Problems on Finite Dimensional Norms (part of HW#5)

1. Suppose $p(x)$ and $q(x)$ are two norms on $\mathbb{R}^1$.

   (a) Show there are positive finite constants A and B so that

   $$(*) \qquad Aq(x) \le p(x) \le Bq(x)$$

   is true for all $x \in \mathbb{R}^1$

   (b) Using $(*)$, show $p(\cdot)$ and $q(\cdot)$ have the same Cauchy sequences and the same convergent sequences (and hence have the same topology).

   (c) Using (b) prove each one dimensional subspace G, of any normed space E is closed.

2. Suppose $\|\cdot\|$ is some norm on $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is some linear functional. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$, $f_1(x, y) = x$ and $f_2(x, y) = y$ be the usual vectors and functionals. Define candidates for norms

   $$p((x_1, x_2)) = \sum_i |x_i| \|e_i\|$$

   $$q((x_1, x_2)) = \max_i \{|x_i| / \|f_i\|\}$$

   (a) Use $(1c)$ to show f is continuous. (And hence $q(\cdot)$ is well-defined.)

   (b) Show $p(\cdot)$ and $q(\cdot)$ are norms and for all (x_1, x_2),

   $$q((x_1, x_2)) \le \|(x_1, x_2)\| \le p((x_1, x_2))$$

   (c) Find $0 < M < \infty$ so that for all (x_1, x_2)

   $$p((x_1, x_2) \le Mq((x_1, x_2))$$

   (d) Show there are positive finite constants A and B so that $A\|x\| \le \sum |x_i| \le B\|x\|$ holds for all $x \in \mathbb{R}^2$

   (e) Repeat $(1b)$ and $(1c)$ to prove each two dimensional subspace G, of any normed space E is closed.

3. Suppose G is a codimension one closed subspace of the normed space E and $x_0 \notin G$. Then $E = \{tx_0 + y : t \in \mathbb{R}, y \in G\}$. Let $\phi : E \to \mathbb{R}$ be given by $\phi(tx_0 + y) = t$. Compute the norm of ϕ in terms of the the distance from x_0 to G which is $\inf\{\|x_0 - g\| : g \in G\}$.