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On Residual Transcendental Extensions of a Valuation on K to $K(x_1, ..., x_n)$

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Abstract

Let v be a valuation of a field K with value group G_v, residue field k_v and w be an extension of v to $K(x_1, ..., x_n)$. w is called residual transcendental extension of v if k_w/k_v is a transcendental extension. A residual transcendental extension of v to $K(x_1, ..., x_n)$ is defined in [5] and in this paper it is proved that if w is a residual transcendental extension of v to $K(x_1, ..., x_n)$ it must be defined as in [5] when $ef = n$ where e is the ramification index and f is the residue degree of the extension w/v.

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1 Introduction

Let K be a field, v be a valuation on K with value group G_v and residue field k_v. Defining all extensions of v to $K(x_1, ..., x_n)$ is an old and important problem. Residual transcendental extensions of v to $K(x)$ are described by N.Popescu, V. Alexandru and A. Zaharescu in [1,2]. All extensions of v to $K(x)$ are classified by them in [3]. Certain residual transcendental and residual algebraic extensions of v to $K(x, y)$ are defined by F. Öke and H. İşcan in [4]. A residual transcendental extension of v to $K(x_1, ..., x_n)$ is defined by F. Öke in [5]. In this paper the opposite of the problem studied in [5] is considered: When w is defined as in [5] if w is an r.t. extension of v to $K(x_1, ..., x_n)$ with $transdeg k_w/k_v = n$? The question is answered in this paper and it is proved that w is defined as the common extension of w_i to $K(x_1, ..., x_n)$ where w_i is a residual transcendental extension of v to $K(x_i)$ defined by a minimal pair $(a_i,\delta_i)\in\overline{K}\times G_{\overline{v}}$ for $i=1,...,n$ if $[G_w:G_v]\,[k:k_v]=n$ where k is an algebraic closure of k_v in k_w.

2 Preliminaries and Some Notations

Throughout this paper, v is a valuation of a field K with value group G_v, valuation ring O_v and residue field k_v, $\overline{K}$ is an algebraic closure of K, $\overline{v}$ is the unique extension of v to $\overline{K}$. The value group of $\overline{v}$ is the divisible closure of G_v and the residue field of $\overline{v}$ is the algebraic closure of k_v. $K(x)$ and $K(x_1,...,x_n)$ are rational function fields over K with one variable and n variables respectively. For any α in O_v, α^* denotes the natural image of α in k_v. If $a_1,...,a_n\in\overline{K}$ then the restriction of $\overline{v}$ to $K(a_1,...,a_n)$ will be denoted by $v_{a_1...a_n}$.

Let w be an extension of v to $K(x)$. w is called residual transcendental (r.t.) extension of v if k_w/k_v is a transcendental extension.

w which is defined for each $F=\sum_i a_ix^i\in K\,[x]$ as

$$w(F)=\inf_i(v(a_i)) \tag{1}$$

is called Gauss extension of v to $K(x)$ and $k_w=k_v(x^*)$ is the simple transcendental extension of k_v where x^* is the residue of x and $G_w=G_v$.

Let $a\in\overline{K}$ and $\delta\in G_{\bar{v}}$. $\overline{w}$ which is defined for each $F=\sum_i a_i(x-a)^i\in K\,[x]$ as

$$\bar{w}(F)=\inf_i(\bar{v}(a_i)+i\delta) \tag{2}$$

satisfies all valuation conditions on $\overline{K}\,[x]$ and it is uniquely extended to $\overline{K}(x)$. w is called a valuation defined by a pair $(a,\delta)\in\overline{K}\times G_{\overline{v}}$ or $(a,\delta)\in\overline{K}\times G_{\overline{v}}$ is called a pair of definition of $\overline{w}$. If $[K(a):K]\leq[K(b):K]$ for every $b\in K$ such that $\bar{v}(b-a)\geq\delta$ then (a,δ) is called minimal pair with respect to K.

If w is an extension of v to $K(x)$ then there exists an extension $\overline{w}$ of w to $\overline{K}(x)$ such that $\overline{w}$ is also an extension of $\bar{v}$.

If w is an r.t. extension of v to $K(x)$ then there exists a minimal pair $(a,\delta)\in\overline{K}\times G_{\overline{v}}$ respect to K where a is separable over K. Let $f=Irr(a,K)$ be a minimal polynomial of a respect to K and $\gamma=w(f)$. If $F\in K\,[x]$ let $F=F_1+F_2f+...+F_nf^n$, $\deg F_i<\deg f,\ i=1,...,n$ be the f-expansion of F.

Define

$$w(F)=\inf_i(v_a(F_i(a))+i\gamma) \tag{3}$$

Then one has:

Theorem 2.1: (see [1]). w defined in (1) satisfies all valuation conditions on $K[x]$ and can be uniquely extended to $K(x)$. Moreover w is an r.t. extension of v to $K(x)$ and the followings hold:

1. $G_w = G_{v_a} + Z\gamma$
2. Let e be the smallest non-zero positive integer such that $e\gamma \in G_{v_a}$. Then there exists $h \in K[x]$ such that $\deg h < \deg f$, $v_a(h(a)) = e\gamma$ and $r = f^e/h$ is an element of $K(x)$ of the smallest order such that $w(r) = 0$ and such that $r^* \in k_w$ is transcendental over k_v. k_{v_a} can be identified canonically with the algebraic closure of k_v in k_w and $k_w = k_{v_a}(r^*)$.
3. If w is an r.t. extension of v to $K(x)$, there exists a minimal pair $(a, \delta) \in \overline{K} \times G_{\bar{v}}$ such that a is separable over K. Two pairs (a, δ) and (a_1, δ_1) define the same valuation w if and only if $\delta = \delta_1$ and $\bar{v}(a - a_1) \geq \delta$.

Now we shall give some notations on r.t. extension of v to $K(x, y)$ which is defined in [4].

Let w_1 be an r.t. extension of v to $K(x)$ defined by a minimal pair $(a, \delta) \in \overline{K} \times G_{\overline{v}}$ and w_2 be an r.t. extensions of v to $K(y)$ defined by a minimal pair $(b, \mu) \in (\overline{K} \times G_{\overline{v}})$. Denote by $f = Irr(a, K)$ and $g = Irr(b, K)$. For each polynomial $F(x, y) \in K[x, y]$ can be uniquely written as $F(x, y) = \sum_{i,j} F_{ij} f^i g^j \in K[x, y]$ where $F_{ij} \in K[x, y]$,

$\deg F_{ij}(x, b) < \deg f$, $\deg F_{ij}(a, y) < \deg g$ then w defined as;

$$w(F(x, y)) = \min_{i,j}(\overline{v}(F_{ij}(a, b)) + i\gamma + j\rho) \tag{4}$$

is a valuation on $K(x, y)$. [4] Then w is the common extension of w_1 and w_2 to $K(x, y)$. The value group of w is $G_w = G_{v_{ab}} + \mathrm{Z}\gamma + \mathrm{Z}\rho$ where v_{ab} is the restriction of $\overline{v}$ to $K(a, b)$. Taking e and e' are the smallest positive integers such that $e\gamma \in G_{v_{ab}}$ and $e'\rho \in G_{v_{ab}}$ respectively,

There exists $h \in K[x]$, $degh < \deg f$, $\overline{v}(h(a)) = e\gamma$ and there exists $p \in K[y]$, $\deg p < \deg g$, $\overline{v}(p(b)) = e'\rho$, $r = f^e/h$, $s = g^{e'}/p$. It is shown that the residue field of w is $k_w = k_{v_{ab}}(r^*, s^*)$. [4]

These are generalized for n valuations in [5].

Let w_i be an r.t. extension of v to $K(x_i)$ defined by a minimal pair $(a_i, \delta_i) \in \overline{K} \times G_{\bar{v}}$ and $f_i = Irr(a_i, K)$, $\gamma_i = w_i(f_i)$ where $i = 1, ..., n$. Each polynomial $F \in K[x_1, ..., x_n]$ can be written uniquely as:

$F = \sum_{i_1,...,i_n} F_{i_1...i_n} f_1^{i_1}...f_n^{i_n}$, where $F_{i_1...i_n} \in K[x_1,...,x_n]$,
$\deg_{x_i} F_{i_1...i_n} < \deg f_i$ for $i = 1,...,n$.
Define;

$$w(F) = \inf_{i_1,...,i_n}(v_{a_1...a_n}(F_{i_1 i_2...i_n}(a_1,...,a_n)) + i_1\gamma_1 + ... + i_n\gamma_n) \tag{5}$$

w satisfies all valuation conditions on $K[x_1,...,x_n]$ and it can be uniquely extended to $K(x_1,..,x_n)$. Here w is the common extension of $w_1,...,w_n$ to $K(x_1,..,x_n)$ and one has:

Proposition 2.2:(See [5]) w defined in (3) satisfies all valuation conditions on $K[x_1,...,x_n]$ and it can be uniquely extended to $K(x_1,..,x_n)$. Moreover w is an r.t. extension of v to $K(x_1,...,x_n)$ and

1. $G_w = G_{v_{a_1...a_n}} + \mathbb{Z}\gamma_1 + ... + \mathbb{Z}\gamma_n$.
2. Let e_i be the smallest number such that $e_i\gamma_i \in G_{v_{a_i}}$ where v_{a_i} is the restriction of $\bar{v}$ to $K(a_i)$. Then there exists $h_i \in K[x_i]$ such that

   $\deg h_i < \deg f_i$ and $v_{a_i}(h(a_i)) = e_i\gamma_i$ and $r_i = f_i^{e_i}/h_i \in O_{w_i}$ such that r_i^* $trans/k_v$ for $i = 1,...,n$.
3. $k_{v_{a_1...a_n}}$ can be identified canonically with the algebraic closure of k_v in k_w and $k_w = k_{v_{a_1...a_n}}(r_1^*,...,r_n^*)$.

are satisfied.

3 Residual Transcendental Extensions of v to $K(x_1,...,x_n)$

Let w be an r.t. extension of v to $K(x_1,...,x_n)$ such that $trans \deg k_w/k_v = n$, $e(w/v) = [G_w : G_v]$, $f(w/v) = [k : k_v]$ where k is the algebraic closure of k_v in k_w. Denote by $\deg(w/v)$ the smallest positive integer d for which there exist elements $s_1,...,s_n \in O_w$ such that $[K(x_1,...,x_n) : K(s_1,...,s_n)] = d(w/v)$ and $s_1^*,...,s_n^*$ are algebrically independent over k_v.

Theorem 3.1: Under the above notations if $e(w/v)f(w/v) = d(w/v)$ then w is the common extension of $w_1,...,w_n$ to $K(x_1,...,x_n)$ where w_i is an r.t. extension of v to $K(x_i)$ defined by a minimal pair $(a_i,\delta_i) \in \overline{K} \times G_{\bar{v}}$ for $i = 1,...,n$.

Proof: The case $n = 2$:

If $e(w/v)f(w/v) = d(w/v)$ then k_w is isomorphic to the rational function field with two variables $k_w = k(r^*, s^*)$ where k is an algebraic closure of k_v in k_w.[6] w is an extension of u_1 to $K(x, y)$ where u_1 is an extension of v to $K(x)$. Since $trans \deg k_{u_1}/k_v \leq 1$ and $trans \deg k_w/k_{u_1} \leq 1$ and $k_w = k(r^*, s^*)$ then u_1 is an r.t. extension of v to $K(x)$ and w is an r.t. extension of u_1 to $K(x, y)$. Then u_1 is defined by a minimal pair $(a, \delta) \in \overline{K} \times G_{\bar{v}}$, $k_{u_1} = k_{v_a}(r^*)$ and w is an r.t. extension of u_1 to $K(x, y)$ defined by a minimal pair (b, μ) where b is an element of an algebraic closure of $K(x)$ and μ is an element of $G_{\bar{u}_1}$. $k_w = k'_{u_1}(s^*)$ where k'_{u_1} is an algebraic extension of k_{u_1} that is k'_{u_1} is an algebraic function field of one variable over k_v. But $k'_{u_1} = k'_v(r^*)$ where k'_v is an algebraic extension of k_v because of $k_w = k(r^*, s^*)$. Then (b, μ) should be choosen as an element of $\overline{K} \times G_{\bar{v}}$. Hence w is the common extension of w_1 and w_2 where w_1 is an r.t. extension of v to $K(x)$ defined by minimal pair $(a, \delta) \in \overline{K} \times G_{\bar{v}}$ and w_2 is an r.t. extension of v to $K(y)$ defined by minimal pair $(b, \mu) \in \overline{K} \times G_{\bar{v}}$.

The case general n:

If $e(w/v)f(w/v) = d(w/v)$ then k_w is isomorphic to the field of rational functions in n variables $k_w = k(s_1^*, ..., s_n^*)$ where k is the algebraic closure of k_v in k_w.[6] Let u_i be an extension of u_{i-1} to $K(x_1, ..., x_i)$ for $i = 1, ..., n$. Thus $u_n = w$ and $u_0 = v$. Since $k_w = k(s_1^*, ..., s_n^*)$ and $trans \deg k_{u_i}/k_{u_{i-1}} \leq 1$ then u_i is an r.t. extension of u_{i-1} for $i = 1, ..., n$. u_i is defined by a minimal pair (b, μ_i) where b_i is an element of algebraic closure of $K(x_1, ..., x_{i-1})$ and μ_i is element of $G_{\overline{u}_{i-1}}$. Hence $k_{u_i} = k'_{u_{i-1}}(s_i^*)$ where $k'_{u_{i-1}}$ is an algebraic extension of $k_{u_{i-1}}$ for $i = 1, ..., n$. But since $k_w = k(s_1^*, ..., s_n^*)$ then $k_{u_i} = k'_{u_{i-1}}(s_i^*) \subseteq k(s_1^*, ..., s_n^*)$. This gives $k'_{u_{i-1}}(s_i^*) = k'_v(s_1^*, ..., s_i^*)$ where k'_v is an algebraic extension of k_v for $i = 1, ..., n$. Whereby $(a_i, \delta_i) \in \overline{K} \times G_{\bar{v}}$ for $i = 1, ..., n$. If we denote an r.t. extension of v to $K(x_i)$ defined by $(a_i, \delta_i) \in \overline{K} \times G_{\bar{v}}$ by w_i it is seen that u_i is the common extension of w_i to $K(x_1, ..., x_i)$. Therefore w can be written as the common extension of $w_1, ..., w_n$ to $K(x_1, ..., x_n)$ where w_i is an r.t. extension of v to $K(x_i)$ defined by a minimal pair $(a_i, \delta_i) \in \overline{K} \times G_{\bar{v}}$ for $i = 1, ..., n$.

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