Analysis Test Bank

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This document is the **Syllabus** for the *Common Core Analysis Comprehensive Exam.* It is followed by the *Test Bank* of sample problems for review. In this document, functions are understood to be defined on an abstract measure space, or else on either the real line or real n–dimensional Euclidean space. Functions are real valued unless stated to the contrary.

Prerequisites

Graduate students are presumed to have preparation in advanced calculus, such as Math 4031, 4032, and 4035 at LSU. Not every student will have learned each of the topics below, so it is a good idea to round out your understanding as needed by means of reading or auditing one or more of these advanced calculus courses.

- From Math 4031: Continuous functions, uniform continuity, pointwise versus uniform convergence, compact sets, and the Riemann integral.
- From Math 4032: Differentiable functions, infinite series of constants and functions, Taylor's theorem, $\mathcal{C}^\infty$–functions, and the Weierstrass approximation theorem.
- From Math 4035: The topology of Euclidean space, derivatives as linear transformations, Jacobians, the inverse– and implicit–function theorems. The topics in this list are particularly helpful as preparation for Math 7550—differential geometry.

Major Test Topics

(a) Measure spaces: Fields, Borel Fields and Measures

- Monotone Classes and Borel Fields
- Additive Measures, Outer Measures
- E. Hopfs Extension Theorem

(b) Lebesgue Measure

- Measurable Sets, Borel Sets, and the Real Line
- Measure Spaces and Completions, Semimetric Space of Measurable Sets
- Lebesgue Measure in $\mathbf{R}^n$; comparison with Jordan Measure in $\mathbf{R}^n$

(c) Measurable Functions (on measure spaces)

- Measurable Functions, Limits of Measurable Functions
- Simple Functions and Egoroffs Theorem; Convergence in Measure; Lusin's Theorem

(d) The Integral (on measure spaces)

- Simple Functions; the Class of Non-Negative Measurable Functions, and the Class L of Lebesgue Integrable Functions
- Lebesgue Dominated Convergence Theorem; the Monotone Convergence theorem, and Fatous lemma
- $L^1(X, \mathfrak{A}, \mu)$ as a Complete Normed Linear Space; Complex Valued Functions

(e) Product Measures and Fubinis Theorem (for products of measure spaces)

- Product Measures, Fubinis Theorem, Comparison of Lebesgue and Riemann Integrals
- Applications of Fubini's theorem to L^1 convolution, Fourier transform, etc.

(f) Functions of a Real Variable

- Functions of Bounded Variation; Fundamental Theorem for the Lebesgue Integral; Lebesgue's Theorem and Vitali's Covering Theorem. The Cantor function.
- Absolutely Continuous & Singular Functions
- Convex functions

(g) General Countably Additive Set Functions

- Hahn Decomposition Theorem
- Radon–Nikodym Theorem
- Lebesgue Decomposition Theorem

Suggested Reading

For Graduate Real Analysis:

- D.S. Bridges, *Foundations of Real and Abstract Analysis*, Springer, 1997
- P. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., 1950.
- A. Browder: *Mathematical Analysis*, Springer, 1996.
- I.P. Natanson, *Theory of Functions of a Real Variable*, Vol. 1, 1955.
- L. Richardson, *Measure and Integration: a Concise Introduction to Real Analysis*, John Wiley & Sons, to appear in 2008-9.
- H.L. Royden, *Real Analysis*, Macmillan, 1988.
- W. Rudin, *Real and Complex Analysis*, McGraw-Hill.
- G.F. Simmons, *Introduction to Topology and Modern Analysis*, McGraw-Hill, 1963.
- E.M. Stein and R. Shakarchi, *Real Analysis-Measure Theory, Integration, & Hilbert Spaces*, Princeton U. Press, 2005.
- R.L. Wheeden and A. Zygmund, *Measure and Integral: An Introduction to Real Analysis.* Marcel Dekker, 1977.

For Advanced Calculus/Undergraduate Analysis Review:

- L. Richardson, *Advanced Calculus: An Introduction to Linear Analysis*, John Wiley & Sons, 2008.
- Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1964.

Analysis Test Bank

Problems representing the core topics of math 7311 (Analysis I) are found in sections II, III, and IV, below.

1 Preliminaries: Advanced Calculus

These topics are found in Math 4031, 4032, and 4035. *Unless otherwise stated, functions are real valued and defined on the real line and/or real n-space. In this section only, the integral is meant exclusively in the sense of Riemann. Students who have not studied these topics in undergraduate analysis may need to learn them independently as graduate students. It is possible also to audit Math 4031, 4032, and 4035.*

I.1 (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$ $(n \in \mathbb{N})$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on $\mathbb{R}$, uniformly convergent on $[0, 1]$, but not uniformly convergent on $\mathbb{R}$.

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = \frac{1}{1+nx^2}$ $(n \in \mathbb{N})$. Show that $(f_n)_{n\in\mathbb{N}}$ is a bounded subset of $C[0, 1]$ and that no subsequence of $(f_n)_{n\in\mathbb{N}}$ converges in $C[0, 1]$.

I.2 Identify all subsets of $[0, 1]$ on which $\sum_{n=0}^{\infty} x^n$ converges uniformly. Explain.

I.3 Recall that a step function is finite linear combination of characteristic functions of intervals.

(a) Show that every continuous function on $[0, 1]$ is a uniform limit of step functions.

(b) Is the converse true?

I.4 Prove or disprove:

(a) The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.

(b) The product of two uniformly continuous functions on $[0, 1]$ is also uniformly continuous.

I.5 Prove or disprove the following two statements:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise everywhere on $\mathbb{R}$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges to a continuous function on $\mathbb{R}$.

I.6 Let $C_{0,0}[0, 1]$ be the space of all continuous real functions f on the interval $[0, 1]$ satisfying $f(0) = f(1) = 0$. Let $\mathcal{P}_{0,0}$ be the subspace of polynomials in $C_{0,0}[0, 1]$. Show that $\mathcal{P}_{0,0}$ is dense in $C_{0,0}[0, 1]$ in the sup norm.

I.7 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Prove that the following are equivalent.

(a) $\lim_{n\to\infty} x_n = a$.

(b) Every subsequence of $(x_n)_{n\in\mathbb{N}}$ contains a subsequence that converges to a.

I.8 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

I.9 Let $f_n : [1,\infty) \to \mathbb{R}$ be defined by $f_n(x) := \frac{n+1}{n}e^{-nx}$ $(n \in \mathbb{N})$. Show that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to a continuous function.

I.10 $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) := (\sin x)^n$ $(n \in \mathbb{N})$. Does $(f_n)_{n\in\mathbb{N}}$ converge uniformly?

I.11 Let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) := (x/n)e^{-(x/n)}$ $(n \in \mathbb{N})$.

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[0,a]$ for any non-negative real number a. Does the sequence converge uniformly to f on $[0,\infty)$? Justify your answer.

(b) Show that

$$\lim_{n\to\infty} \int_0^a f_n(x)\,dx = \int_0^a f(x)\,dx,$$

but that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty f(x)\,dx.$$

I.12 Let $f_n(x) = \displaystyle\sum_{i=0}^{n-1} \frac{1}{n} f\left(x + \frac{i}{n}\right)$, where f is a continuous function on $I\!R$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on every finite segment $[a,b]$ to the function $F(x) = \int_x^{x+1} f(s)\,ds$.

I.13 Does $e^z(x^2+y^2+z^2) - \sqrt{1+z^2} + y = 0$ have a solution $z = f(x,y)$, where f is continuous at $(x=1, y=0)$ and $f(1,0) = 0$? Explain carefully.

I.14 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

(a) Use Taylor's formula with remainder to show that, given x and h, $f'(x) = (f(x+2h) - f(x))/2h - hf''(\xi)$ for some ξ.

(b) Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

I.15 Prove or give a counterexample: If f is a uniform limit of polynomials on $[-1,1]$, then the Maclaurin series of f converges to f in some neighborhood of 0.

I.16 Let $f(x) = x^k \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

(a) If $k = 2$, show that f is differentiable everywhere but f' fails to be continuous at some point.

(b) If $k = 3$, does f have a second derivative at all points? If so, is f'' a continuous function? Give your reasons.

I.17 Let f be defined on $\mathbb{R}^3$ by $f(x,y,z) = x^2+4y^2-2yz-z^2$. Show that $f(2,1,-4) = 0$ and $f_z(2,1,-4) \neq 0$, and that there exists therefore a differentiable function g in a neighborhood of $(2,1)$ in $\mathbb{R}^2$, such that $f(x,y,g(x,y)) = 0$. Find $g_x(2,1)$ and $g_y(2,1)$. What is the value of $g(2,1)$?

I.18 Suppose that a function f is defined on $(0,1]$ and has a finite derivative with $|f'(x)| < 1$. Define $a_n := f(1/n)$ for $n = 1,2,3,\ldots$. Show that $\lim_{n\to\infty} a_n$ exists.

I.19 Define a function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0 & , \text{if } x \leq 0 \end{cases}$$

(a) Check whether f is infinitely differentiable at 0, and, if so, find $f^{(n)}(0)$, $n = 1,2,3,\cdots$. Show details.

(b) Does f have a power series expansion at 0?

(c) Let $g(x) = f(x)f(1-x)$. Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0,1)$.

I.20 Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ if the partial derivatives $f_{x_1}, \ldots, f_{x_n}$ exist and are bounded in a neighborhood of x.

I.21 Let $f_N(x) = \sum_{n=1}^{N} a_n \sin(nx)$ for $a_n, x \in \mathbb{R}$. If $\sum_{n=1}^{\infty} na_n$ converges absolutely, show that $(f_N)_{N\in\mathbb{N}}$ converges uniformly to a function f on $\mathbb{R}$, and that $(f'_N)_{N\in\mathbb{N}}$ converges uniformly to f' on $\mathbb{R}$.

I.22 Let f be a twice continuously differentiable real-valued function on $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is a critical point of f if all partial derivatives of f vanish at x (i.e., $\bigtriangledown f(x) = 0$), a critical point x is nondegenerate if the $n \times n$ matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$$

is nonsingular. Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points. (i.e., the nondegenerate critical points are isolated.)

I.23 Show that a function $f(x) = e^{-x} + 2e^{-2x} + \ldots + ne^{-nx} + \ldots$ is continuous on $(0,\infty)$.

I.24 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded first and second derivatives. By considering the Taylor expansion of f, show that: $\|f'\|_\infty \leq \frac{1}{h}\|f\|_\infty + h\|f''\|_\infty$ for every $h > 0$. By minimizing over h, show that $\|f'\|_\infty \leq 2\sqrt{\|f\|_\infty \|f''\|_\infty}$, where $\|g\|_\infty$ denotes $\sup_{x\in\mathbb{R}} |g(x)|$.

I.25 Let f be a real-valued differentiable function on an interval (a, b). Show that f is Lipschitz continuous if and only if f has bounded derivative.

I.26 Let E be a nonempty subset of $\mathbb{R}$. Let $c \in \mathbb{R}$. Show $\sup E = c$ if and only if $x \leq c$ for all $x \in E$ and for each $\epsilon > 0$, there is an $x \in E$ with $x + \epsilon > c$.

I.27 Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences in $[-\infty, \infty]$. Show

(a) $\limsup a_n = -\liminf(-a_n)$

(b) $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$

I.28 Show every open subset U of $\mathbb{R}$ can be written has a countable disjoint union $\cup_{i\in I}(a_i, b_i)$ of disjoint open intervals.

I.29 Let E be a subset of $\mathbb{R}^d$. Recall a subset S of E is said to be dense in E if for each $x \in E$ and each $\epsilon > 0$, one has $B_\epsilon(x) \cap S \neq \emptyset$. Using the countability of the set of points in $\mathbb{R}^d$ with rational entries, show every subset E of $\mathbb{R}^d$ has a countable dense subset.

I.30 Let F_1 and F_2 be disjoint compact subsets of $\mathbb{R}^d$. Show $d(F_1, F_2) > 0$.

I.31 Show if F_1 and F_2 are nonempty compact subsets of $\mathbb{R}^d$, there are points $x_1 \in F_1$ and $x_2 \in F_2$ with $d(x_1, x_2) = d(F_1, F_2)$.

I.32 Show if $F \subseteq O \subseteq \mathbb{R}^d$ and F is compact and O is open, then there is a continuous function f such that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}^d$ and $f(x) = 1$ on F and $f(x) = 0$ for $x \notin O$. **Hint:** Consider $f(x) = \min\left\{\frac{d(x,O^c)}{d(F,O^c)}, 1\right\}$.

I.33 Assume that $A \geq 0$, $B > 0$, and f is continuous and nonnegative on $[a, b]$. Assume that $f(t) \leq A + B\int_a^t f(s)\,ds$ for $a \leq t \leq b$. Prove that $f(t) \leq Ae^{B(t-a)}$ for $a \leq t \leq b$.

I.34 Let $f : [a, b] \to \mathbb{R}$ be monotonically increasing. Show the set of points in $[a, b]$ where f is discontinuous is countable.

2 Lebesgue measure, σ-algebras, Borel sets, measurable functions

Problems concerning $\mathbb{R}$, on $\mathbb{R}^n$ refer to Lebesgue measure, l. Other problems are about arbitrary measure spaces $(X, \mathfrak{A}, \mu)$.

II.1 Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{X}$, and let $f : \mathbb{X} \to \mathbb{X}$ be arbitrary. Prove that the set

$$\{f^{-1}[S] \mid S \in \mathcal{A}\}$$

is a σ-algebra.

II.2 Suppose that $g : X \to \mathbb{R}$ is μ-measurable and that $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, meaning that $f^{-1}[(a, \infty)]$ is a Borel set for each real number a. Prove that $f \circ g$ is μ-measurable.

II.3 Prove that every Lebesgue measurable subset of $\mathbb{R}$ is sandwiched between an F_σ-set and a G_δ-set of the same Lebesgue measure.

II.4 Prove that the Cantor set is a Borel set.

II.5

(a) Show the set of irrational numbers is a G_δ set but is not an F_σ set. **Hint:** Show $\mathbb{Q}$ is not a G_δ, for otherwise you could obtain a decreasing sequence G_n of dense open sets that have empty intersection. Then use the decomposition of each G_n into a disjoint countable union of open intervals.

(b) Using the fact that the set of rational numbers in any closed interval $a \leq x \leq b$ where $a < b$ is not a G_δ set, give an example of a Borel subset of $\mathbb{R}$ which is neither an F_σ or a G_δ set.

(c) Let f be any function from $\mathbb{R}$ to $\mathbb{R}$. Prove that the set of points of discontinuity of f is of type F_σ.

(d) Can a function from $\mathbb{R}$ to $\mathbb{R}$ be continuous on the rationals and discontinuous on the irrationals? What if the roles of the rationals and irrationals are interchanged?

II.6 Let f be a measurable function that is not almost everywhere infinite. Prove that there exists a subset $S \subset \mathbb{R}$ of positive measure such that f is bounded on S.

II.7 If E is a measurable subset of $[0, 1]$ prove that there is a measurable subset $A \subset E$ such that $l(A) = \frac{1}{2} l(E)$.

II.8 Let $E \subset \mathbb{R}$ be a measurable set with the property that

$$l(E \cap I) \leq \frac{l(I)}{2},$$

for every open interval I. Prove that $l(E) = 0$.

II.9 Let E be a measurable set in $[0, 1]$ and let $c > 0$. If $l(E \cap I) \geq c l(I)$, for all open intervals $I \subset [0, 1]$ show that $l(E) = 1$.

II.10 Let $A \subset (0, 1)$ be a measurable set and $l(A) = 0$. Show that

$$l\{x^2 : x \in A\} = 0 \quad \text{and} \quad l\{\sqrt{x} : x \in A\} = 0.$$

II.11 Show the Cantor ternary set is totally disconnected; that is show it contains no nonempty open interval.

II.12 Show the Cantor ternary set consists of all x in $[0,1]$ that can be written in form

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where each a_k is either 0 or 2. Use the digits a_n to define the value of Cantor function ϕ at x.

II.13 Let $\delta_1, \delta_2, \ldots, \delta_d$ be nonzero real numbers. Let E be a subset of $\mathbb{R}^d$. Define

$$\phi(x_1, x_2, \ldots, x_d) = (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_d x_d).$$

Show that the *outer measure*

$$l^*(\phi(E)) = |\delta_1 \delta_2 \cdots \delta_d|\, l^*(E).$$

Then show E is Lebesgue measurable if and only if $\phi(E)$ is Lebesgue measurable and then

$$l(\phi(E)) = |\delta_1 \delta_2 \cdots \delta_d|\, l(E).$$

II.14 An extended real valued function $f : X \to [-\infty, \infty]$ is *measurable* if $f^{-1}[-\infty, \alpha)$ is measurable for each $\alpha \in \mathbb{R}$. Show $f^{-1}(\alpha) = \{x \mid f(x) = \alpha\}$ is measurable for $\alpha \in [-\infty, \infty]$.

II.15 Let $(X, \mathfrak{A}, \mu)$ be a complete measure space. Show that a subset $E \subseteq X$ is measurable if and only if for each $\epsilon > 0$ there is a measurable set W with outer measure $\mu^*(E \Delta W) < \epsilon$.

II.16 Show that there is a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{R}$ that converges pointwise everywhere to a function f having the property that there is not a subset A of $[0,1]$ of Lebesgue measure 0 such that f_n converges uniformly to f on the complementary set $[0,1] \setminus A$.

II.17 Let $Q = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and suppose $f : Q \to \mathbb{R}$ is a continuous function. Show the Lebesgue measure of the compact set $\Gamma = \{(x, f(x)) \mid x \in Q\}$ in $\mathbb{R}^{d+1}$ is zero. (*Hint:* Use uniform continuity.)

II.18 Let μ be a *finite* and *finitely additive* measure on a field $\mathfrak{A} \subseteq \mathfrak{P}(X)$. Prove that μ is countably additive *on* $\mathfrak{A}$ if and only if for each decreasing sequence

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$$

with $\bigcap_{n=1}^{\infty} A_n = \emptyset$ we have $\lim_{n\to\infty} \mu(A_n) = 0$.

II.19 Prove that Lebesgue measure l is translation-invariant on the real line.

II.20 In the semimetric space $(\mathfrak{L}, d)$ formed from $([0,1], \mathfrak{L}, l)$, where l is Lebesgue measure, prove that each of the following sets is dense.

(a) $\mathcal{O}$, the set of all *open* sets in $[0,1]$.

(b) $\mathcal{K}$, the set of all *closed* sets in $[0,1]$.

II.21 Suppose A and B are measurable subsets of $\mathbb{R}^n$, each one of strictly positive but finite measure. Prove that there exists a vector $c \in \mathbb{R}^n$ such that $l((A+c) \cap B) > 0$. (Hint: Consider the outer measure of A and B.)

II.22 Let $f : X \to \mathbb{R}$. Prove that f is measurable (meaning that $f^{-1}(-\infty, a] \in \mathcal{L}$ for all $a \in \mathbb{R}$) if and only if $f^{-1}(B) \in \mathcal{L}$ for each Borel set B. (Hint: Show that the family

$$\mathfrak{S} = \left\{ A \in \mathfrak{P}(\mathbb{R}) \;\middle|\; f^{-1}(A) \in \mathcal{L} \right\}$$

is a σ-field.)

II.23 Suppose $f_n : X \to \mathbb{R}$ is a measurable function for each $n \in \mathbb{N}$, where $(X, \mathfrak{A}, \mu)$ is a measure space. Prove that the set

$$S = \left\{ x \;\middle|\; \lim_{n \to \infty} f_n(x) \text{ exists} \right\}$$

is a measurable set.

II.24 Suppose E_n is a sequence of measurable sets in X such that $\sum_{n \in \mathbb{N}} \mu(E_n) < \infty$. Set $F_N = \bigcup_{n \geq N} E_n$.

(a) Show that

$$\lim_{N \to \infty} \mu(F_N) = 0.$$

(b) One defines $\limsup E_n$ by

$$\limsup_{n \to \infty} E_n = \bigcap_{N \in \mathbb{N}} F_N.$$

Show that $\limsup E_n$ consists of all points p which are in infinitely many of the sets E_n.

(c) Part (a) shows that $\mu(\limsup E_n) = 0$ if $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Give an example of a sequence of measurable sets E_n such that $\mu(E_n) \to 0$ as $n \to \infty$, but $\mu(\limsup E_n) \neq 0$.

II.25 Suppose f_n is a sequence of real valued Lebesgue measurable functions on X. Suppose that for each $\epsilon > 0$, $\mu\{x \mid |f_n(x)| \geq \epsilon\} \to 0$ as $n \to \infty$. Prove the following conclusions.

(a) there is a subsequence f_{n_k} of f_n such that $f_{n_k}(x) \to 0$ as $k \to \infty$ almost everywhere.

(b) Show for each $\epsilon > 0$, there is a measurable subset E_ϵ of X and a subsequence f_{n_k} such that $\mu(E_\epsilon) < \epsilon$ and f_{n_k} converges uniformly to 0 off E_ϵ.

II.26 Prove or give a counterexample: $E \times F$ is $\mu \otimes \nu$-measurable in $(X, \mathfrak{A}, \mu) \bigotimes (Y, \mathfrak{B}, \nu)$ if and only if both E and F are measurable with respect to μ and ν respectively.

II.27 Let $f \geq 0$ be a Lebesgue measurable function on $\mathbb{R}^d$. Define $\mu(E) = \int_E f\, dl$.

(a) Show that μ is a measure on the Lebesgue measurable sets.

(b) Show that $\mu \prec l$.

(c) Show that μ is purely infinite on the set $E = \{x \mid f(x) = \infty\}$. (Note μ is purely infinite on E if and only if $\mu(F) = 0$ or $\mu(F) = \infty$ on any measurable subset F of E.)

(d) Show that μ is σ-finite on the set $E = \{x \mid f(x) < \infty\}$. Note this means there is a sequence E_n of measurable sets with $E = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for all n.

II.28 It is known the ball of radius 1 in $\mathbb{R}^{2m}$ has volume (i.e., Lebesgue measure)

$$\frac{\pi^m}{m!}.$$

Find the volume of any ball in $\mathbb{R}^{2m}$ which has radius $r > 0$.

II.29 Show that if f is a Lebesgue measurable function on $\mathbb{R}^d$, then there exists a Borel measurable function h on $\mathbb{R}^d$ such that $f = h$ a.e. **Hint:** Let $q \in \mathbb{Q}$ and let $E_q = \{x \mid f(x) < q\}$. The set E_q is Lebesgue measurable. Choose a Borel set $B_q \supseteq E_q$ with $l(B_q - E_q) = 0$. Set $F_q = \bigcup_{q'<q} B_q$. Note $F_q \subseteq F_{q'}$ if $q < q'$. Define $h(x) = \inf\{q \mid x \in F_q\}$. Show h is a Borel function and equals f a.e. **Hint:** Show $f = h$ on the complement of

$$\bigcup_{q\in\mathbb{Q}} (B_q - E_q)$$

and show h is a Borel function.

II.30 Let F be a measurable subset of $\mathbb{R}$. Let E be the set of x in F such that there is a $\delta > 0$ with $m((x-\delta, x+\delta) \cap F) = 0$. Show that E has measure 0.

II.31 Show there are no countably infinite σ-algebras.

3 The Integral: Convergence Theorems, Product Measures & Fubini's Theorem.

Integrals on $\mathbb{R}$ or $\mathbb{R}^n$ refer to Lebesgue measure unless stated explicitly to the contrary. Other problems concern integrals on an arbitrary measure space $(X, \mathfrak{A}, \mu)$.

III.1 Let $\chi_{[-n,n]}(\cdot)$ denote the characteristic function of the interval $[-n, n]$ ($n \in \mathbb{N}$). Consider the sequence of functions $f_n(x) := \chi_{[-n,n]}(x)\sin(\frac{\pi x}{n})$ ($x \in \mathbb{R}$).

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

(b) Show that

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx.$$

Are the assumptions of Lebesgue's dominated convergence theorem satisfied?

III.2 Let f be a positive function on $(0, 1]$ such that f is Riemann integrable on $[t, 1]$ for all $t \in (0, 1)$, but $\lim_{x\to 0^+} f(x) = \infty$. Assume that the improper (Riemann) integral $(R)\int_0^1 f(x)dx$ exists. Show that f is a measurable Lebesgue integrable function and that

$$\int_{[0,1]} f(x)\,dx \;=\; (R)\int_0^1 f(x)\,dx.$$

III.3 For each of the following problems, check whether the limit exists. If so, find its value.

(a) $\lim_{n\to\infty}\int_1^n (1-\frac{x}{n})^n\,dx$,

(b) $\lim_{n\to\infty}\int_1^{2n} (1-\frac{x}{n})^n\,dx$.

III.4 (a) Characterize those bounded functions on $[0, 1]$ that are Riemann integrable.

(b) Define g on $[0, 1]$ by

$$g(x) = \begin{cases} 0 & \text{if x is irrational,} \\ 1/q & \text{if } x = p/q \text{ in lowest terms.} \end{cases}$$

Is g a Riemann integrable function? Give a proof of your assertion.

III.5 Show that if

$$f(x) = \begin{cases} 1 & \text{if if } x \text{ is rational,} \\ -1 & \text{if if } x \text{ is irrational,} \end{cases}$$

then f is not Riemann integrable on the interval $[0, 1]$. Is f Lebesgue integrable? Explain.

III.6 Show there are no bounded sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ for which

$$f_n(x) = a_n \sin(2\pi n x) + b_n \cos(2\pi n x)$$

converges to 1 almost everywhere on $[0, 1]$.

III.7 Let $f(x)$ be a real-valued measurable function on a finite measure space $(X, \mathfrak{A}, \mu)$. Show that

$$\lim_{n\to\infty} \int_0^1 \cos^{2n}(\pi f(x))\, d\mu = \mu\{x \mid f(x) \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the set of integers.

III.8 (a) Show that $f(x) = x^{-r}$ is a Lebesgue integrable function on $[0, 1]$ if $0 \leq r < 1$.

(b) If $0 \leq r < 1$ let

$$a_n = \int_0^1 \frac{dx}{n^{-1} + x^r}.$$

Compute $\lim_{n\to\infty} a_n$. Be sure to quote the appropriate integration theorems to justify your calculations.

III.9 Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \leq n, \\ 0 & \text{if } |x| > n. \end{cases}$$

(a) Show that f_n converges to 0 uniformly on $\mathbb{R}$, and that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = 2$$

while

$$\int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n\right)(x)\, dx \neq 2.$$

(b) Explain why this example does not contradict the Lebesgue dominated convergence theorem.

III.10 (a) Show that $f(x) = \frac{1}{\sqrt{x}}$ is Lebesgue integrable on $(0, 1)$.

(b) Find

$$\inf \left\{ \int_0^1 \psi(x)\, dx \,\middle|\, \psi \in \mathfrak{S},\ \psi(x) \geq \frac{1}{\sqrt{x}}\ \forall\, x \in (0,1) \right\}.$$

(Here $\mathfrak{S}$ is the set of simple functions, which are finite linear combinations of characteristic functions of measurable sets with extended real-valued coefficients.)

III.11 Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, measurable functions on $[0, 1)$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 0$$

but such that f_n converges pointwise nowhere.

III.12 Prove that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}(\sin nt)f(t)dt = 0$$

for every Lebesgue integrable function f on $\mathbb{R}$. (Hint: Do the problem first for step functions.)

III.13 Let

$$f_n(x) = \frac{n}{x(\ln x)^n}$$

for $x \geq e$ and $n \in \mathbb{N}$.

(a) For which $n \in \mathbb{N}$ does the Lebesgue integral $\int_e^\infty f_n(x)\,dx$ exist?

(b) Determine $\lim_{n\to\infty} f_n(x)$ for $x > e$.

(c) Does the sequence $(f_n)_{n\in\mathbb{N}}$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

III.14 Define $f(x) = \int_{\mathbb{R}} \cos(xy)\,g(y)\,dy$ for $x \in \mathbb{R}$ where g is an integrable function on $\mathbb{R}$. Show that f is continuous.

III.15 Let f be a differentiable function on $[-1,1]$. Prove that $\lim_{\epsilon\to 0}\int_{\epsilon<|x|\leq 1}\frac{1}{x}f(x)\,dx$ exists.

III.16 Let $f_n(x) = \dfrac{x^n}{n!}e^{-x}$ for all integers $n \geq 0$.

(a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$.

(b) Show that $f_n \in L^1(0,\infty)$ with $\|f_n\|_1 = 1$ for all $n \geq 0$.

(c) Show that $\lim_{k\to\infty}\int_0^k \frac{x^n}{n!}(1-\frac{x}{k})^k\,dx = 1$ for all $n \geq 0$.

III.17 Prove that $\displaystyle\lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\,dx$ exists but that the function $\dfrac{\sin x}{x}$ is not integrable over $(0,\infty)$.

III.18 Compute the following limit and justify your calculations:

$$\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)\,dx$$

III.19 Find and justify the limits:

(a) $\displaystyle\lim_{n\to\infty}\int_0^n \frac{\sin x}{1+nx^2}\,dx$

(b) $\displaystyle\lim_{n\to\infty}\int_0^{e^n} \frac{x}{1+nx^2}\,dx.$

III.20 Let $f \in L^1(0,\infty)$, and suppose that $\int_0^\infty x|f(x)|\,dx < \infty$. Prove that the function

$$g(y) = \int_0^\infty e^{-xy} f(x)dx$$

is differentiable at every $y \in (0,\infty)$.

III.21 Let $f \in L^1(\mathbb{R})$ with respect to Lebesgue measure, and suppose that

$$\int_{\mathbb{R}} |x||f(x)|\,dx < \infty.$$

Show that the function

$$g(y) = \int_{\mathbb{R}} e^{ixy} f(x)dx$$

is differentiable at every $y \in \mathbb{R}$.

III.22 Prove that, if f is a real-valued Lebesgue integrable function on $\mathbb{R}$, then

$$\lim_{x\to 0} \int_{\mathbb{R}} |f(x+t) - f(t)|\,dt = 0.$$

III.23 Let $f \in L^1(X, \mathfrak{A}, \mu)$, a σ-finite measure space with $X = \dot{\bigcup}_{n\in\mathbb{N}} X_n$ such that $\mu(X_n) < \infty$ for each n. Let $A_n \in \mathfrak{A}$ be such that $\mu(A_n) \to 0$.

(a) Prove: $\displaystyle\lim_{n\to\infty} \int_{A_n} f\,d\mu = 0.$

(b) Prove or disprove: $\displaystyle\lim_{N\to\infty} \int_{\bigcup_{n\geq N} X_n} f(x)\,dx = 0.$

III.24 Give an example of a sequence of uniformly bounded measurable functions f_n on $[0,1]$ that converges in measure to 0 as $n \to \infty$, yet the sequence $f_n(x)$ does not converge for any $x \in [0,1]$.

III.25 For each $n \geq 3$ let

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x < \frac{1}{n} \\ 2n - n^2 x, & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} < x \leq 1. \end{cases}$$

Sketch the graphs of f_3 and f_4. Prove that if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = g(0).$$

(Hint: First show that $\int_0^1 f_n(x)\,dx = 1$.)

III.26 Assume that the real-valued measurable function $f(t,x)$ and its partial derivative $\frac{\partial}{\partial t} f(t,x)$ are bounded on $[0,1]^2$. Show that for $t \in (0,1)$

$$\frac{d}{dt}\left[\int_0^1 f(t,x)\,dx\right] = \int_0^1 \frac{\partial}{\partial t} f(t,x)\,dx.$$

(Hint: Consider the difference quotient for the derivative on the left.)

III.27 Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$, and

$$\sum_{n\in\mathbb{N}} \int_0^1 |f_n(x)|\,dx < \infty,$$

then $\sum_{n\in\mathbb{N}} f_n(x)$ is convergent almost everywhere, and

$$\int_0^1 \sum_{n\in\mathbb{N}} f_n(x)\,dx = \sum_{n\in\mathbb{N}} \int_0^1 f_n(x)dx.$$

III.28 Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{n\to\infty} \int_0^n e^{-nx} f(x)\,dx = 0$.

III.29 Let f be a nonnegative integrable function on $[0,1]$. Suppose that for every $n \in \mathbb{N}$

$$\int_0^1 (f(x))^n\,dx = \int_0^1 f(x)\,dx.$$

Show that f is almost everywhere the characteristic function for some measurable set.

III.30 Provide an example of a sequence $\{f_n\}$ of measurable functions on $[0,1]$ such that $f_n \to f$ almost everywhere, and $f_n \geq 0$, yet $\liminf \int_0^1 f_n\,dl \neq \int_0^1 f\,dl$.

III.31 Determine

$$\lim_{n\to\infty} \int_1^\infty \sin\left(\frac{x}{n}\right) \frac{n^3}{1+n^2x^3}\,dx.$$

III.32 Give an example of a sequence of measurable real valued functions f_n such that

$$\liminf \int f_n(x)\,dx < \int \liminf f_n(x)\,dx.$$

Also give a sequence where $f_n \geq 0$ and

$$\int \liminf f_n(x)\,dx < \liminf \int f_n(x)\,dx.$$

III.33 Prove the function

$$\phi(x) = \int_0^\infty \frac{t^{x-1}}{1+t}\,dt$$

is continuous for $0 < x < 1$. Is $\phi(x)$ differentiable on $0 < x < 1$?

III.34 Let f be a real valued function on $\mathbb{R} \times \mathbb{R}$. Suppose that

$$x \mapsto f(x, y)$$

is continuous for each y, and that

$$y \mapsto f(x, y)$$

is Lebesgue measurable for each x. Show f is Lebesgue measurable as a function on the plane. (Hint: Express f as a pointwise limit of measurable functions on the plane.)

III.35 Let $p(x_1, x_2)$ be a polynomial in two variables, but not the zero polynomial. Prove that the set of points $x \in \mathbb{R}^2$ with $p(x) = 0$ has 2-dimensional Lebesgue measure $(l \times l)\left(p^{-1}(0)\right) = 0$.

III.36 Let f be an integrable function on a measure space $(X, \mathfrak{A}, \mu)$, where $\mathfrak{A}$ is the σ-field generated by a field $\mathfrak{E}$ of elementary sets, as in the Carathéodory Extension Theorem. Show if $\int_E f\, d\mu = 0$ for every elementary set $E \in \mathfrak{E}$, then $f(x) = 0$ for almost all x.

III.37 Prove that the family $\mathcal{S}$ of step-functions having finite carrier on the real line is dense in $L^1(\mathbb{R})$.

III.38 Suppose $f \in L^1(X, \mathfrak{A}, \mu)$. Show if $\int_E f\, d\mu \geq 0$ for all Lebesgue measurable sets E, then $f(x) \geq 0$ for almost all x.

III.39 Show if f is an integrable function on $\mathbb{R}^d$ and one has

$$\int_{\mathbb{R}^d} fh\, dl = 0$$

for every bounded continuous function h on $\mathbb{R}^d$, then $f = 0$ almost everywhere.

III.40 Let f be a Lebesgue integrable function on $\mathbb{R}$ and let E be a bounded measurable set. Let x_n be a sequence in $\mathbb{R}$ with $|x_n| \to \infty$ as $n \to \infty$. Show

$$\int_{x_n + E} f\, dl \to 0$$

as $n \to \infty$.

III.41 Let f be a real valued measurable function on $(X, \mathfrak{A}, \mu)$. If $g = f$ almost everywhere, prove that g is measurable.

The following three problems are related, but each has sufficient substance to stand alone as a problem.

III.42 Suppose $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by $h(x, y) = x - y$. Let N be *both* a G_δ-set and a Lebesgue null set. Prove that $h^{-1}(N)$ is both a G_δ-set and a null set in $\mathbb{R}^2$. (Hint: Use Fubini's Theorem. Be sure to justify that your integrand for Fubini's Theorem is measurable.)

III.43 Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $\phi \in L^1(\mathbb{R})$, with g *Borel measurable*. Show that $\phi \circ g$ is *Lebesgue measurable* on $\mathbb{R}^2$, provided that g^{-1} carries Lebesgue null sets to Lebesgue null sets.

III.44 Suppose f and h are in $L^1(\mathbb{R})$, and let $H(x, y) = f(x - y)h(y)$. Show that $H \in L^1\left(\mathbb{R}^2\right)$. Use this to show the function

$$f * h(x) = \int_{\mathbb{R}} f(x - y)h(y)\, dy$$

is defined almost everywhere and is an integrable function on $\mathbb{R}$. Then show that $\|f * h\|_1 \leq \|f\|_1 \|h\|_1$. (Hint: Use Fubini's Theorem. You may *assume* the measurability of the integrand of the double integral over the plane.)

III.45 Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue measurable and suppose $H(x, y) = f(x) - f(y)$ is integrable on $[0, 1]^2$. Show f is integrable on $[0, 1]$ and determine the integral of H.

III.46 Suppose f is integrable on $\mathbb{R}^d$. Prove

$$H(\alpha) = m\{x \mid f(x) > \alpha\} - m\{x \mid f(x) < -\alpha\}$$

is integrable for $\alpha \geq 0$ and show

$$\int_{[0,\infty)} H\, dl_1 = \int_{\mathbb{R}^d} f\, dl.$$

III.47 Let $(X, \mathfrak{A}, \mu)$ be a measure space for which $\mu(X) < \infty$. Show

$$L^q(X, \mathfrak{A}, \mu) \subseteq L^p(X, \mathfrak{A}, \mu)$$

whenever $1 \leq p \leq q \leq \infty$.

III.48 Let $(X, \mathfrak{A}, \mu)$ be a probability space, meaning that the nonnegative measure $\mu(X) = 1$. Suppose that $f \in L^p(X, \mathfrak{A}, \mu)$.

(a) Use Jensen's inequality to prove that

$$\left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^q\, d\mu\right)^{\frac{1}{q}},$$

provided that $1 \leq p < q < \infty$.

(b) Prove that $\|f\|_p \leq \|f\|_q$ for all $1 \leq p \leq q \leq \infty$.

III.49 Let $f \in L^1(\mathbb{R}^2)$. Show $\int_0^1 f(x, y+n)\, dl(y)$ exists a.e. in x and defines a function H_n in $L^1(\mathbb{R})$. Then determine if the sequence H_n has a limit in $L^1(\mathbb{R})$.

III.50 Let f be defined on a measure space, $(X, \mathfrak{A}, \mu)$. Suppose that for each $\epsilon > 0$, there is a set $W \in \mathfrak{A}$ such that $\mu(W) < \epsilon$ and $f\big|_{W^c}$ is measurable on W^c. Prove that f is Lebesgue measurable.

4 Bounded Variation, Absolutely Continuous Functions and Measures, Convex Functions, the Fundamental Theorem of Calculus, and the Radon-Nikodym Derivative.

Integrals on $\mathbb{R}$ or $\mathbb{R}^n$ refer to Lebesgue measure. Other problems concern integrals on an arbitrary measure space $(X, \mathfrak{A}, \mu)$.

IV.1 Let F and G be functions on $[a, b]$ into $\mathbb{R}$ and let c be a constant. If F and G have bounded variation, prove that

$$V_a^b(F+G) \le V_a^b F + V_a^b G,$$

and that $V_a^b cF = |c| V_a^b F$.

IV.2 Show f has bounded variation if and only if both f^+ and f^- have bounded variation. **Hint:** Show that $|f|$ has bounded variation.

IV.3 Show if f is absolutely continuous on $[a, b]$, then the total variation $V_a^x f$ where $a \le x \le b$ is given by

$$V_a^x f = \int_a^x |f'(t)| \, dl(t).$$

Hint: First show $F'(x) \ge |f'(x)|$ a.e by using $V_x^{x+h} f \ge |f(x+h) - f(x)|$ and then conclude $\int_a^c |f'(x)| \, dx \le \int_a^c F'(x) \, dx \le V_a^c f$. Then show $V_a^c f \le \int_a^c |f'(x)| \, dx$.

IV.4 Show $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if f^+ and f^- are absolutely continuous on $[a, b]$.

IV.5 Let $V_a^x g$ denote the total variation of g from a to x. Show if f is absolutely continuous on $[a, b]$, then $V_a^x f = V_a^x |f|$ for $a \le x \le b$ and then give an example of a function f of bounded variation where $V_a^x f$ and $V_a^x |f|$ are different. You may use the fact that $V_a^x f = \int_a^x |f'| \, dl$ if f is absolutely continuous.

IV.6 If f is continuous on an interval $[a, b]$ and has a bounded derivative in (a, b), show that f is of bounded variation on $[a, b]$. Is the boundedness of f' necessary for f to be of bounded variation? Justify your answer.

IV.7 Let $f(x) = x^2 \sin(\frac{1}{x})$, $g(x) = x^2 \sin(\frac{1}{x^2})$ for $x \ne 0$, and $f(x) = g(x) = 0$ for $x = 0$.

(a) Show that f and g are differentiable everywhere (including at $x = 0$).

(b) Show that f is bounded variation on the interval $[-1, 1]$, but g is not.

(c) Let $f(x) = x \sin(1/x)$ for $x \ne 0$ and $f(x) = 0$ for $x = 0$. Is f of bounded variation on $[-1, 1]$?

IV.8 Show that the product of two absolutely continuous functions on a closed finite interval $[a, b]$ is absolutely continuous.

IV.9 A real-valued function f on an interval I for which there exists a constant C such that

$$|f(x) - f(y)| \le C|x - y|$$

for all x and y in I is called a *Lipschitz function.*

(a) Show that a Lipschitz function is absolutely continuous.

(b) Show that an absolutely continuous function f on an interval is Lipschitz if and only if f' is essentially bounded.

IV.10 Let f be a nonnegative integrable function on $[0,1]$ and let $I = \int_0^1 f(x)\,dx$. Show that

$$\sqrt{1+I^2} \le \int_0^1 \sqrt{1+f^2(x)}\,dx \le 1 + I.$$

IV.11 Suppose f is a nonnegative integrable function on $[0,1]$. Prove that

$$\sqrt{\int_0^1 f(t)\,dt} \ge \int_0^1 \sqrt{f(t)}\,dt.$$

IV.12 (a) Provide an example of a function on $[0,1]$ that is not absolutely continuous but is of bounded variation.

(b) Provide examples of two different continuous functions on $[0,1]$ that have the same derivative *a.e.* and that are both equal to zero at 0.

IV.13 Suppose F is absolutely continuous on $[0,1]$ and that $g \in L^1([0,1])$, with $\int_0^1 g = 0$. Prove the "integration by parts" law:

$$\int_0^1 F(x)\,g(x)\,dx \quad = \quad -\int_0^1 \left[F'(x)\int_0^x g\right]\,dx$$

IV.14 (a) Provide an example of a function of unbounded variation on $[0,1]$ that has a derivative equal to zero at almost all $x \in [0,1]$.

(b) Provide an example of a function that is absolutely continuous on $[0,1]$ but has an unbounded derivative.

IV.15 Prove that, if f is differentiable a.e. on $[0,1]$ and f' is not in $L^1[0,1]$, then f is not of bounded variation on $[0,1]$.

IV.16 Suppose that f is a real-valued function of bounded variation on $[0,1]$. Prove that

(a) f has a right- and left-hand limit at each point in $(0,1)$;

(b) f can have only countable many points of discontinuity;

(c) If, in addition to being of bounded variation on $[0,1]$, f is absolutely continuous on $[0,T]$ for each $T < 1$, then there exists an absolutely continuous function g on $[0,1]$ that coincides with f on $[0,1)$.

IV.17 Suppose that both f and $\frac{\partial f}{\partial y}$ lie in $L^1([a, b] \times [c, d])$. Suppose also that $f(x, y)$ is absolutely continuous as a function of y for almost all fixed values of x. Prove that

$$\frac{\partial}{\partial y} \int_a^b f(x, y)\, dl(x) = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dl(x)$$

for almost all y. Take care to establish that both sides exist. Hint: Use Fubini's Theorem to prove that

$$g(y) = \int_a^b f(x, y)\, dl(x) - \int_c^y \left(\int_a^b \frac{\partial f}{\partial t}(x, t)\, dl(x) \right) dl(t)$$

is a constant function of y.

IV.18 Suppose that we have two decompositions of the measure space for a *signed measure* μ, as in the Hahn Decomposition Theorem. That is

$$X = P \cup N = P' \cup N',$$

where each measurable subset of P or P' is nonnegative and each measurable subset of N or N' is nonpositive. Prove that every measurable subset of

$$(P \vartriangle P') \cup (N \vartriangle N')$$

is a μ-null set.

IV.19 † Suppose that the measures λ, μ, ν on a σ-field $\mathfrak{A} \subset \mathfrak{P}(X)$ have the relationship

$$\lambda \prec \mu \prec \nu$$

where λ and μ are finite and ν is σ-finite. Prove that $\lambda \prec \nu$ and that

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu} \frac{d\mu}{d\nu}.$$

(Hints: Let $f = \frac{d\lambda}{d\mu}$, $g = \frac{d\mu}{d\nu}$, $h = \frac{d\lambda}{d\nu}$ and show that there is a sequence f_n of special simple functions such that $f_n \nearrow f$ pointwise almost everywhere. Show that $\left|\lambda(A) - \int_A f_n\, d\mu\right| \to 0$ for all $A \in \mathfrak{A}$ as $n \to \infty$. Show that $\int_A f_n\, d\mu = \int_A f_n\, g\, d\nu$ for all $A \in \mathfrak{A}$. Use the essential uniqueness of the Radon-Nikodym derivative to complete the proof that $f_n g_n \to h$.)

IV.20 Let $E_1 \supset E_2 \supset \ldots \supset E_n \supset \ldots$ be a decreasing nest of measurable sets in the complete measure space $(X, \mathfrak{A}, \mu)$. Let f be integrable on $(X, \mathfrak{A}, \mu)$ and suppose

$$\mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = 0.$$

Use the Radon-Nikodym derivative to prove that $\int_{E_n} f\, d\mu \to 0$ as $n \to \infty$ by letting ν be defined by $\frac{d\nu}{d\mu} = f$.

IV.21 We consider the relationship between the concept of absolute continuity of functions on the real line with that of absolute continuity of measures.

(a) If λ and μ are any two finite measures on a σ-field $\mathfrak{A} \subset \mathfrak{P}(X)$, prove that $\lambda \prec \mu$ if and only if they satisfy the following condition: for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(A) < \delta$ implies that $\lambda(A) < \epsilon$. (Hint: For one direction, use the Radon-Nikodym theorem.)

(b) Suppose now that the finite measure λ is defined on the Lebesgue measurable sets of $([a, b), \mathfrak{L})$. Define $f(x) = \lambda[a, x)$ for all $x \in [a, b]$. Prove that f is an absolutely continuous function on $[a, b]$ if and only if $\lambda \prec l$. (Hint: From right to left is easy by part (a). For the other direction, prove that $\lambda(A) = \int_A f'\, dl$ for all $A \in \mathfrak{L}$.)

IV.22 Let μ and ν be non-negative finite measures on $(X, \mathfrak{A})$. If $\nu \perp \mu$ and $\nu \prec \mu$, prove that $\nu = 0$, the identically zero measure on $\mathfrak{A}$.

IV.23 Let f be a monotone increasing function on $[a, b]$ and define a measure μ by letting it assign to an interval $[a, x)$ the measure $\mu[a, x) = f(x) - f(a)$.

(a) Let μ_1 be the absolutely continuous part of μ with respect to Lebesgue measure, and find the Radon-Nikodym derivative

$$\frac{d\mu_1}{dl}.$$

(b) Show that the singular part μ_0 and the absolutely continuous part μ_1 of μ_f can be used to define absolutely continuous and singular parts of the function f.

5 L^p, L^2, Dual Spaces, Topics from either 7311 or 7330.

Integrals on $\mathbb{R}$ or $\mathbb{R}^n$ refer to Lebesgue measure. Other problems concern integrals on an arbitrary measure space $(X, \mathfrak{A}, \mu)$.

V.1 Let $C([a,b])$ be the space of real continuous functions on a closed interval $[a,b]$ equipped with the sup norm. Let $\mathcal{M} = \{f \in C([a,b]) : f(x) > 0$ for all $x \in [a,b]\}$. Show that $\mathcal{M}$ is an open subset of $C([a,b])$.

V.2 Let X be the normed linear space obtained by putting the norm $\|f\|_1 = \int_0^1 |f(t)|\,dt$ on the set of real continuous functions on $[0,1]$.

(a) Show that X is not a Banach space.

(b) Show that the linear functional $\Lambda f = f(1/2)$ is not bounded.

V.3 Show that $L^p[0,1]$ is separable for $1 \le p < \infty$, but not separable for $p = \infty$.

V.4 Consider a Lebesgue measurable function f on $\mathbb{R}$ with $\int_{\mathbb{R}} f(t)^2\,dt < \infty$. Show that the function

$$g(x) = \int_{\mathbb{R}} f(t-x)f(t)\,dt$$

is continuous.

V.5 Show that $L^p(0,1) \subset L^q(0,1)$ for any $p > q \ge 1$. Here the integrability is with respect to the Lebesgue measure. Is the inclusion map for $L^p(0,1)$ into $L^q(0,1)$ continuous?

V.6 Prove or disprove the equality $L^\infty[0,1] = \bigcap_{1 \le p < \infty} L^p[0,1]$.

V.7 Let $f \in L_p(\mathbb{R})$ with $1 \le p < \infty$. Show that $\displaystyle\int_{|x|>n} |f(x)|^p\,dx \to 0$ for $n \to \infty$.

V.8 Let $g_n = n\chi_{[0,n^{-3}]}$. Show that $\int_0^1 f(x)g_n(x)\,dx \to 0$ as $n \to \infty$ for all $f \in L^2[0,1]$, but not all $f \in L^1[0,1]$.

V.9 Provide an example of the following:

(a) A nonzero bounded linear functional on $L^p[0,1]$, $1 < p < \infty$.

(b) A nonzero bounded linear functional on ℓ^∞.

V.10 Describe precisely how the dual of ℓ^1 is represented concretely.

V.11 Why is the dual of L^∞ not equal to L^1, in other words, why is L^1 not reflexive?

V.12 What is the completion of the space of continuous functions on $[0,1]$ in the p-norm ($1 \le p < \infty$)? In the ∞-norm?

V.13 Let $p + q = pq$. For $g \in L^q(E)$, define $\hat{g} \in (L^p(E))^*$ as $\hat{g}(f) = \int_E gf$. Prove that $\|\hat{g}\| = \|g\|_{L^q(E)}$.

V.14 Prove that the linear space of finite sequences is dense in ℓ^p for $1 \leq p < \infty$, but it is not dense in ℓ^∞.

V.15 Prove that $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

V.16 Let $1 < p < q < r < \infty$. If $f \in L^p(\mathbb{R})$ and $f \in L^r(\mathbb{R})$, then $f \in L^q(\mathbb{R})$.

V.17 Prove that ℓ^∞ is not separable, that is, it has no countable dense set.

V.18 Let $f \in L^\infty[0,1]$ and $\int_0^1 x^n f(x)\,dx = 0$ for $n \in \mathbb{N}$. Show that $f = 0$ a.e.

V.19 Let f be a nonnegative Lebesgue measurable function on $(0,\infty)$ such that f^2 is integrable. Let $F(x) = \int_0^x f(t)\,dt$ where $x > 0$. Show that $\lim_{x\to 0^+} \frac{F(x)}{\sqrt{x}} = 0$.

V.20 Let f be a measurable nonnegative function on $(X, \mathfrak{A}, \mu)$ $0 < \mu(X) < \infty$. Let

$$\|f\|_\infty = \sup\left\{M \mid \mu(f^{-1}(M-\delta, M)) > 0\,\forall \delta > 0\right\}.$$

Show that

$$\lim_{n\to\infty} \left(\int_X f(x)^n\,d\mu(x)\right)^{\frac{1}{n}} = \|f\|_\infty.$$

V.21 Let $1 \leq p < \infty$. Suppose $f_n \in L^p([0,1])$, $\|f_n\|_p \leq 1$, and $f_n \to f$ almost everywhere.

(a) Show that $f \in L^p([0,1])$ and $\|f\|_p \leq 1$.

(b) Let $1 < p < \infty$ and $g \in L^q([0,1])$ where $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$\int_0^1 f_n g \to \int_0^1 fg.$$

(c) Give an example to show that the conclusion in item b would be false if $p = 1$.

V.22 Let $1 < p < \infty$. Suppose $1 \leq q \leq \infty$ where $q \neq p$. Give an example of a measurable function f on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ but $f \notin L^q(\mathbb{R})$.

V.23 Suppose $1 \leq p \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Let $f \in L^p(\mathbb{R})$ where $1 \leq p < \infty$. Find a function $g \in L^q(\mathbb{R})$ so that $|g|_q = 1$ and $\int fg\,dm = |f|_p$.

(b) Give an example of an $f \in L^\infty(\mathbb{R})$ for which there is no $g \in L^1(\mathbb{R})$ such that $|g|_1 = 1$ and $\int fg\,dm = |f|_\infty$.

V.24 Let f and g be functions in $L^2(\mathbb{R}^d)$. Suppose $\|x_n\| \to \infty$ as $n \to \infty$. Show

$$\int f(x)g(x + x_n)\,dl(x) \to 0 \text{ as } n \to \infty.$$

V.25 Let $1 \leq p < \infty$. Suppose $f_n \in L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ and

$$\|f_n - f\|_p \to 0 \text{ as } n \to \infty.$$

Show that there is a subsequence $f_{n(k)}$ such that $f_{n(k)}(x)$ converges a.e. to $f(x)$ as $n \to \infty$.

V.26 A function $f : [0,1] \to L^1[0,1]$ is called Lipschitz continuous if there exists $M > 0$ such that $\|f(t) - f(s)\|_1 \leq M|t-s|$ for all $t, s \in [0,1]$. It is called differentiable at a point $s \in (0,1)$ if the quotients $\dfrac{f(t)-f(s)}{t-s}$ converge in $L^1[0,1]$ as $t \to s$. Let $f : [0,1] \to L^1[0,1]$ be given by $f(t) = \chi_{[0,t]}$, where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0,t]$. Show that f is Lipschitz continuous and nowhere differentiable.