M. Math 2003-4, Functional Analysis, 2nd semestral examination. Total 60.

1. Let X be a finite dimensional normed linear space. Show that X is a Banach space. [5]

2. Let $M = \{f \in C([0,1]) : f([0, \frac{1}{2}]) = 0\}$. Let $\Phi : C([0,1])/M \to C([0, \frac{1}{2}])$ be defined by $\Phi(\pi(f)) = f|[0, \frac{1}{2}]$ where π is the quotient map. Show that Φ is an onto isometry. [10]

3. For $f \in C([0,1])$ define $||f||_1 = \int |f| dx$. Show that this is a norm. Show that this norm is not equivalent to the supremum norm. [5]

4. Let X and Y be Banach spaces. Let $\{T_n\}_{n \geq 1}$ $\subset L(X,Y)$, be a sequence of compact operators. Suppose $T \in L(X,Y)$ and $||T - T_n|| \to 0$. Show that T is a compact operator. [10]

5. Let $\{f_n\}_{n \geq 1} \subset L^2([0,1])$ be an ortho normal sequence. Define $\Psi : L^2([0,1]) \to \ell^2$ by $\Psi(f) = (\int f \bar{f}_n dx)_{n \geq 1}$. Show that Ψ is an onto map and Ψ^* is a one-to-one map. [10]

6. Let H be a Hilbert space. Show that $N \in L(H)$ is a normal operator if and only if $||N(x)|| = ||N^*(x)||$ for all $x \in H$. Hence or otherwise show that there exists a $S \in L(H)$ such that $SN = N^*$. [10]

7. Let X be a Banach space. Let $P \neq Q \in L(X)$ be projections. Show that P^*, Q^* are projections in $L(X^*)$. If $PQ = QP$ show that $||P - Q|| \geq 1$. [10]