p-adic Hodge Theory, MATH 726 Fall 2008

Assignment 1

1. Let I be a directed set and $\{G_i\}_{i\in I}$ an inverse system of finite groups with projection maps $\phi_{ij} : G_i \to G_j$ for all $i, j \in I$ satisfying $j \leq i$. Give each G_i the discrete topology and denote by π the product $\pi := \prod_{i\in I} G_i$ endowed with the product topology. Define

$$G := \varprojlim_{i\in I} G_i := \{(g_i)_{i\in I} \mid \phi_{ij}(g_i) = g_j \text{ for all } j \leq i\} \subseteq \pi$$

   (a) Show that G is a closed subset of π.

   (b) Give G the subspace topology. Show that G is compact and totally disconnected for this topology.

   (c) Prove that the natural projection maps $\phi_i : G \to G_i$ are continuous, and that the (open) subgroups $K_i := \ker \phi_i$ for a basis of open neighborhoods of the identity.

   (d) Show that a subgroup of G is open if and only if it is closed and of finite index.

2. Let $I \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the inertia subgroup and $W \subseteq I$ the wild inertia subgroup. Show that there is a non-canonical isomorphism of topological groups

$$I/W \simeq \prod_{\ell\neq p} \mathbf{Z}_\ell.$$

   What can be said if one replaces $\mathbf{Q}_p$ with a general p-adic field K?

3. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{Q}_p)$ be a continuous representation. Show that for all $\ell \neq p$, the image under ρ of any wild inertia group W_ℓ at ℓ is finite. Is the same necessarily true of the image of any I_ℓ?

4. Let F be a finite extension of $\mathbf{Q}_\ell$, and suppose $\rho : G_F \to \mathrm{GL}_n(\mathbf{Q}_p)$ is a continuous representation. Show that $\overline{F}^{\ker(\rho)}$ is infinitely (wildly) ramified if and only if the image of (wild) inertia under ρ is infinite.

5. Do Exercise 1.2.5 in the notes.

6. Do Exercise 1.3.2 in the notes.

7. Let K be a p-adic field. Show that the image of the p-adic cyclotomic character $\chi : G_K \to \mathbf{Z}_p^\times$ is closed.

8. Show that the two definitions of *continuous representation* given in Definition 1.2.1 of the notes really are equivalent.

9. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C})$ be a continuous representation.

   (a) Prove that up to conjugation by an element of $\mathrm{GL}_n(\mathbf{C})$, the representation ρ factors through $\mathrm{GL}_n(K)$ for some field K of finite degree over $\mathbf{Q}$. (You may use the fact that any compact, totally disconnected subgroup of $\mathrm{GL}_n(\mathbf{C})$ is finite).

   (b) Prove that we may take K above to be an abelian extension of $\mathbf{Q}$.

   (c) For a prime p, is it the case that any continuous $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C}_p)$ must factor through $\mathrm{GL}_n(K)$ for some $K/\mathbf{Q}_p$ of finite degree?