0. In the Handout *Equivalence relations defined by level sets of functions* we see that a function, say, $f : A \longrightarrow B$ defines, in a natural way, an equivalence relation in A, through which we may *dis-regard the distinction* between two distinct elements of A when they belong to the same non-empty level set of f. Besides, A is also naturally 'partitioned' into the non-empty level sets of f, giving rise to the quotient by the equivalence relation induced by f.

   Here we shall see that this 'correspondence' goes 'the other way'.

1. **Definition. (Disjoint sets.)**

Let S, T be sets. We say that S, T are **disjoint** *if $S \cap T = \emptyset$.*

Definition. (Generalized union, and generalized disjoint union).

Let A be a set, and Γ be a subset of $\mathfrak{P}(A)$.

(a) *The* **(generalized) union of** Γ *is defined to be the set*

$$\{x \in A : x \in S \text{ for some } S \in \Gamma\}.$$

It is denoted by $\bigcup_{S \in \Gamma} S$.

(b) *Further suppose that the elements of Γ are pairwise disjoint (as sets).*

Then we say the generalized union $\bigcup_{S \in \Gamma} S$ of Γ is called the **generalized disjoint union of** Γ.

We may denote it by $\bigsqcup_{S \in \Gamma} S$ (when we want to emphasize on the disjointness of the distinct elements of Γ).

2. **Definition. (Partition.)**

Let A be a set, and Ω be a subset of $\mathfrak{P}(A)$.

Ω is called a **partition of** *A if the statements (N), (O), (P) hold:*

(N) *For any $S \in \Omega$, $S \neq \emptyset$.*

(O) $\{x \in A : x \in S$ *for some* $S \in \Omega\} = A$.

(P) *For any $S, T \in \Omega$, exactly one of the statements $(P1)$, $(P2)$ holds:*

$(P1)$ $S = T$. $(P2)$ $S \cap T = \emptyset$.

Remarks.

(a) In terms of the notion of generalized union, the statement (O) reads:

‘*A equals the generalized union of* Ω.’

(b) In terms of the notion of disjointness, the statement (P) reads:

‘*The elements of* Ω *are pairwise disjoint.*’

With the validity of (P) assumed, we can re-state (O) as:

‘*A equals the disjoint union of* Ω.’

Further remarks.

We shall now see that how a partition, say, Ω, of a set, say, A, will define, in a natural way, an equivalence relation in A and a function with domain A which induces the same equivalence relation in A and whose quotient by such an equivalence relation is Ω itself.

3. **Lemma (a).**

Let A be a set. Suppose Ω is a subset of $\mathfrak{P}(A)\backslash\{\emptyset\}$.

Then the statements below are logically equivalent:

(a) *Ω is a partition of A.*

(b) *For any $x \in A$, there exists some unique $S \in \Omega$ such that $x \in S$.*

Proof of Lemma (a). Exercise in set language.

Remark on symbols and terminologies.

The validity of Lemma (a) allows us to introduce, for each $x \in A$, the specific symbol $[x]_\Omega$, for denoting the uniquely determined element of Ω which, as a set, contains x as an element.

4. **Theorem (2).**

Let A be a set, and Ω be a partition of A.

Suppose $E_\Omega = \left\{ (x,y) \;\middle|\; \begin{array}{l} x, y \in A, \text{ and there exist some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S. \end{array} \right\}$, *and* $R_\Omega = (A, A, E_\Omega)$.

Then the statements below hold:

(a) *R_Ω is an equivalence relation in A, with graph E_Ω.*

(b) *For any $x \in A$, define $[x]_{R_\Omega} = \{y \in A : (x,y) \in E_\Omega\}$.*

Then, for any $w \in A$, $[w]_{R_\Omega} = [w]_\Omega$. Moreover, for any $u, v \in A$, the statements below are logically equivalent:

(a) $(u,v) \in E_\Omega$. (b) $u \in [v]_{R_\Omega}$. (c) $v \in [u]_{R_\Omega}$. (d) $[u]_{R_\Omega} = [v]_{R_\Omega}$.

(c) *Suppose $f_\Omega : A \longrightarrow \Omega$ is the function defined by $f_\Omega(x) = [x]_\Omega$ for any $x \in A$.*

Then the statements below hold:

i. *f_Ω is surjective.*

ii. *The equivalence relation R_{f_Ω} in A is the same as the equivalence relation R_Ω.*

iii. *For any $x \in A$, $[x]_\Omega = [x]_{f_\Omega}$.*

iv. *The quotient by R_{f_Ω} in A is the same as Ω.*

Proof of Theorem (2). Exercise in set language.

Remark on terminologies.

(a) R_Ω is called the **equivalence relation in A induced by the partition Ω.**
In this context, for each $x \in A$, the set $[x]_{R_\Omega}$ (which equals $[x]_\Omega$ as sets) is referred to as the **equivalence class of x under R_Ω.**

(b) f_Ω is called the **natural surjective function induced by the partition Ω.**

Further remark.

The key to Theorem (2) is the synonymity, for each $x \in A$, of these three sets:

- the unique element $[x]_\Omega$ of the partition Ω which, as a set, contains x as an element,
- the level set ${f_\Omega}^{-1}(\{f_\Omega(x)\})$ of the function f_Ω at $f(x)$,
- the equivalence class $[x]_{R_\Omega}$ under the equivalence relation R_Ω.

What is Theorem (2) about?

- Assumption:

  Ω is a partition of A.

  $E_\Omega = \left\{ (x, y) \;\middle|\; \begin{array}{l} x, y \in A \text{ and} \\ \text{there exists some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S \end{array} \right\}$.

  $R_\Omega = (A, A, E_\Omega)$.

  So, by definition of Ω as a partition of A:

(N) For any $S \in \Omega$, $S \neq \emptyset$.

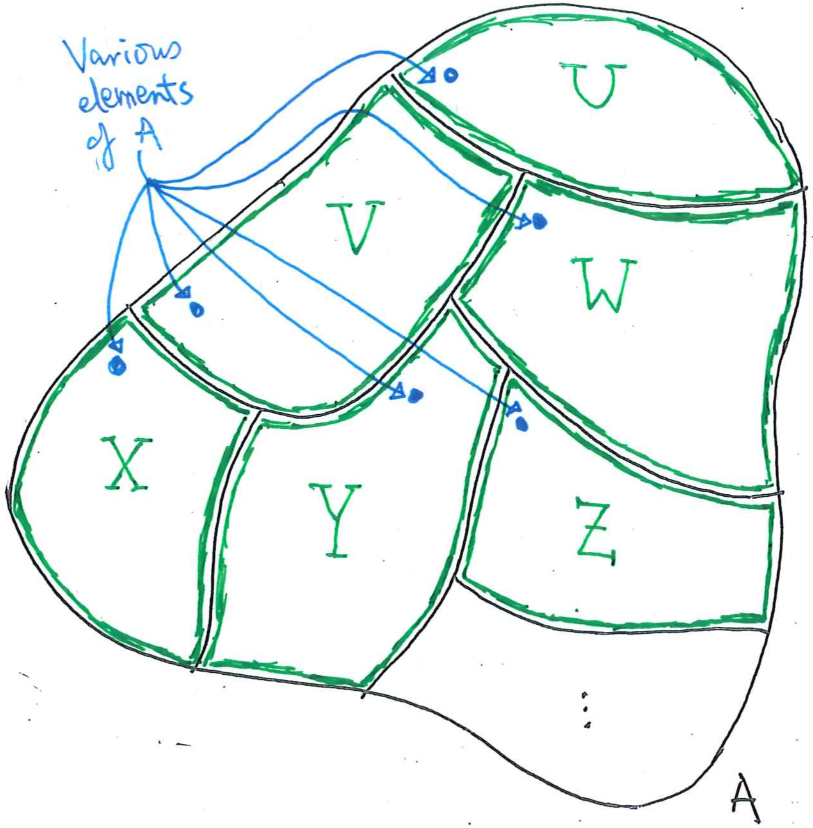
(O) $\{x \in A : x \in S \text{ for some } S \in \Omega\} = A$

(This is called the generalized union of Ω.)

(P) For any $S, T \in \Omega$, exactly one of '$S = T$', '$S \cap T = \emptyset$' is true.

(S, T are 'disjoint'.)

$\Omega = \{U, V, W, X, Y, Z, \ldots\}$



(N) Each of U, V, W, X, Y, Z is non-empty.

(O) '$U \cup V \cup W \cup X \cup Y \cup Z \cup \ldots$' $= A$.

(P) $U \cap V = \emptyset$, $U \cap W = \emptyset$, $U \cap X = \emptyset$, $\ldots$;
$V \cap W = \emptyset$, $V \cap X = \emptyset$, $\ldots$;
$W \cap X = \emptyset$, $\ldots$; $\ldots$.

What is Theorem (2) about?

- **Assumption:**

  Ω is a partition of A.

  $$E_\Omega = \left\{ (x,y) \;\middle|\; \begin{array}{l} x, y \in A \text{ and} \\ \text{there exists some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S. \end{array} \right\}.$$

  $$R_\Omega = (A, A, E_\Omega).$$

  So, by definition, for any $x, y \in A$,

  $(x,y) \in E_\Omega$ iff there exists some $S \in \Omega$ such that $x \in S$ and $y \in S$.

  Or simply,

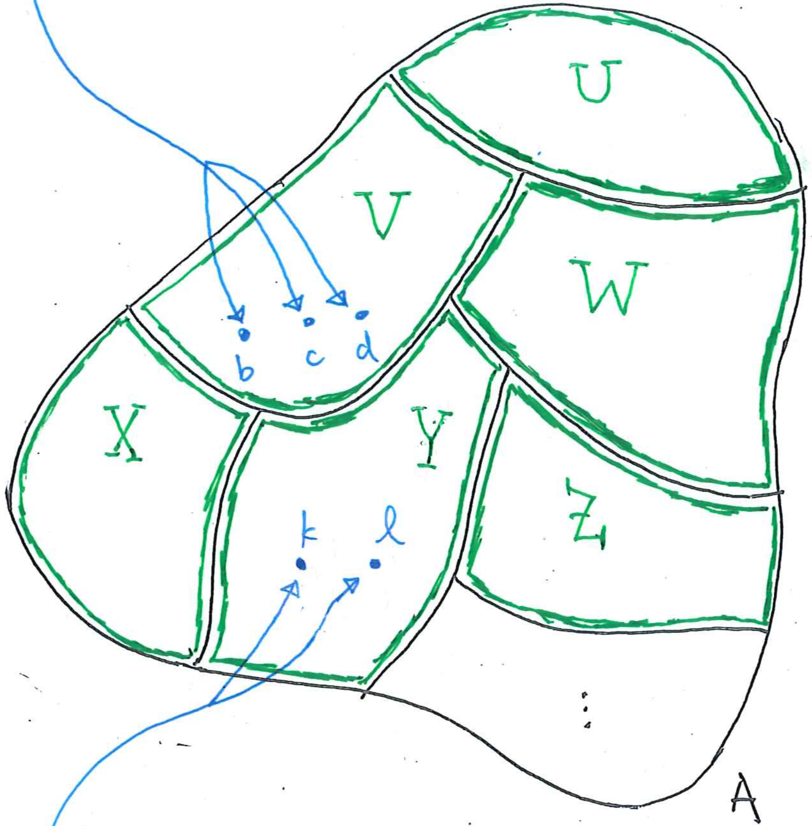
  $(x,y) \in E_\Omega$ iff x, y belong to the same (non-empty) subset of A 'collected' as some element of Ω.

- **Conclusion:**

  R_Ω is an equivalence relation in A.

  (Proof? Exercise in 'definition of partition' and 'there exist'.)

$(b, c) \in E_\Omega$ because $b \in V$, $c \in V$ and $V \in \Omega$
$(c, d) \in E_\Omega$ because $c \in V$, $d \in V$ and $V \in \Omega$
$(b, d) \in E_\Omega$ because $b \in V$, $d \in V$ and $V \in \Omega$



$(k, l) \in E_\Omega$ because $k \in Y$, $l \in Y$ and $Y \in \Omega$.

$(b, k) \notin E_\Omega$ because $b \in V$, $k \in Y$, $V \in \Omega$, $Y \in \Omega$ and $V \neq Y$.
$(c, k) \notin E_\Omega$ because $c \in V$, $k \in Y$, $V \in \Omega$, $Y \in \Omega$ and $V \neq Y$.
$(b, l) \notin E_\Omega$ because $b \in V$, $l \in Y$, $V \in \Omega$, $Y \in \Omega$ and $V \neq Y$.

What is Theorem (2) about?

- Assumption:

  Ω is a partition of A.

  $$E_\Omega = \left\{ (x,y) \;\middle|\; \begin{array}{l} x, y \in A \text{ and} \\ \text{there exists some } S \in \Omega \\ \text{such that } x \in S \text{ and } y \in S. \end{array} \right\}.$$

  $$R_\Omega = (A, A, E_\Omega).$$

- Conclusion:

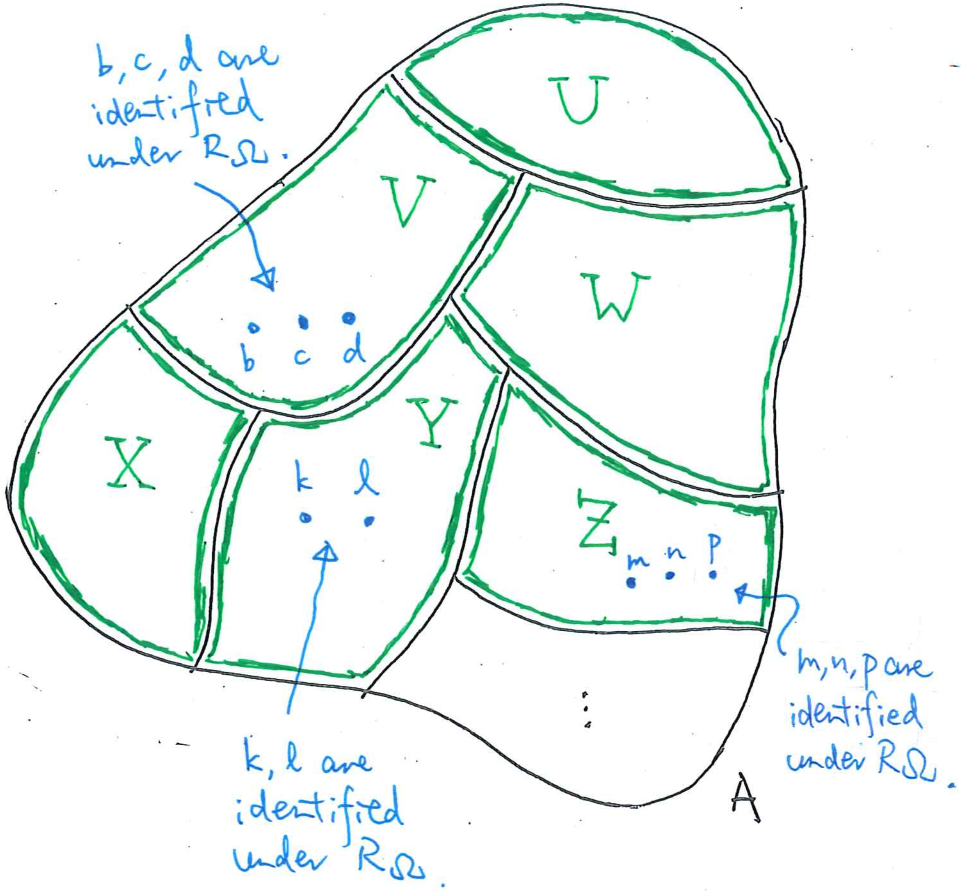
  R_Ω is an equivalence relation in A.

- Consequences of the conclusion:

  Through R_Ω, we dis-regard (possible) distinction between $x, y \in A$ exactly when they belong to the same element of Ω.

  Then any two elements of A are identified under R_Ω exactly when they belong to the same (non-empty) subset of A 'collected' as an element of Ω.

$$\Omega = \{U, V, W, X, Y, Z, \ldots\}$$



5. We shall now see that how an equivalence relation, say, R, in a set, will define, in a natural way, a partition which induces the equivalence relation R in A, and also a function with domain A which also induces the equivalence relation R.

As preparation, we introduce the notion of *set of destinations of an object under a relation.*

Definition. (Set of destinations of an object under a relation.)

Let H, K be sets, and L be a subset of $H \times K$. Let R be the relation from H to K with graph L. Let $x \in H$.

The **set of destinations of x under the relation** R *is defined to be the set*

$$\{y \in K : (x, y) \in L\}.$$

It is denoted by $R[x]$.

6. **Lemma (b).**

Let A be a set. Suppose R is an equivalence relation in A with graph E.

Then, for any $x \in A$, $x \in R[x]$.

Moreover, for any $u, v \in A$, the statements below are logically equivalent:

(a) $(u, v) \in E$. (b) $u \in R[v]$. (c) $v \in R[u]$. (d) $R[u] = R[v]$

Proof of Lemma (b). Exercise in set language.

Remark on Terminology.

For any $x \in A$, the set $R[x]$ is called the **equivalence class of x under the equivalence relation R**.

7. **Theorem (3).**

Let A be a set, and R be an equivalence relation in A with graph E.

Suppose

$$\Omega_R = \{S \in \mathfrak{P}(A) : S = R[x] \text{ for some } x \in A\}.$$

Suppose $q_R : A \longrightarrow \Omega_R$ is the function by $q_R(x) = R[x]$ for any $x \in A$.

Then the statements below hold:

(a) *Ω_R is a partition of A.*

(b) *q_R is a surjective function.*

(c) *The equivalence relation R_{Ω_R} in A induced by the partition Ω_R is R itself.*

(d) *The equivalence relation R_{q_R} in A induced by the function q_R is R itself.*

Proof of Theorem (3). Exercise in set language.

Remark on terminologies.

Note that Ω_R is a special partition of A induced by the equivalence relation R, and q_R is a special surjective function with domain A induced by the equivalence relation R.

(a) Ω_R is called the **quotient in A by the equivalence relation** R, and is denoted by A/R.

(b) q_R is called the **quotient mapping of the equivalence relation** R.

Further remark.

An equivalence relation can be visualized through its quotient and its quotient mapping, in the sense that the information about the equivalence relation is carried in full in both its quotient and its quotient mapping. This is the point of the equalities

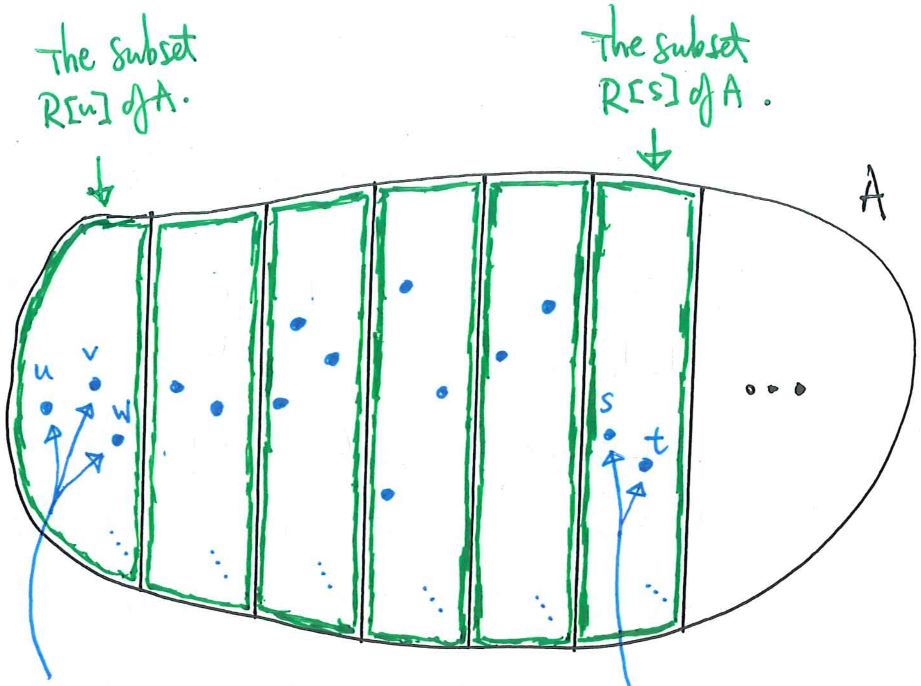
$$\text{`}R_{A/R} = R = R_{q_R}\text{'}.$$

What is Theorem (3) about?

- Assumption:

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{y \in A : (x,y) \in E\}$.



$$\begin{cases} (u,u)\in E, (u,v)\in E, (u,w)\in E, \ldots \\ (v,u)\in E, (v,v)\in E, (v,w)\in E, \ldots \\ (w,u)\in E, (w,v)\in E, (w,w)\in E, \ldots \\ \vdots \end{cases}$$

Then

$R[u] = R[v] = R[w] = \{u, v, w, \ldots\}$.

$$\begin{cases} (s,s)\in E, (s,t)\in E, \ldots \\ (t,s)\in E, (t,t)\in E, \ldots \\ \vdots \end{cases}$$

Then

$R[s] = R[t] = \{s, t, \ldots\}$.

What is Theorem (3) about?

- Assumption:

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{ y \in A : (x, y) \in E . \}$.

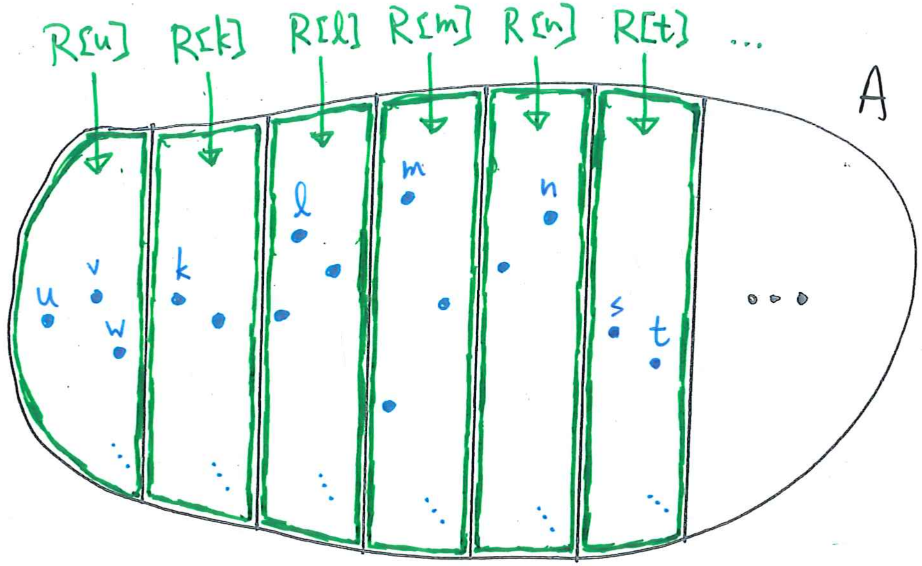
  $\Omega_R = \{ S \in \mathfrak{P}(A) : S = R[x]$ for some $x \in A . \}$.

- Conclusion:

  Ω_R is a partition of A.

  The equivalence relation in A induced by the partition Ω_R in A is R itself.

  (Proof? Exercise in playing with definitions.)



Ω_R '=' $\{ R[u], R[k], R[l], R[m], R[n], R[t], \ldots \}$

Why is such an Ω_R a partition of A?

- For any $S \in \Omega_R$, $S \neq \emptyset$. Why?

  $u \in R[u]$ because $(u, u) \in E$;

- $\{ x \in A : x \in S$ for some $S \in \Omega_R \} = A$. Why?

  $u \in R[u]$ and $R[u] \in \Omega_R$, $v \in R[v]$ and $R[v] \in \Omega_R, \ldots$;

  $k \in R[k]$ and $R[k] \in \Omega_R, \ldots$; ...

- For any $S, T \in \Omega_R$, exactly one of '$S = T$', '$S \cap T = \emptyset$' is true. Why?

  $(u, t) \notin E$ gives 'disjoint-ness' of $R[u], R[t]$;

  $(u, v) \in E$ gives 'identity' of $R[u], R[v]$; ...

What is Theorem (3) about?

- **Assumption:**

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{ y \in A : (x, y) \in E. \}$.

  $\Omega_R = \{ S \in \mathfrak{P}(A) : S = R[x] \text{ for some } x \in A. \}$.

  $q_R : A \to \Omega_R$ is the function given by

  $q_R(x) = R[x]$ for any $x \in A$.

- **Conclusion:**

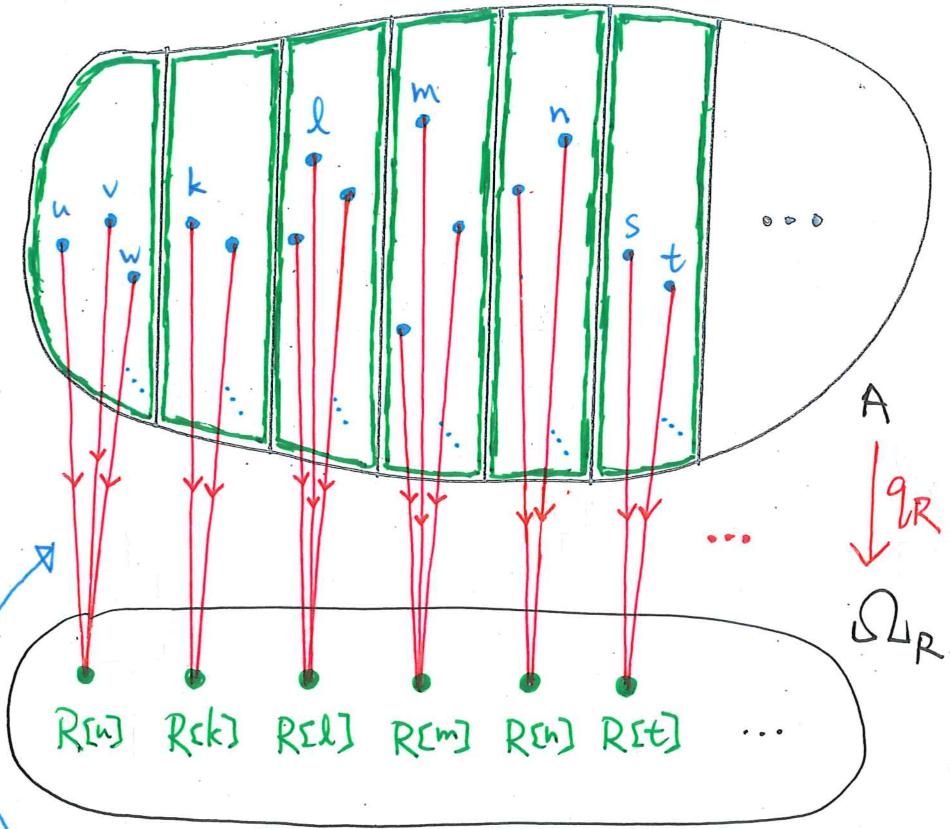
  Ω_R is a partition of A.

  The equivalence relation in A induced by the partition Ω_R in A is R itself.

  q_R is a surjective function.

  The equivalence relation in A induced by the function q_R in A is R itself.

  (Proof? Exercise in playing with definitions.)



$q_R(u) = q_R(v) = q_R(w) = \ldots = R[u]$
$= R[v]$
$= R[w] = \ldots$

$q_R(s) = q_R(t) = \ldots = R[t]$
$= R[s] = \ldots$

Why is q_R surjective?
Every $S \in \Omega_R$ is $R[x]$ for some $x \in A$.
Then for this $x \in A$, $q_R[x] = R[x] = S$.

What is Theorem (3) about?

- Assumption:

  R is an equivalence relation in A with graph E.

  For any $x \in A$, $R[x] = \{y \in A : (x, y) \in E.\}$.

  $\Omega_R = \{S \in \mathcal{P}(A) : S = R[x] \text{ for some } x \in A.\}$.

  $q_R : A \to \Omega_R$ is the function given by $q_R(x) = R[x]$ for any $x \in A$.

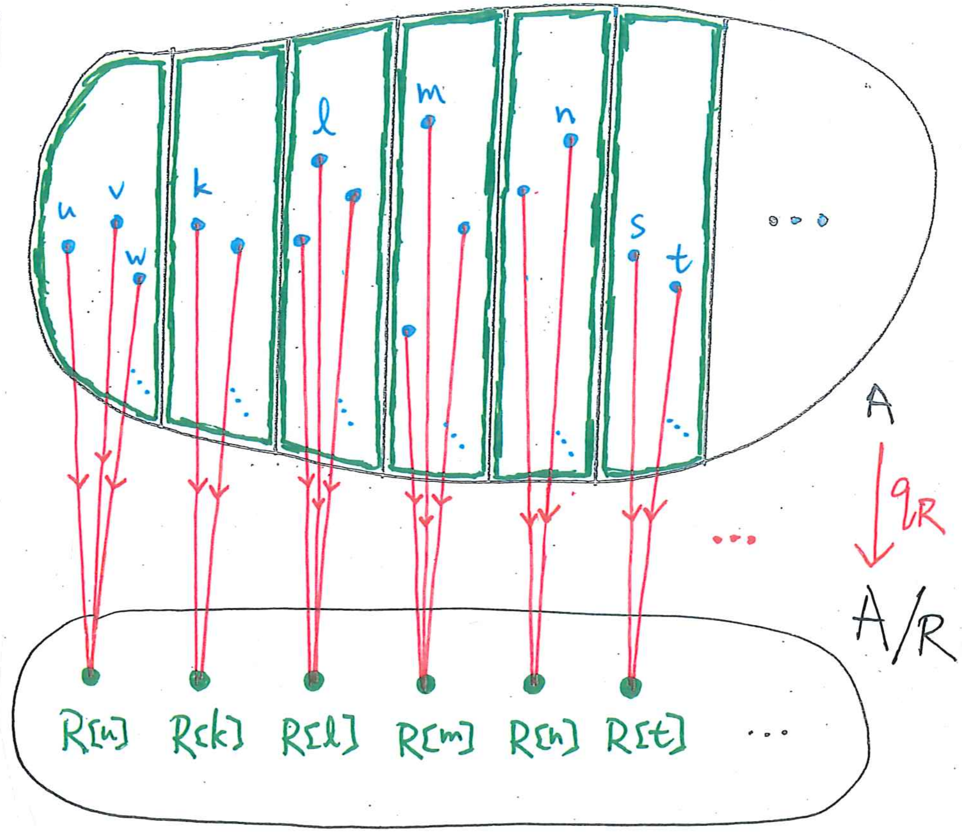
- Conclusion:

  Ω_R is a partition of A.

  The equivalence relation in A induced by the partition Ω_R in A is R itself.

  q_R is a surjective function.

  The equivalence relation in A induced by the function q_R in A is R itself.



Common terminologies and notations.

$R[x]$: Equivalence class of x under R.

Ω_R : Quotient of A by R, usually written as A/R.

q_R : Quotient mapping of R.

8. **Definition. (Systems of representatives for an equivalence relation.)**

Let A be a set, and R be an equivalence relation in A with graph E.

Let H be a subset of A.

H is said to be a **system of representatives for the equivalence relation** *R if the statement (SR) holds:*

(SR) *For any $S \in A/R$, there exists some unique $x \in H$ such that $S = R[x]$.*

9. **Lemma (c).**

Let A be a set, and R be an equivalence relation in A with graph E.

Let H be a subset of A.

Suppose

$$\Gamma = \{S \in \mathfrak{P}(A) : S = R[u] \text{ for some } u \in H\}.$$

Then H is a system of representatives of R iff the statements (SR') holds: :

(SR') *For any $x \in A$, there exists some unique $u \in H$ such that $(u, x) \in E$.*

Proof of Lemma (c). Exercise in set language.

Example (D) re-visited.

- Assumption:

$$E = \left\{ (A,B) \,\middle|\, \begin{array}{l} A, B \in Mat_{p\times q}(\mathbb{R}) \text{ and} \\ A \text{ is row-equivalent to } B \end{array} \right\}.$$

$$R = (Mat_{p\times q}(\mathbb{R}), Mat_{p\times q}(\mathbb{R}), E).$$

- Conclusion:

R is an equivalence relation in $Mat_{p\times q}(\mathbb{R})$.

Question (☆):

- what can we say about the equivalence classes under R?
- what can we say about the quotient $Mat_{p\times q}(\mathbb{R})/R$?

Recall (#):

For any $A \in Mat_{p\times q}(\mathbb{R})$, there exists some unique $B \in Mat_{p\times q}(\mathbb{R})$ such that A is row-equivalent to B and B is a reduced row-echelon form.

Re-interpretation of (#) gives an answer for Question (☆):

- Each equivalence class under R is the set of all $(p\times q)$-matrices with real entries which are row-equivalent to a uniquely determined reduced row-echelon form.
- $Mat_{p\times q}(\mathbb{R})/R$ is the partition of $Mat_{p\times q}(\mathbb{R})$ into subsets of row-equivalent matrices, each such subset containing exactly one reduced row-echelon form.

Illustration on Example (D) re-visited.

How does R partition $Mat_{2\times 2}(\mathbb{R})$?