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On Some Properties of Baire-1 Functions

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Abstract

In this paper, we give alternative proofs of some of the properties of Baire-1 functions with respect to the new characterization of Baire-1 functions due to P.Y. Lee, W.K. Tang and D. Zhao. Some well-known functions were given to illustrate some of these properties.

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1 Introduction

Recently, P.Y. Lee, W.K. Tang and D. Zhao formulated a new characterization of Baire-1 functions which is similar to the epsilon-delta definition of continuous functions. A function $f : \mathbb{R} \to \mathbb{R}$ is Baire-1 if for every $\epsilon > 0$ there is a positive real valued function $\delta(\cdot)$ on $\mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$|x - y| < \min\{\delta(x), \delta(y)\} \Longrightarrow |f(x) - f(y)| < \epsilon.$$

Due to this new definition, old results in the theory of Baire-1 functions can be viewed once more from a new perspective. A respected mathematician Russell A. Gordon has this to say regarding alternative proofs: "An alternate proof of a theorem provides a new way of looking at the theorem and this fresh perspective is often enough to justify the new approach"[3]. Certainly, providing a new proof for an old result has immediate advantages. New methods are learned which may turn out useful in solving other problems as well. Moreover, some old and difficult results may now have direct and straightforward proofs. For instance, the usual proof for showing that the class of Baire-1 functions is closed under uniform convergence is quite long and involved. However, with the new characterization the proof becomes short and easy. More than that one can see more clearly now that indeed Baire-1 functions are the natural extensions of the continuous functions. In fact, the proofs of some of the properties of Baire-1 functions are similar to the proofs for continuous functions involving similar properties. Moreover, to appreciate this new approach we gave well-known functions particularly Dirac, Riemann and Thomae functions to illustrate some of these properties and to see some of the theorems in action.

2 Definitions and Preliminaries

We shall denote $\min\{a, b\}$ by $a \wedge b$ and $\max\{a, b\}$ by $a \vee b$ for any two real numbers a and b. We need the following definitions.

Definition 1 *([4]) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Baire-1 if for every $\epsilon > 0$ there is a positive function δ on $\mathbb{R}$ such that for any $x, y \in \mathbb{R}$,*

$$|x - y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x) - f(y)| < \epsilon.$$

Definition 2 *([5]) Let $\mathcal{D}$ be a class of positive real valued functions defined on $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called $\mathcal{D}$-continuous if for any $\epsilon(\cdot) \in \mathcal{D}$ there is a positive real valued function $\delta(\cdot)$ such that for any $x, y \in \mathbb{R}$,*

$$|x - y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x) - f(y)| < \epsilon(f(x)) \wedge \epsilon(f(y)).$$

The following propositions are crucial in the succeeding sections.

Proposition 1 *([5]) Let $\mathbb{R} = \bigcup_{n=1}^{+\infty} F_n$ where F_n's are disjoint F_σ sets. Then there is a positive function $\delta(\cdot)$ on $\mathbb{R}$ such that $x \in F_n$, $y \in F_m$ and $n \neq m$ imply*

$$|x - y| \geq \delta(x) \wedge \delta(y).$$

Proposition 2 *([5]) Let $\mathcal{D}$ be the set of all positive real valued continuous functions defined on $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is Baire-1 then f is $\mathcal{D}$-continuous.*

Proposition 3 *([1]) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function then for every $\epsilon > 0$ there is a positive continuous function δ on $\mathbb{R}$ such that for any $x, y \in \mathbb{R}$,*

$$|x - y| < \max\{\delta(x), \delta(y)\} \Longrightarrow |f(x) - f(y)| < \epsilon.$$

3 Determination of the Positive Function $\delta(\cdot)$

In this section we find explicitly the positive function $\delta(\cdot)$ in the ϵ-δ characterization of Baire-1 functions for some well-known functions. We shall start by showing that any function with a finite set of discontinuity points is Baire-1. We then provide an example to see the theorem in action.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with finite set of discontinuities. Then f is Baire-1.*

Proof: Let $D_f = \{x_1, x_2, \ldots, x_n\}$ be the set of discontinuities of f.
Let $N = \min\{|x_i - x_j| : 1 \leq i, j \leq n \quad \text{and} \quad i \neq j\}$. For each real number $\xi \notin D_f$ there is a corresponding positive number δ_ξ such that for any $y \in \mathbb{R}$,

$$|\xi - y| < \delta_\xi \Longrightarrow |f(\xi) - f(y)| < \epsilon.$$

Define

$$\delta(x) = \begin{cases} N, & x \in D_f; \\ \delta_x, & x \notin D_f. \end{cases}$$

Suppose $|x - y| < \delta(x) \wedge \delta(y)$. It is clear that x and y cannot be both in D_f. Hence, either $x \in \mathbb{R} - D_f$ or $y \in \mathbb{R} - D_f$. In either case, $|f(x) - f(y)| < \epsilon$. Hence, f is Baire-1. $\square$

Example 1 *Consider the well-known Dirac function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} 0, & x \neq 0; \\ 1, & \textit{otherwise.} \end{cases}$$

We will show that f is Baire class one. Let $\epsilon > 0$. One can verify that f is continuous on the set of all real numbers except at $x = 0$. Therefore, for each real number $\xi \neq 0$ there is a corresponding positive number δ_ξ such that for any $y \in \mathbb{R}$,

$$|\xi - y| < \delta_\xi \Longrightarrow |f(\xi) - f(y)| < \epsilon.$$

One can take $\delta_\xi = |\xi|$. Define

$$\delta(x) = \begin{cases} 1, & x = 0; \\ \delta_x, & x \neq 0. \end{cases}$$

Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta(x) \wedge \delta(y)$. It can easily be seen that $|f(x) - f(y)| < \epsilon$ and hence f is Baire-1.

The next theorem extends Theorem 1. Note however that the method of constructing the positive function $\delta(\cdot)$ in Theorem 1 cannot be applied in the next theorem.

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function with discrete set of discontinuities. Then f is Baire-1.*

Proof: Let $D_f = \{r_1, r_2, \ldots, r_n, \ldots\}$ be the set of discontinuities of f. Since D_f is discrete then for every i there is an open interval U_{r_i} of r_i such that $U_{r_i} \cap D_f = \{r_i\}$. Let $l(U_{r_i})$ denotes the length of the interval U_{r_i} for each i. Again, for every $\xi \notin D_f$ there exists a positive number δ_ξ such that for any $y \in \mathbb{R}$,

$$|\xi - y| < \delta_\xi \Longrightarrow |f(\xi) - f(y)| < \epsilon.$$

Put

$$\delta(x) = \begin{cases} \frac{1}{2} l(U_{r_i}), & \text{if } x = r_i,\ i \in \mathbb{N}; \\ \delta_x, & \text{otherwise.} \end{cases}$$

Suppose $|x - y| < \delta(x) \wedge \delta(y)$. Observe that x and y cannot both belong to D_f at the same time. Hence, either $x \in \mathbb{R} - D_f$ or $y \in \mathbb{R} - D_f$. Clearly, f is Baire-1. □

Example 2 *Let $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} 1, & x = \frac{1}{n},\ n \in \mathbb{N}; \\ 0, & \textit{otherwise.} \end{cases}$$

One can show that the set of discontinuities of f is the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. For every $\xi \notin A$ there exists a positive number δ_ξ such that for any number $y \in \mathbb{R}$,

$$|\xi - y| < \delta_\xi \Longrightarrow |f(\xi) - f(y)| < \epsilon.$$

Put

$$\delta(x) = \begin{cases} \frac{1}{n}, & x = \frac{1}{n},\ n \in \mathbb{N}; \\ 1, & x = 0; \\ \delta_x, & \textit{otherwise.} \end{cases}$$

Suppose that $|x - y| < \delta(x) \wedge \delta(y)$. One can check that the only possibilities are the following: (1) both x and y belong to $A - \{0\}$ or (2) x or y is outside A. In both cases, $|f(x) - f(y)| < \epsilon$. Hence, f is Baire-1.

Two more interesting examples are given below the Riemann and Thomae functions. These functions have interesting properties in the sense that both functions are discontinuous on a countable set that is dense in $\mathbb{R}$.

Example 3 *Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ be an enumeration of the set of rational numbers. Define $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} \frac{1}{n}, & x = r_n; \\ 0, & \textit{otherwise}. \end{cases}$$

This is the well-known Riemann function. One can show that f is continuous on the set of irrational numbers and discontinuous on the set of rational numbers. Let $\epsilon > 0$. For every irrational number ξ there is a positive number δ_ξ such that for any $y \in \mathbb{R}$,

$$|\xi - y| < \delta_\xi \Longrightarrow |f(\xi) - f(y)| < \epsilon.$$

We can find a natural number n such that $\frac{1}{n} < \epsilon$. Consider $r_1, r_2, \ldots r_n$. Let $N = \min\{|r_i - r_j| : 1 \leq i, j \leq n \quad \text{and} \quad i \neq j\}$. Fixed $k > n$ and let $M_k = \min\{|r_k - r_i| : 1 \leq i \leq n\}$. Define

$$\delta(x) = \begin{cases} N, & x = r_k, \quad k \leq n; \\ M_k, & x = r_k, \quad k > n; \\ \delta_x, & \textit{otherwise}. \end{cases}$$

Suppose $|x - y| < \delta(x) \wedge \delta(y)$. Notice that x and y cannot be both in $\{r_1, r_2, \ldots r_n\}$. Furthermore, x cannot be in $\{r_1, r_2, \ldots, r_n\}$ and $y = r_k$, $k > n$. There are two possibilities left: (1) both x and y belong to $\{r_{n+1}, r_{n+2}, \ldots\}$ or (2) at least one of x and y is irrational. In both cases, $|f(x) - f(y)| < \epsilon$. Therefore, f is Baire-1.

We will now show that the Thomae's function is Baire-1 using ϵ-δ characterization of Baire-1 functions.

Example 4 *Let $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q},\ p, q \in \mathbb{Z} \text{ and } \gcd(p,q) = 1,\ q > 0; \\ 1, & x = 0; \\ 0, & \textit{otherwise}. \end{cases}$$

It is continuous on the set of irrational numbers but discontinuous on the set of rational numbers. Let $\epsilon > 0$. For every irrational number ξ there exists a corresponding positive number δ_ξ such that for any $y \in \mathbb{R}$,

$$|\xi - y| < \delta_\xi \Longrightarrow |f(\xi) - f(y)| < \epsilon.$$

We can also find a natural number n such that $\frac{1}{n} < \epsilon$. Fix an integer k. We will find a positive function $\delta_k : [k, k+1) \to \mathbb{R}^+$ such that for any $x, y \in [k, k+1)$,

$$|x - y| < \delta_k(x) \wedge \delta_k(y) \Longrightarrow |f(x) - f(y)| < \epsilon.$$

One can verify that there are only finitely many rationals in $[k, k+1)$ with denominators less than n. Let $A = \{r_1, r_2, \ldots, r_s\}$ be the set of rational numbers in $[k, k+1)$ with denominator less than n. Let $N = \min\{|r_i - r_j| : 1 \leq i, j \leq s,\ i \neq j\}$. Let x be a rational number in $[k, k+1)$ but $x \notin A$. Let $M_x = \min\{|x - r_i| : 1 \leq i \leq s\}$. Define

$$\delta_k(x) = \begin{cases} N, & x \in A; \\ M_x, & x \in \mathbb{Q} \cap [k, k+1) \cap A^c; \\ \delta_x, & \textit{otherwise.} \end{cases}$$

Suppose $x, y \in [k, k+1)$ and $|x - y| < \delta_k(x) \wedge \delta_k(y)$. Notice that x and y cannot both belong to A. Furthermore, x cannot be in A and at the same time $y \in \mathbb{Q} \cap [k, k+1) \cap A^c$. If $x, y \in \mathbb{Q} \cap [k, k+1) \cap A^c$ then the denominators of x and y are greater than or equal to n. Thus, $|f(x) - f(y)| < \frac{1}{n} < \epsilon$. If x or y is an irrational number then it follows immediately that $|f(x) - f(y)| < \epsilon$. Thus, we have shown the existence of the positive function δ_k on $[k, k+1)$ with the desired property. We will proceed now to prove that f is Baire-1. Note that $\mathbb{R} = \bigcup_{k=-\infty}^{+\infty} [k, k+1)$ and $[k, k+1) \cap [j, j+1) = \emptyset$ for $k \neq j$. By Proposition 1, there is a positive function δ_0 such that if $x \in [k, k+1)$ and $y \in [j, j+1)$, $k \neq j$ then $|x - y| < \delta_0(x) \wedge \delta_0(y)$ does not hold. Define

$$\delta(x) = \delta_0(x) \wedge \delta_k(x), \quad x \in [k, k+1).$$

Suppose $x, y \in \mathbb{R}$ and $|x - y| < \delta(x) \wedge \delta(y)$. By definition of δ_0 there is a k such that $x, y \in [k, k+1)$. Since $|x - y| < \delta_k(x) \wedge \delta_k(y)$ then $|f(x) - f(y)| < \epsilon$. Hence, f is Baire-1.

So far,we have shown that specific functions with countable set of discontinuities are Baire-1. However, the techniques used in each example are quite different. We shall unify all the results in this section by proving that every function with a countable set of discontinuity points is Baire-1 by using the fact that $\mathbb{R}$ is a Lindelof space.

Theorem 3 *Let f be a real valued function on $\mathbb{R}$ with a countable set of discontinuities. Then f is Baire-1.*

Proof: Let $D_f = \{r_1, r_2, \ldots, r_n, \ldots\}$ be the set of discontinuities of f. We may assume without loss of generality that $D_f = \mathbb{Q}$. For every irrational number ξ there is a corresponding open interval I_ξ centered at ξ such that for any y in $\mathbb{R}$,

$$y \in I_\xi \Longrightarrow |f(\xi) - f(y)| < \frac{\epsilon}{2}.$$

Observe that the family of open sets $\{I_\xi\}_{\xi\in\mathbb{Q}'}$ is an open cover for $\mathbb{R}$. Since $\mathbb{R}$ is Lindelof under the usual metric then there is countable subset $\{I_{\xi_i}\}_{i=1}^{+\infty}$ of $\{I_\xi\}_{\xi\in\mathbb{Q}'}$ that covers $\mathbb{R}$. We can find disjoint countable collection of F_σ sets $\{F_i\}$ such that $\mathbb{R} = \bigcup_{i=1}^{+\infty} F_i$ and $F_i \subseteq I_{\xi_i}$ for each i. By Proposition 1, there exists a positive function δ_0 such that for any $x \in F_m$ and $y \in F_n$, $m \neq n$ implies $|x-y| \geq \delta_0(x) \wedge \delta_0(y)$. Put $\delta(x) = \delta_0(x)$. Suppose $|x-y| < \delta(x) \wedge \delta(y)$. By definition of δ_0 there is an n such that $x, y \in F_n$. Since $F_n \subseteq I_{\xi_n}$ then $x, y \in I_{\xi_n}$. Now,

$$|f(x) - f(y)| \leq |f(x) - f(\xi_n)| + |f(\xi_n) - f(y)| < \epsilon.$$

Hence, f is Baire-1. $\square$

4 Properties of Baire-1 Functions

In this section, we prove some of the basic properties of Baire-1 functions using the new characterization. Although most of the proofs here are quite straightforward, however for the sake of completeness, we provide proof for each of the property. We will start by showing that every continuous function is Baire-1.

Proposition 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then f is Baire-1.*

Proof: Let $\epsilon > 0$ and let f be continuous. For each $x \in \mathbb{R}$ there is a real number $\delta_x > 0$ such that for all $y \in \mathbb{R}$ we have $|x-y| < \delta_x$ implies $|f(x) - f(y)| < \epsilon$. Define a positive function $\delta : \mathbb{R} \to \mathbb{R}^+$ such that $\delta(x) = \delta_x$. If $|x-y| < \delta(x) \wedge \delta(y) < \delta(x) \vee \delta(y)$ implies $|f(x) - f(y)| < \epsilon$. Hence, f is a Baire-1. $\square$

Proposition 5 *If f is continuous and g is Baire-1 then $f \circ g$ is Baire-1.*

Proof: Let $\epsilon > 0$. Since f is continuous there exists a positive function $\delta : \mathbb{R} \to \mathbb{R}^+$ such that for any x, y in $\mathbb{R}$,

$$|x-y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x) - f(y)| < \epsilon.$$

By Proposition 3 the function $\delta : \mathbb{R} \to \mathbb{R}^+$ can be arranged to be a continuous function. Since g is Baire-1 then by Proposition 2, there exists a positive function $\lambda : \mathbb{R} \to \mathbb{R}^+$ such that for any x, y in $\mathbb{R}$,

$$|x-y| < \lambda(x) \wedge \lambda(y) \Longrightarrow |g(x) - g(y)| < \delta(g(x)) \wedge \delta(g(y)).$$

It follows that

$$|x-y| < \lambda(x) \wedge \lambda(y) \Longrightarrow |f(g(x)) - f(g(y))| < \epsilon.$$

All these show that $f \circ g$ is Baire-1. $\square$

Proposition 6 *If $f : \mathbb{R} \to \mathbb{R}$ is Baire-1 then so is the function $|f|$.*

Proof: Let $\epsilon > 0$. There exists a positive function $\delta : \mathbb{R} \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{R}$,

$$|x - y| < \delta(x) \wedge \delta(y) \Longrightarrow |f(x) - f(y)| < \epsilon.$$

By a well-known inequality, we have $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$. Hence, $|f|$ is Baire-1. $\square$

Proposition 7 *If f and g are Baire-1 functions then so is $f + g$.*

Proof: Let $\epsilon > 0$. Since f and g are Baire-1 functions there exist positive functions δ_1 and δ_2 that correspond to f and g respectively such that for any x,y in $\mathbb{R}$,

$$|x - y| < \delta_1(x) \wedge \delta_1(y) \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$$

and

$$|x - y| < \delta_2(x) \wedge \delta_2(y) \Longrightarrow |g(x) - g(y)| < \frac{\epsilon}{2}.$$

Put $\delta(x) = \delta_1(x) \wedge \delta_2(x), \quad x \in \mathbb{R}$. Suppose $|x - y| < \delta(x) \wedge \delta(y)$. Then

$$\begin{aligned} |f(x) + g(x) - (f(y) + g(y))| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $f + g$ is Baire-1. $\square$

The idea of the proof in the next proposition is borrowed from [5].

Proposition 8 *If f and g are Baire-1 functions then so is the product fg.*

Proof: Let $\epsilon > 0$. Let $E_n = \{x \in \mathbb{R} : |f(x)| < n \text{ and } |g(x)| < n\}$, $n \in \mathbb{N}$. It is straightforward to show that $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ and each E_n is an F_σ set. We can find a sequence $\{F_n\}$ of disjoint F_σ sets such that $F_n \subseteq E_n$ for each n and $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$. By Proposition 1 there is a positive function δ_0 such that if $x \in F_m$ and $y \in F_n$ with $m \neq n$ then $|x - y| \geq \delta_0(x) \wedge \delta_0(y)$. Since f and g are Baire-1 functions then there exist positive functions λ_n and μ_n such that

$$|x - y| < \lambda_n(x) \wedge \lambda_n(y) \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{2n}$$

and

$$|x - y| < \mu_n(x) \wedge \mu_n(y) \Longrightarrow |g(x) - g(y)| < \frac{\epsilon}{2n}.$$

Define $\delta : \mathbb{R} \to \mathbb{R}^+$ as follows

$$\delta(x) = \lambda_n(x) \wedge \mu_n(x) \wedge \delta_0(x), \quad x \in \mathbb{R}.$$

Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta(x) \wedge \delta(y)$. Now,

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &\leq |f(x)|\,|g(x) - g(y)| + |g(y)|\,|f(x) - f(y)| \\
&\leq n\left(|g(x) - g(y)| + |f(x) - f(y)|\right) \\
&< n\left(\frac{\epsilon}{2n}\right) + n\left(\frac{\epsilon}{2n}\right) \\
&= \epsilon.
\end{aligned}$$

Thus, fg is Baire-1. □

Corollary 1 *If $f : \mathbb{R} \to \mathbb{R}$ is Baire-1 then for every constant c, $c + f$ and cf are Baire-1 functions.*

Corollary 2 *If f and g are Baire-1 then the functions $\max\{f, g\}$ and $\min\{f, g\}$ are both Baire-1.*

The classical proof of the theorem below is quite long and involved(See [2]). However, with the new characterization of Baire-1 functions the proof becomes short and easy.

Theorem 4 *If $\{f_n\}$ is a sequence of Baire-1 functions that converges uniformly to f then f is Baire-1.*

Proof: Let $\epsilon > 0$. Since $\{f_n\}$ converges uniformly to f then there exists a natural number N such that for all $n > N$ we have $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in \mathbb{R}$. Pick a natural number $n_0 > N$. Since f_{n_0} is Baire-1, there exists a positive function $\delta : \mathbb{R} \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{R}$,

$$|x - y| < \delta(x) \wedge \delta(y) \Longrightarrow |f_{n_0}(x) - f_{n_0}(y)| < \frac{\epsilon}{3}.$$

Let $x, y \in \mathbb{R}$ such that $|x - y| < \delta(x) \wedge \delta(y)$.
Then

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Therefore, f is Baire-1. □

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