JÓZEF MARCINKIEWICZ: ANALYSIS AND PROBABILITY

N. H. BINGHAM, Imperial College London

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JÓZEF MARCINKIEWICZ

Life

Born 4 March 1910, Cimoszka, Bialystok, Poland
Student, 1930-33, University of Stefan Batory in Wilno (professors Stefan Kempisty, Juliusz Rudnicki and Antoni Zygmund)
1931-32: taught Lebesgue integration and trigonometric series by Zygmund
MA 1933; military service 1933-34
PhD 1935, under Zygmund
1935-36, Fellowship, U. Lwów, with Kaczmarz and Schauder
1936, senior assistant, Wilno; dozent, 1937

Spring 1939, Fellowship, Paris; offered chair by U. Poznań

August 1939: in England; returned to Poland in anticipation of war (he was an officer in the reserve); already in uniform by 2 September

Second lieutenant, 2nd Battalion, 205th Infantry Regiment

Defence of Lwoẃ 12 - 21 September 1939; Lwoẃ surrendered to Red (Soviet) Army

Prisoner of war 25 September (”temporary internment” by USSR); taken to Starobielsk

Presumed executed Starobielsk, or Kharkov, or Kozielsk, or Katyń; Katyń Massacre commemorated on 10 April

Work

We outline (most of) the main areas in which M’s influence is directly seen today, and sketch the current state of (most of) his areas of interest – all in a very healthy state, an indication of M’s (and Z’s) excellent mathematical taste.

55 papers 1933-45 (the last few posthumous)

Collaborators: Zygmund 15, S. Bergman 2, B. Jessen, S. Kaczmarz, R. Salem

Papers (analysed by Zygmund) on:

Functions of a real variable

Trigonometric series

Trigonometric interpolation

Functional operations

Orthogonal systems

Functions of a complex variable

Calculus of probability

MATHEMATICS IN POLAND BETWEEN THE WARS

K. Dabrowski and E. Hensz-Chadyńska, Józef Marcinkiewicz (1910-40) in commemoration of the 60th anniversary of his death, *Fourier Analysis and Related Topics*, Banach Centre Publications 56 (2002), 1-5.

Kazimierz Kuratowski: *A half-century of Polish mathematics: Remembrances and reflections*, PWN, 1980

Chapters on:

1. Before the restoration of independence
2. The twenty years' period between the Wars
3. The period of Nazi occupation
4. Mathematics in post-War Poland
5. Profiles of the creators of the Polish school of mathematics (Banach, Janiszewski, Mazurkiewicz, Sierpiński, Steinhaus, Zaremba)
6. Profiles of post-War leaders (d. after 1970) (Ważewski, Mostowski, Marczewski)

Founding of *Fundamenta Mathematicae*, 1920

Monografje Matematyczne, Warszawa, 1932

I. S. Banach, *Théorie des opérations linéaires*, 1932

II. S. Saks, *Théorie de 'intégrale*, 1933 [VII, Theory of the integral, 1937]

V. A. Zygmund, *Trigonometrical series*, 1935

VI. S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, 1935

Mark Kac, *Enigmas of chance*, Harper and Row, New York, 1987

Stefan Banach: Remarkable life, brilliant mathematics. Biographical materials ed. Emilia Jakimowicz and Adam Miranowicz, Gdańsk UP and Adam Mickiewicz UP, 2007
P. Holgate, Studies in the history of probability and statistics **XLV**. The late Philip Holgate's paper 'Independent functions: Probability and analysis in Poland between the Wars'. *Biometrika* **84** (1997), 159-173 (Introduction, Bingham, 159-160; text, Holgate, 161-173).
N. H. Bingham, Studies in the history of probability and statistics **XLVI**. Measure into probability: from Lebesgue to Kolmogorov. *Biometrika* **87** (2000), 145-156.

ANALYSIS

Antoni Zygmund, *Trigonometrical series*:
M cited twice (aged 25)
Antoni Zygmund, *Trigonometric series, Volumes I, II*, CUP, 1959/68/79:

IV.2, Theorem of Marcinkiewicz (M 1936/37/38): for F closed in (a,b), $f \in L_1(a,b)$, $\chi(.)$ the distance from F, $\lambda > 0$,

$$J_\lambda(x) := \int_a^b f(t)\chi^\lambda(t)dt/|t-x|^{\lambda+1} < \infty$$

a.e. on F, and

$$\int_F |J_\lambda| \leq (2/\lambda) \int_a^b |f|.$$

VIII.3 (M 1936): There exists $f \in L_1$ the partial sums of whose Fourier series oscillates finitely a.e.
IX, Miscellaneous Theorems and Examples 6, 16 (M 1935, MZ 1937): Riemann summability of trig. series
XII.4 (M 1939): Marcinkiewicz interpolation theorem (weak type $(.,.)$; cf. Riesz-Thorin)
XII, MTE 1-4 (MZ 1937): Interpolation
XIII.3 (M 1935/36): If $f \in L_1$ satisfies

$$\frac{1}{h}\int_0^h |f(x+t) - f(x)|dt = O(1/\log 1/|h|)$$

as $h \to 0$ for all $x \in E$, then the Fourier series of f converges a.e. on E.
XIII, MTE 8 (M 1936): Lagrange interpolation
XIV.5 (M 1938): Marcinkiewicz function $\mu(.)$ (analogue of Littlewood-Paley function $g(.)$)
XV.4 (M 1939): Multipliers; Littlewood-Paley function
R. E. A. C. Paley (1907-33), obituary by Hardy (*JLMS* 9 (1934), 76-80; *Works VII*, 744-748): Paley collaborated with Zygmund (1930 – non-continuability of $\sum c_n \phi_n(t) z^n$ for almost all t, with ϕ_n the Rademacher functions and $\sum c_n z^n$ of unit radius of convergence), and with Wiener and Zygmund (1933)
E. M. Stein, On the functions of Littlewood-Paley, Lusin and Marcinkiewicz, *Trans. AMS* **88** (1958), 430-466
E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton UP, 1970
I.2.3: Integral of Marcinkiewicz (cf. Zygmund

IV.2 above)
IV. The Littlewood-Paley theory; multipliers;
IV.3, The Marcinkiewicz multiplier theorem (cf. Z XV.4 above)
Appendix B, Marcinkiewicz Interpolation Th.
E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princton UP, 1971 (dedicated to Zygmund)
IV.2, Marcinkiewicz Interpolation Theorem
E. M. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*, Princeton UP, 1993
II.5.E, multi-dimensional maximal functions, Jessen, M and Z 1935.
Maximal inequalities
Hardy & Littlewood, A maximal inequality with function-theoretic applications, *Acta Math.* **54** (1930), 81-116 [HLP 10.18]
This is one of the papers M cited most often, and maximal inequalities run right through his work, alone and with Zygmund. This theme is

continued in the Calderón-Zygmund collaboration (1950 on), in Stein's work (above), and in probability (below).

Hardy spaces [31], MZ

P. L. Duren, *Theory of H^p spaces*, Acad. Press, 1970 [M Interpolation theorem]

Stein and Weiss, 1960: $H^p(R^n_+)$ in place of $L^p(R^n)$, via Cauchy-Riemann systems in n variables. See e.g.

C. Fefferman, Selected theorems of Eli Stein, *Essays on Fourier anaysis in honor of Elias M. Stein*, Princeton UP, 1995, Ch. 1.

Convergence of Fourier series

M and Z were writing before the Carleson-Hunt theorem on convergence of Fourier series (L. Carleson in 1966, R. A. Hunt in 1968). For their results, see e.g.

A. M. Garsia, *Topics in almost everywhere convergence*, Markham, 1970,

C. P. Mozzochi, *On the pointwise convergence of Fourier series*, LNM 199, 1971.

Gap theorems [29], [30]

N. Levinson, *Gap and density theorems*, AMS, 1940

Multipliers [26], [40]

R. E. Edwards and G. I. Gaudry, *Littlewood-Paley theory and multipliers*, Springer, 1977.

Higher dimensions [7], M-Jessen-Z

In R^k: condition $|f|(\log_+|f|)^{k-1} \in L$

Non-absolute integrals

The Lebesgue integral is absolute (f is integrable iff $|f|$ is), and one needs non-absolute integrals for Fourier series, and to link differentiation more closely with integration, both favourite themes of Marcinkiewicz. For the Denjoy and Perron integrals, see Zygmund, XI.6, Saks VIII. For the Burkill integral, see H. R. Pitt's obituary of J. C. Burkill (BLMS 30 (1998), 85-98). For the Henstock and Kurzweil integrals, see e.g.

R. M. McLeod, *The generalized Riemann integral*, MAA, 1980.

PROBABILITY

The Kolmogorov axiomatics of the *Grundbegriffe der Wahrscheinlichkeitsrechnung* of 1933 were still quite recent. We speak naturally nowadays of *independent random variables*, taking 'random variable' as 'measurable function'. The Marcinkiewicz-Zygmund (MZ) papers of 1937 and 1938, and the M papers of 1938, speak of *independent functions*:

'Les résultats obtenus peuvent être traduits en langage de la théorie des probabilités, ce que nous laissons au lecteur' [Sur les fonctions indépendantes I, *Fund. Math.* 30; Works 328].

For background to this Polish work, see Kac's autobiography *Enigmas of chance*, and

M. Kac, *Statistical independence in probability, analysis and number theory*, MAA, 1959.

For a modern textbook treatment of probability, see e.g.

Y. S. Chow and H. Teicher, *Probability theory: Independence, interchangeabiity, martingales*, Springer, 1978.

10.3: *MZ inequality*. For $p \geq 1$, there are constants A_p, B_p such that for X_n independent, 0-mean random variables,

$$A_p \|(\Sigma_1^n X_j^2)^{1/2}\|_p \leq \|\Sigma_1^n X_j\|_p \leq B_p \|(\Sigma_1^n X_j^2)^{1/2}\|_p.$$

5.2: *MZ Law of Large Numbers*. For X, X_n independent and identically distributed (iid), $S_n := \sum_i^n X_i$, and $0 < p < 2$, the following are equivalent:

(i) $X \in L_p$, i.e. $E|X|^p < \infty$;

(ii) there exists a constant c with

$$(S_n - nc)/n^{1/p} \to 0 \quad a.s. \quad (n \to \infty)$$

(and then w.l.o.g. $c = EX$ if $1 \leq p < 2$, while c is arbitrary if $0 < p < 1$).

MZ Law of the Iterated Logarithm (LIL). According to Kolmogorov's LLN, if X_n have mean 0 and finite variances, $S_n := \sum_1^n X_k$, $s_n^2 := var(S_n) \to \infty$, then if

$$|X_n| = o(s_n/\sqrt{\log \log s_n}) \quad a.s.$$

then

$$limsup\ S_n/\sqrt{2s_n \log\ \log\ s_n} = +1 \quad a.s.$$

Marcinkiewicz and Zygmund (1937) showed that this is sharp: one cannot replace o here by O. For a textbook reference, see e.g.
W. F. Stout, *Almost sure convergence*, Academic Press, 1974, 5.2.

Maximal inequalities

Kolmogorov, 1928 on: use of maximal inequalities (before HL!) to prove strong (a.s.) limit theorems. For details, see e.g.
N. H. Bingham, The work of A.N. Kolmogorov on strong limit theorems. *Theory of Probability and Applications* **34** (1989), 129-139;
N. H. Bingham, Kolmogorov's work on probability, particularly limit theorems, *Bull. LMS* **22** (1990), 51-58.
Use of maximal inequalities for martingales (below) was pioneered by Doob:
J. L. Doob, *Stochastic processes*, Wiley, 1953,

VII.3.

Martingales

One can generalize beyond the sum of 0-mean random variables to martingales. The MZ inequalities become the *Burkholder-Davis-Gundy* (BDG) inequalities; see e.g.

O. Kallenberg, *Foundations of modern probability*, Springer, 1997/2002, Prop. 15.7, Th. 23.12.

D. L. Burkholder has many papers on martingales from 1966 on, often with R. F. Gundy or B. J. Davis. I have heard Burkholder speak many times, always mentioning the name Marcinkiewicz (which I first heard from him).

A. M. Garsia, *Martingale inequalities*, Benjamin, 1973

Random series

Jean-Pierre Kahane, *Some random series of functions*, 2nd ed., CUP, 1985.

Probability in Banach spaces

The ideas of *type* p and *cotype* p have proved

useful in the geometry of Banach spaces. It was shown by A. de Acosta (1981, *Ann. Prob.*) that for a Banach space B and $p \in [1, 2)$, the following are equivalent:
(i) the MZ LLN holds for B;
(ii) B has type p.
See e.g.
M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer, 1991, 7.2, 9.3.
Infinite divisibility [22], [23], [27]
B. V. Gnedenko and A. N. Kolmogorov, *Limit theorem for sums of independent random variables*, Addison-Wesley, 1954.
Analytic characteristic functions [27], [35]
E. Lukacs, *Characteristic functions*, 2nd ed., Griffin, 1970
Brownian motion [41]
This is in the spirit of later work on diffusions and Markov processes, e.g. by E. B. Dynkin.

ANALYSIS AND PROBABILITY

M's work begins in analysis, and develops naturally into probability. His use of maximal inequalities in analysis has inspired much work in probability, e.g. the work of Burkholder and collaborators on martingales.

Ideas also move in the reverse direction. See e.g.

R. Durrett, *Brownian motion and martingales in analysis*, Wadsworth, 1984,

R. F. Bass, *Probabilistic techniques in analysis*, Springer, 1995,

J.-P. Kahane, A century of interplay between Taylor series, Fourier series and Brownian motion, *BLMS* **29** (1997), 257-279.

Kahane's masterly survey (which contains many references) refers to the Paley-Wiener-Zygmund paper of 1933, on random series, on which are based the last two chapters of the Paley-Wiener book *Fourier transforms in the complex domain* (AMS, 1934). Here one finds Brownian motion represented as the sum of random series of various kinds – e.g. Rademacher series (PZ) and Fourier series (W). Lévy's 'broken-line' construction of Brownian motion (in his book *Processus stochastiques et mouvement brownien*, 1948) can be seen to give a similar series expansion of Brownian motion, using the Schauder functions) (Juliusz Schauder (1899-1943)). We now use *wavelet* expansions:

J.-P. Kahane and P.-G. Lemarié-Rieusset, *Fourier series and wavelets*, Gordon & Breach, 1995.

Martingales and differentiation

C. A. Hayes and C. Y. Pauc, *Derivation and martingales*, Springer, 1970.

Martingales and the geometry of Banach space

The Radon-Nikodym theorem and the martingale convergence theorem are the *same theorem*, in that if one holds for a Banach space B, so does the other:

S. D. Chatterji, Martingale convergence and the Radon-Nikodym theorem in Banach spaces. *Math. Scand.* **22** (1968), 21-41.

J. Diestel and J. J. Uhl, *Vector measures*, AMS, 1977.