Math155: Calculus for Biological Scientists Analysis of Discrete-Time Dynamical Systems

David Eklund

Colorado State University

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Cobwebbing: A graphical solution technique

Consider a discrete-time dynamical system with updating function f:

$$m_{t+1} = f(m_t).$$

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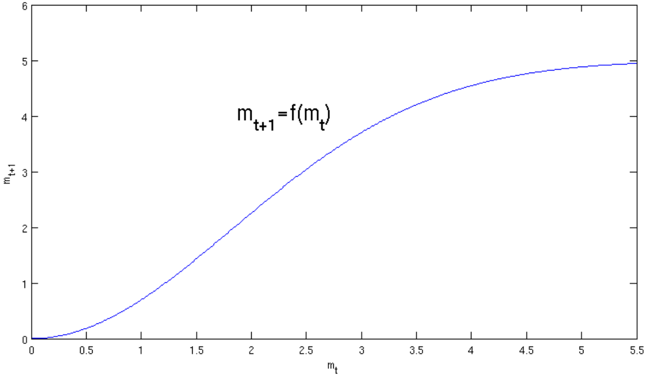
For example, the graph of f might look like this:



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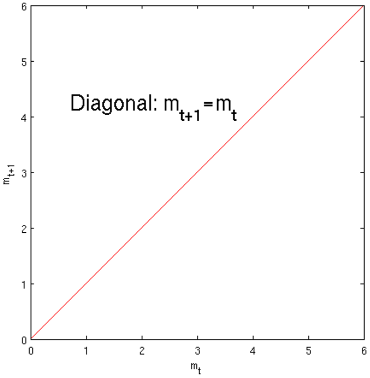
The diagonal is the line below defined by $m_{t+1} = m_t$:



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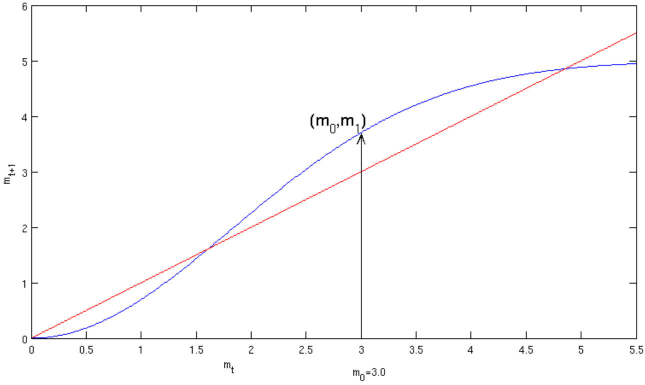
Note that the point (m_0, m_1) lies on the graph of f, $m_1 = f(m_0)$. For example, if $m_0 = 3.0$:



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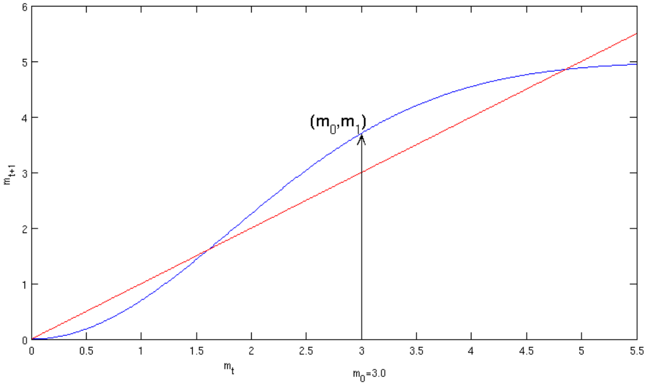

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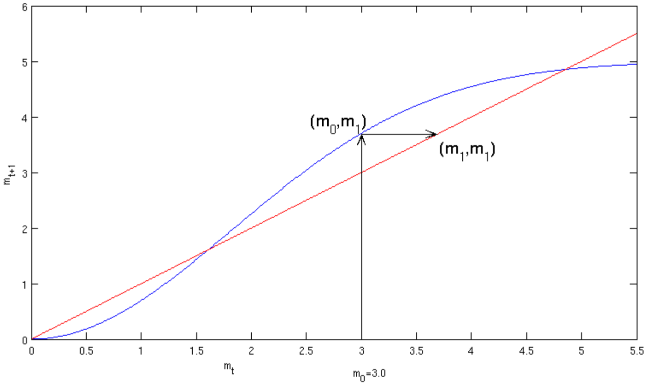
Step 2: move horizontally to the diagonal, ending at (m_1, m_1).



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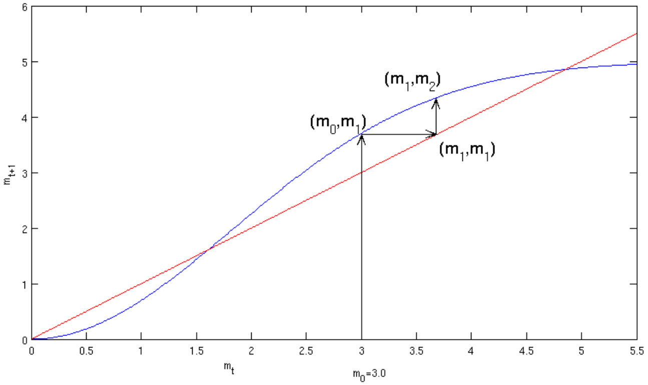
Step 3: move vertically to (m_1, m_2) on the graph of f.



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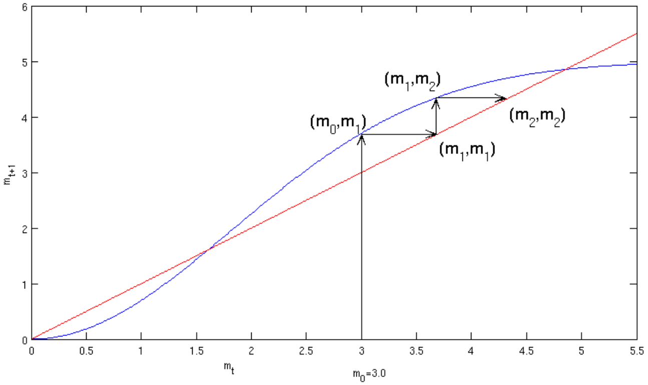
Move horizontally to the diagonal again: move to (m_2, m_2).



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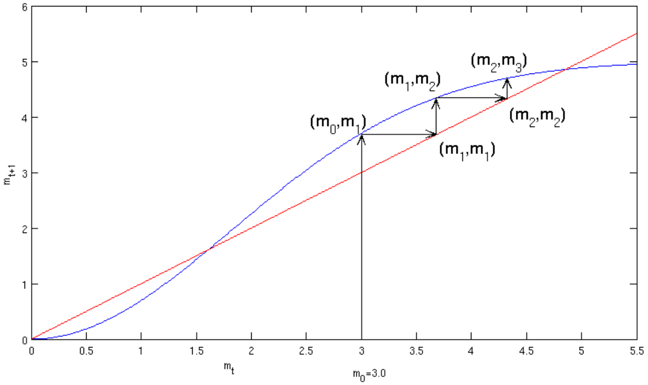
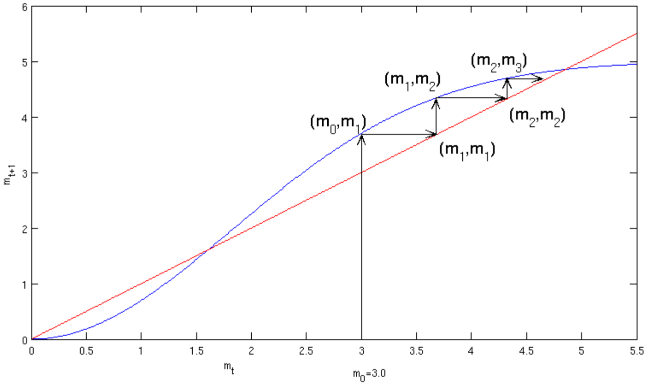
And so on...move vertically to (m_2, m_3).



Move horizontally to (m_3, m_3).



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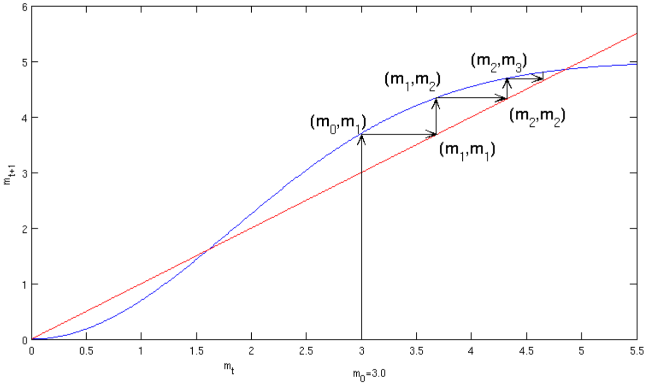
Move vertically to (m_3, m_4).



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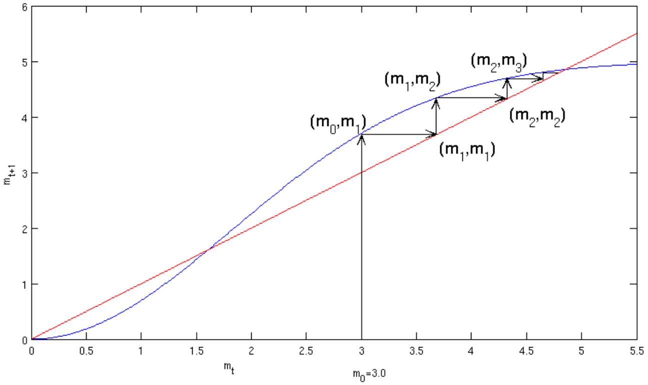
Move horizontally to (m_4, m_4)



Cobwebbing: A graphical solution technique

We approach a point where the graph of f intersects the diagonal, that is where $f(m_t) = m_t$. This is called an **equilibrium**.

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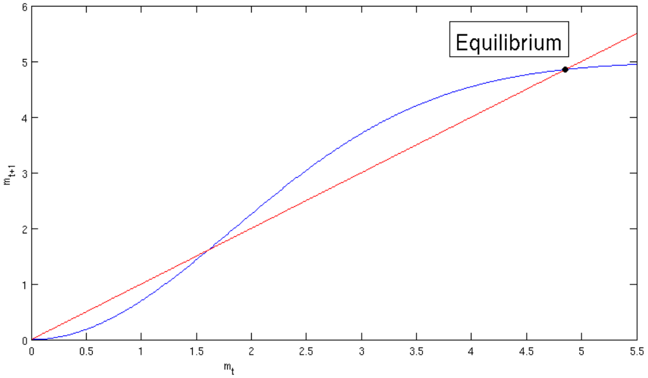

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- In the present context this makes sense: if m_t is an equilibrium then $f(m_t) = m_t$ and therefore

$$m_{t+1} = f(m_t) = m_t,$$

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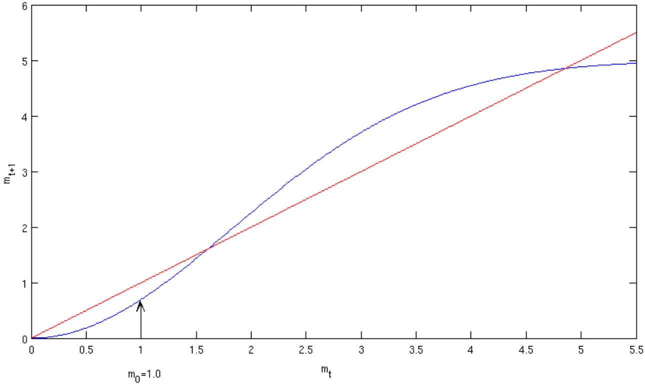
- Physical systems often tend toward some equilibrium over time.

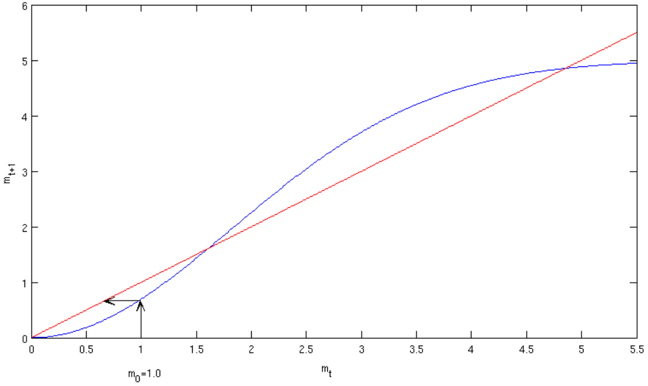
Cobwebbing: A graphical solution technique

What happens if we start at $m_0 = 1.0$?



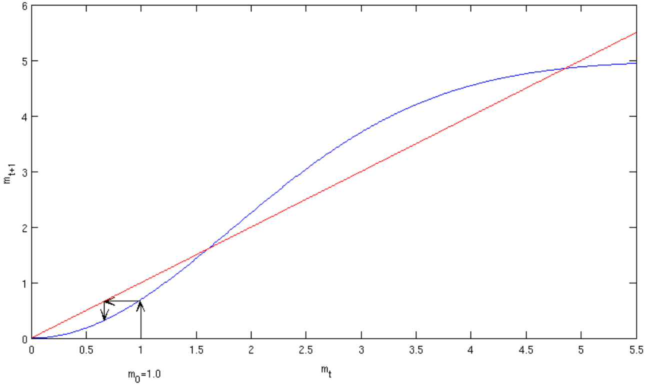
Cobwebbing: A graphical solution technique

Cobweb as before. First move horizontally to the diagonal.



Cobwebbing: A graphical solution technique

Move vertically to the graph of f.



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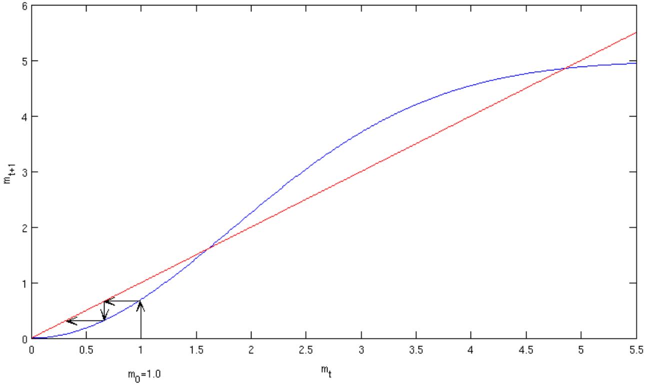

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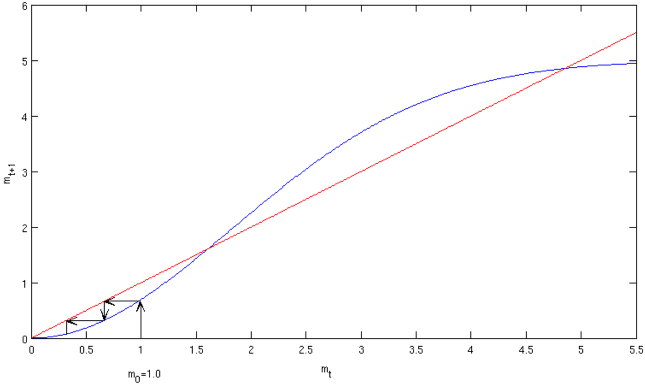

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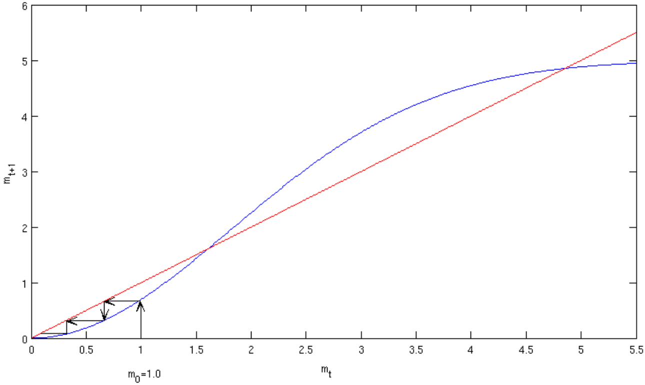

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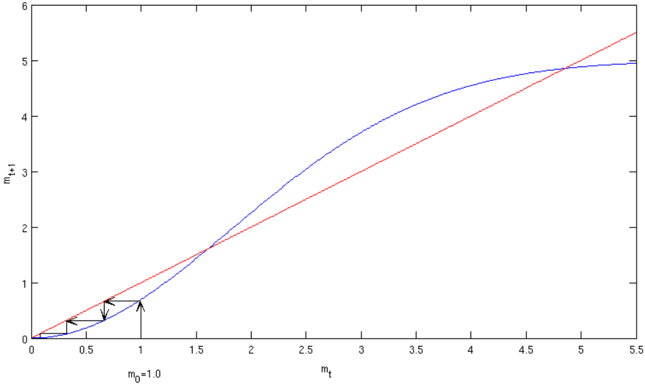

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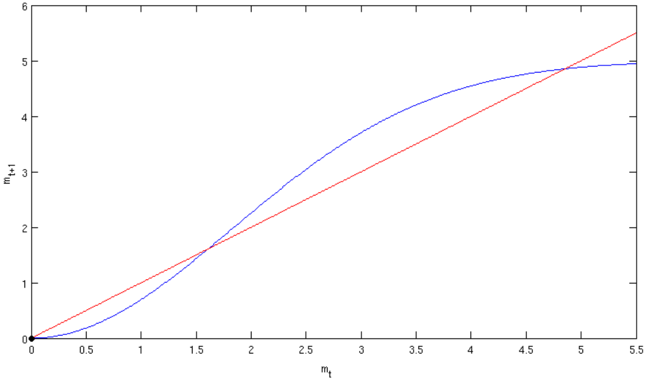

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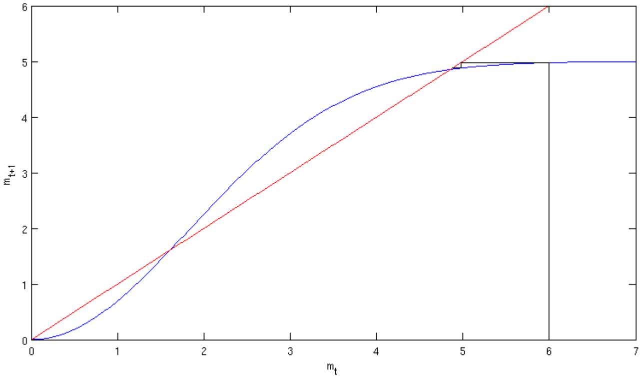
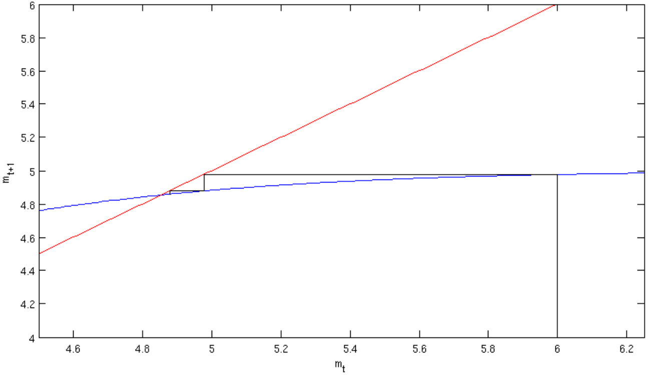


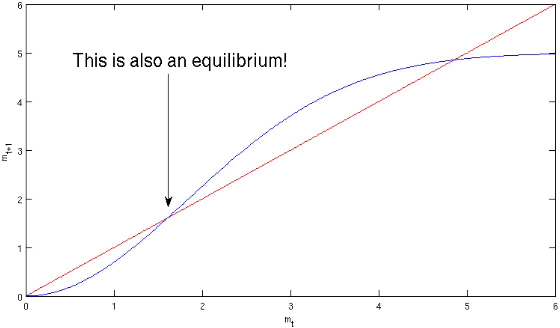
Cobwebbing: A graphical solution technique

If we start at $m_0 = 6.0$, we approach the same equilibrium as before. Zooming in:



Cobwebbing: A graphical solution technique

The equilibrium in the middle is special. If we start there we never leave, but if we merely start close to it then we move away from it!



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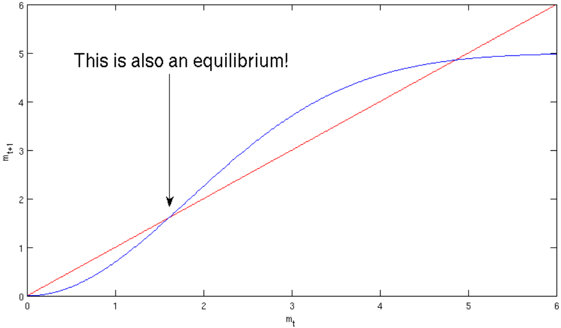

The middle equilibrium is called **unstable**, and the other two are called **stable**.

Examples

Example 1: Consider the medication dynamics from a previous lecture. Let M_t denote the concentration (in mg/l) of medication in a persons bloodstream and suppose this concentration is measured every day and that the updating function is $M_{t+1} = 0.5M_t + 1.0$.

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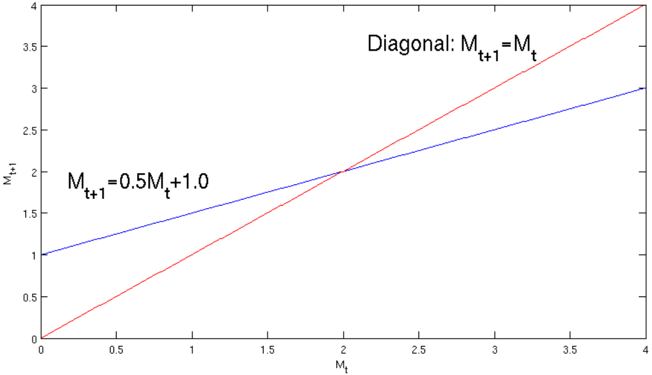

Examples

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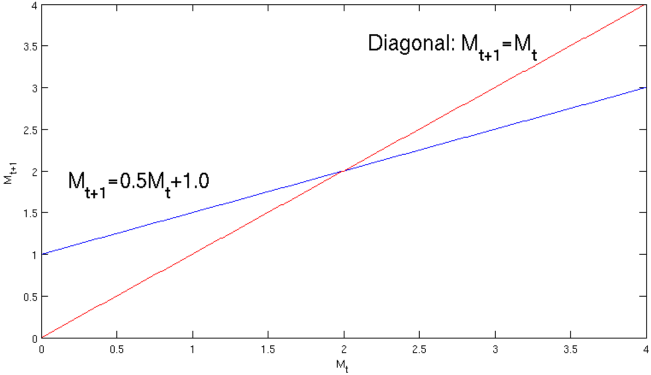

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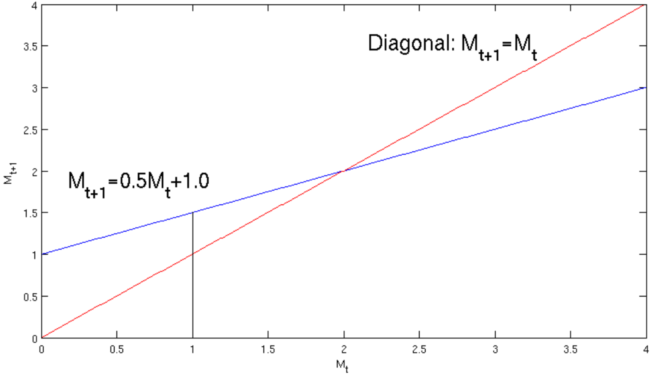

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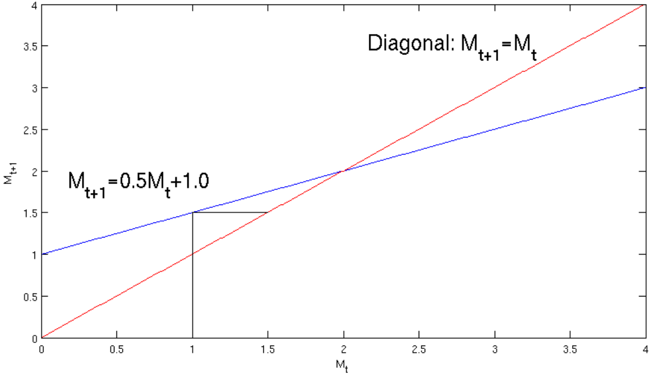

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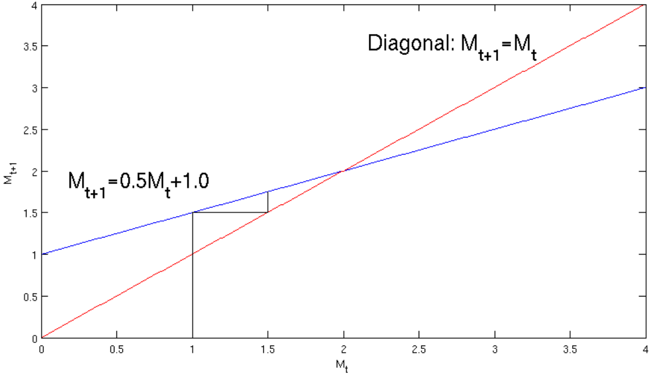

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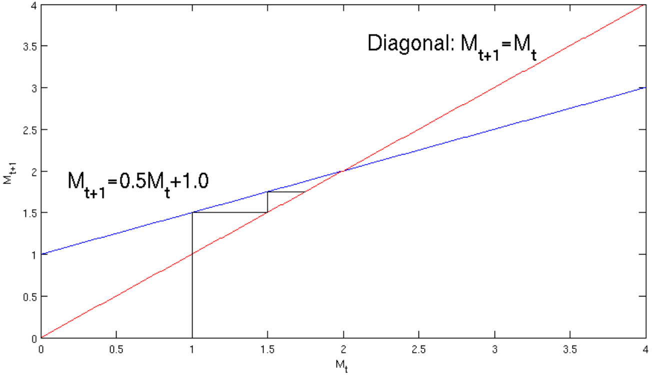

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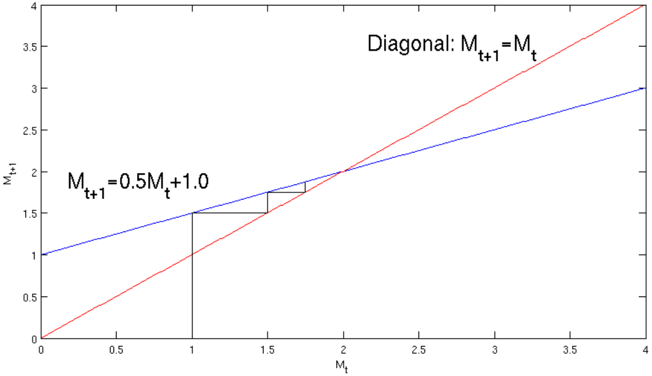

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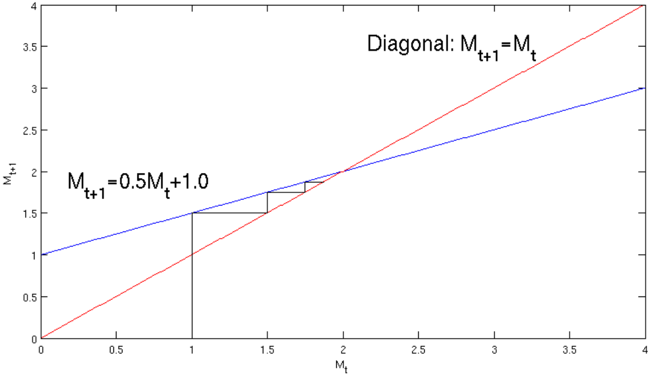

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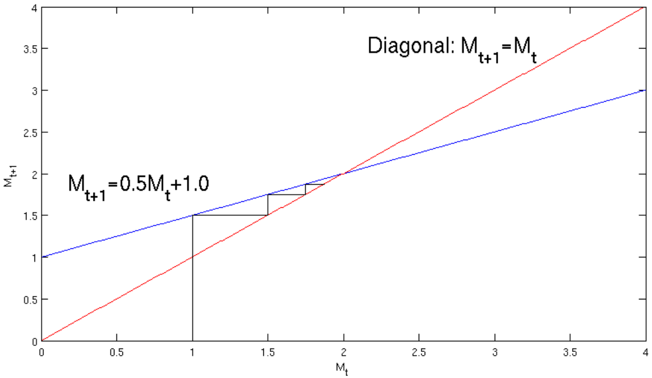

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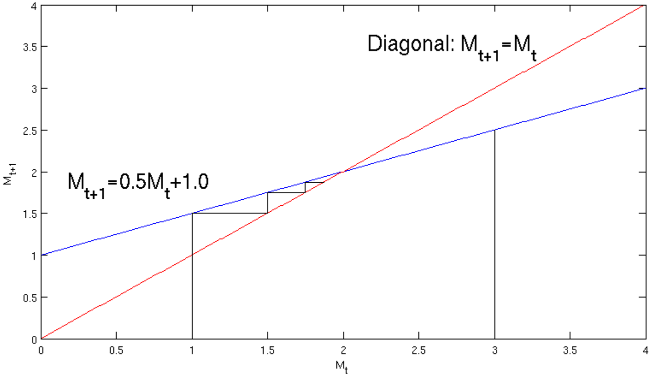

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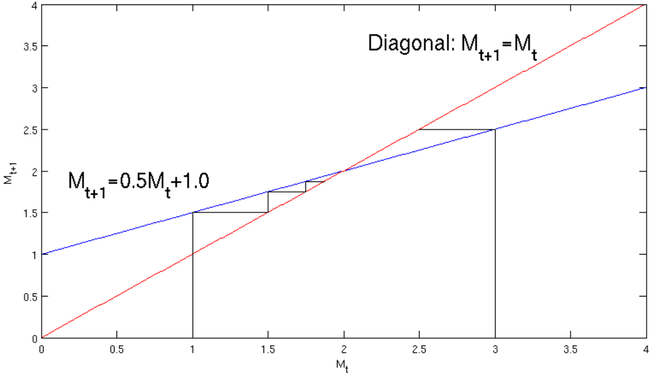

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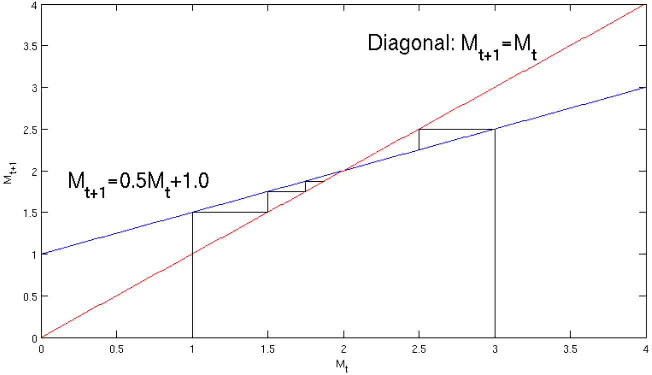

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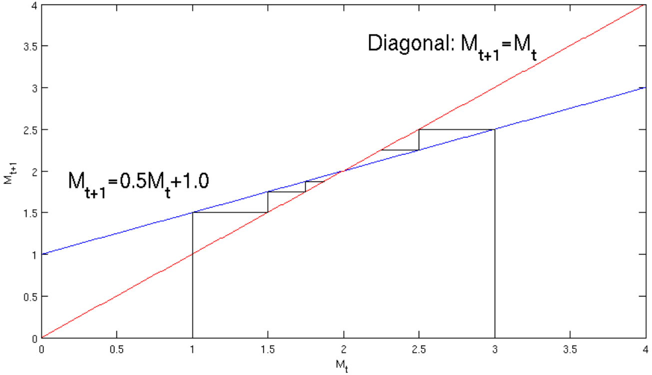

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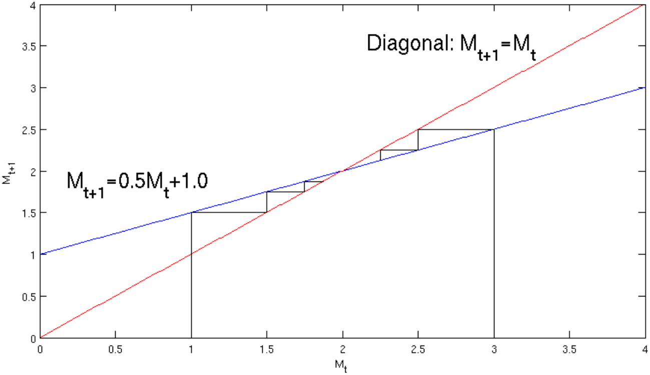

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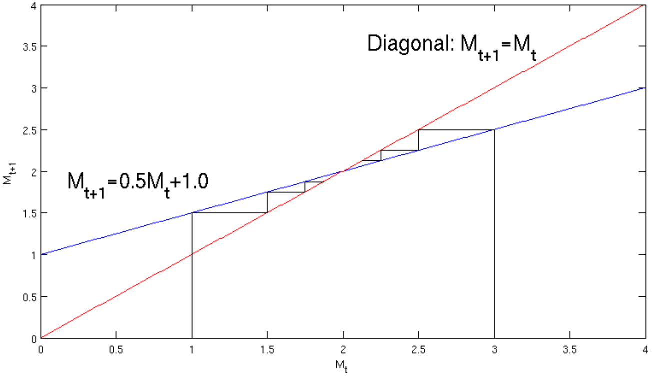

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In both cases we approach the equilibrium 2.0! Recall that we have seen this before.

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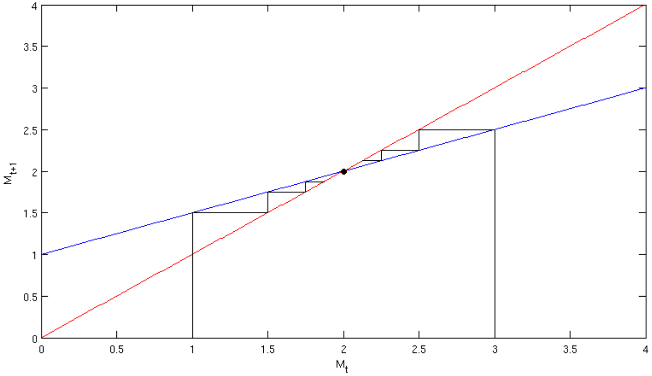

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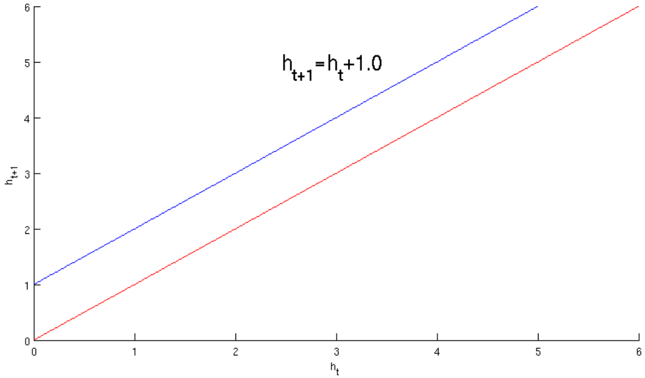

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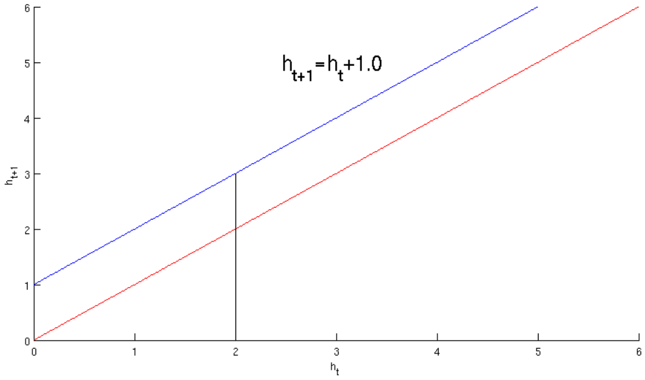

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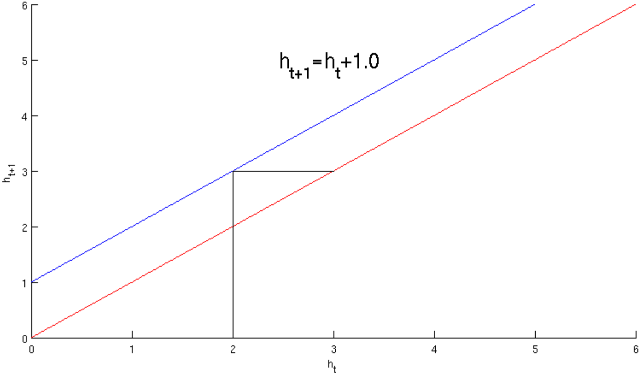

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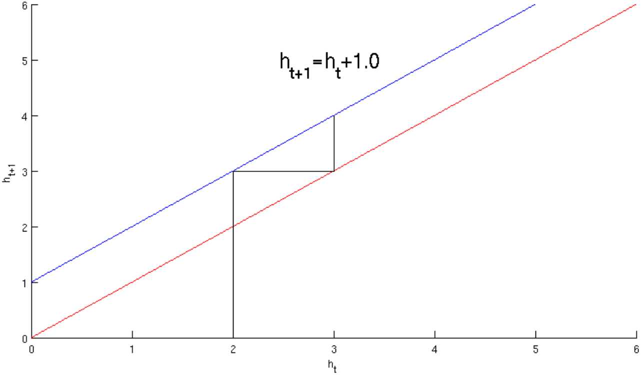

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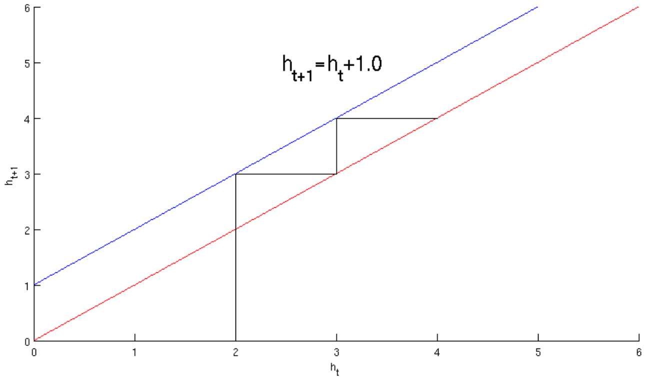

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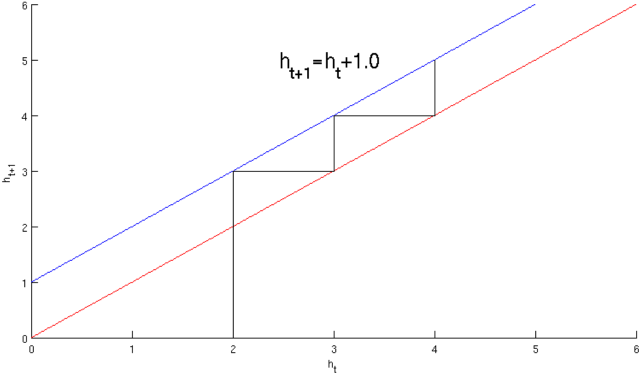
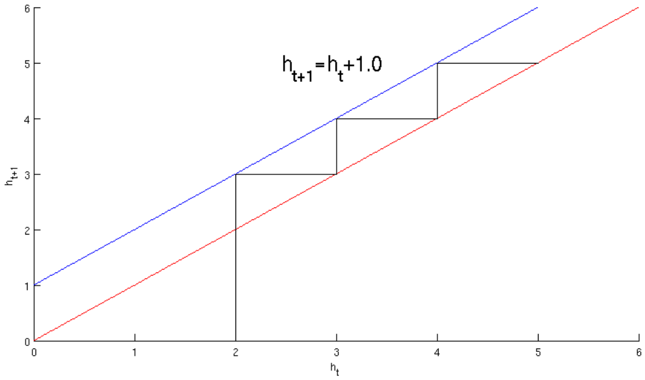


Examples

The sequence keeps growing! No equilibrium is approached.



Finding equilibria

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If M^* is an equilibrium, then $0.5M^* + 1.0 = M^*$.

This implies that $1.0 = 0.5M^*$, and hence $M^* = 2.0$.

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If h^* is an equilibrium, then $h^* = h^* + 1.0$. But this equation has no solutions and hence there is no equilibrium.

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We will find the equilibria of the dynamical system $x_{t+1} = \frac{cx_t}{x_t+1}$, where c is some number with $c \neq 0$.

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These points are really equilibria, as long as $x^*+1 \neq 0$, which is true since $c \neq 0$.

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This implies that $x^*(x^*+1) = cx^*$ and hence $x^*(x^*+1-c) = 0$.

Therefore $x^* = 0$ or $x^*+1-c = 0$, which gives $x^* = 0$ or $x^* = c-1$.

These points are really equilibria, as long as $x^*+1 \neq 0$, which is true since $c \neq 0$.

Note that if $c = 1$, then $c-1 = 0$ and hence the equilibria coincide in this case.

Finding equilibria

Example

We will find the equilibria of the dynamical system $x_{t+1} = \frac{cx_t}{x_t+1}$, where c is some number with $c \neq 0$.

If x^* is an equilibrium then $x^* = \frac{cx^*}{x^*+1}$.

This implies that $x^*(x^*+1) = cx^*$ and hence $x^*(x^*+1-c) = 0$.

Therefore $x^* = 0$ or $x^*+1-c = 0$, which gives $x^* = 0$ or $x^* = c-1$.

These points are really equilibria, as long as $x^*+1 \neq 0$, which is true since $c \neq 0$.

Note that if $c = 1$, then $c-1 = 0$ and hence the equilibria coincide in this case.

Conclusion: When $c \neq 1$ there are two equilibria, $x^* = 0$ and $x^* = c-1$. When $c = 1$ there is only one equilibrium, $x^* = 0$.