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Filters of Heyting Algebras on Soft Set Theory

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Abstract

The notions of intersection soft filer(IS-filter), Boolean intersectional soft filter(Boolean IS-filter) and ultra inetersectional soft filter(ultra IS-filter) in Heyting algebras are introduced and their characterizations and relations are investigated. We discuss characterizations of IS-filter and Boolean IS-filter and consider relations between IS-filters and Boolean IS–filters. Also we introduce the concept of prime IS-filter and investigate the relation between ultra IS-filter and prime Boolean IS-filter.

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1 Introduction

In mathematics, Heyting algebras are special bounded lattice that constitute a generalization of Boolean algebras. In the 19th century, Luitzen Brouwer founded the mathematical philosophy of intuitionism. Intuitionism is based on the idea that mathematics is a creation of the mind and believed that a statement could only be demonstrated by direct proof. Arend Heyting, a student of Brouwer's, formalized this thinking into his namesake algebras

(Heyting algebra). Heyting-algebras have played an important role and have its comprehensive applications in many aspects including genetic code of biology, dynamical systems and algebraic theory [2, 3, 4, 5, 6, 7, 8, 14].

The complexities of modeling uncertian data in in economics, engineering, environment and many other fields can not successfully use classical methods because of various uncertainties typical for those problems.

To overcome these difficulties, Molodtsov [17] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Maji et al. [15] also studied several operations on the theory of soft sets. Since then, soft set theory has wide range of application in economics, engineering, environment, information science ,inteligence system and algebraic structure [10, 11, 16]

In this paper, we define intersection soft filer(IS-filter), Boolean intersectional soft filter(Boolean IS-filter) and ultra inetersectional soft filter(ultra IS-filter) and investigates related properties.

2 Preliminary Notes

In this section, we recall the definition of heyting algebra and investigate several properties of Heyting algebras. Also we introduce filter and soft set.

Definition 2.1. [1] Heyting algebra is defined to be a bounded lattice $\mathcal{H}$ such that for any pair of elements $x, y \in \mathcal{H}$, there is a largest element $z \in \mathcal{H}$ such that $z \wedge x \leq y$. This element denoted by $x \to y$ and is called implication. The operation which sends each element x to the element $x' = x \to 0$ is called negation.

The definition of implication is equivalent to the existence of an element $x \to y$ such that

$$z \wedge x \leq y \iff z \leq x \to y$$

Proposition 2.2. [1] *For elements x, y, z in a Heyting algebra:*

(hp1) $x \wedge (x \to y) \leq y$,

(hp2) $x \wedge y \leq z \iff y \leq x \to z$,

(hp3) $x \leq y \iff x \to y = 1$,

(hp4) $y \leq x \to y$,

(hp5) $x \leq y \implies z \to x \leq z \to y$ and $y \to z \leq x \to z$,

(hp6) $x \to (y \to z) = (x \wedge y) \to z$,

(hp7) $x \wedge (y \to z) = x \wedge \{(x \wedge y) \to (x \wedge z)\}$, (hp8) $x \wedge (x \to y) = x \wedge y$,

(hp9) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(hp10) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

Corollary 2.3. *For elements x, y, z in a Heyting algebra:*

(hp11) $x \to (y \to z) = y \to (x \to z)$,

(hp12) $x \to 1 = 1, 1 \to x = x, x \to x = 1$,

(hp13) $x \to (y \to x) = 1$,

(hp14) $(x \vee y) \leq (x \to y) \to y$.

Proof. (hp11) Using (hp6) we have $x \to (y \to z) = (x \wedge y) \to z = (y \wedge x) \to z = y \to (x \to z)$. (hp12) $x \leq 1 \Rightarrow x \to 1 = 1$ By (hp8),we have $1 \to x = 1 \wedge (1 \to x) = 1 \wedge x = x$. $x \leq x \Rightarrow x \to x = 1$. (hp13) Using (hp11) and (hp12), we have $x \to (y \to x) = y \to (x \to x) = y \to 1 = 1$. (hp14) Using (hp6) and (hp9), we get $(x \vee y) \to ((x \to y) \to y)$ $=(x \to ((x \to y) \to y)) \wedge (y \to ((x \to y) \to y))$ $=((x \to y) \to (x \to y)) \wedge ((x \to y) \to (y \to y))$ $=1 \wedge ((x \to y) \to 1)$ $=1 \wedge 1$ $=1$ and so $(x \vee y) \leq (x \to y) \to y$ by (hp3). □

Example 2.4. [8] (1)Every Boolean algebra is a Heyting algebra and every Heyting algebra is a distributive lattice.

(2)Every bounded chain lattice $\mathcal{H}$ is a Heyting algebra. Indeed, for any $a, b \in \mathcal{H}$

$$a \to b := \begin{cases} a & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

Hence a Heyting algebra need not be a Boolean algebra.

Definition 2.5. [7] A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called a *filter* of $\mathcal{H}$ if it satisfies

(1) $(\forall x, y \in \mathcal{H}) \quad (x \in \mathcal{F}, x \leq y \;\Rightarrow\; y \in \mathcal{F})$,

(2) $(\forall x, y \in \mathcal{H}) \quad (x, y \in \mathcal{F}, x \wedge y \in \mathcal{F})$.

Proposition 2.6. [7] A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called a *filter* of $\mathcal{H}$ if it satisfies

(1) $1 \in \mathcal{F}$,

(2) $(\forall x, y \in \mathcal{H}) \; (x \in \mathcal{F}, x \to y \in \mathcal{F} \;\Rightarrow\; y \in \mathcal{F})$.

Definition 2.7. [7] Let $\mathcal{F}$ be a filter of $\mathcal{H}$. $\mathcal{F}$ is called a Boolean filter of $\mathcal{H}$ if it satisfies $(x \wedge x') \in \mathcal{F}$ for all $x \in \mathcal{H}$.

Definition 2.8. [7] Let $\mathcal{F}$ be a filter of $\mathcal{H}$. $\mathcal{F}$ is called an ultra filter of $\mathcal{H}$ if it satisfies $x \in \mathcal{F}$ or $x' \in F$ for all $x \in \mathcal{H}$.

Molodtsov [17] introduced the concept of soft set as a new mathematical tool, and Çağman et al. [10] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$

Definition 2.9. [10, 17] A soft set (f, A) of E (over U) is defined to be the set of ordered pairs

$$(f, A) := \{(x, f_A(x)) : x \in E,\ f_A(x) \in \mathscr{P}(U)\},$$

such that $f_A(x) = \emptyset$ if $x \notin A$. The soft set (f, A) is simply denoted by f_A.

For a soft set (f, A) of E over U and a subset τ of U, the set

$$i_A(f_A; \tau) = \{x \in A \mid f_A(x) \supseteq \tau\}$$

is called the *τ-inclusive set* of (f, A).

3 Main Results

We first introduce the definition of IS-filter and investigate several properties.

Definition 3.1. *A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is called an IS-filter of $\mathcal{H}$ if it satisfies:*

(f1) $(\forall x, y \in \mathcal{H}) \quad (x \leq y \ \Rightarrow \ f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}(y))$,

(f2) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \wedge y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(y))$.

Proposition 3.2. *A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies:*

(f3) $(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x))$,

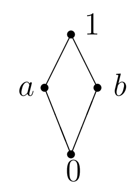
(f4) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y))$.

Proof. Suppose that $f_{\mathcal{H}}$ of $\mathcal{H}$ is an IS-filter of $\mathcal{H}$. Since $x \leq 1$ for all $x \in \mathcal{H}$, it follows from Definition 3.1(f1) that $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x)$ for all $x \in \mathcal{H}$. This proves (f3) hold. By (hp1), we have $x \wedge (x \to y) \leq y$. Hence $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \wedge (x \to y))$ By Definition 3.1(f2),$f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \wedge (x \to y)) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y)$. This proves (f4) hold. Conversely, assume that $f_{\mathcal{H}}$ satisfies conditions (f3) and (f4). Let $x, y \in \mathcal{H}$ such that $x \leq y$ then $x \to y = 1$ by (h3). By condition (f4) and (f3), we have $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x)$.

which implies, $f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}(y)$.This prove (f1). By (hp6) and (hp12), we have $x \to (y \to (x \wedge y)) = (x \wedge y) \to (x \wedge y) = 1$. By Definition 3.1(f2), we have $f_{\mathcal{H}}(x \wedge y) \supseteq f_{\mathcal{H}}(y) \cap f_{\mathcal{H}}(y \to (x \wedge y)) \supseteq f_{\mathcal{H}}(y) \cap (f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to (y \to (x \wedge y)))) = f_{\mathcal{H}}(y) \cap (f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(1)) = f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(y)$. for all $x, y \in \mathcal{H}$. This proves (f2) hold, and so $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Example 3.3. *Let $\mathcal{H} = \{0, a, b, 1\}$ be a set with the following Cayley table and Hasse diagram.*

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1



Then $\mathcal{H}$ is a Heyting algebra. Let $f_{\mathcal{H}}$ be a soft set over $U = \mathbf{Z}$ in $\mathcal{H}$ given as follows:

$$f_{\mathcal{H}}(x) = \begin{cases} 2\mathbf{Z} & \text{if} \quad x \in \{a, 1\} \\ 2\mathbf{N} & \text{if} \quad \text{otherwise} \end{cases}$$

Theorem 3.4. *A soft set $f_{\mathcal{H}}$ in $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if*

(f5) $(\forall a, b, c \in \mathcal{H}) \quad (a \to (b \to c) = 1 \Longrightarrow f_{\mathcal{H}}(c) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b))$.

Proof. Assume that $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$. Let $a, b, c \in \mathcal{H}$ be such that $a \to (b \to c) = 1$. By (hp3), we have $a \leq b \to c$. Then $f_{\mathcal{H}}(b \to c) \supseteq f_{\mathcal{H}}(a)$ by (f1), and so $f_{\mathcal{H}}(c) \supseteq f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(b \to c) \supseteq f_{\mathcal{H}}(b) \cap f_{\mathcal{H}}(a)$. Conversely, let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ satisfying (f5). By $x \leq 1$ and (hp12) we have $x \to (x \to 1) = 1$ it follows from (f5) that $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x)$ for all $x \in \mathcal{H}$. Using (hp12), we know that $(x \to y) \to (x \to y) = 1$ for all $x, y \in \mathcal{H} = 1$. It follows from (f5) that $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y)$ for all $x, y \in \mathcal{H}$. Therefore $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$. □

Corollary 3.5. *A soft set $f_{\mathcal{H}}$ in $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if*

(f6) $(\forall a, b, c \in \mathcal{H}) \quad ((a \wedge b) \leq c \Longrightarrow f_{\mathcal{H}}(c) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b))$.

Proof. Using (hp2) and (hp3), we have $(a \wedge b) \to c = (a \to (b \to c)) = 1$. Therefore Corollary is valid by Theorem 3.4. □

Theorem 3.6. *Let $f_{\mathcal{H}}$ be a soft set in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies conditions* (f3) *and*

(f7) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((x \to (y \to z)) \cap f_{\mathcal{H}}(y))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Since $x \to (y \to z) = y \to (x \to z)$, we have $(x \to (y \to z)) \to (y \to (x \to z)) = 1$ by (hp3). It follows from Theorem 3.4, we have $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (y \to z)) \cap f_{\mathcal{H}}(y)$ for all $x, y, z \in \mathcal{H}$. Convesely, suppose that $f_{\mathcal{H}}$ satisfies condition (f3) and (f7). Putting $x = 1$ in (f7) and using (hp12), we have $f_{\mathcal{H}}(z) = f_{\mathcal{H}}(1 \to z) \supseteq f_{\mathcal{H}}(1 \to (y \to z)) \cap f_{\mathcal{H}}(y) = f_{\mathcal{H}}(y \to z) \cap f_{\mathcal{H}}(y)$ for all $x, y \in \mathcal{H}$. Therefore $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$. □

Theorem 3.7. *Let $f_{\mathcal{H}}$ be a soft set in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of H if and only if it satisfies conditions* (f3) *and*

(f8) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (y \to z)) \cap f_{\mathcal{H}}(x \to y))$.

Proof. Assume $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Since $y \wedge (y \to z) \leq z$, we have $x \to z \geq x \to ((y \to z) \wedge y) = (x \to (y \to z)) \wedge (x \to y)$. By (hp5) and (hp10) It follows from Corollary 3.5, we have $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((x \to (y \to z)) \cap f_{\mathcal{H}}(x \to y)$. Conversely, suppose that $f_{\mathcal{H}}$ satisfies condition (f1) and (f8). Taking $x = 1$ in (f8) and using (f2), we have $f_{\mathcal{H}}(z) \supseteq f_{\mathcal{H}}(y \to z) \cap f_{\mathcal{H}}(y)$ for all $x, y \in \mathcal{H}$. Hence $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Theorem 3.8. *Let $f_{\mathcal{H}}$ be a IS-filter in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies conditions* (f3) *and*

(f9) $(\forall x, y, z \in \mathcal{H})$ $(f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(y \to z))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. By (hp6), (hp8) and (hp11), we have $(x \to y) \to ((y \to z) \to (x \to z)) = (y \to z) \to ((x \to y) \to (x \to z)) = (y \to z) \to (((x \to y) \wedge x) \to z) = (y \to z) \to ((x \wedge y) \to z) = (y \to z)(x \to (y \to z)) = x \to ((y \to z) \to (y \to z)) = x \to 1 = 1$. It follows from Theorem 3.4, we have we have $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(y \to z)\}$. This proves (f9) hold. Suppose that $f_{\mathcal{H}}$ satisfies conditions (f3) and (f9). Obviously $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x)$. Taking $x = 1$ in (f9) and using (hp12), we have $f_{\mathcal{H}}(z) \supseteq f_{\mathcal{H}}(y \to z) \cap f_{\mathcal{H}}(y)$ for all $x, y \in f_{\mathcal{H}}$. This proves (f4) hold, and so $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$ by Proposition 3.2 □

Theorem 3.9. *Let $f_{\mathcal{H}}$ be a soft set in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ if and only if it satisfies the following conditions:*

(f10) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(x))$,

(f11) $(\forall x, a, b \in \mathcal{H}) \quad (f_{\mathcal{H}}((a \to (b \to x)) \to x) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b))$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. Using (hp13), we get $f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(x \to (y \to x)) \cap f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x)$ for all $x, y \in \mathcal{H}$. By (hp11) and $a \to ((a \to (b \to x)) \to (b \to x)) = (a \to (b \to x)) \to (a \to (b \to x)) = 1$, then $a \subseteq ((a \to (b \to x)) \to (b \to x))$. It follows from (f3) that $f_{\mathcal{H}}((a \to (b \to x)) \to (b \to x)) \supseteq f_{\mathcal{H}}(a)$. By Theorem 3.6 we have $f_{\mathcal{H}}((a \to (b \to x)) \to x) \supseteq f_{\mathcal{H}}((a \to (b \to x)) \to (b \to x)) \cap f_{\mathcal{H}}(b) \supseteq f_{\mathcal{H}}(a) \cap f_{\mathcal{H}}(b)$. Conversely, let $f_{\mathcal{H}}$ be a IS-filter in $\mathcal{H}$ satisfying conditions (f10) and (f11). If we take $y = x$ in (f11), then $f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \to x) \supseteq f_{\mathcal{H}}(x)$ for all $x \in \mathcal{H}$. Using (f11), we obtain $f_{\mathcal{H}}(y) = f_{\mathcal{H}}(1 \to y) = f_{\mathcal{H}}(((x \to y) \to (x \to y)) \to y) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x)$ for all $x, y \in \mathcal{H}$. Therefore $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$. □

Theorem 3.10. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$. Then the following are equivalent:*

(f12) $(\forall x, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (z' \to z))$,

(f13) $(\forall x, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$,

(f14) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(y \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y)$.

(f15) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq \{f_{\mathcal{H}}(x \to (z' \to y)) \cap f_{\mathcal{H}}(y \to z)\})$.,

Proof. (f12) $\Rightarrow$ (f13) Assume that $f_{\mathcal{H}}$ satisfies the condition (f12) and let $x, y, z \in \mathcal{H}$. Using (hp5) and (hp11), we know that $x \to z \leq z' \to (x \to z) = x \to (z' \to z)$. Using (f1), we have $f_{\mathcal{H}}(x \to z) \subseteq f_{\mathcal{H}}(x \to (z' \to z).)$ Therefore $f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$. (f13) $\Rightarrow$ (f14) Assume that $f_{\mathcal{H}}$ satisfies the condition (f13) and let $x, y, z \in \mathcal{H}$. Since $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$, we have $f_{\mathcal{H}}(x \to (z' \to z)) \supseteq f_{\mathcal{H}}(y \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y)$. Using (f13), then we have $f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z)) \supseteq f_{\mathcal{H}}(y \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y)$. (f14) $\Rightarrow$ (f15) Assume that $f_{\mathcal{H}}$ satisfies the condition (f14) and let $x, y, z \in \mathcal{H}$. By (hp5) and $(z' \to y) \leq ((y \to z) \to (z' \to z))$ then we have $x \to (z' \to y) \leq x \to ((y \to z) \to (z' \to z))$. It follows from (f1) that $f_{\mathcal{H}}(x \to ((y \to z) \to (z' \to z))) \supseteq f_{\mathcal{H}}(x \to (z' \to y))$. Using (f14), (hp11) and (f3), we have $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((y \to z) \to (x \to (z' \to z))) \cap f_{\mathcal{H}}(y \to z) = f_{\mathcal{H}}(x \to ((y \to z) \to (z' \to z))) \cap f_{\mathcal{H}}(y \to z) \supseteq f_{\mathcal{H}}(x \to (z' \to y)) \cap f_{\mathcal{H}}(y \to z)$. for all $x, y \in \mathcal{H}$. (f15) $\Rightarrow$ (f12) Assume that $f_{\mathcal{H}}$ satisfies the condition (f12) and let $x, y, z \in \mathcal{H}$. Taking $y = z$ in condition (f15) and using (f3), we obtain $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (z' \to z)) \cap f_{\mathcal{H}}(z \to z) = f_{\mathcal{H}}(x \to (z' \to z)) \cap f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \to (z' \to z)$. Therefore $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}(x \to (z' \to y))$. □

Theorem 3.11. *A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is an IS-filter of $\mathcal{H}$ if and only if the nonempty τ-inclusive set $i_{\mathcal{H}}(f_{\mathcal{H}}; \tau)$ is a filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$.*

Proof. Suppose that $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$ and for each $\tau \in \mathscr{P}(U)$ be such that $i_{\mathcal{H}}(f_{\mathcal{H}}; \tau) \neq \emptyset$, then there exists $a \in i_{\mathcal{H}}(f_{\mathcal{H}}; \tau)$ such that $f_{\mathcal{H}}(a) \supseteq \tau$. By (f3) we have $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(a) \supseteq \tau$ and $1 \in i_{\mathcal{H}}(f_{\mathcal{H}}; \tau)$. Let $x, y \in \mathcal{H}$ be such that $x \to y \in i_{\mathcal{H}}(f_{\mathcal{H}}; \tau)$ and $x \in i_{\mathcal{H}}(f_{\mathcal{H}}; \tau)$. Then $f_{\mathcal{H}}(x \to y) \supseteq \tau$ and $f_{\mathcal{H}}(x) \supseteq \tau$. It follows from (f4) that $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) \supseteq \tau$, that is, $y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Thus $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)\,(\neq \emptyset)$ is a filter of $\mathcal{H}$ by Proposition 2.6. Conversely, suppose that τ-inclusive set $i_H(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$ with $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)\,(\neq \emptyset)$. For any $x \in \mathcal{H}$, let $f_{\mathcal{H}}(x) = \tau$. Then $x \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Since $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$, hence $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. It follows that $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x) = \tau$. Let $x, y \in \mathcal{H}$ such that $f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) = \tau$. Then $x, x \to y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Since τ-inclusive set $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$, then we have $y \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. It follows that $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) = \tau$. Therefore $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$ by Proposition 3.2. □

Theorem 3.12. *If $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$, then the set*

$$\Gamma_a := \{x \in \mathcal{H} \mid f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a)\}$$

is a filter of $\mathcal{H}$ for every $a \in \mathcal{H}$.

Proof. Assume that $f_{\mathcal{H}}$ is an IS-filter. For any $x \in \mathcal{H}$, since $f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}(x)$, then $1 \in \Gamma_a$. Let $x, y \in \mathcal{H}$ be such that $x \in \Gamma_a$ and $x \to y \in \Gamma_a$. Then $f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a) \;\; and \;\; f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a)$. It folllow from (f1) that $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a)$. Hence $y \in \Gamma_a$, and so Γ_a is a filter of $\mathcal{H}$. □

Theorem 3.13. *Let $a \in \mathcal{H}$ and let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$. Then*

(1) *If Γ_a is a filter of $\mathcal{H}$, then $f_{\mathcal{H}}$ satisfies the following condition:*

(f12) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a) \;\Rightarrow\; y \in \Gamma_a)$.

(2) *If $f_{\mathcal{H}}$ satisfies* (f1) *and* (f12)*, then Γ_a is a filter of $\mathcal{H}$.*

Proof. (1)Assume that Γ_a is a filter of $\mathcal{H}$. Let $x, y \in \mathcal{H}$ be such that $f_{\mathcal{H}}(x \to y) \cap f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a)$. Then we have the following consequence $x \to y \in \Gamma_a$ and $x \in \Gamma_a$. Since Γ_a is filter, we have $y \in \Gamma_a$. (2) Suppose that $f_{\mathcal{H}}$ satisfies (f3) and (f12). From (f1) it follows that $1 \in \Gamma_a$. Let $x, y \in \mathcal{H}$ be such that $x \in \Gamma_a$ and $x \to y \in \Gamma_a$. We have $f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(a)$ and $f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a)$, This implies that $f_{\mathcal{H}}(x) \cap f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(a)$. By the assumed condition (f12), we get $y \in \Gamma_a$. Therefore Γ_a is a filter of $\mathcal{H}$ by Proposition 2.6 □

we introduce the concept of Boolean IS-filter and and investigate some of the properties.

Definition 3.14. *A IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$ is said to be Boolean IS-filter if the following assertion is valid.*

$$(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)).$$

Remark 3.15. *Every Boolean IS-filter is IS-filter of $\mathcal{H}$, but the converse may not be true a shown in the following example.*

Example 3.16. *Let $\mathcal{H} = [0,1]$ and define $\wedge, \vee$ and implication $\rightarrow$ on $\mathcal{H}$ as follows:*

$$\begin{cases} x \wedge y & = \min\{x,y\}, \\ x \vee y & = \max\{x,y\} \end{cases} \quad x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y \end{cases}$$

for all $x, y \in \mathcal{H}$. Then $\mathcal{H}$ is a Heyting-algebra. Let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ in which

$$f_{\mathcal{H}}(x) := \begin{cases} \tau & \text{if } x \in [0.5, 1], \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\tau(\neq \emptyset) \in \mathscr{P}(U)$. Then $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$.

But it is not a Boolean IS-filter of $\mathcal{H}$ over U since $f_{\mathcal{H}}(1) = \tau$ and

$$f_{\mathcal{H}}(\frac{1}{3} \vee \frac{1}{3}') = f_{\mathcal{H}}(\frac{1}{3} \vee (\frac{1}{3} \rightarrow 0)) = f_{\mathcal{H}}(\frac{1}{3} \vee 0) = f_{\mathcal{H}}(\frac{1}{3}) = \emptyset$$

Proposition 3.17. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$, then the following are equivalent:*

(1) $(\forall x, z \in \mathcal{H}) \quad f_{\mathcal{H}}(x \rightarrow z) = f_{\mathcal{H}}(x \rightarrow (z' \rightarrow z))$,

(2) $(\forall x \in \mathcal{H}) \quad f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x' \rightarrow x)$,

(3) $(\forall x, y \in \mathcal{H}) \quad f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}((x \rightarrow y) \rightarrow x)$,

(4) $(\forall x, y \in \mathcal{H}) \quad f_{\mathcal{H}}(x) = f_{\mathcal{H}}((x \rightarrow y) \rightarrow x)$,

(5) $(\forall x, y, z \in \mathcal{H}) \quad f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}(z \rightarrow ((x \rightarrow y) \rightarrow x)) \cap f_{\mathcal{H}}(z)$,

Proof. (1) $\Rightarrow$ (2) Assume that $f_{\mathcal{H}}$ satisfies the condition (1) and let $x \in \mathcal{H}$. Using condition (1), we have $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1 \rightarrow x) = f_{\mathcal{H}}(1 \rightarrow (x' \rightarrow x)) = f_{\mathcal{H}}(x' \rightarrow x)$. (2) $\Rightarrow$ (3) Since $x' \leq x \rightarrow y$, then $(x \rightarrow y) \rightarrow x \leq x' \rightarrow x$, and so $f_{\mathcal{H}}(x' \rightarrow x) \supseteq f_{\mathcal{H}}((x \rightarrow y) \rightarrow x)$. Thus, from (2), we can deduce that $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(x' \rightarrow x) \supseteq f_{\mathcal{H}}((x \rightarrow y) \rightarrow x)$. (3) $\Rightarrow$ (4) On the other hand, since $x \leq (x \rightarrow y) \rightarrow x$, we have $f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}((x \rightarrow y) \rightarrow x)$. Thus, we can get $f_{\mathcal{H}}(x) = f_{\mathcal{H}}((x \rightarrow y) \rightarrow x)$. (4) $\Rightarrow$ (5) Since $f_{\mathcal{H}}$ is an IS-filter of $\mathcal{H}$, then $f_{\mathcal{H}}((x \rightarrow y) \rightarrow x) \supseteq f_{\mathcal{H}}(z \rightarrow ((x \rightarrow y) \rightarrow x)) \cap f_{\mathcal{H}}(z)$. It follows from (4) that $f_{\mathcal{H}}(x) = f_{\mathcal{H}}((x \rightarrow y) \rightarrow x) \supseteq f_{\mathcal{H}}(z \rightarrow ((x \rightarrow y) \rightarrow x)) \cap f_{\mathcal{H}}(z)$. (5) $\Rightarrow$ (1) Since $z \leq x \rightarrow z$, we have $(x \rightarrow z)' \leq z'$ and $z' \rightarrow (x \rightarrow z) \leq (x \rightarrow z)' \rightarrow (x \rightarrow z)$. Thus, we have $f_{\mathcal{H}}((x \rightarrow z)' \rightarrow (x \rightarrow z)) \supseteq f_{\mathcal{H}}(z' \rightarrow (x \rightarrow z))$. It follows from (5) that $f_{\mathcal{H}}(x \rightarrow z) \supseteq f_{\mathcal{H}}(1 \rightarrow (((x \rightarrow z) \rightarrow 0) \rightarrow (x \rightarrow z))) \cap f_{\mathcal{H}}(1) = f_{\mathcal{H}}((x \rightarrow z)' \rightarrow (x \rightarrow z)) \cap f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}((x \rightarrow z)' \rightarrow (x \rightarrow z)) \supseteq f_{\mathcal{H}}(z' \rightarrow (x \rightarrow z))$ which implies, $f_{\mathcal{H}}(x \rightarrow z) \supseteq f_{\mathcal{H}}(z' \rightarrow (x \rightarrow z))$. Therefore, it follows from Theorem 3.10 that $f_{\mathcal{H}}(x \rightarrow z) = f_{\mathcal{H}}(z' \rightarrow (x \rightarrow z))$. □

Theorem 3.18. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$, then the following are equivalent:*

(1) $f_{\mathcal{H}}$ *is a Boolean IS-filter of* $\mathcal{H}$,

(2) $(\forall x, z \in \mathcal{H}) \quad f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$,

Proof. (1) $\Rightarrow$ (2) Suppose that $f_{\mathcal{H}}$ is Boolean IS-filter and let $x, y \in \mathcal{H}$. Using (f2) we have $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)) \cap f_{\mathcal{H}}(z \vee z') \supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)) \cap f_{\mathcal{H}}(1) \supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z))$ From (hp9), (hp11) and (hp12) and Definition 3.1 we get $(z \vee z') \to (x \to z) = (z \to (x \to z)) \wedge (z' \to (x \to z)) = (x \to (z \to z)) \wedge (z' \to (x \to z)) = (x \to 1) \wedge (z' \to (x \to z)) = 1 \wedge (z' \to (x \to z)) = z' \to (x \to z) = x \to (z' \to z)$ Thus $f_{\mathcal{H}}(x \to z) \supseteq f_{\mathcal{H}}((z \vee z') \to (x \to z)) = f_{\mathcal{H}}(x \to (z' \to z))$ (2) $\Rightarrow$ (1) Assume that $f_{\mathcal{H}}$ satisfies (2). Using Theorem 3.10 (3) and (hp12), we have $f_{\mathcal{H}}((x' \to x) \to x) = f_{\mathcal{H}}((x' \to x) \to (x' \to x)) = f_{\mathcal{H}}(1)$ Using (hp5), (hp9), (hp11) and (hp12), we have $(x' \to x) \to x \leq (x' \to x) \to (x \vee x') = (1 \wedge (x' \to x)) \to (x \vee x') = ((x \to x) \wedge (x' \to x)) \to (x \vee x') = ((x \vee x') \to x) \to (x \vee x')$, It follow from Definition 3.1 and Proposition 3.17 $f_{\mathcal{H}}(1) = f_{\mathcal{H}}((x' \to x) \to x) \subseteq f_{\mathcal{H}}(((x \vee x') \to x) \to (x \vee x')) = f_{\mathcal{H}}(x \vee x')$, and so $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is Boolean IS-filter. □

Theorem 3.19. *Let* $f_{\mathcal{H}}$ *be an IS-filter of* $\mathcal{H}$. *Then the following are equivalent:*

(1) $f_{\mathcal{H}}$ *is a Boolean IS-filter of* $\mathcal{H}$,

(2) $(\forall x, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) = f_{\mathcal{H}}(x \to (z' \to z))$,

(3) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}((x \to y) \to x))$,

(4) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \to z) \supseteq \{f_{\mathcal{H}}(x \to (z' \to y)) \cap f_{\mathcal{H}}(y \to z)\})$.,

Lemma 3.20. *In Heyting algebra* $\mathcal{H}$ *, the following are hold:*

(1) $(\forall x, y, z \in \mathcal{H}) \quad (x \to y \leq (y \to z) \to (x \to z))$

(2) $(\forall x, y, z \in \mathcal{H}) \quad (x \to y \leq (z \to x) \to (z \to y))$

(3) $(\forall x, y \in \mathcal{H}) \quad ((x \to y) \to y \leq (x \to (x \to y)) \to (x \to y))$

Proof. (1) Since $x \wedge y \leq y$, we have $x \wedge (x \to y) \leq x \wedge y \leq y$ by (hp8). It follows from that (hp8) $(x \wedge (x \to y)) \wedge (y \to z) \leq y \wedge (y \to z) \leq (y \wedge z) \leq z$ and so from (hp2) $(x \to y) \wedge (y \to z) \leq x \to z$ Thus, we have $x \to y \leq (y \to z) \to (x \to z)$ (2) Since $z \wedge x \leq x$, we have $z \wedge (z \to x) \leq z \wedge x \leq x$ by (hp8). It follows from that (hp5) and (hp6) $x \to y \leq (z \wedge (z \to x)) \to y = ((z \to x) \wedge z) \to y \leq (z \to x) \to (z \to y)$ (3) Using (hp12) and (hp6) we get $(x \to y) \to y \leq 1 = (x \to y) \to (x \to y) = ((x \wedge x) \to y) \to (x \to y) = (x \to (x \to y)) \to (x \to y)$ □

Theorem 3.21. *Let $f_{\mathcal{H}}$ be an IS-filter of $\mathcal{H}$. Then the following are equivalent:*

(1) *$f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$,*

(2) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(((x \to y) \to y) \to x) \supseteq f_{\mathcal{H}}(y \to x))$

(3) $(\forall x, y, z \in \mathcal{H}) \quad (f_{\mathcal{H}}(((x \to y) \to y) \to x) \supseteq f_{\mathcal{H}}(z) \cap f_{\mathcal{H}}(z \to (y \to x)))$.

(4) $(\forall x, y \in \mathcal{H}) \quad (f_{\mathcal{H}}(x) \supseteq f_{\mathcal{H}}((x \to y) \to x))$,

Proof. (1) $\Rightarrow$ (2) Suppose that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. Since $x \leq ((x \to y) \to y) \to x$ we have $(((x \to y) \to y) \to x) \to y \leq x \to y$ by Lemma 3.20. Using Lemma 3.20 and (hp 11), we get $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$ and so $f_{\mathcal{H}}((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \supseteq f_{\mathcal{H}}(y \to x)$ for all $x, y \in H$ by Definition 3.1(f1). It follows from Proposition 3.17 that $f_{\mathcal{H}}(((x \to y) \to y) \to y) \supseteq f_{\mathcal{H}}((((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \supseteq f_{\mathcal{H}}(y \to x)$. (2) $\Rightarrow$ (3) Assume that the condition (2) holds in $\mathcal{H}$ and let $x, y \in, \mathcal{H}$. Since $f_{\mathcal{H}}$ is a IS-filter, we have $f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(z) \cap (z \to (y \to x))$. By appling to (2), we get $f_{\mathcal{H}}(((x \to y) \to y) \to x) \supseteq f_{\mathcal{H}}(y \to x) \supseteq f_{\mathcal{H}}(z) \cap (z \to (y \to x))$ (3) $\Rightarrow$ (4) Assume that $f_{\mathcal{H}}$ satisfies condition (3) and let assume that $f_{\mathcal{H}}$ is fantastic filter of $\mathcal{H}$. Since $x \to (((x \to y) \to y) = (x \to y) \to (x \to y) = 1$, we have $x \leq (x \to y) \to y$. Using (hp5) and (hp6), we get $((x \to y) \to x) \leq (x \to y) \to ((x \to y) \to y) = ((x \to y) \wedge (x \to y)) \to y = (x \to y) \to y$ By Definition 3.1 (f1), we have $f_{\mathcal{H}}(x \to (((x \to y) \to y)) \subseteq f_{\mathcal{H}}((x \to y) \to y)$ By Lemma 3.20 (3), we have $((x \to y) \to y \leq (x \to (x \to y)) \to (x \to y))$. By (hp5), $(x \to (x \to y)) \to (x \to y)) \to x \leq ((x \to y) \to y) \to x$. By condition (3) $f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}(((x \to (x \to y)) \to (x \to y)) \to x) = f_{\mathcal{H}}(((x \to y) \to y) \to x)$ Hence $f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}((x \to y) \to y) \cap f_{\mathcal{H}}(((x \to y) \to y) \to x)$ By Proposition 3.2(f3), $f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}(x)$ Since $x \to ((x \to y) \to y) = (x \to y) \to (x \to y) = 1$, we have $x \leq (x \to y) \to y$. Using (hp5) and (hp6), we get $((x \to y) \to x) \leq (x \to y) \to ((x \to y) \to y) = ((x \to y) \wedge (x \to y)) \to y = (x \to y) \to y$ By Definiton 3.1 (f1), we have $f_{\mathcal{H}}(x \to (((x \to y) \to y)) \subseteq f_{\mathcal{H}}((x \to y) \to y)$ By Lemma 3.20 (3), we have $((x \to y) \to y \leq (x \to (x \to y)) \to (x \to y))$. By (hp5), $(x \to (x \to y)) \to (x \to y)) \to x \leq ((x \to y) \to y) \to x$. By condition (2) $f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}(((x \to (x \to y)) \to (x \to y)) \to x) = f_{\mathcal{H}}(((x \to y) \to y) \to x)$ Hence $f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}((x \to y) \to y) \cap f_{\mathcal{H}}(((x \to y) \to y) \to x)$ By Proposition 3.2 (f4), $f_{\mathcal{H}}((x \to y) \to x) \subseteq f_{\mathcal{H}}(x)$ (4) $\Rightarrow$ (1) By Theorem 3.19 $\square$

Theorem 3.22. *A soft set $f_{\mathcal{H}}$ on $\mathcal{H}$ is a Boolean IS-filter of $\mathcal{H}$ if and only if the nonempty τ-inclusive set $i_{\mathcal{H}}(f_{\mathcal{H}}; \tau)$ on $\mathcal{H}$ is a Boolean filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$.*

Proof. Suppose that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. Let $\tau \in \mathscr{P}(U)$ with $i_{\mathcal{H}}(f_{\mathcal{H}};\tau) \neq \emptyset$. Then $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$ by Theorem 3.11. Hence, $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$, and so $\tau \subseteq f_{\mathcal{H}}(1)$. For all $x \in \mathcal{H}$. It follows from Definition 3.14 that $\tau \subseteq f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \vee x')$ and so that $x \vee x' \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Therefore $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a Boolean filter of $\mathcal{H}$ Conversely suppose that $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a Boolean filter of $\mathcal{H}$ for all $\tau \in \mathscr{P}(U)$ with $i_{\mathcal{H}}(f_{\mathcal{H}};\tau) \neq \emptyset$. Then $i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$ is a filter of $\mathcal{H}$, and so $f_{\mathcal{H}}$ is a IS-filter of $\mathcal{H}$. Note that $1 \in i_{\mathcal{H}}(f_{\mathcal{H}};\tau)$. Since $i_{\mathcal{H}}(f_{\mathcal{H}}; f_{\mathcal{H}}(1))$ is a Boolean filter of $\mathcal{H}$, we get $x \vee x' \in i_{\mathcal{H}}(f_{\mathcal{H}}; f_{\mathcal{H}}(1))$ for all $x \in \mathcal{H}$. Hence $f_{\mathcal{H}}(x \vee x') \supseteq f_{\mathcal{H}}(1)$. This implies that $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. □

Theorem 3.23. (Extension property) *Let $f_{\mathcal{H}}$ and $g_{\mathcal{H}}$ be IS-filters of $\mathcal{H}$ such that $f_{\mathcal{H}}(1) = g_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(x) \subseteq g_{\mathcal{H}}(x)$ for all $x \in \mathcal{H}$. If $g_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$, then so is $f_{\mathcal{H}}$.*

Proof. Assume that $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$. Then $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$ for all $x \in \mathcal{H}$. Hence $f_{\mathcal{H}}(x \vee x') \supseteq g_{\mathcal{H}}(x \vee x') = g_{\mathcal{H}}(1) = f_{\mathcal{H}}(1)$ for all $x \in \mathcal{H}$. This implies that $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is a Boolean IS-filter of $\mathcal{H}$ □

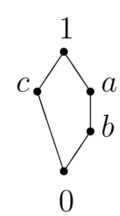
Finally,we introduce the concept of ultra IS-filter and investigate some of the properties. Also we introduce the concept of prime IS-filter and investigate the relation between ultra IS-filter and prime Boolean IS-filter.

Definition 3.24. *A soft set $f_{\mathcal{H}}$ of $\mathcal{H}$ is called an ultra IS-filter of $\mathcal{H}$ if it is a IS-filter of $\mathcal{H}$ that satisfies:*

$$(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1) \quad or \quad f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)).$$

Example 3.25. *Let $\mathcal{H} = \{0, a, b, c, 1\}$ be a set with the following Cayley table and Hasse diagram:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	0	1	1	c	1
c	0	a	a	1	1
1	0	a	b	c	1



Then $\mathcal{H}$ is a Heyting algebra. Let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ in which

$$f_{\mathcal{H}}(x) := \begin{cases} \tau_1 & \text{if } x \in \{1, a, b\}, \\ \tau_2 & \text{otherwise,} \end{cases}$$

where $\tau_2 \subsetneq \tau_1 \in \mathcal{H}$. Then $f_{\mathcal{H}}$ is an ultra IS-filter of $\mathcal{H}$.

Theorem 3.26. *For an IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$, the following assertions are equivalent:*

(1) *$f_{\mathcal{H}}$ is ultra IS-filter,*

(2) $(\forall x, y \in \mathcal{H})$ $(f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ *and* $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1) \Rightarrow f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(1)$ *and* $\quad f_{\mathcal{H}}(y \to x) = f_{\mathcal{H}}(1))$.

Proof. Suppose that $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1)$. Then $f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y') = f_{\mathcal{H}}(1)$ by Hypothesis. Since $f_{\mathcal{H}}(x \to y) \supseteq f_{\mathcal{H}}(x \to 0) = f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$ we get $f_{\mathcal{H}}(x \to y) \geq f_{\mathcal{H}}(1)$ and so $f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(1)$. Similary, it follows from $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1)$ that $f_{\mathcal{H}}(y \to x) = f_{\mathcal{H}}(1)$. Conversely, let $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y) \neq f_{\mathcal{H}}(1)$ imply $f_{\mathcal{H}}(x \to y) = f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(y \to x) = f_{\mathcal{H}}(1))$. Assume that $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$. Since $0 \leq x$, we have $f_{\mathcal{H}}(0) \subseteq f_{\mathcal{H}}(x)$. If $f_{\mathcal{H}}(0) = f_{\mathcal{H}}(1)$ then $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1)$. This is contradiction. So $f_{\mathcal{H}}(x \to 0) = f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$. Therefore $f_{\mathcal{H}}$ is ultra IS-filter. □

Definition 3.27. *A IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$ is said to be prime IS-filter if the following assertion is valid.*

$$(\forall x \in \mathcal{H}) \quad (f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(y)).$$

Theorem 3.28. *Every ultra IS-filter is a prime IS-filter.*

Proof. Suppose that $f_{\mathcal{H}}$ is ultra IS-filter and let $x, y \in \mathcal{H}$. By (hp14), we get $(x \vee y) \leq (x \to y) \to y$. By $f_{\mathcal{H}}$ is IS-filter of $\mathcal{H}$, we have $f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}((x \to y) \to y)$. From $0 \leq y$ and Proposition 2.2 hp(5), we get $(x \to y) \to y \leq x' \to y$. Thus, $f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}((x \to y) \to y) \subseteq f_{\mathcal{H}}(x' \to y)$ by Definition 3.1. So $f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(x' \to y)$. For any $x \in \mathcal{H}$, if $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1)$. then $f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(y)$ If $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$ then $f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$ by Hypthesis. Thus, $f_{\mathcal{H}}(y) \supseteq f_{\mathcal{H}}(x') \cap f_{\mathcal{H}}(x' \to y) = f_{\mathcal{H}}(1) \cap f_{\mathcal{H}}(x' \to y) = f_{\mathcal{H}}(x' \to y)$ by Definition 3.1. Therefore, $f_{\mathcal{H}}(x \vee y) \subseteq f_{\mathcal{H}}(x' \to y) \subseteq f_{\mathcal{H}}(y) \subseteq f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(y)$ This means that $f_{\mathcal{H}}$ is a prime IS-filter of $\mathcal{H}$. □

Example 3.29. *Let $\mathcal{H} = [0, 1]$ and define $\wedge, \vee$ and implication $\to$ on $\mathcal{H}$ as follows:*

$$\begin{cases} x \wedge y & = \min\{x, y\}, \\ x \vee y & = \max\{x, y\} \end{cases} \quad x \to y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y \end{cases}$$

for all $x, y \in \mathcal{H}$. Then $\mathcal{H}$ is a Heyting-algebra.(In Example 3.16) Let $f_{\mathcal{H}}$ be a soft set of $\mathcal{H}$ in which

$$f_{\mathcal{H}}(x) := \begin{cases} \tau_1 & \text{if } x \in [0, 0.5], \\ \tau_2 & \text{if } x \in (0.5, 1]. \end{cases}$$

where $\tau_1 \subsetneq \tau_2$ in $\mathcal{H}$. Then $f_{\mathcal{H}}$ is a prime IS-filter of $\mathcal{H}$. But it is not an ultra IS-filter of $\mathcal{H}$ over U since $f_{\mathcal{H}}(0.5) \neq f_{\mathcal{H}}(1)$ and $f_{\mathcal{H}}(0.5') \neq f_{\mathcal{H}}(1)$.

Definition 3.30. *An IS-filter $f_{\mathcal{H}}$ of $\mathcal{H}$ is said to be prime Boolean IS-filter if it is both prime IS-filter and Boolean IS-filter.*

Theorem 3.31. *In a Heyting-algebra $\mathcal{H}$, the notion of a ultra IS-filter coincides with the notion of prime Boolean IS-filter.*

Proof. In Theorem 3.28, we show that every ultra IS-filter is a prime IS-filter. For any $x \in \mathcal{H}$, since $x \leq x \vee x'$, $x' \leq x \vee x'$, we get $f_{\mathcal{H}}(x) \subseteq f_{\mathcal{H}}(x \vee x')$, $f_{\mathcal{H}}(x') \subseteq f_{\mathcal{H}}(x \vee x')$. According to the Definition of ultra IS-filter, we have $f_{\mathcal{H}}(x) = f_{\mathcal{H}}(1)$ *or* $f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$. Thus, $f_{\mathcal{H}}(1) \subseteq f_{\mathcal{H}}(x \vee x')$. From this and Definition 3.1(f1), we get $f_{\mathcal{H}}(1) = f_{\mathcal{H}}(x \vee x')$. This means that $f_{\mathcal{H}}$ is an Boolean IS-filter of $\mathcal{H}$. Conversely, suppose that $f_{\mathcal{H}}$ is an Boolean prime IS-filter of $\mathcal{H}$. For any $x \in \mathcal{H}$, $f_{\mathcal{H}}(x \vee x') = f_{\mathcal{H}}(1) \leq f_{\mathcal{H}}(x) \cup f_{\mathcal{H}}(x')$ by Definition 3.14 and Definition 3.24 If $f_{\mathcal{H}}(x) \neq f_{\mathcal{H}}(1)$, since $f_{\mathcal{H}}(x) \leq f_{\mathcal{H}}(1), f_{\mathcal{H}}(x') \leq f_{\mathcal{H}}(1)$, by Definition 3.1 (f1) we have $f_{\mathcal{H}}(x') = f_{\mathcal{H}}(1)$. Thus, $f_{\mathcal{H}}$ is an ultra IS-filter of $\mathcal{H}$.

□

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