Lecture 2 (adapted from Quantum Optics: an Introduction by M. Fox and from K. Jöns, Optical and Quantum Optical Properties of Site-controlled and Strain-tuned Quantum Dots. Verlag Dr. Hut; 2013)

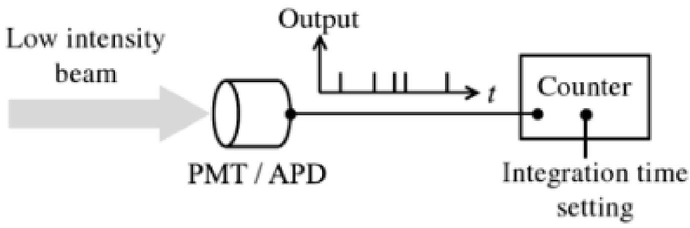
Photon Statistics

Introduction

We know that light is made of photons: Maxwell showed light is a wave, quantum physics tells us about a particle-wave duality.

But how are these photons be distributed? How can the distributions be measured?

Different light sources generate different light statistics, the distribution of time intervals among successive photons. Something that can conceptually be measured with this experimental setup:



The photon flux Φ is defined as the average number of photons passing through a cross section of the beam per unit time. It is obtained by dividing the energy flux by the energy of the individual photons:

$$\Phi = \frac{IA}{\hbar\omega} \equiv \frac{P}{\hbar\omega} \text{ photons s}^{-1}$$

Where A is the area of the beam and P is the power.

Photon detectors are defined by their quantum efficiency (among other parameters) η, defined as the ratio of detection events to the number of incident photons. The average number of detection events measured by our detector in a time interval T is given by:

$$N(T) = \eta\Phi T = \frac{\eta P T}{\hbar\omega}.$$

The corresponding average count rate R is given by:

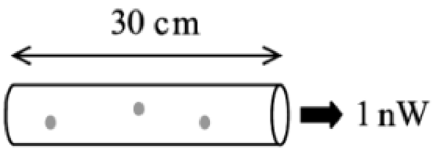
$$\mathcal{R} = \frac{N}{T} = \eta\Phi = \frac{\eta P}{\hbar\omega} \text{ counts s}^{-1}.$$

The maximum detection rate, the saturation level, is the maximum rate of photons detection event that a detector is able to measure. This is usually limited by the dead time of the detector, the length of time the detector needs to reset following a detection event.

Consider a beam of light with energy 2.0 eV with an average power of 1 nW. We could realize this with a 1 mW HeNe laser and an attenuation of a factor of 10^6. The average photon flux is then:

$$\Phi = \frac{10^{-9}}{2.0 \times (1.6 \times 10^{-19})} = 3.1 \times 10^{9} \text{ photons s}^{-1}.$$

Now consider that light travels with approximately $3 \times 10^8\ m/s$ you will realize that the photons are quite spread in space during this one second. (A beam segment with a length of $3 \times 10^8\ m$ contains 3.1×10^9 photons). If we now reduce the volume we are interested, let us say down to one meter instead of $3 \times 10^8\ m$ we have on average 10.33 photons in that volume.



There would be 3 photons on average in 30 cm of a 1 nW light beam.

Since photons are the smallest quantum of light and are discrete 10.33 mean photons does not make sense physically. Instead there will be fluctuations in the number of photons and these fluctuations in mean photon numbers will increase as we make the volume we are looking at smaller. Similarly speaking one can also look at shorter and shorter time windows. Let us look at a segment which on average contains 31 photons. We will divide the segment in 31 sub-segments. There are very many possibilities how the photons will be distributed within these sub-segments:

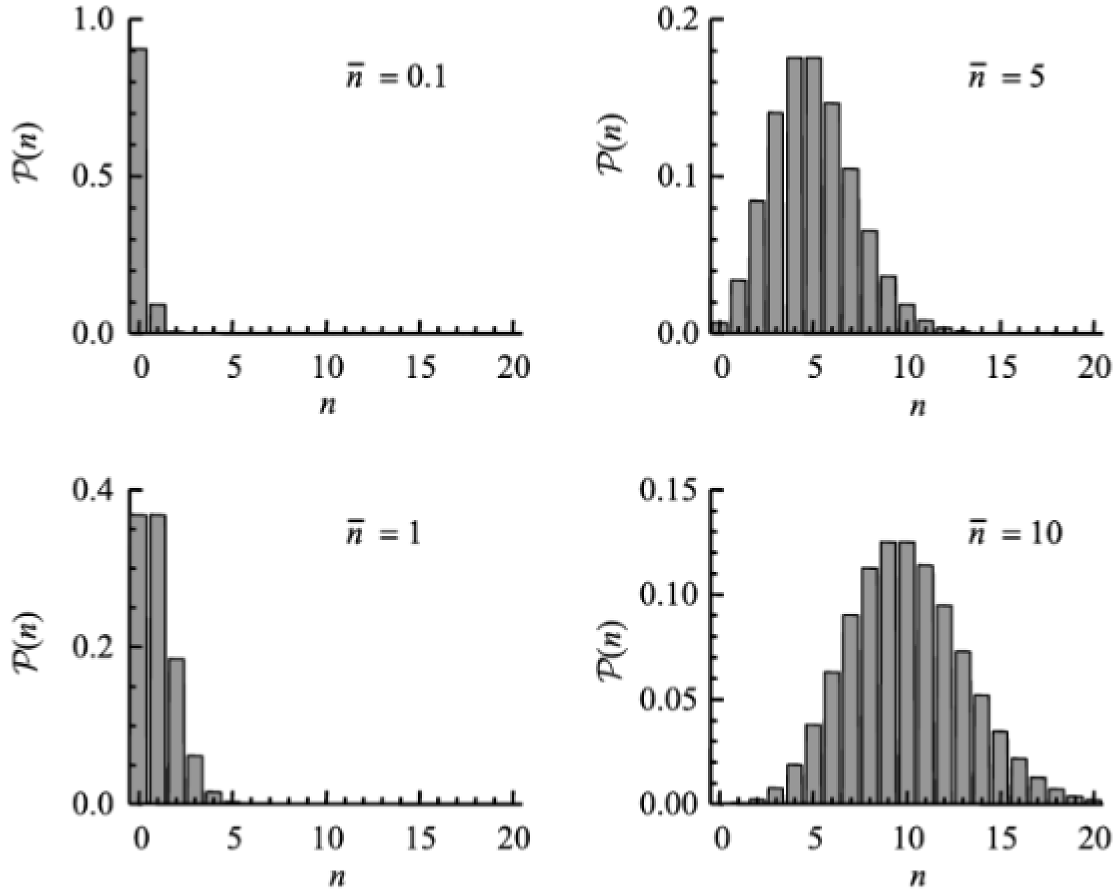
0	0	2	1	0	4	0	3	1	0	1	0	2	0	2	0	1	0	0	2	1	0	3	0	0	5	0	1	1	0	1
0	1	0	0	2	1	0	2	1	4	0	2	1	0	1	0	0	1	1	0	1	2	1	1	0	2	1	3	0	2	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Even though in all 3 cases the mean photon number <n> for each sub-segment is 1, the distribution function for each case is fundamentally different. In the first row it is super-Poissonian, second row Poissonian, and in the third row sub-Poissonian. We will now define these distributions.

<u>Poisson distribution</u>

Poissonian statistics applies to random processes that give integer values., such as the number of counts from a Geiger counter measuring a radioactive source. Th actual count rate fluctuates above and below the mean value due to the random nature of the radioactive decay and the probability for registering n counts is given by the Poissonian formula:

$$\mathcal{P}(n) = \frac{\overline{n}^n}{n!} e^{-\overline{n}}, \quad n = 0,\ 1,\ 2,\ \cdots$$



Poisson distribution for mean values of 0.1, 1, 5 and 10.

For Poisson distribution, the variance is equal to the mean value:

$$(\Delta n)^2 = \overline{n}.$$

The standard deviation for the fluctuations of the photon number above and below the mean value is given by:

$$\Delta n = \sqrt{\overline{n}}.$$

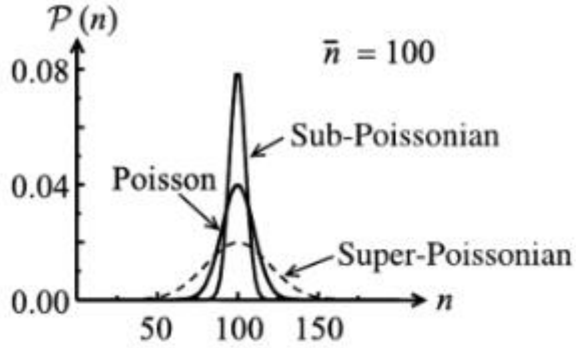
What we see here is that with Poissonian distribution, we have a widespread distribution of photon numbers, we can't have only single photons for instance.

This tells us that we can't attenuate a laser and get single photons only, at best we can get laser pulses that are mostly empty, sometimes have one photon and rarely more than one photon. For many applications, this approximation of single photons is not acceptable.

Note however that for a single photon source to beat a attenuated laser, for some applications, is demanding.

Compared to this laser (coherent) beam, we can classify all other types of light sources, we have three possibilities:

- **sub-Poissonian statistics**: $\Delta n < \sqrt{\overline{n}}$,
- **Poissonian statistics**: $\Delta n = \sqrt{\overline{n}}$,
- **super-Poissonian statistics**: $\Delta n > \sqrt{\overline{n}}$.



Comparison of photon statistics for light with an average photon number 100.

What could give us these different statistics?
Intensity fluctuations would result in larger photon number fluctuations than for the case with a constant intensity: all classical light beams with time varying intensities have super-Poissoninan photon number distribution. One example is thermal light from a black body source. This light is noisier than perfectly coherent light, it has larger intensity fluctuations.
Sub-Poissonian light has a narrower distribution than Poissonian light, it is 'quieter' and has no classical counterpart, it is an example of non-classical light.

Photon statistics	Classical equivalents	$I(t)$	Δn
Super-Poissonian	Partially coherent (chaotic), incoherent, or thermal light	Time-varying	$> \sqrt{\bar{n}}$
Poissonian	Perfectly coherent light	Constant	$\sqrt{\bar{n}}$
Sub-Poissonian	None (non-classical)	Constant	$< \sqrt{\bar{n}}$

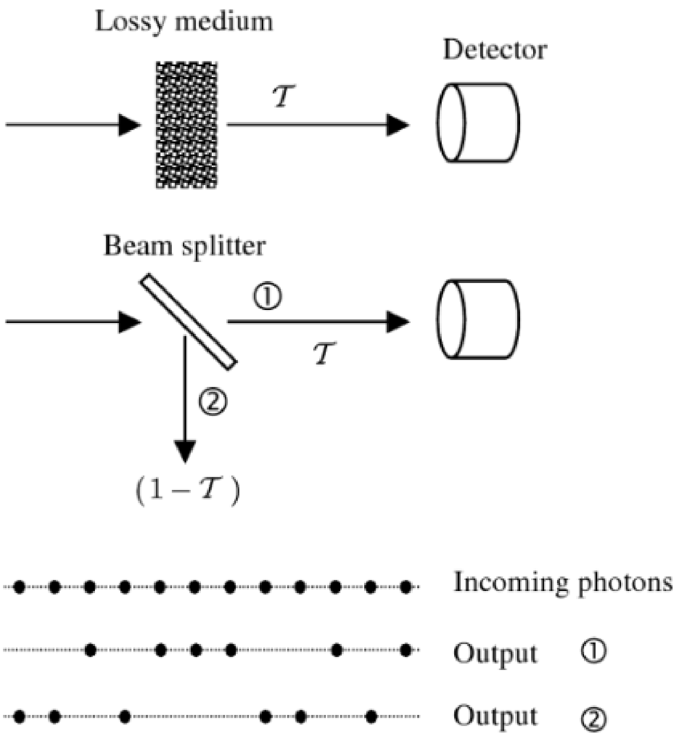
Sub-Poissonian light is defined by

$$\Delta n < \sqrt{\bar{n}}.$$

While there is no classical counterpart to sub-Poissonian light, we can think of conditions that would give rise to sub-Poissonian statistics such as a beam of light where the time interval between the photons are identical, 'a photon crystal'. The number of photons in a given time interval would be the same for every measurement (interesting in metrology, for example to calibrate detector efficiency).

What happens to photon statistics with losses?

Suppose we have a perfect sub-Poissonian light beam where each photon is at a fixed distance from the next. If we now have losses, we lose our perfect periodicity:



The lossy medium can be modeled by a beam splitter with transmission T. The regularity of the photon stream is now reduced compared to the original beam. This effect is almost unavoidable as there are always losses in optics and the detection efficiency is never 100%.

The g(2) function

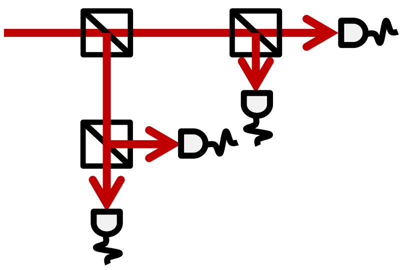
The second order correlation function is defined as:

$$g^{(2)}(\tau) = \frac{\langle : \hat{n}(t)\hat{n}(t+\tau) : \rangle}{\langle \hat{n} \rangle^2}$$

- $g^{(2)}(0)$=1 – **random**, no correlation
- $g^{(2)}(0)$>1 – **bunching**, photons arrive together
- $g^{(2)}(0)$<1 – **anti-bunching**, photons "repel"
- $g^{(2)}(\tau) \rightarrow 1$ at long times for all fields

The term antibunched light applies to the case where there is always a minimum time interval between photons.

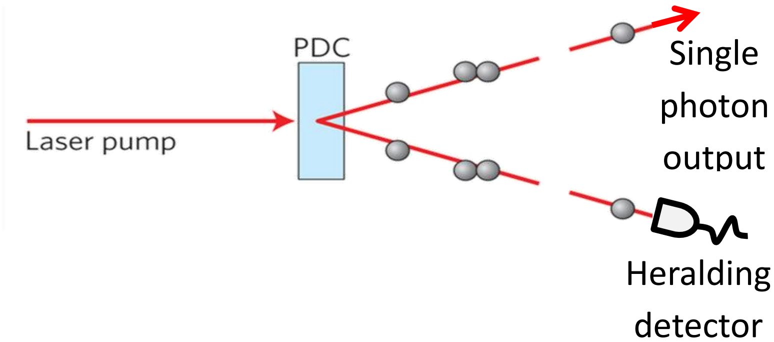
We can go beyond the g(2) function and measure g(n) which are correlations among n detectors. While this is possible, it still lacks a use and implies longer measurements.



This setup would allow for the measurement of g(4).

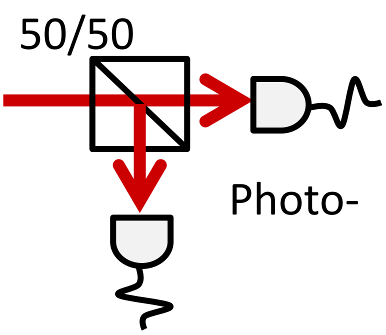
Heralded single photons

By shining light on a non-linear crystal, we can have a down conversion process: each blue photon is turned into two red photons. When we detect one red photon in one output, we know that there is another photon in the other arm, that photon is 'heralded'. While the statistics of the light is the same as that of the incoming laser (Poissonian), we can use this approach to work with single photons. This has its limits as Poissonian statistics implies that two photons could very well be simultaneous.

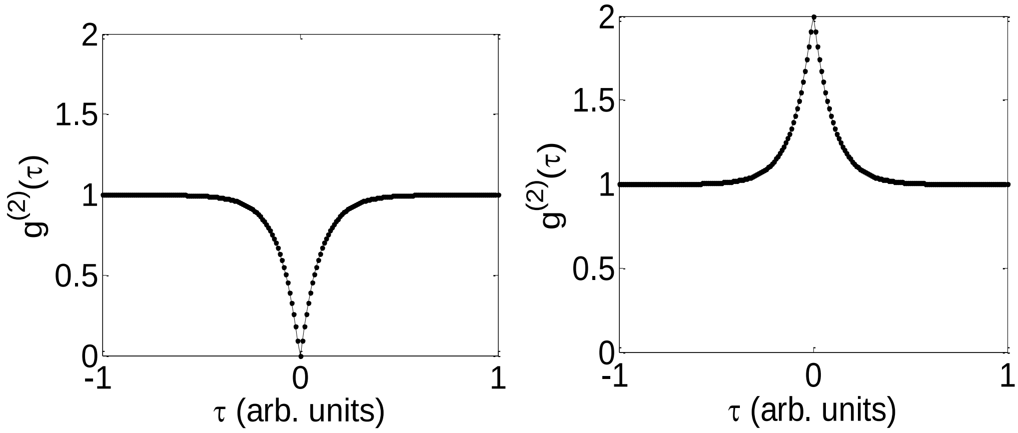


The Hanbury-Brown Twiss interferometer

How do we measure photon statistics? We use a Hanbury-Brown Twiss interferometer and measure time intervals between detection events on each detector. After acquiring a long list of events, we build a histogram of the time intervals.



Trick question: *why not use only one single photon detector?* This has to do with deadtime, after a detection event a detector is not able to detect another photon for a certain time. We need to rely on another detector during that time.

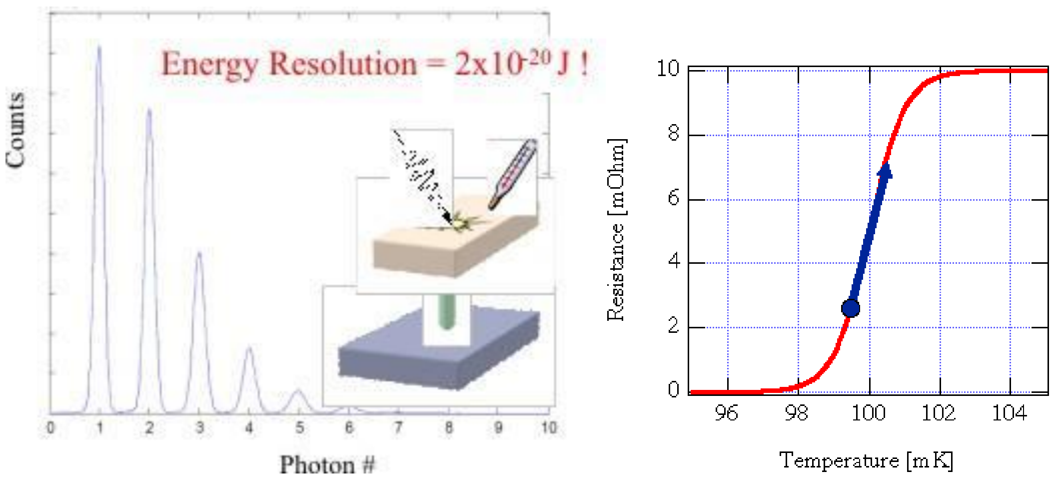


Left: antibunching histogram of time intervals, this shows that photons are separated in time. Right: bunching, this shows that photons tend to come together.

A practical **photon number detector** remains to be invented, all we have at our disposal today are ‘click’ detectors: if one photon or more impinge the detector, it generates a detection pulse, irrespective of the number of impinging photons.

Note that resolving the number of photons is not impossible, a bolometer does just that, it measures the amount of heat deposited by a pulse of light on a detector. When operated very precisely, it is possible to distinguish the heat associated with one photon from the heat associated with two photons at optical frequencies.

Trick question: *if the detection efficiency of a photon number resolving detector is less than unity, what happens to the fidelity of the measurement?*



Left: Transition edge sensor: the heat deposited in a nanoscale superconductor is measured precisely to yield the number of impinging photons. (Image from NIST) Right: the superconducting transition offers a very high resistance slope vs temperature. (Image from MIT).

<u>Quantization of the electromagnetic field</u>

We give a brief introduction to the quantization of the electromagnetic field. In simplified terms we can replace the field amplitudes in the classical electrodynamics description with bosonic creation $\hat{a}^\dagger$ and annihilation operators $\hat{a}$. These operators follow the bosonic commutator relation. Each mode of the light field (k, λ) can be independently described by a harmonic oscillator. The consecutive operation of first the annihilation and then the creation operator is equal to an operation of a new quantum mechanical operator $\hat{n}$, the so-called photon number operator which gives the number of photons in one mode: $\hat{n} = \hat{a}^\dagger\hat{a}$. The quantum mechanical approach allows to sum up all modes as independent quantum mechanical harmonic oscillators. The Hamiltonian of the electromagnetic field $\hat{H}_{em}$ for an arbitrary number of field modes is given by:

$$\hat{H}_{\mathrm{em}} = \sum_k \sum_\lambda \frac{1}{2}\hbar\omega_k \left(\hat{a}_{k\lambda}\hat{a}^\dagger_{k\lambda} + \hat{a}^\dagger_{k\lambda}\hat{a}_{k\lambda}\right)$$

If we consider only the fundamental mode, we can rewrite the Hamiltonian with the help of the photon number operator:

$$\hat{H}_{\mathrm{em}} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

Similar to the quantum mechanical harmonic oscillator, there is a ground state with finite energy, called vacuum state. The photon number operator $\hat{n}$ states how many photons with an energy $\hbar\omega$ are in the fundamental mode. The mean photon number of a mode $\langle n \rangle$ is an important figure of merit for the characterization of the light states. Together with the photon number variance $(\Delta n)^2$, we will later identify different light states and use it to categorize light sources.

Different light states

Glauber state

The Glauber state, or coherent state $|\hat{\alpha}_i\rangle$, describes the electromagnetic wave of a laser mode *i*. In 1963, Glauber provided a complete quantum mechanical description of these light states [180]. The Glauber state is an eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a}_i\,|\alpha_i\rangle = \alpha_i\,|\alpha_i\rangle$$

Being an eigenstate of $\hat{a}$, a coherent state remains unchanged by the annihilation of a photon. Additionally, since the vacuum state can be written as an eigenstate of the annihilation operator with $\alpha = 0$, all coherent states have the same minimal uncertainty as the vacuum state. Linear superposition of these states allows for an expression of the Glauber state in the basis of the photon number operator $\hat{n}$ [Loudon1983]:

$$|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^2}{\sqrt{n!}}\,|n_i\rangle$$

We can calculate the probability distribution P for n photons in a given mode i.

$$
\begin{aligned}
P_{coherent}(n) &= |\langle n \mid \alpha_i \rangle|^2 \\
&= \left| \exp\left(-\tfrac{1}{2}|\alpha_i|^2\right) \cdot \sum_m \frac{\alpha_i^m}{\sqrt{m!}} \langle n \mid m \rangle \right|^2 \\
&= \exp\left(-|\alpha_i|^2\right) \cdot \frac{|\alpha_i|^{2n}}{n!}
\end{aligned}
$$

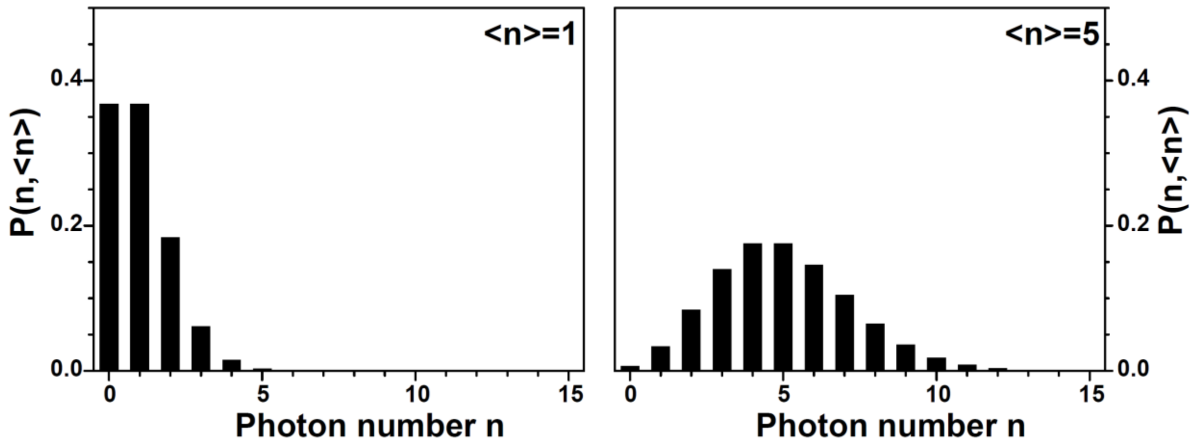
This is the characteristic Poisson statistics used to describe coherent light states. The expected value of the photon number in one mode of a coherent state is therefore:

$$\langle n \rangle = \langle \alpha | \, \hat{n} \, | \alpha \rangle = |\alpha|^2$$

The variance of a Glauber (coherent) state is given as:

$$(\Delta n)^2_{\text{Glauber}} = \left\langle n^2 \right\rangle - \langle n \rangle^2 = |\alpha|^2 = \langle n \rangle$$

For the coherent state the maximum probability to find *n* photons in a mode is at the expected value $\langle n \rangle$. The probability distribution of the photon number obeys a Poisson distribution. Any light state with larger (smaller) variance is called super (sub)- Poissonian light. The photon number distribution for a Glauber state is plotted below for 2 different mean photon numbers.



<u>Fock state</u>

The Fock or photon number state $|\hat{n}_i\rangle$ results directly from the quantization of the electromagnetic field, since the Fock state is the eigenstate of the photon number operator $\hat{n}_i$:

$$\hat{n}_i \, |n_i\rangle = n_i \, |n_i\rangle$$

The eigenvalue n_i of the photon number operator describes the number of photons in a specific mode *i*. The probability $P_{Fock}(n)$ to find n_i photons in one mode is either 1 for $n = n_i$ or 0 for $n \neq n_i$. This is a special characteristic of the Fock state: The photon number is fully determined. Thus. the probability distribution of the photon number follows a δ-distribution.

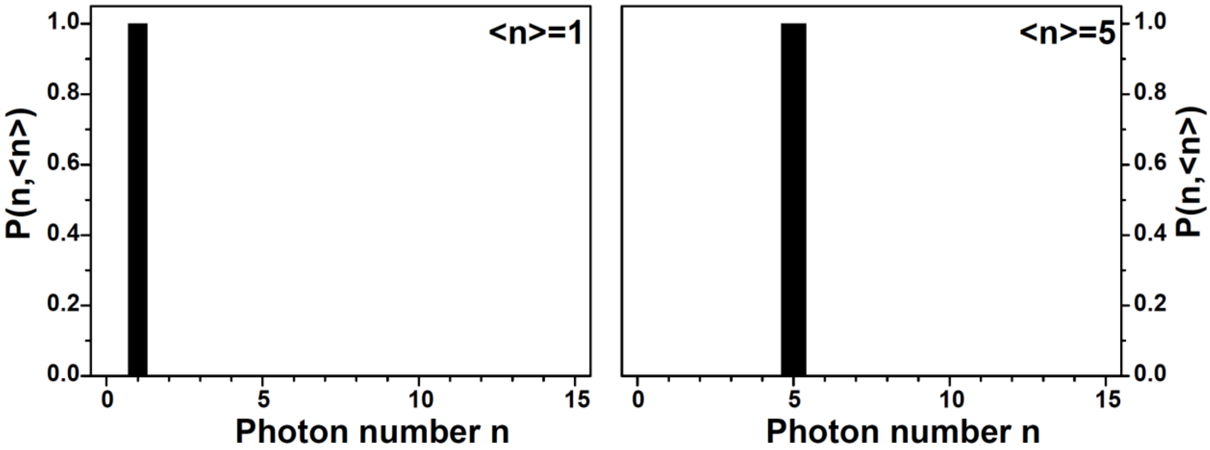
The expected value of the photon number in a Fock state is equal to the number of photons in the state:

$$\langle n \rangle = \langle n_i | \, \hat{n} \, | n_i \rangle = n_i$$

For the Fock state the variance is therefore:

$$(\Delta n)^2_{\text{Fock}} = \left\langle n^2 \right\rangle - \langle n \rangle^2 = 0$$

The Fock state fulfills the inequality $\Delta n < \sqrt{n}$, showing a variance smaller than the Glauber state. Such sub-Poisson statistics cannot be described by classical electromagnetic theory; thus such light is classified as non-classical light. The figure below shows the photon number distribution for two Fock states. Light emitters with a Fock state n = 1 are called single photon sources, since they can only emit one single photon at a time.



Thermal state

The thermal state which is well described by the black-body radiation [Planck1900,Planck1901], is an incoherent mixture of different photon number states $|\hat{n}_i\rangle$. A quantum mechanical description of the thermal state takes advantage of the density matrix notation. The thermal state, being a quantum mechanical mixed state, can be written as the sum over all possible photon number states $|\hat{n}_i\rangle$ weighted with their occurrence probability:

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n(n) \, |n\rangle \, \langle n|$$

where $\hat{\rho}$ is the density matrix operator and P(n) gives the probability of finding n photons in a certain mode *i* of the thermal state. This is identical to the probability of having a certain photon number state occupied. P(n) can be expressed as a function of n and $\langle n \rangle$ for a single mode:

$$P(n, \langle n \rangle) = \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+1}}$$

P(n, $\langle n \rangle$) has the form of a Bose-Einstein distribution; the state with the maximum probability is always the vacuum state with n = 0. The variance for a thermal state is given by:

$$(\Delta n)^2_{\mathrm{Thermal}} = \langle n \rangle^2 + \langle n \rangle$$

It follows that the fluctuation in the photon number is typically larger than the mean photon number. Therefore, thermal light states are also called chaotic light. Since $\Delta n > \sqrt{n}$ one often describes thermal state statistics as super-Poissonian statistics. The thermal state distribution for 2 different mean photon numbers is plotted below. Finding zero photons in the mode always has the highest probability.

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