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NOTE ON REGULAR SEQUENCES ON SEMIGROUPS

RYÛKI MATSUDA

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ABSTRACT. We determine what are Macaulay semigroups which satisfy Property (*).

Let G be a torsion-free abelian (additive) group, and let S be a subsemigroup of G which contains 0. Then S is called a grading monoid (or a g-monoid)[N].

Let S be a g-monoid, and let A be a non-empty set. Assume that, for every $s \in S$ and $a \in A$, there is defined $s + a \in A$ so that, for every $s_1, s_2 \in S$ and $a \in A$, we have $(s_1 + s_2) + a = s_1 + (s_2 + a)$ and $0 + a = a$. Then A is called an S-module. Clearly, S is an S-module.

Throughout the paper, S denotes a g-monoid. If every ideal of S is finitely generated, then S is called a Noetherian semigroup.

Let A be an S-module and $s \in S$. If $s + a_1 = s + a_2$ (for $a_1, a_2 \in A$) implies $a_1 = a_2$, then s is called a non-zerodivisor on A. If s is not a non-zerodivisor, then s is called a zerodivisor on A. The set of zerodivisors on A is denoted by $Z(A)$. Let B be a submodule of an S-module A, and $s \in S$. If $s + a \in B$ (for $a \in A$) implies $a \in B$, then s is called a non-zerodivisor on A modulo B (or a non-zerodivisor on A/B). If s is not a non-zerodivisor on A/B, then s is called a zerodivisor. The set of zerodivisors on A/B is denoted by $Z(A/B)$.

For a subset T of S, the ideal generated by T is denoted by (T). The ordered sequence $x_1, \cdots, x_n$ of elements of S is called a regular sequence on A, if $(x_1, \cdots, x_n) + A \subsetneqq A$ and if $x_1 \notin Z(A)$, $x_2 \notin Z(A/((x_1) + A))$, $\cdots$, $x_n \notin Z(A/((x_1, \cdots, x_{n-1}) + A))$.

Let I be an ideal of S, and let $x_1, \cdots, x_n$ be a regular sequence in I on A. If $x_1, \cdots, x_n, x$ is not a regular sequence on A for each $x \in I$, then $x_1, \cdots, x_n$ is called a maximal regular sequence in I on A. The maximum of lengths of all regular sequences in I on A is called the grade of I on A, and is denoted by $G(I, A)$.

If S is a Noetherian semigroup with maximal ideal M, and if $G(M, S)$ equals to the dimension of S, then S is called a Macaulay semigroup.

In [TM] and [M, Section 3], we studied regular sequences on semigroups and Macaulay semigroups.

Remark. Let S be a Noetherian semigroup. Then two maximal regular sequences on S need not have the same length.

For example, let Z_0 be the g-monoid of non-negative integers and let $S = Z_0 \oplus Z_0$. Set $p = (1, 0)$, $q = (0, 1)$ and $x = (1, 1)$. Then the sequence p, q is a maximal regular sequence on S. Also, the sequence x is a maximal regular sequence on S.

Let A be an S-module. If any two maximal regular sequences in I on A have the same length for every ideal I with $I + A \subsetneqq A$, then we say that A satisfies Property (*).

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In [M], we obtained the following two Propositions.

Proposition 1. Let S be a Macaulay semigroup which satisfies Property (*). Then $G(I, S)$ equals to the height of I for every ideal I of S.

Let X be an indeterminate. Then the g-monoid $S + Z_0X$ is denoted by $S[X]$, and is called the polynomial semigroup of X over S.

Proposition 2. The polynomial semigroup $S[X]$ is a Macaulay semigroup if and only if S is a Macaulay semigroup.

The aims of this note are to determine what are Macaulay semigroups which satisfy Property (*), and when the $S[X]$-module $S[X]$ satisfies Property (*).

Theorem 1. S is a Macaulay semigroup which satisfies Property (*) if and only if S is a Noetherian semigroup with dimension ≤ 1.

Proof. The sufficiency: If S is of 0-dimension, we have nothing to prove.

Assume that S is of 1-dimension and let M be a maximal ideal of S. Let x be any element of M. Then the sequence x is a regular sequence. Suppose that x is not a maximal regular sequence. There is an element $y \in M$ such that x, y is a regular sequence.

There is a positive integer n such that $ny \in (x)$. For, suppose the contrary. Put $T = \{ny \mid n \in N\}$. Let $\{J_\lambda \mid \lambda\}$ be the set of ideals of S between (x) and M which are disjoint from T.

By Zorn's Lemma, the family $\{J_\lambda \mid \lambda\}$ has a maximal member J. Suppose that $s_1 \notin J$ and $s_2 \notin J$ for elements s_1, s_2 of S. Since the ideal (J, s_1) of S properly contains J, there exists a positive integer n_1 such that $(J, s_1) \ni n_1y$. Since J is disjoint from T, we have $n_1y \in (s_1)$. Similarly, we have $n_2y \in (s_2)$ for some positive integer n_2. It follows that $(n_1 + n_2)y \in (s_1 + s_2)$. By assumption, we have $s_1 + s_2 \notin J$. Therefore J is a prime ideal that is properly contained in M. Hence $dim(S) \geq 2$; a contradiction.

Let m be the least positive integer n such that $ny \in (x)$. Since $my \in (x)$ and $(m-1)y \notin (x)$, we see that y is a zerodivisor modulo (x); a contradiction.

We have proved the sufficiency.

The necessity: We may assume that $dim(S) > 0$. Let M be a maximal ideal of S, and let $d = dim(S)$.

First, every element of S is the sum of a finite number of irreducible elements of S. For, suppose that there is an element of S which is not the sum of a finite number of irreducible elements. Let $\{x_\lambda \mid \lambda\}$ be the set of such elements of S. Let (x) be a maximal member in the family $\{(x_\lambda) \mid \lambda\}$ of ideals. Then x is not an irreducible element of S. Hence we have $x = x_1 + x_2$ for some elements x_1 and x_2 of M. Since $(x_i) \supsetneqq (x)$ for $i = 1$ and 2, we see that x_i is the sum of irreducible elements of S. Hence x is the sum of irreducible elements of S ; a contradiction.

Next, there exists only a finite number of irreducuible elements of S. For, suppose that there exists an infinite number of irreducible elements $x_1, x_2, x_3, \cdots$. Then we have the ascending chain of ideals $(x_1) \subsetneqq (x_1, x_2) \subsetneqq (x_1, x_2, x_3) \subsetneqq \cdots$; a contradiction.

Let $x_1, x_2, \cdots, x_n$ be the set of irreducible elements of S. Set $x = x_1 + \cdots + x_n$.

Then x is a maximal regular sequence. For, suppopse the contrary. There is an element $y \in M$ such that x, y is a regular sequence. Then we may assume that y is of the form $n_1x_{i_1} + \cdots + n_mx_{i_m}$ with positive integers $n_1, \cdots, n_m$. If $\{1, \cdots, n\} - \{i_1, \cdots, i_m\} = \emptyset$, we see that y is a zerodivisor modulo (x); a contradiction. If $\{1, \cdots, n\} - \{i_1, \cdots, i_m\} =$

$\{j_1, \cdots, j_h\}$, let $z = x_{j_1} + \cdots + x_{j_h}$. Then $y + z \in (x)$ and $z \notin (x)$; that is, y is a zerodivisor modulo (x); a contradiction.

There does not exists a regular sequence of length $> d$. For, we rely on induction on d. Thus let $x_1, \cdots, x_d, y$ be a regular sequence on S, and set $T = \{ny \mid n \in N\}$. Since y is a non-zerodivisor on $S/(x_1, \cdots, x_d)$, we see that T is disjoint from $(x_1, \cdots, x_d)$. Take a prime ideal Q which contains $(x_1, \cdots, x_d)$ and disjoint from T. Set $S_Q = \{s - s' \mid s, s' \in S, s' \notin Q\}$.Then $x_1, \cdots, x_d$ is a regular sequence on S_Q of length $> dim(S_Q)$; a contradiction.

Since S satisfies property (*), we have $G(M, S) = 1$. Since S is a Macaulay semigroup, we have $dim(S) = 1$.

Every element f of $S[X]$ is of the form $s + dX$ with an element s of S and a non-negative integer d. We call d the degree of f.

Theorem 2. Let S be a Noetherian semigroup. Then the $S[X]$-module $S[X]$ satisfies Property (*) if and only if S is a group.

Proof. We see that $S[X]$ is a Noetherian semigroup. For, suppose the contrary. There exists an ideal J of $S[X]$ which is not finitely generated. Let $f_1 \in J$ be an element whose degree is minimal in J. Let $f_2 \in J - (f_1)$ be an element whose degree is minimal in $J - (f_1)$. Inductively, let $f_i \in J - (f_1, f_2, \cdots, f_{i-1})$ be an element whose degree is minimal in $J - (f_1, f_2, \cdots, f_{i-1})$ for each i. For each i, set $f_i = s_i + d_i X$ with an element s_i of S and a non-negative integer d_i. Since the ideal $(s_1, s_2, s_3, \cdots)$ of S is finitely generated, we have $(s_1, s_2, s_3, \cdots) = (s_1, s_2, \cdots, s_n)$ for some positive integer n. There exist an element s of S and a non-negative integer m with $m \leq n$ such that $s_{n+1} = s_m + s$. It follows that $f_{n+1} = s_m + d_m X + s + (d_{n+1} - d_m)X \in (f_1, f_2, \cdots, f_m)$; a contradiction to the choice of f_{n+1}.

If S is not a group, S has a maximal ideal M. Take an element $x \in M$. Then the sequence x, X is a regular sequence on $S[X]$. Since $S[X]$ has a maximal regular sequence of length 1 by the proof of the necessity of Theorem 1, we see that $S[X]$ does not satisfy Property (*).

The sufficiency is obvious.

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Department of Mathematics,Ibaraki University,Mito 310,Japan
Tel: 029-228-8336
matsuda@mito.ipc.ibaraki.ac.jp