Conjoint Measurement without Additivity and Transitivity

Denis Bouyssou

LAMSADE - Paris - France

Marc Pirlot

Faculté Polytechnique de Mons - Mons - Belgium

Outline

- **Introduction and Motivation**
- **Models**
  - **Inter-Attribute Decomposable Models**
  - **Intra-Attribute Decomposable Models**
- **Extensions/Applications**
- **Conclusion and Open Problems**

Introduction

Context = **Conjoint Measurement**

- **Set of Objects:** $X \subseteq X_1 \times X_2 \times \ldots \times X_n$
- **Binary relation on this set:** $\succcurlyeq$

Objective = **Study/Build/Axiomatise numerical representations of** $\succcurlyeq$

Introduction

Interest of Numerical Representations

- Manipulation of $\succcurlyeq$
- Construction of numerical representations

Interest of Axiomatic Analysis

- Tests of models
- Understanding models

Examples: Cartesian Product Structure

$x = (x_1, x_2, \ldots, x_n) \in X$

- **MCDM**
  - **x is an "alternative" evaluated on "attributes"**

Other examples

- **DM under uncertainty**
  - **x is an "act" evaluated on "states of nature"**
- **Economics**
  - **x is a "bundle" of "commodities"**
- **Dynamic DM**
  - **x is an "alternative" evaluated at "several moments in time"**
- **Social Choice**
  - **x is a "distribution" between several "individuals"**
- **$x \succcurlyeq y$ means "x is at least as good as y"**

Additive Transitive Representation

$$x \succcurlyeq y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

Basic Model = Additive Utility

- **Examples:**

MCDM

Weighted sum, Additive utility, Goal programming, Compromise Programming

DM under uncertainty: SEU

Dynamic DM: Discounting

- **Properties (among others !)**

$\succcurlyeq$ **is complete**

$\succcurlyeq$ **is transitive**

$\succcurlyeq$ **is independent**

Independence

Independence:
A common consequence on attribute i does not affect preference

$$(a_{-i}, x_i) \succcurlyeq (b_{-i}, x_i) \Rightarrow (a_{-i}, y_i) \succcurlyeq (b_{-i}, y_i)$$

Necessity:

$$(a_{-i}, x_i) \succcurlyeq (b_{-i}, x_i) \Rightarrow \sum_{j \neq i} u_j(a_j) + u_i(x_i) \geq \sum_{j \neq i} u_j(b_j) + u_i(x_i) \Rightarrow$$

$$\sum_{j \neq i} u_j(a_j) + u_i(y_i) \geq \sum_{j \neq i} u_j(b_j) + u_i(y_i) \Rightarrow (a_{-i}, y_i) \succcurlyeq (b_{-i}, y_i)$$

Weak Independence:
Common consequences on attributes other than i does not affect preference

$$(a_{-i}, x_i) \succcurlyeq (a_{-i}, y_i) \Rightarrow (b_{-i}, x_i) \succcurlyeq (b_{-i}, y_i)$$

- **Independence $\Rightarrow$ Weak Independence**
- **Weak Independence allows to define "partial preference relations" $\succcurlyeq_i$**

Triple Cancellation

$$\left.\begin{array}{r}(x_i, a_{-i}) \succcurlyeq (y_i, b_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \succcurlyeq (w_i, a_{-i}) \\ \text{and} \\ (w_i, c_{-i}) \succcurlyeq (z_i, d_{-i})\end{array}\right\} \Rightarrow (x_i, c_{-i}) \succcurlyeq (y_i, d_{-i})$$

Remarks

- **possible generalization to subsets of attributes (replace i by J and –i by –J)**
- **TC and $\succcurlyeq$ reflexive $\Rightarrow$ Independence**
- **C_m : Cancellation condition of order m**

Cancellation Condition of Order m (C_m)

$x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$

If for all $i \in \{1, 2, \ldots, n\}$

$(x_i^1, x_i^2, \ldots, x_i^m)$ is a permutation of $(y_i^1, y_i^2, \ldots, y_i^m)$ then

$x^j \succcurlyeq y^j$ for $j = 1, 2, \ldots, m-1 \Rightarrow y^m \succcurlyeq x^m$

Necessity $\sum_{j=1}^{m} \sum_{i=1}^{n} u_i(x_i^j) = \sum_{j=1}^{m} \sum_{i=1}^{n} u_i(y_i^j)$

Remarks

- $C_{m+1} \Rightarrow C_m$
- For no finite m , $C_m \Rightarrow C_{m+1}$
- $C_2 \Rightarrow$ Independence
- $C_3 \Rightarrow$ Transitivity
- $C_4 \Rightarrow$ TC

Axiomatic Analysis: 2 cases

- **X finite** (Scott-Suppes 1958, Scott 1964)
  - ☞ **Necessary and sufficient Conditions**
  - ☞ **Denumerable Set of "Cancellation Conditions"**
  - ☞ **No nice uniqueness results**

- **X has a "rich structure"**
  and ≽ behaves consistently in this "continuum"
  (Debreu 1960, Luce-Tukey 1964)
  - ☞ **(Topological assumptions + continuity) or (solvability assumption + Archimedean condition)**
  - ☞ **A finite (and limited) set of "Cancellation Conditions" entails the representation (independance, TC)**
  - ☞ **u_i define "interval scale" with common unit**

Sample Result on Additive Utility

Theorem(Scott 1964): If X is finite then

$\succcurlyeq$ is complete and satisfies C_m for m = 2, 3, …

<u>iff</u>

the additive utility models holds

Remarks

- **No nice uniqueness result**
- **Proof rests on the "theorem of the alternative"**
- **Extension to general sets Jaffray 1974**
- **Fishburn (1997) : bounds on m given |X|**

Sample Result on Additive Utility

Theorem(Luce-Tukey 1964):

If n ≥ 3 (three essential components) and

≽ is an independent weak order

≽ satisfies restricted solvability

≽ satisfies an archimedean axiom

<u>then</u>

the additive utility model holds and

u_i define interval scales with common unit

Remarks

- **independence may be replaced by Triple Cancellation**
- **With Triple Cancellation result is valid for n = 2**

Problems

- **Transitivity and completeness of $\succsim$**
  - Experimental violations (May 1954, Luce 1969)
  - Aggregation models in MCDM violating these hypothesis
  - Decision Theory can be conceived without transitivity (Fishburn 1991)
  - ⇒ Find a more flexible framework
- **Axiomatic Problems**
  - Finite case: Axioms hardly interpretable and testable
  - "Rich case"
    - ☞ Respective roles of unnecessary structural conditions and necessary "cancellation" conditions (Furkhen and Richter 1991)
    - ☞ Asymmetry $n = 2$ vs. $n \geq 3$ cases ($n = 2$ more difficult)
  - Asymmetry Finite vs. "Rich" case
- **Few Results outside this case (MCDM contribution ?)**

Possible extensions

Additive utility = Additive Transitive Conjoint Measurement

(1) (2)

- **Extensions**

Drop additivity

Drop transitivity and/or completeness

Decomposable Transitive Models

Keep transitivity and completeness – Drop additivity

Krantz et al (1971)

$$x \succsim y \Leftrightarrow F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n)) \geq F(u_1(y_1), u_2(y_2), \ldots, u_n(y_n))$$

F increasing

Advantages

- **Simple axiomatic analysis**
- **Simple proofs**
- **Allows to "understand" the "pure consequences" of weak independence + transitivity and completeness**

Drawbacks

- **Transitivity and completeness**
- **No nice unicity results**
- **Too general ? (F is not specified)**

Sample Result on Decomposable Transitive Models

Theorem(Krantz et al 1971):

$\succcurlyeq$ is a <u>weakly</u> independent weak order

(having a numerical representation)

<u>iff</u>

the decomposable transitive model holds

Remarks

- **Necessary and Sufficient conditions for all X**
- **Simple proof**
- **No asymmetry "Rich" vs. finite, $n = 2$ vs. $n \geq 3$**
- **No nice uniqueness result (u_i are "related" ordinal scales)**

Additive Non Transitive Models

Keep Additivity – Drop transitivity and completeness

Bouyssou 1986, Fishburn 1990, 1991, Vind 1991

$x \succcurlyeq y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0$ **(with additional properties)**

p_i **skew symmetric or** $p_i(x_i,x_i)= 0$

Advantages

- Flexible towards transitivity and completeness
- Classical results are particular cases
- Interpretation in terms of "preference differences"
- Nice unicity results with rich structure: p_i define ratio scales with common unit

Drawbacks

- Asymmetry: Finite vs. Rich, $n \geq 3$ vs. $n = 2$ ($n = 2$ simpler !)
- Complex proofs

Cancellation Condition of Order m (S_m)

$x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$

If for all $i \in \{1, 2, \ldots, n\}$

$|\{(x_i^j, y_i^j)\}| = |\{(y_i^j, x_i^j)\}|$

then

$x^j \succsim y^j$ for $j = 1, 2, \ldots, m-1 \Rightarrow y^m \succsim x^m$

Necessity : $\sum_{j=1}^m \sum_{i=1}^n p_i(x_i^j, y_i^j) = 0$ if p_i are skew symmetric

Remarks

- $S_{m+1} \Rightarrow S_m$
- For no finite m , $S_m \Rightarrow S_{m+1}$
- $S_4 \Rightarrow TC$

Sample Results on Additive Non Transitive Models (with skew symmetry)

Theorem (Fishburn 1991)

If X is finite then

$\succcurlyeq$ is complete and satisfies S_m for m = 1, 2, 3, …

<u>iff</u>

the non transitive additive model holds

with p_i skew symmetric

Remarks

- **No nice uniqueness result**
- **Proof rests on the "theorem of the alternative"**

Sample Results on Additive Non Transitive Models (with skew symmetry)

Theorem (Fishburn 1991)

If $n \geq 3$ (three essential components) and

$\succcurlyeq$ satisfies restricted solvability

$\succcurlyeq$ complete and satisfies S_4

$\succcurlyeq$ satisfies an archimedean axiom

<u>then</u>

the non transitive additive model holds

(with skew symmetric p_i) and

p_i define ratio scales with common unit (if non extremality)

Remarks

- **$n = 2$ is a simpler case**
- **$S_4 \Rightarrow$ Triple Cancellation on subsets**

Cancellation Condition of Order m (T_m)

$x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m, z^1, z^2, \ldots, z^m, w^1, w^2, \ldots, w^m \in X$

If for all $i \in \{1, 2, \ldots, n\}$

$[(x_i^1, y_i^1), (x_i^2,, y_i^2), \ldots, (x_i^m, y_i^m)]$ is a permutation of

$[(z_i^1, w_i^1), (z_i^2,, w_i^2), \ldots, (z_i^m, w_i^m)]$

Not[$x^j \succcurlyeq y^j$ and Not($z^j \succcurlyeq w^j$)] for $j = 1, 2, \ldots, m$

Necessity

$\sum_{j=1}^{m} \sum_{i=1}^{n} p_i(x_i^j, y_i^j) = \sum_{j=1}^{m} \sum_{i=1}^{n} p_i(z_i^j, w_i^j)$

Sample Results on Additive Non Transitive Models (with $p_i(x_i, x_i) = 0$)

Theorem (adapted from Fishburn 1992)

If X is finite then

$\succcurlyeq$ is reflexive and independent

$\succcurlyeq$ satisfies T_m for m = 1, 2, 3, …

<u>iff</u>

the non transitive additive model holds

with $p_i(x_i, x_i) = 0$

Remarks

- No nice uniqueness result
- Proof rests on the "theorem of the alternative"
- Rich case Vind 1991
- $T_2 \Rightarrow$ RC1

Keep additvity Relax Transitivity

Additive Transitive

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

Keep Transitivity
Relax Additivity

Additive Non Transitive

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0$$

Decomposable Transitive

$$x \succsim y \Leftrightarrow F(u_i(x_i)) \geq F(u_i(y_i))$$

?

Keep additvity Relax Transitivity →

Additive Transitive

$$x \succcurlyeq y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

↓ Keep Transitivity
Relax Additivity

Decomposable Transitive

$$x \succcurlyeq y \Leftrightarrow F(u_i(x_i)) \geq F(u_i(y_i))$$

Additive Non Transitive

$$x \succcurlyeq y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0$$

↓

Non Transitive Decomposable

Non Transitive Decomposable models

Trivial model: $x \succcurlyeq y \Leftrightarrow F(x, y) \geq 0$

$$F(x,y) = \begin{cases} 1 \text{ if } x \succcurlyeq y \\ -1 \text{ otherwise.} \end{cases}$$

- **Inter-attribute Decomposability**

Decompose F along the various attribute

$$F(x,y) = F(p_i(x_i, y_i))$$

- **Intra-attribute decomposability**

Decompose $p_i(x_i, y_i)$ to build “criteria”

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$$

Inter-Attribute Decomposability

General Model

$$x \succcurlyeq y \Leftrightarrow F(p_i(x_i, y_i)_{i=1,2,\ldots,n}) \geq 0$$

Problems

- **This model is trivial (under a mild cardinality assumption)**
- **$\succcurlyeq$ is not independent**
- **$\succcurlyeq$ is not reflexive !!**

Care should be taken in the definition of the models !!

Inter-Attribute Decomposable Models

(M)	**$x \succcurlyeq y \Leftrightarrow F(p_1(x_1,y_1), p_2(x_2,y_2), \ldots, p_n(x_n,y_n)) \geq 0$**
(M0)	**(M) with $p_i(x_i, x_i) = 0$ and $F(0) \geq 0$**
(M1)	**(M0) with F non decreasing in all arguments**
(M1′)	**(M0) with F increasing in all arguments**
(M2)	**(M1) with p_i skew symmetric**
(M2′)	**(M1′) with p_i skew symmetric**
(M3)	**(M2) with F odd**
(M3′)	**(M2′) with F odd**

Remarks

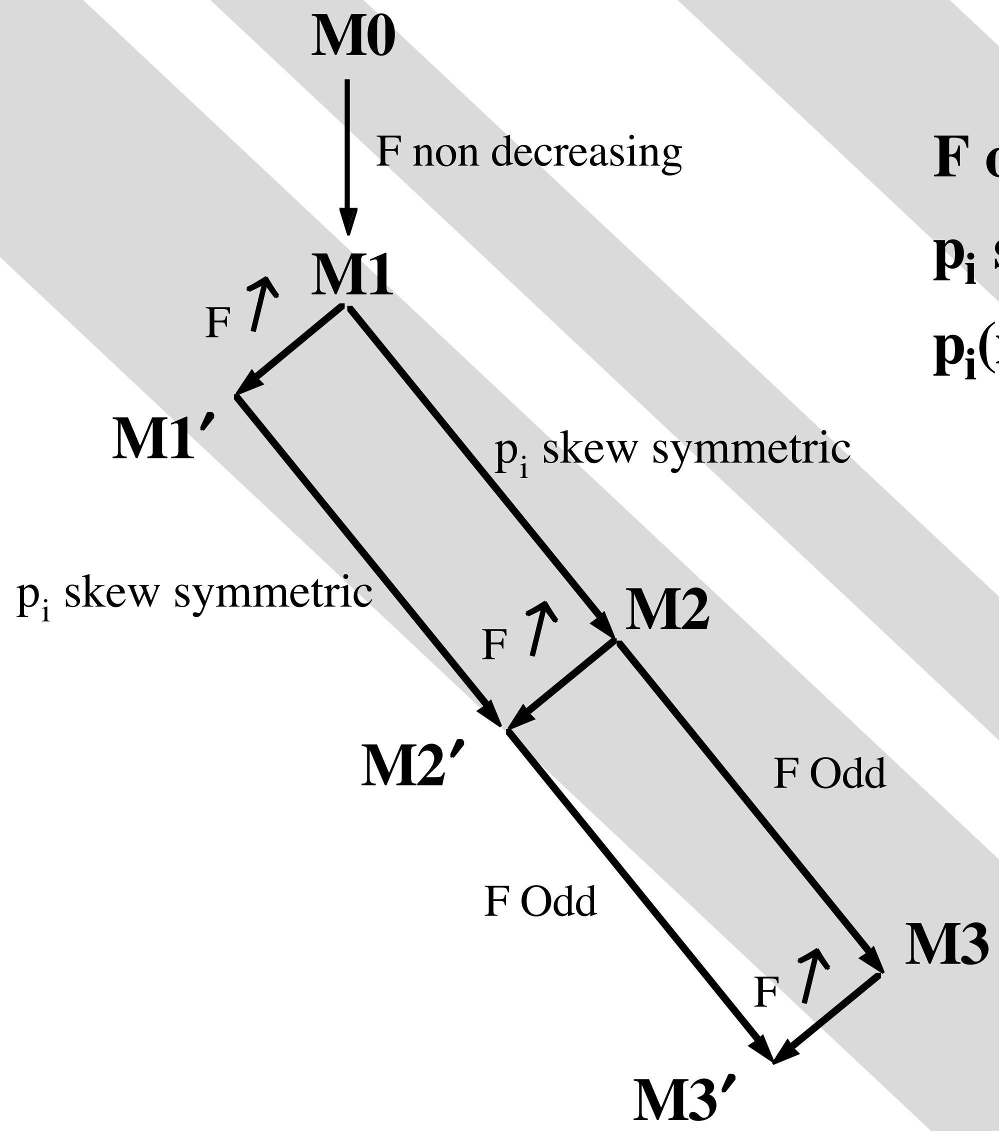
- **(Mk′) $\Rightarrow$ (Mk–1′), (Mk) $\Rightarrow$ (Mk–1)**
- **All Models are particular cases of (M0)**

Intuition

- p_i captures “preference differences” between levels of X_i
- F combines these “differences” in a consistent way

- F increasing and odd brings it “closer” to addition
- Skew symmetry of p_i $\Rightarrow$

  the ™difference∫(x_i, y_i) is linked to
  the ™opposite difference∫(y_i, x_i)



F odd : $F(-x) = -F(x)$

p_i skew symmetric :

$p_i(x_i, y_i) = -p_i(y_i, x_i)$

Basic Properties

(i) If $\succcurlyeq$ satisfies model (M0) then it is reflexive and independent.

(ii) If $\succcurlyeq$ satisfies model (M1) or (M1′) then:

- $[x_i \succ_i y_i$ for all $i \in J \subseteq \{1, 2, \ldots, n\}] \Rightarrow [\text{Not}(y_J \succcurlyeq_J x_J)]$

(iii) If $\succcurlyeq$ satisfies model (M2) or (M2′) then:

- $\succcurlyeq_i$ is complete,
- $[x_i \succ_i y_i$ for all $i \in J] \Rightarrow [x_J \succ_J y_J]$

(iv) If $\succcurlyeq$ satisfies model (M3) then it is complete

(v) If $\succcurlyeq$ satisfies model (M3′) then:

- $[x_i \succcurlyeq_i y_i$ for all $i \in J] \Rightarrow [x_J \succcurlyeq_J y_J]$
- $[x_i \succcurlyeq_i y_i$ for all $i \in J$, $x_j \succ_j y_j$, for some $j \in J] \Rightarrow [x_J \succ_J y_J]$

Axioms: RC1

RC1$_i$

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \begin{cases}(z_i, a_{-i}) \succsim (w_i, b_{-i})\\ \text{or}\\ (x_i, c_{-i}) \succsim (y_i, d_{-i})\end{cases}$$

Interpretation

(x_i, y_i) is either larger or smaller than (z_i, w_i)

Consequence

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow$$

$$[(z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i})]$$

is complete and therefore a weak order

Axioms: RC2

$$\text{RC2}_i \quad \left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (w_i, b_{-i}) \\ \text{or} \\ (w_i, c_{-i}) \succsim (z_i, d_{-i})\end{array}\right.$$

Interpretation

(x_i, y_i) is "linked" to (y_i, x_i)

Consequence

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow \text{for all } a_{-i}, b_{-i},$$

$$[(z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i})] \text{ and}$$

$$[(y_i, c_{-i}) \succsim (x_i, d_{-i}) \Rightarrow (w_i, c_{-i}) \succsim (z_i, d_{-i})]$$

is complete and therefore a weak order

Results - Denumerable case

If X is finite or countably infinite:

- **(M0) <u>iff</u> reflexivity, independence,**
- **(M1′) <u>iff</u> reflexivity, independence, RC1,**
- **(M2′) <u>iff</u> reflexivity, RC1, RC2,**
- **(M3) <u>iff</u> completeness, RC1, RC2,**
- **(M3′) <u>iff</u> completeness, TC.**

Non Denumerable case

Add a necessary Order Density condition: OD*

Remarks

- (M1)$\Leftrightarrow$ (M1′), (M2)$\Leftrightarrow$ (M2′)
- Necessary and Sufficient conditions for all X
- Axioms are independent
- No nice uniqueness results and Irregular representations
- Allow to study the “pure consequences” of classical cancellation conditions
  - TC vs. Independence
- Adding “rich structure” + axioms on subsets implies F is additive and uniqueness results (Fishburn 1991, Vind 1991)
- Adding transitivity and completeness on M0 implies the Decomposable Transitive Model

More (technical) remarks

- $RC1_i \Leftrightarrow$ biorder between X_i^2 and X_{-i}^2

Adding an order density condition implies

$$x \succsim y \Leftrightarrow p_i(x_i, y_i) + P_{-i}(x_{-i}, y_{-i}) \geq 0$$

- **n = 2 is a very particular case**
- TC_i **+ completeness implies**

$$x \succsim y \Leftrightarrow p_i(x_i, y_i) + P_{-i}(x_{-i}, y_{-i}) \geq 0$$

with p_i and P_{-i} skew symmetric

- **RC2 $\Rightarrow$ Independence**
- **TC, completeness $\Rightarrow$ RC1, RC2**

Remarks

- **RC1 is NS for:**
  $x \succcurlyeq y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n) \geq 0$
  with F nondecresing

- **In all models the function p_i can be chosen so as to represent $\succcurlyeq^*_i$ (or $\succcurlyeq^{**}_i$)**

Example: Additive Utility

- **Additive utility**

  $x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$

- **Interpretation**

  $x \succsim y \Leftrightarrow F(p_i(x_i, y_i)_{i=1,2,\ldots,n}) \geq 0$

  with

  $F = \sum$ **and**

  $p_i(x_i, y_i) = u_i(x_i) - u_i(y_i)$

Example: ELECTRE I
(Roy 1968)

$$x \succcurlyeq y \Leftrightarrow \begin{cases} (\sum_{i:\, x_i \succcurlyeq_i y_i} k_i)/(\sum_{i=1}^{n} k_i) \geq s \\ \text{and} \\ \text{Not}(x_i V_i y_i) \end{cases}$$

$$x_i \succcurlyeq_i y_i \Leftrightarrow u_i(x_i) - u_i(y_i) \geq -q$$

$$x_i V_i y_i \Leftrightarrow u_i(x_i) - u_i(y_i) < v$$

$$s \geq 1/2$$

- **Interpretation**

$$x \succcurlyeq y \Leftrightarrow F(p_i(x_i, y_i)_{i=1,2,\ldots,n}) \geq 0$$

with $F = \sum$ and

$$p_i(x_i, y_i) = \begin{cases} k_i \text{ if } u_i(x_i) - u_i(y_i) \geq -q \\ -\dfrac{s}{1-s} k_i \text{ if } -v \leq u_i(x_i) - u_i(y_i) < -q \\ -M \text{ otherwise} \end{cases}$$

TACTIC

Vansnick 1986

(Adaptation)

$$x \succsim y \Leftrightarrow \begin{cases} \rho \sum_{i: x_i \succ_i y_i} k_i \geq \sum_{i: y_i \succ_i x_i} k_i \\ \text{and} \\ \text{Not}(x_i V_i y_i) \end{cases}$$

$$x_i \succ_i y_i \Leftrightarrow u_i(x_i) - u_i(y_i) > q$$

$$x_i V_i y_i \Leftrightarrow u_i(x_i) - u_i(y_i) < v$$

$$\rho \geq 1$$

Application: Compensation vs. Noncompensation

$RC1_i$ implies $[(x_i, y_i) \sim_i^* (z_i, w_i), \text{for all } i] \Rightarrow [x \succcurlyeq y \Leftrightarrow z \succcurlyeq w]$

Fishburn's 1976 Definition of Noncompensation

$$[(x_i \succcurlyeq_i y_i) \Leftrightarrow (z_i \succcurlyeq_i w_i), (y_i \succcurlyeq_i x_i) \Leftrightarrow (w_i \succcurlyeq_i z_i)] \Rightarrow [x \succcurlyeq y \Leftrightarrow z \succcurlyeq w]$$

Formally very similar definitions

- **Fishburn's Noncompensation**
  - **only at most three distinct equivalence classes of $\sim_i^*$**
  - **the comparison of preference differences is only based on $\succcurlyeq_i$**
- **All Methods are "Noncompensatory" in our more general sense**
- **Clue = number of equivalence classes of $\sim_i^*$**

ELECTRE I

PI: $u_i(x_i) - u_i(y_i) \geq -q$

↓

P^-: $-v \leq u_i(x_i) - u_i(y_i) < -q$

↓

V: $u_i(x_i) - u_i(y_i) < -v$

$\succcurlyeq$ is reflexive, independent and satisfies RC1 and RC2

TACTIC

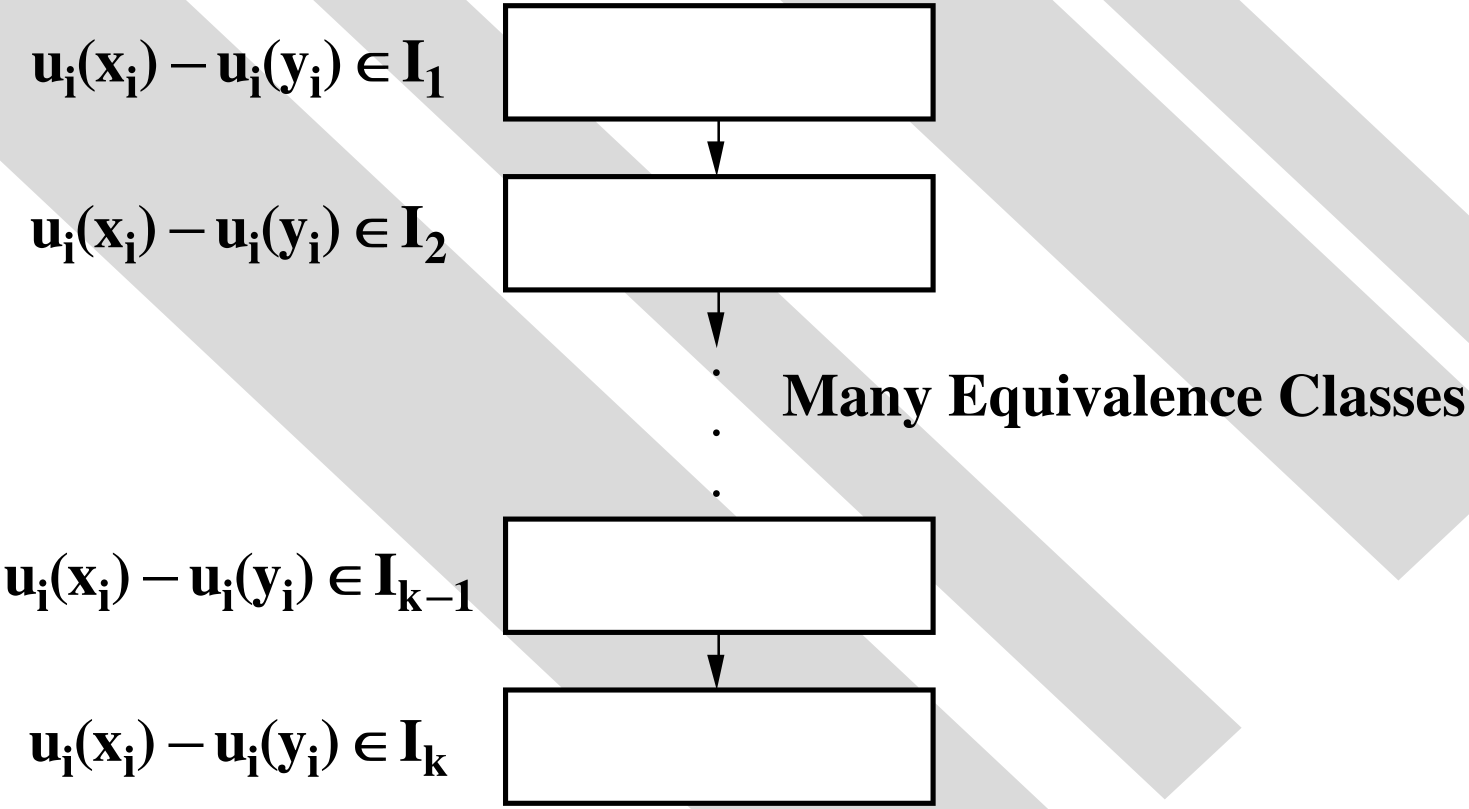
P: $u_i(x_i) - u_i(y_i) > q$

I: $-q \leq u_i(x_i) - u_i(y_i) \leq q$

P^-: $-v \leq u_i(x_i) - u_i(y_i) < -q$

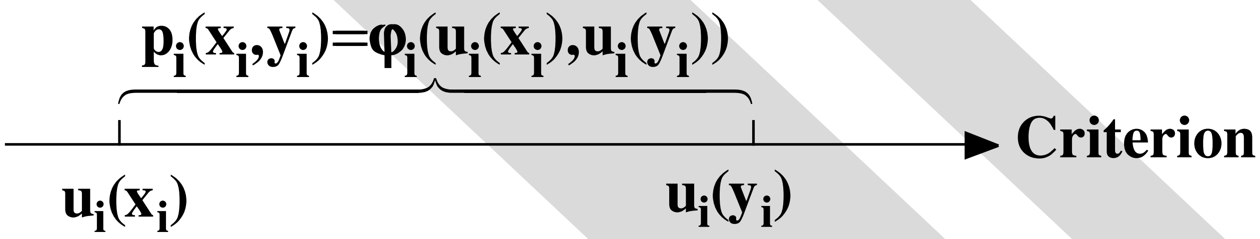
V: $u_i(x_i) - u_i(y_i) < -v$

Additive Utility



Difficulty

- In all models the "weight" of the preference difference is computed with respect to an underlying "criterion"
- $\succsim_i$ has nice properties (semi order)
- Study "Intra-Attribute Decomposability"

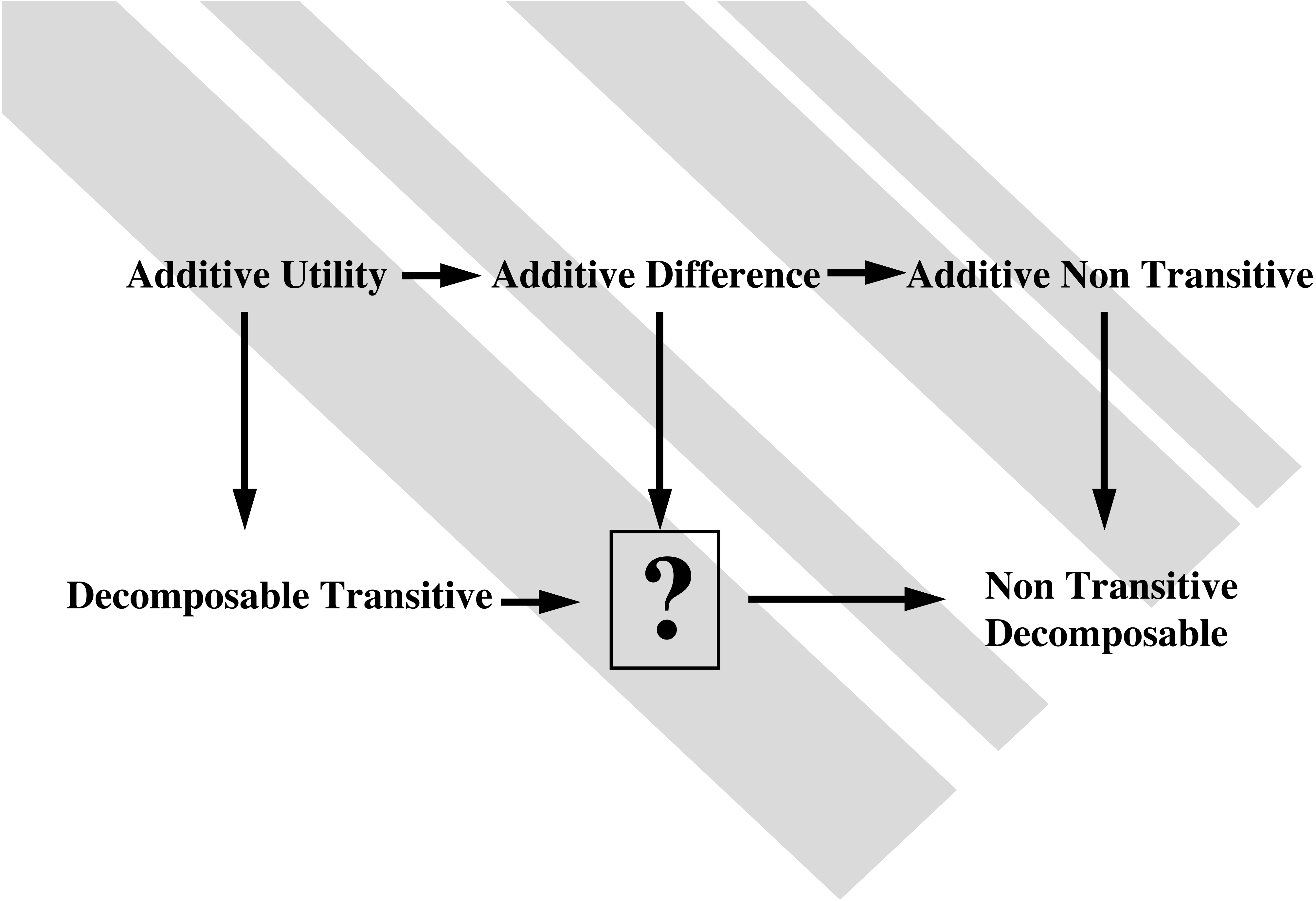


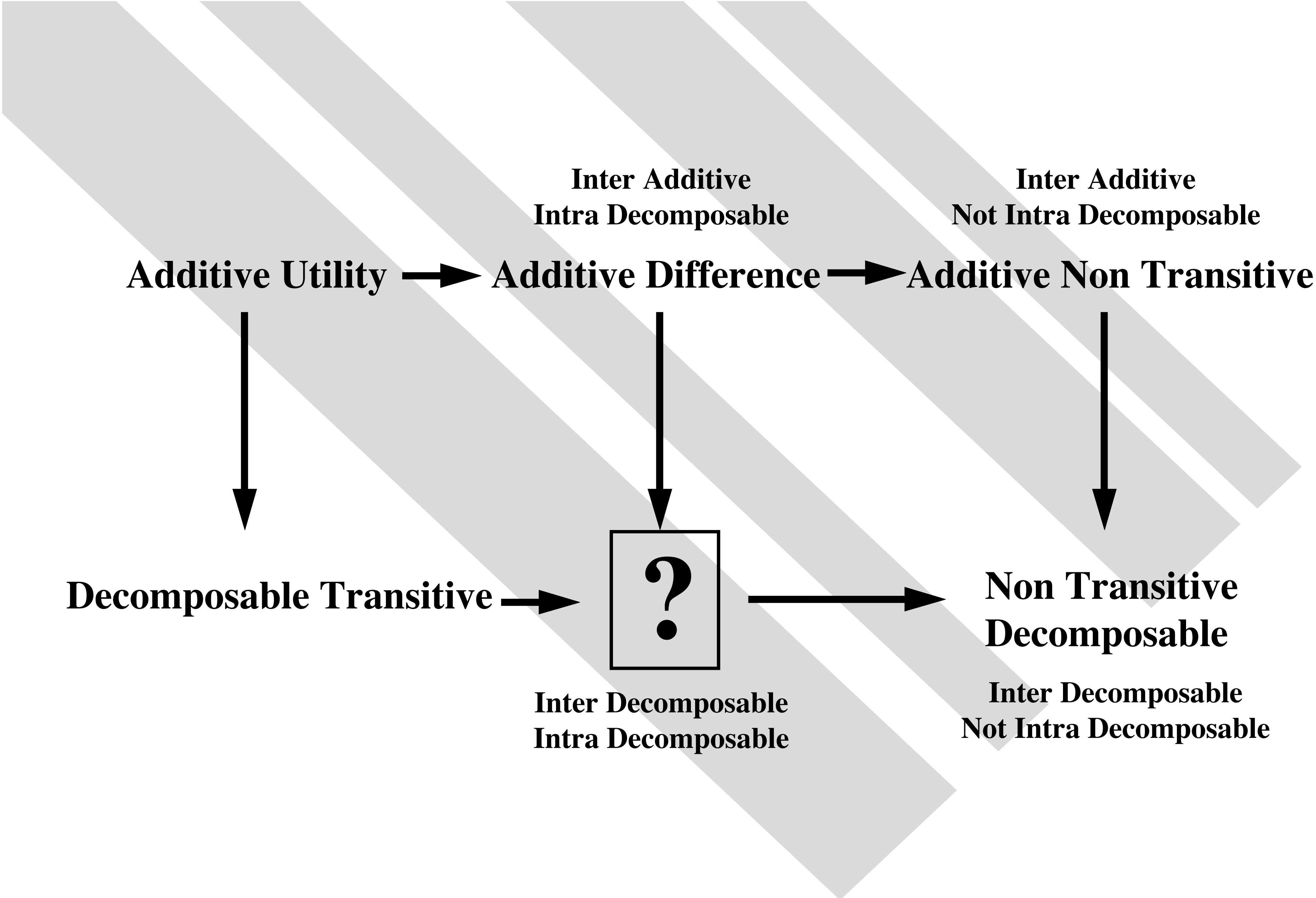
Additive Difference Model (Tversky 1969)

$$x \succcurlyeq y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0$$

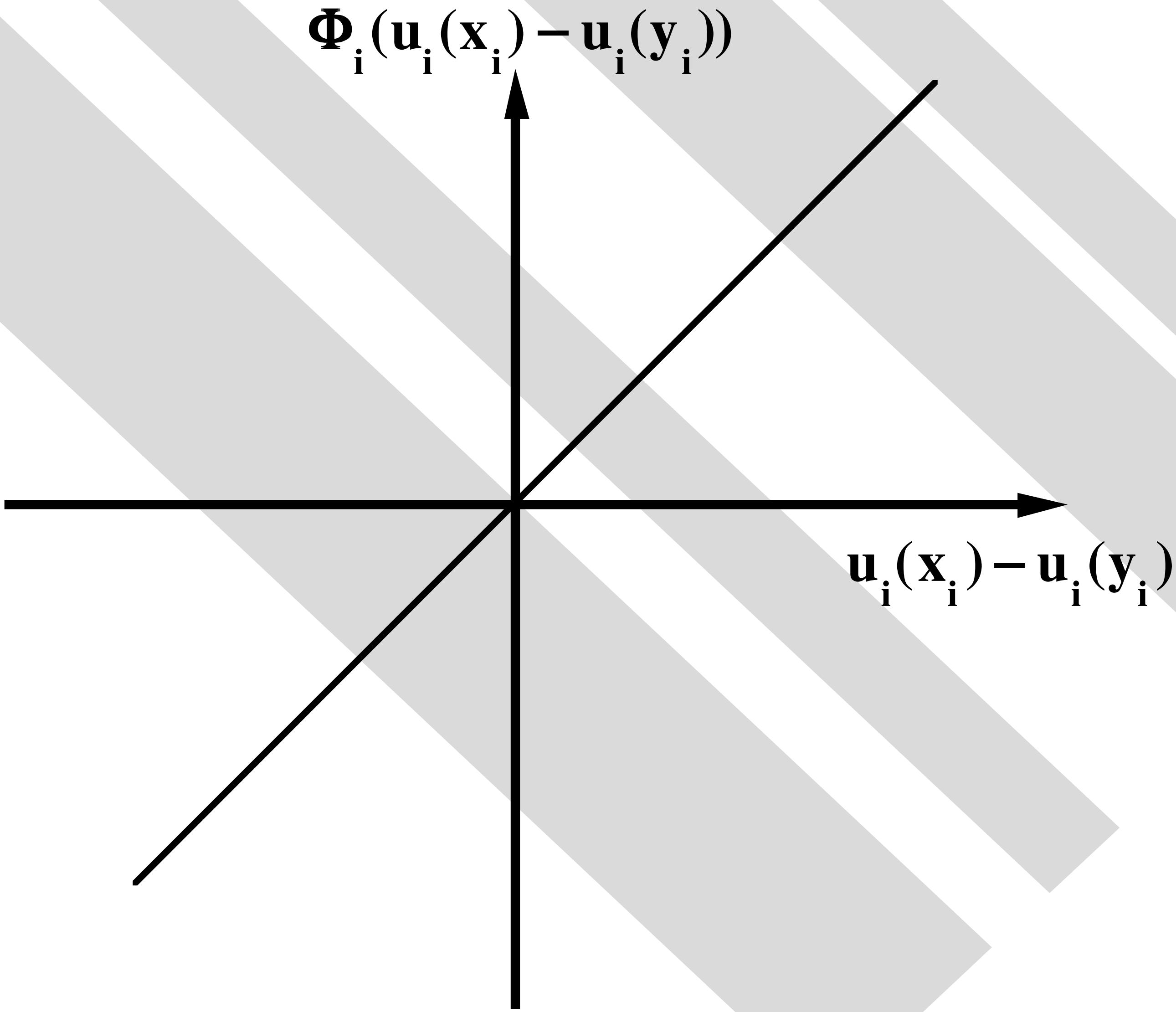
Φ_i increasing and odd

- **Introduction of Intra-Attribute Decomposability**
  - **$\succcurlyeq$ may be intransitive (but is complete)**
  - **$\succcurlyeq_i$ are weak orders**
- **Axioms (Fishburn 1992)**
  - **Rich Structure**
  - **Complex proofs**
  - **Nice uniqueness results (u_i define interval scale)**
- **Extensions (Bouyssou 1986)**
  - **Non decreasing Φ_i (allows for semi orders on each attribute)**
  - **Φ_i not odd but $\Phi_i(0) = 0$ (allows for incomplete $\succcurlyeq$)**

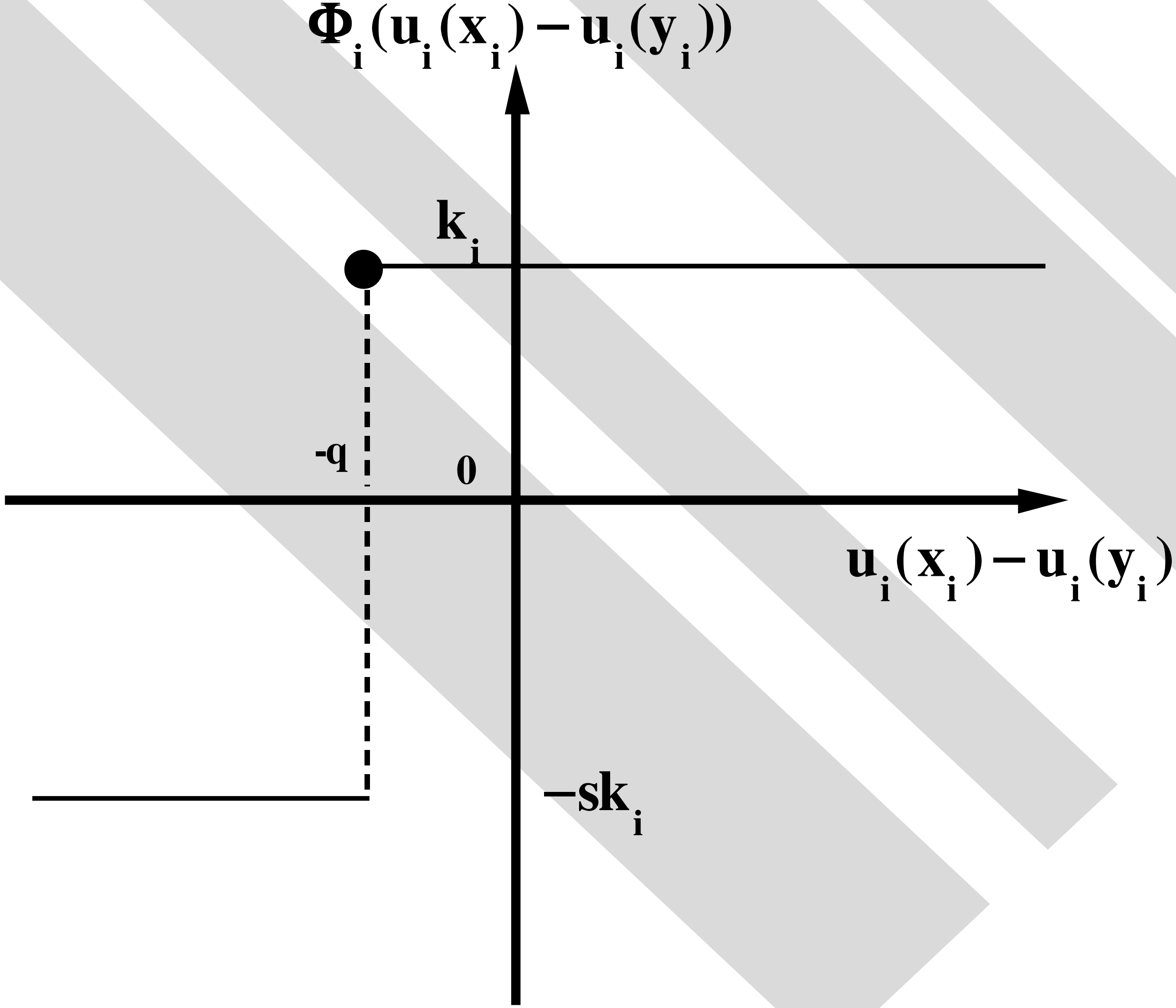

Additive Utility
Additive Difference
Additive Non Transitive
Decomposable Transitive
?
Non Transitive
Decomposable

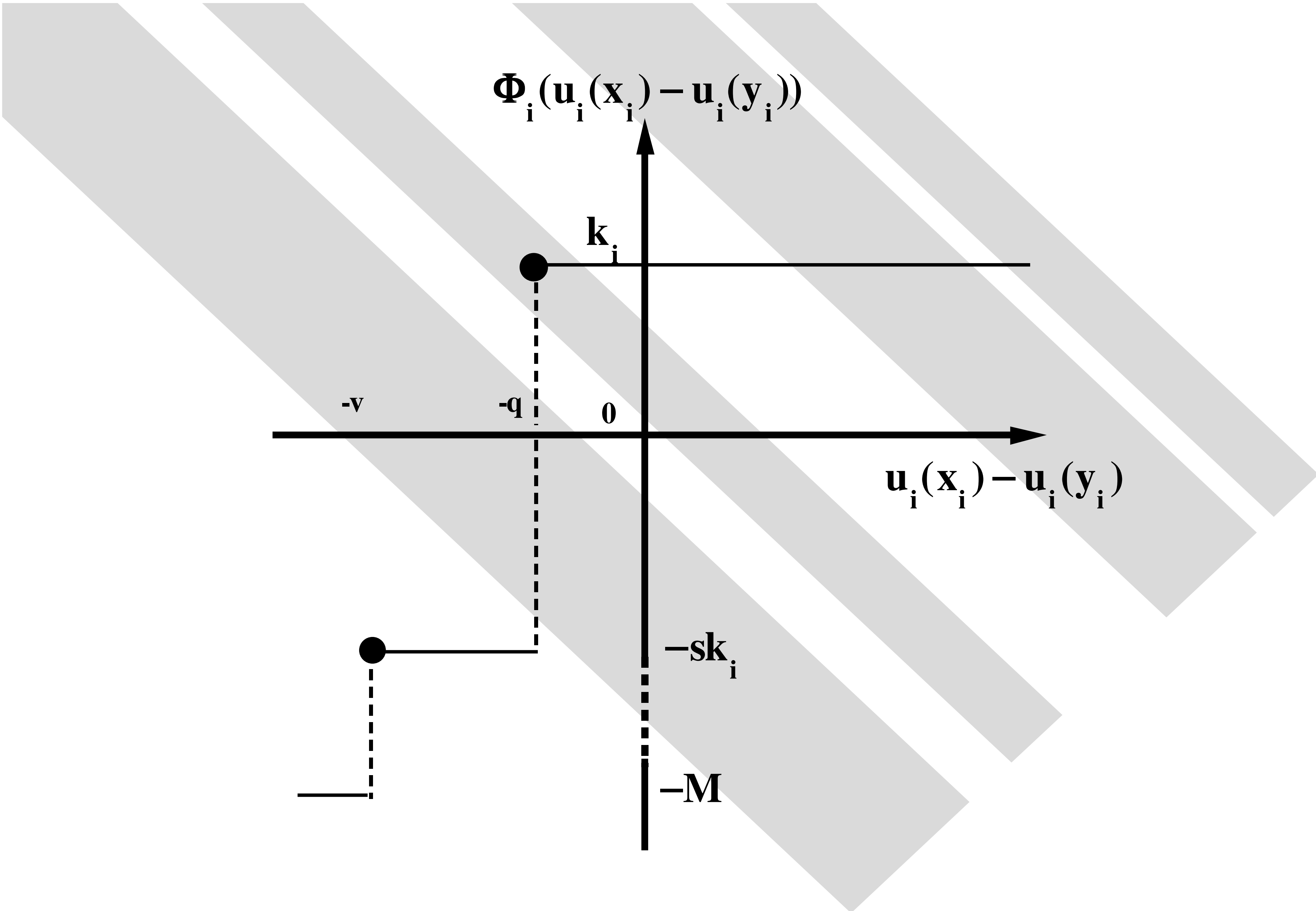

Inter Additive
Intra Decomposable
Inter Additive
Not Intra Decomposable
Additive Utility
Additive Difference
Additive Non Transitive
Decomposable Transitive
?
Non Transitive
Decomposable
Inter Decomposable
Intra Decomposable
Inter Decomposable
Not Intra Decomposable

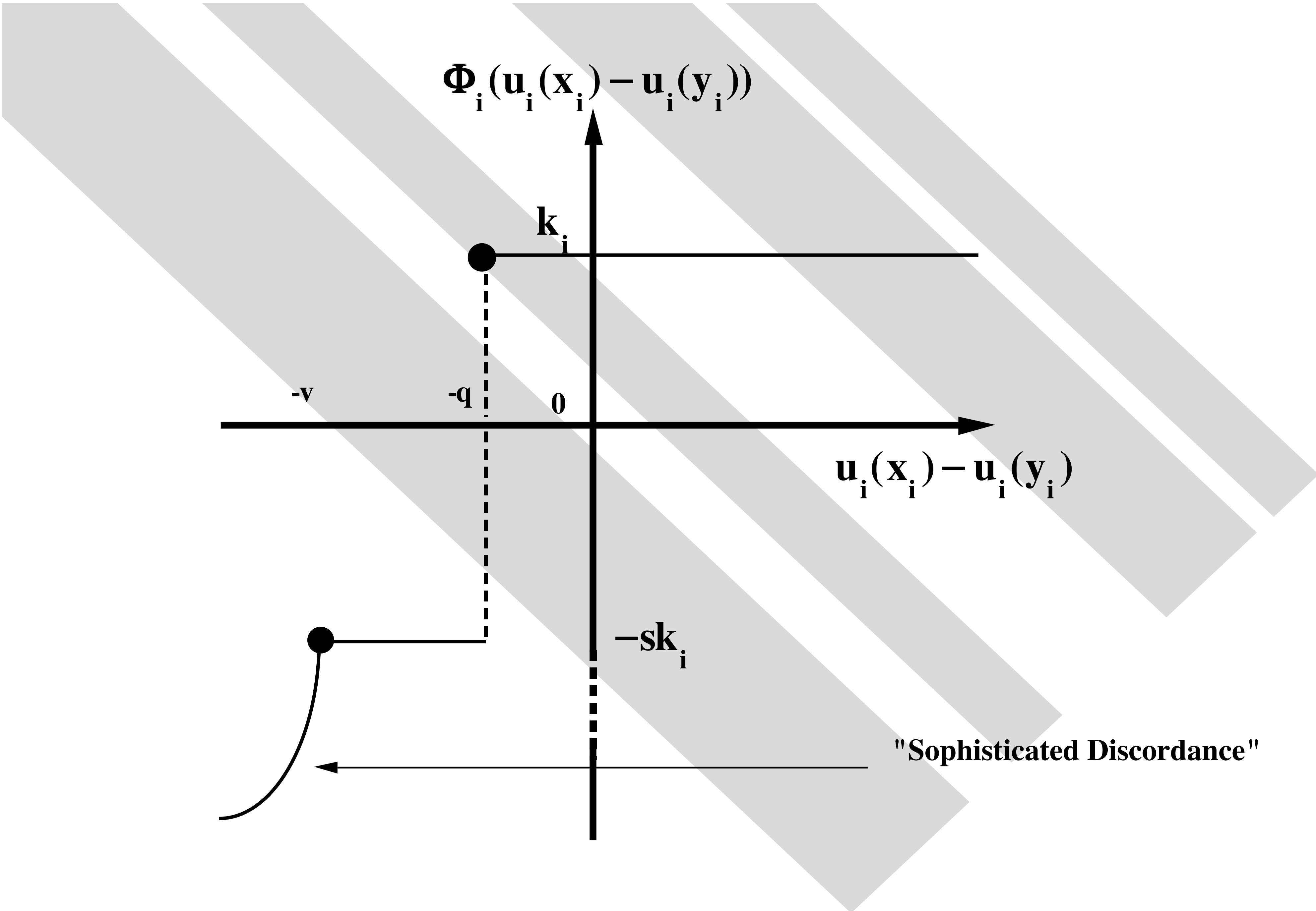
Additive Transitive Model



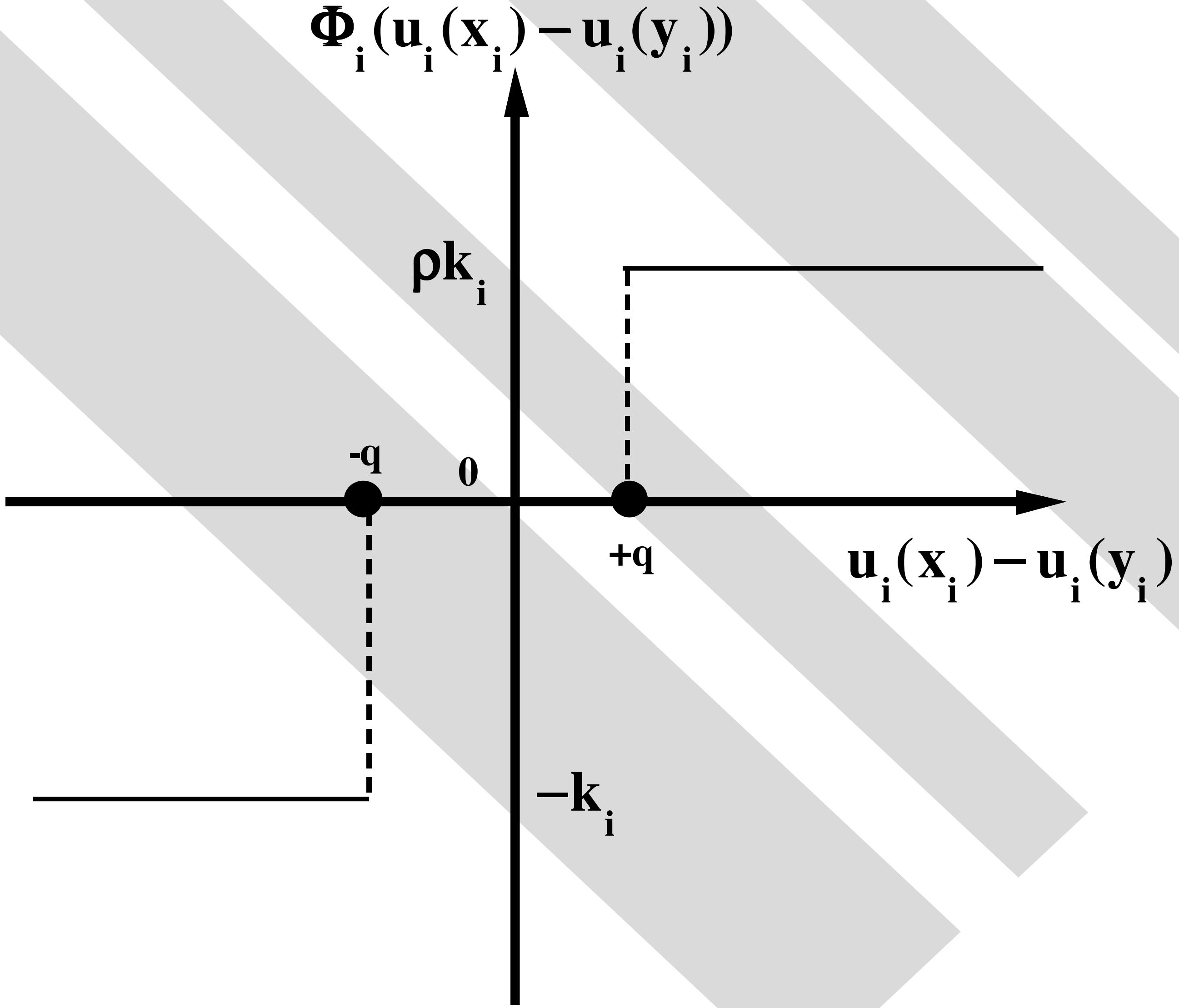
ELECTRE I without Discordance



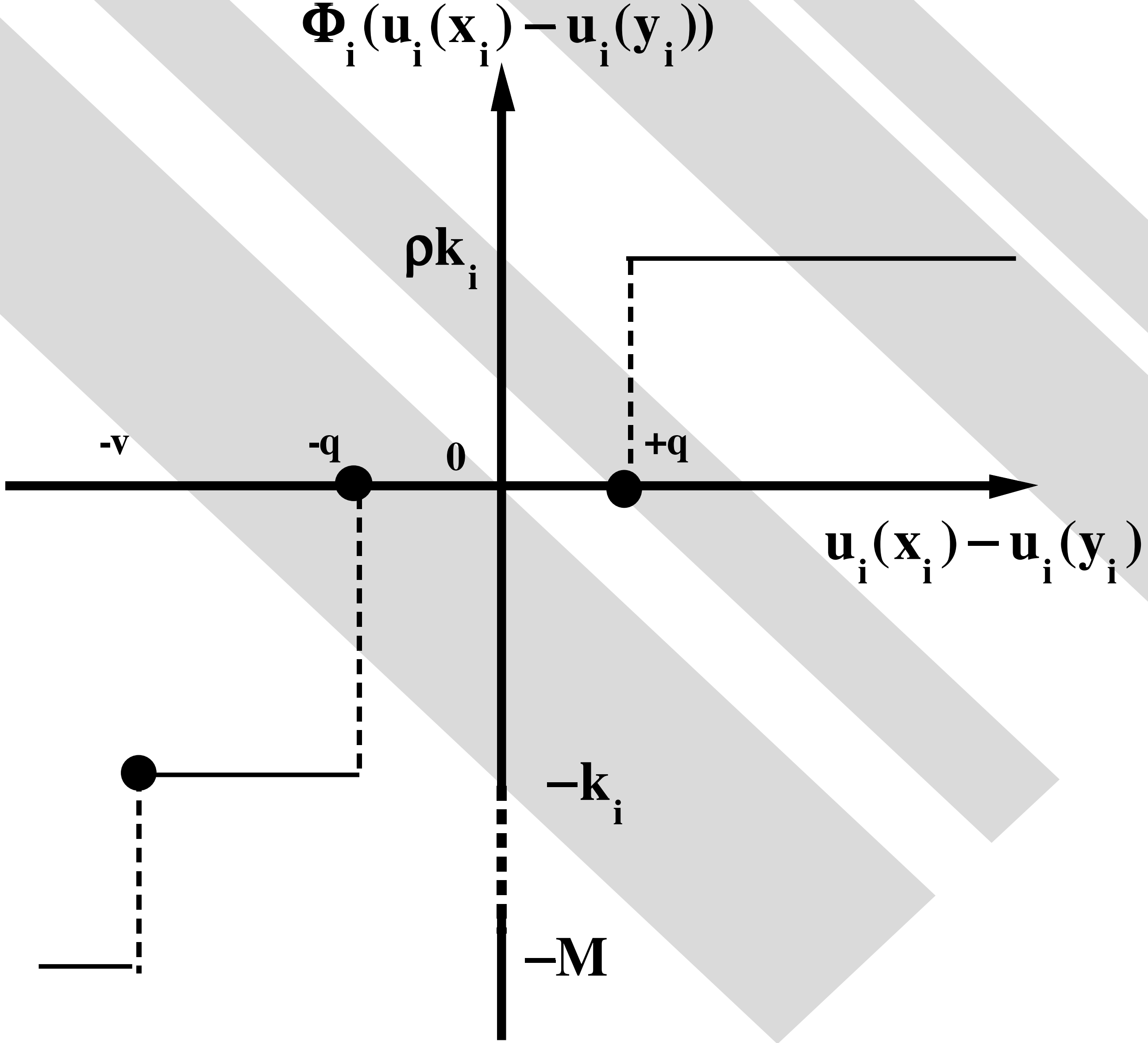

$\Phi_i(u_i(x_i) - u_i(y_i))$
k_i
-v
-q
0
$u_i(x_i) - u_i(y_i)$
$-sk_i$
$-M$


$\Phi_i(u_i(x_i) - u_i(y_i))$
k_i
-v
-q
0
$u_i(x_i) - u_i(y_i)$
$-sk_i$
"Sophisticated Discordance"

TACTIC without Discordance



TACTIC with Discordance



Intra-Attribute Decomposability

- **Idea: Use previous theorems and find conditions such that**

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$$

where φ_i is non decreasing in 1st and non increasing in 2nd

- **Coming close to the additive difference model without implying subtractivity**
- **Use of general theorem on numerical representation of valued relations (Doignon et al 1986) (generated by p_i)**
- **Difficulty = Irregularity of the representations in the previous models**

Models

- **(D)** $p_i(x_i, y_i) = \varphi_i(u_i(x_i), v_i(y_i))$
- **(D1)** **(D) with φ_i non decreasing in 1st argument**
- **(D2)** **(D) with φ_i non increasing in 2nd argument**
- **(D12)** **(D1) and (D2)**
- **(D12=)** **(D12) with $u \equiv v$**

Preliminary Results

- **Model (D) is trivial (under mild cardinality assumptions)**
- **With (M2′), (M3) and (M3′)**

  (D1)⇔ (D2)⇔ (D12) ⇔ (D12=)
- **Seven models of interest**

(M1′) with (D1), (D2), (D12) and (D12=)

(M2′) with (D12=)

(M3) with (D12=)

(M3′) with (D12=)

Axioms

$$\text{AC1}_i \text{ if } \left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right.$$

$$\text{AC2}_i \text{ if } \left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, a_{-i}) \succsim (w_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \succsim (y_i, d_{-i})\end{array}\right.$$

$$\text{AC3}_i \text{ if } \left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(w_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right.$$

Axioms

$AC1_i$ if

Upward Dominance

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \begin{cases}(z_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \succsim (w_i, d_{-i})\end{cases}$$

$AC2_i$ if

Downward Dominance

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \begin{cases}(x_i, a_{-i}) \succsim (w_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \succsim (y_i, d_{-i})\end{cases}$$

$AC3_i$ if

Not incompatible

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \begin{cases}(w_i, a_{-i}) \succsim (y_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{cases}$$

Consequences

$AC1_i \Leftrightarrow \succcurlyeq_i^*$ **is right linear** $\Leftrightarrow [Not(y_i, z_i) \succcurlyeq_i^* (x_i, z_i) \Rightarrow (x_i, w_i) \succcurlyeq_i^* (y_i, w_i)]$

$AC2_i \Leftrightarrow \succcurlyeq_i^*$ **is left linear** $\Leftrightarrow [Not(z_i, x_i) \succcurlyeq_i^* (z_i, y_i) \Rightarrow (w_i, y_i) \succcurlyeq_i^* (w_i, x_i)]$

$AC1_i, AC2_i, AC3_i \Leftrightarrow \succcurlyeq_i^*$ **is strongly linear** $\Leftrightarrow \succcurlyeq_i^{**}$ **is strongly linear**

Remark

- **In all Inter-Attribute Models these axioms are independent**

Results - Denumerable case

If X is finite or countably infinite

- **(M1′-D1) <u>iff</u> reflexivity, independence, RC1, AC1,**
- **(M1′-D2) <u>iff</u> reflexivity, independence, RC1, AC2,**
- **(M1′-D12) <u>iff</u> reflexivity, independence, RC1, AC12,**
- **(M1′-D12=) <u>iff</u> reflexivity, independence, RC1, AC123**

Non Denumerable case:

Add necessary Order Density conditions:

OD* and (ODT^{R}* or ODT^{L}* or ODT*)

Results - Denumerable case

If X is finite or countably infinite

- **(M2′-D12=) <u>iff</u> reflexivity, RC12, AC123**
- **(M3-D12=) <u>iff</u> completeness, RC12, AC123,**
- **(M3′-D12=) <u>iff</u> completeness, TC, AC123.**

Non Denumerable case:

Add necessary Order Density conditions: OD*, ODT*

Remarks

- **Necessary and sufficient conditions**
- **Independent axioms**
- **$AC3_i$ + ($AC1_i$ or $AC2_i$) $\Rightarrow$ $\succsim_i$ is a semi order**
- **With all Inter-Attribute Models**
  $AC1_i, AC2_i, AC3_i \Rightarrow$
  $\succsim_i^*$ defines an homogeneous family of semi orders

Summary of Results (Denumerable case)

		(D1)	(D2)	(D12)	(D12=)
		AC1	AC2	AC12	AC123
(M1′)	ref+indep+RC1	x	x	x	x
(M2′)	ref+RC12				x
(M3)	RC12+comp.				x
(M3′)	TC+comp.				x

Non denumerable case: add necessary order density conditions

Extensions: Rough Sets

- **Greco-Matarazzo-Slowinski: Nontransitive Conjoint Measurement is the theoretical framework for rough approximations of outranking relations**
  - **formal definition of "preference differences"**
  - **relation $\succcurlyeq$ is build through rules combining preference differences**

Extensions: Particular Cases

- **Nontransitive Decomposable Conjoint Measurement models contain as particular cases :**
  - **MCDM methods aiming at building a crisp preference relation (MAUT, ELECTRE I, TACTIC, etc.)**
  - **rules of thumb put forward by experimental psychologists**
    - » **lexicographic semiorder**
    - » **conjunctive**
    - » **disjunctive**
    - » **etc.**

Extensions: Valued relations

- **Non Transitive Decomposable models can easily be extended to the valued case**

  $$f(x,y) = F(p_1(x_1,y_1), p_2(x_2,y_2), \ldots, p_n(x_n,y_n))$$

  $$x \succcurlyeq_{\alpha} y \Leftrightarrow f(x,y) \geq \alpha$$

- **Example: PROMETHEE**

  $$f(x,y) = \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i))$$

Sample Model for (ordinal) Valued Relations

$(\succcurlyeq_\alpha)_{\alpha \in A}$

$x \succcurlyeq_\alpha y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n) \geq \alpha$

with F nondecreasing, $p_i(x_i, y_i) = 0$

Similar models corresponding to (M2), (M3) (M3′) + D12=

Sample Result on valued relations

- The preceding models obtains when X and A are finite <u>iff</u>

  $(\succsim_{\alpha})_{\alpha \in A}$ is a nonincresaing family of independent relations (A is a well-ordered set of indices) satisfying $RC_{\alpha\alpha'}$

$$
RC\alpha\alpha' \quad \left.\begin{array}{r}(x_i, a_{-i}) \succsim_{\alpha} (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succsim_{\alpha'} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim_{\alpha} (w_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \succsim_{\alpha'} (y_i, d_{-i})\end{array}\right.
$$

for all $i \in \{1, 2, \ldots, n\}$ and all $\alpha, \alpha' \in A$

- Add Order Density when X is not denumerable + Lower semi-continuity when A is not finite.

Extensions

- **(Dis)similarity indices**

  $$\sigma(x,y) = \sigma(y,x) = F(p_1(x_1,y_1), p_2(x_2,y_2), \ldots, p_n(x_n,y_n))$$

- **Perny (1998) : Filtering by indifference**

Summary

Non Transitive Decomposable models

- imply substantive requirements on $\succcurlyeq$
- may be axiomatised in a simple way avoiding the use of a denumerable number of conditions in the finite case and of unnecessary structural assumptions in the infinite case
- allow to study the “pure consequences” of cancellation conditions in the absence of transitivity, completeness and structural requirements on X
- are sufficiently general to include as particular cases most aggregation rules that have been proposed in the literature
- provide insights on the links and differences between methods

Future Work

- **Many technical Open Problems**
  - **Additive non transitive**
  - **Rich Structure**
  - **Intra-Attribute Decomposability and increasingness**
  - **Other interesting models ?**
  - **Specific form of F (Min, Max, OWA, etc.)**
- **Aggregation Theory of Homogeneous families of semi orders**
  - **valued relations**
  - **difference measurement**
  - **conjoint measurement**
- **“Model-free” tests of MCDM: RC12, AC123**