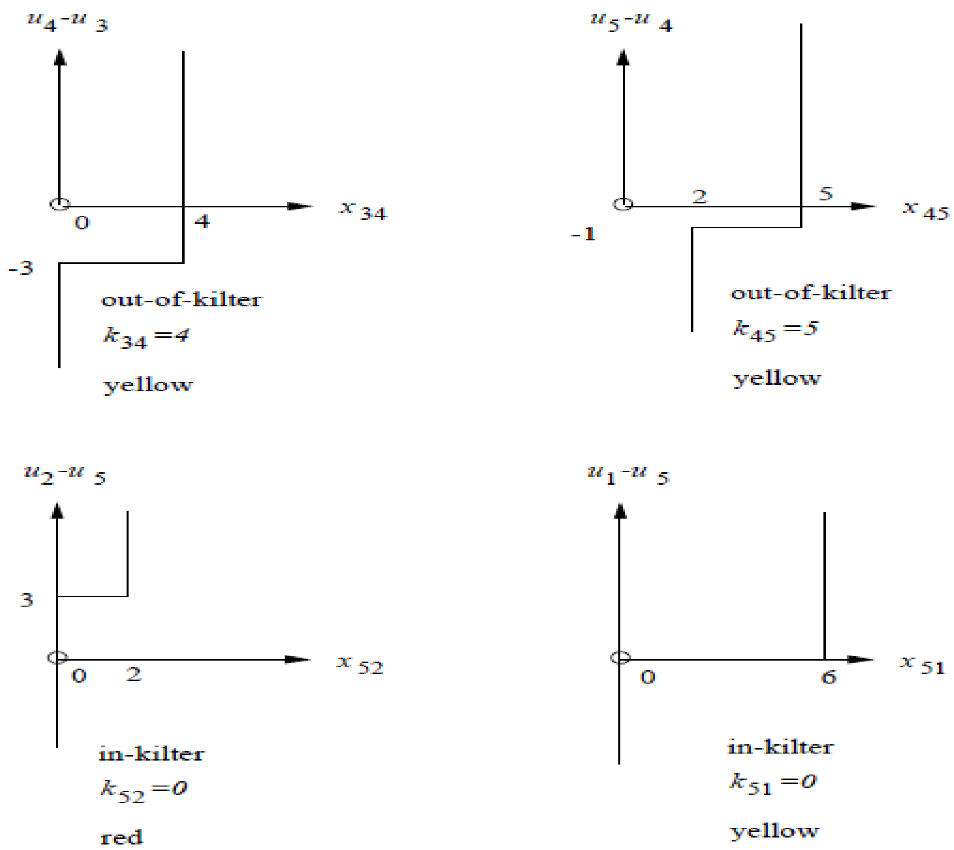


in-kilter

$k_{24}=0$

yellow

Example



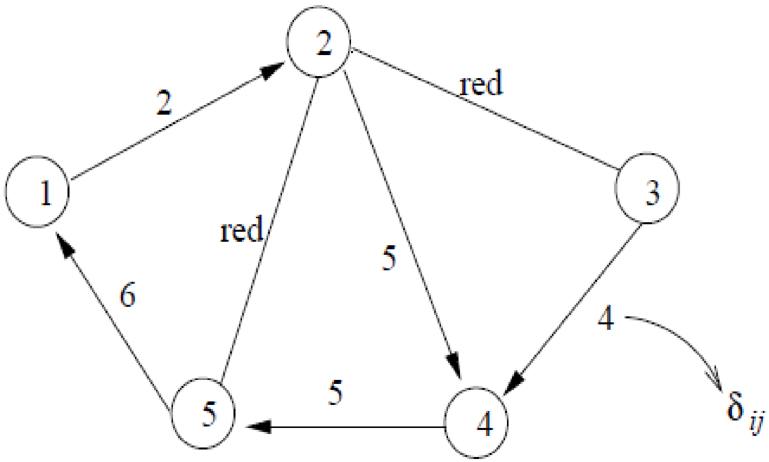
$$K = 2 + 0 + 0 + 4 + 5 + 0 + 0 = 11$$

Example

Minty's Painting

Pick an out-of-kilter yellow arc (1,2).

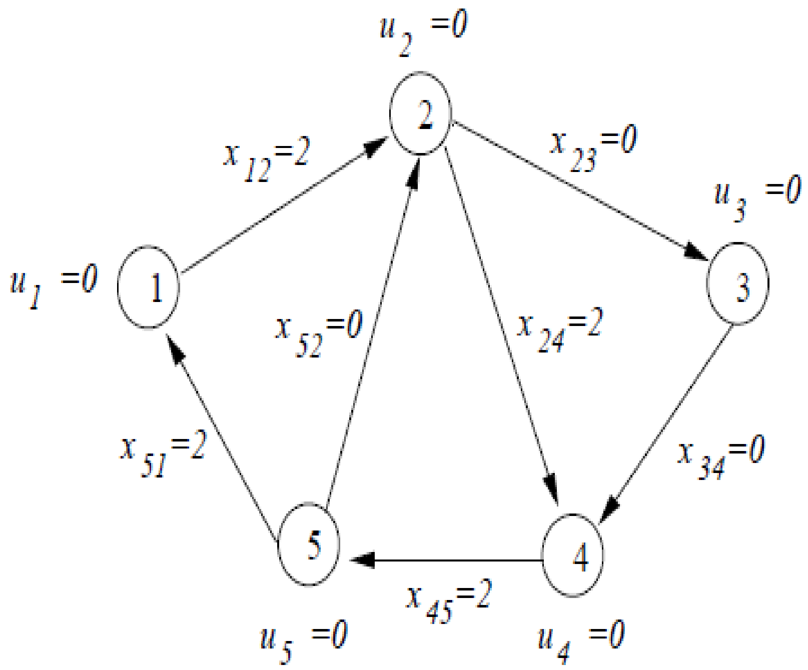
Find $C = \{(1,2),(2,4),(4,5),(5,1)\}$.



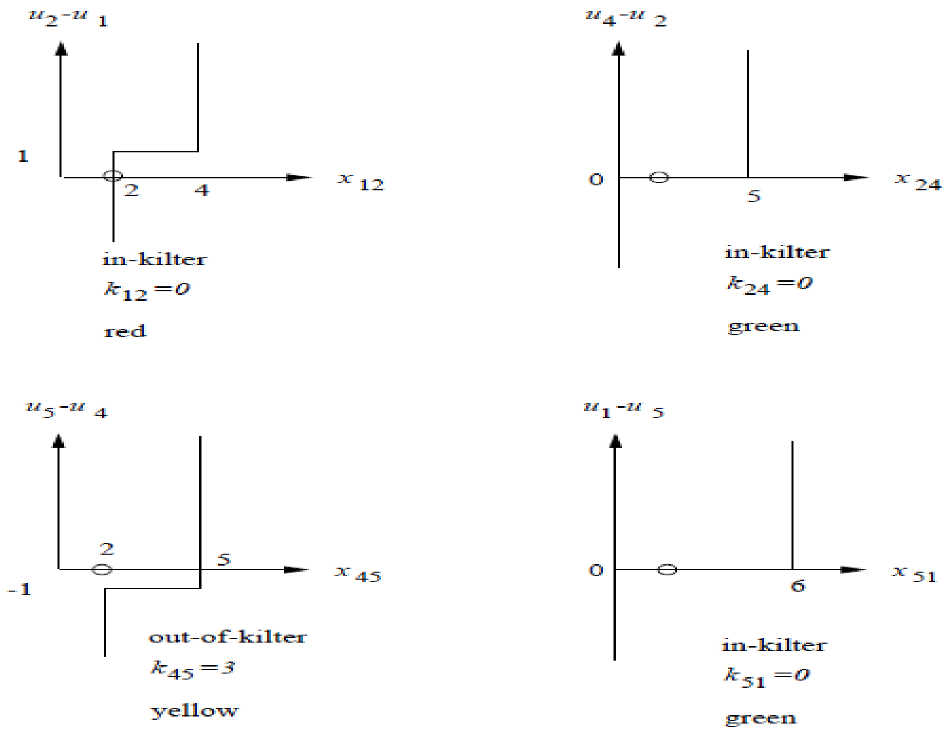
$\delta = 2$

Example

Change in Circulation



Example



x_{23} : no change.

x_{34} : no change.

x_{52} : no change.

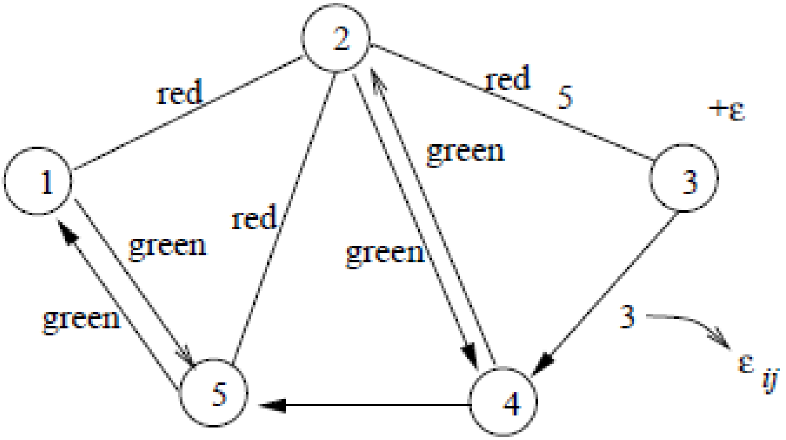
$K = 0 + 0 + 0 + 4 + 3 + 0 + 0 = 7 < 11.$

Example

<u>Minty's Painting</u>

Pick an out-of-kilter yellow arc (3,4).

Find a cocycle $S = \{4, 2, 5, 1\}, T = \{3\}$.



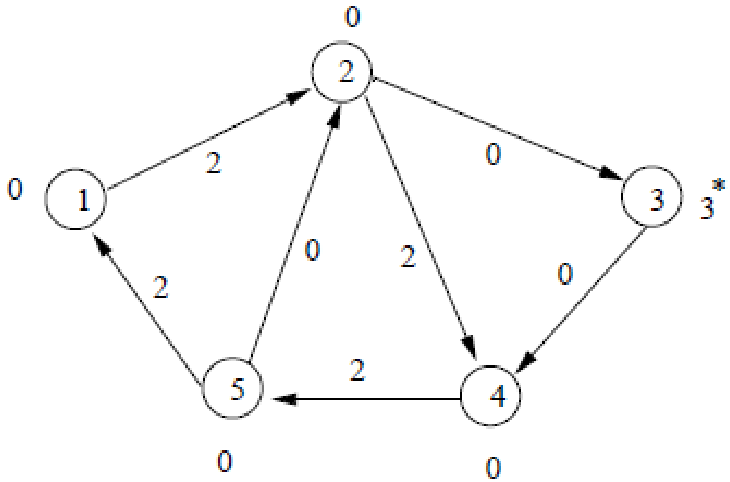
Example

Change in Node Numbers

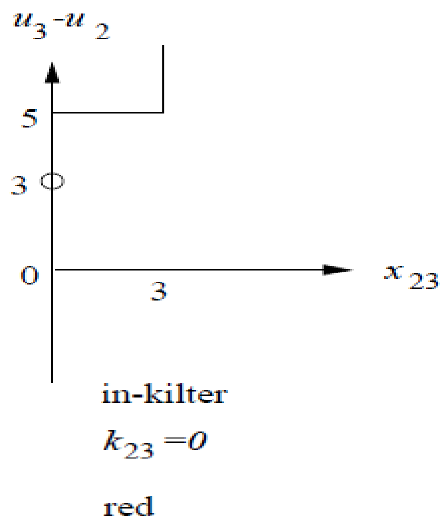
$$\varepsilon = 3$$

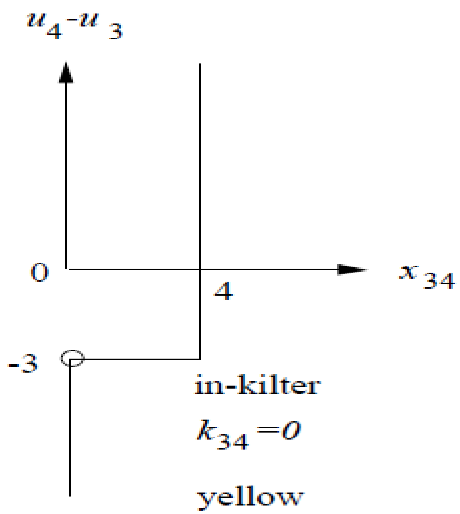
$$u_1 = u_2 = u_4 = u_5 = 0$$

$$u_3 = 3$$



Example





x_{12} : no change.

x_{24} : no change.

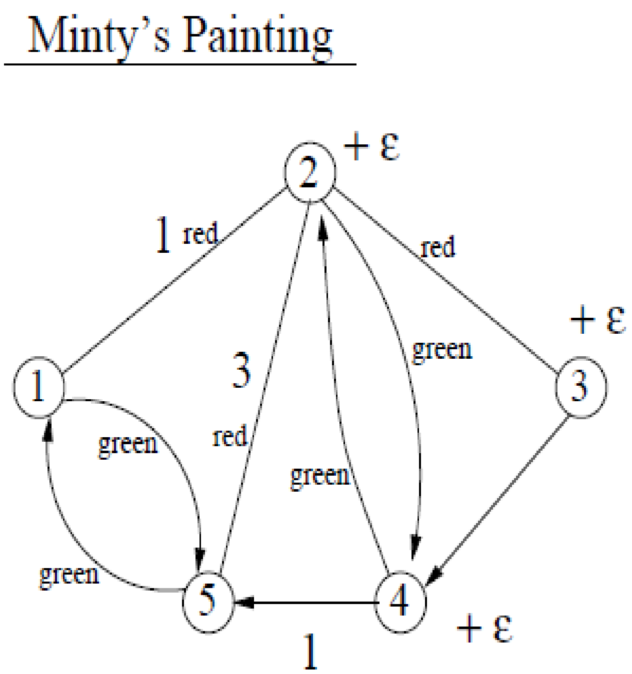
x_{45} : no change.

x_{52} : no change.

x_{51} : no change.

$K = 3 < 7.$

Example



Pick an out-of-kilter arc (4, 5)

Find a cocycle

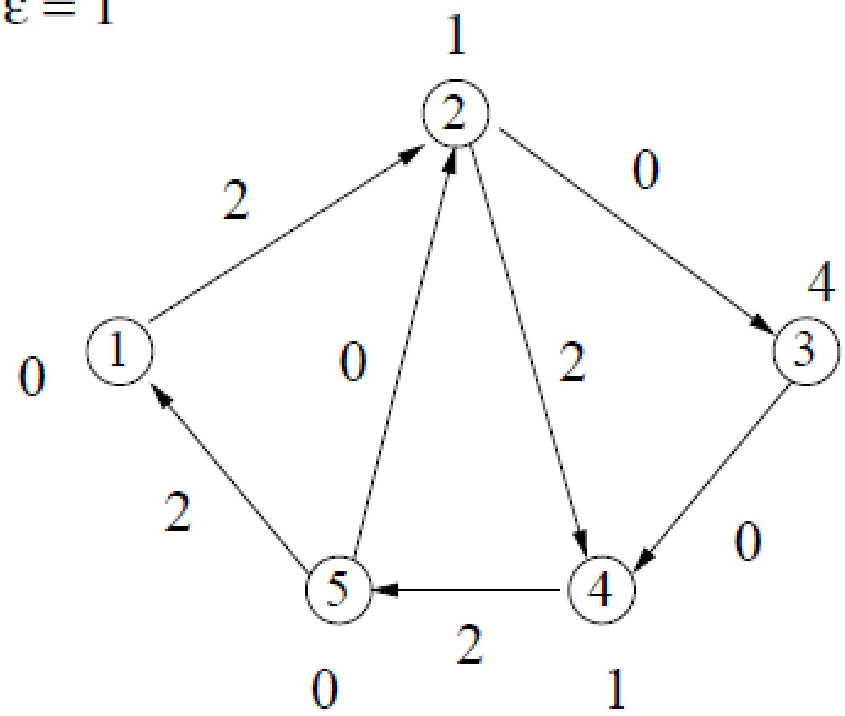
$S = \{5, 1\}$

$T = \{4, 2, 3\}$

Example

Change in Node Numbers

$\varepsilon = 1$



$u_1 = 0$
$u_2 = 1$
$u_3 = 4$
$u_4 = 1$
$u_5 = 0$

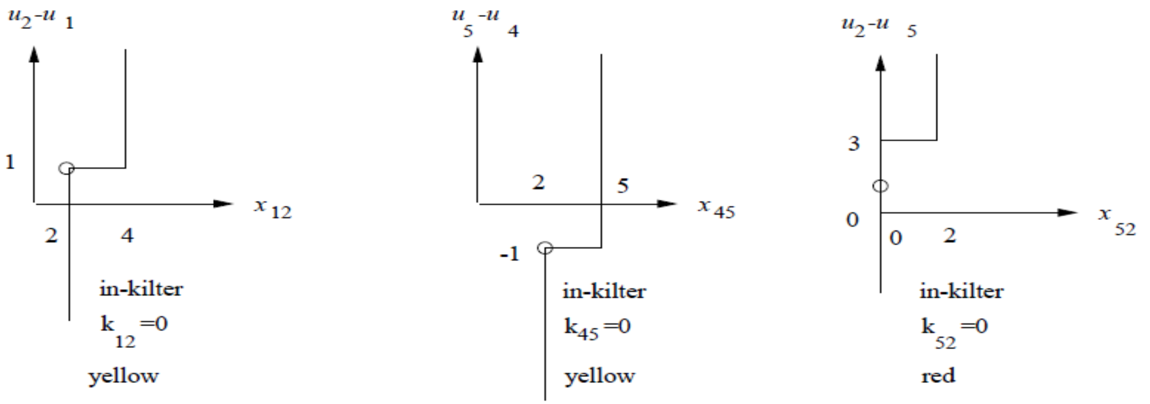
Example

x_{23} : no change $k_{23} = 0$

x_{34} : no change $k_{34} = 0$

x_{24} : no change $k_{24} = 0$

x_{51} : no change $k_{51} = 0$



$K = 0$ => Optimal Solution, STOP!

min-cost = 0

Integral Flows and Total Unimodularity

- The nature of the out-of-kilter method is such that it provides a constructive proof of the following theorem.

- **Theorem 12.1** *(Integral Circulation Theorem)* If all lower bounds and capacities are integers and there exists a finite optimal circulation, then there exists an integral optimal circulation (whether or not arc costs are integers).

  Some insight into Theorem 12.1 is obtained by an examination of the algebraic structure of the circulation problem from the viewpoint of linear programming.