LATTICES IN LIE GROUPS

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Abstract. These lectures are concerned with Construction and Classification of Lattices in Lie Groups.

1. Introduction

In this exposition, we consider construction and classification of lattices (i.e. discrete subgroups with finite Co-volume) in (connected) Lie Groups.

In Section 2, we will collect some general results on lattices in locally compact groups.

The first main theorem is that the discrete subgroup $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$. This will be proved in section 4.

We then prove the Mahler criterion, which will enable us to prove the co-compactness of many discrete groups.

In sections 6 and 7, we specialise to the case $G = SO(n,1)$ and construct compact hyperbolic manifolds whose fundamental groups are arithmetic subgroups of $SO(n,1)$. .

2. Lattices in Locally Compact Groups

2.1. **Haar Measure on groups and the Modular Function.** Fix a locally compact Hausdorff topological group G. A **left invariant Haar measure** on G, is by definition a regular Borel measure μ on G such that for all $g \in G$ and all Borel sets E in G, we have $\mu(gE) = \mu(E)$. We can similarly define a right invariant Haar measure.

We recall the fundamental theorem on Haar measures.

Theorem 1. *Every locally compact Hausdorff topological group G has a left invariant Haar measure.*

A left invariant Haar measure is unique up to scalar multiples.

A similar result holds for right invariant Haar measures.

If μ is a left invariant Haar measure on G and $g \in G$, then the function $E \mapsto \mu(Eg)$ on the algebra of Borel sets gives another left invariant Haar measure on G. By the uniqueness part of Theorem 1, there exists a constant $\Delta_G(g)$ such that for all Borel sets $E \subset G$, we have

$$\mu(Eg) = \Delta_G(g)\mu(E).$$

The function $\Delta_G : G \to R_{>0}$ is easily seen to be a homomorphism from G into the multiplicative group $\mathbb{R}_{>0}$ of positive real numbers.

Definition 1. If G is a locally compact Hausdorff topological group and Δ_G is the homomorphism as in the preceding, the function Δ_G is called the **modular function** of G.

A locally compact Hausdorff topological group is said to be **unimodular** if the modular function Δ_G is identically 1. In that case, a left invariant Haar measure is also a right invariant Haar measure as well.

Example 1. The additive group $\mathbb{R}^n$ has left and right invariant Haar measure, namely the Lebesgue measure

$$d\mu = dx_1 dx_2 \cdots dx_n.$$

In particular, the group $\mathbb{R}^n$ is unimodular.

Example 2. If Γ is a discrete group with the discrete topology, then **the counting measure** μ is defined to be the measure which assigns to any set (note that in a discrete space, any set is a Borel set), its cardinality. Then μ is a left and right invariant Haar measure, and Γ is unimodular.

Example 3. Let $G = \{g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in GL_2(\mathbb{R}) : a \neq 0,\ b \in \mathbb{R}\}$. Then, $d\mu(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = \frac{da}{|a|} db \frac{1}{|a|}$ is a left invariant Haar measure on G. The modular function is given by

$$\Delta_G(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) = \frac{1}{\mid a \mid^2},$$

and hence G is not unimodular.

Example 4. If μ is a left invariant Haar measure on G and Δ_G is the modular function, then $d\nu(x) = \frac{d\mu(x)}{\Delta_G(x)}$ is a **right** invariant Haar measure on G.

2.2. **Haar measures on quotients.** If G is as in the preceding subsection and $H \subset G$ is a closed subgroup, then the quotient space G/H of left cosets of H in G is a topological space under the quotient topology, which declares a set in G/H to be open if and only if its preimage under the quotient map $G \to G/H$ is open. Then it is easy to see that G/H is a locally compact Hausdorff space. Further, G acts by left translations on G/H and the action is continuous.

Let Δ_G and Δ_H be the modular functions of G and H respectively.

We will say that a regular Borel measure μ on the quotient G/H is a **left invariant Haar measure** if for all Borel sets $E \subset G/H$ and all $g \in G$ we have $\mu(gE) = \mu(E)$.

We prove an analogue of Theorem 1 for quotient spaces G/H. Note that uniqueness still holds, but not the existence of a left invariant measure.

Theorem 2. *A left invariant Haar measure on G/H is unique up to scalar multiples.*

The quotient G/H carries a left invariant Haar measure if and only if we have

$$\Delta_G(h) = \Delta_H(h),$$

for all $h \in H$.

Definition 2. If G is a locally compact group and Γ is a discrete subgroup such that the quotient G/Γ carries a **finite** left G-invariant Haar measure, then we say that Γ is a lattice in G.

Corollary 1. *If a locally compact topological group G admits a lattice, then G is unimodular.*

Proof. Suppose that G contains a lattice Γ. Being discrete, Γ is closed. By Theorem 2, the modular function of G and that of Γ coincide on Γ. From Example 2 it follows that the modular function of Γ is trivial, hence $\Delta_G(\gamma) = 1$ for all $\gamma \in \Gamma$.

We therefore get a map $\Delta_G : G/\Gamma \to \mathbb{R}_{>0}$ which is a homomorphism on G and is trivial on Γ. The image $\Delta_G^*(\mu)$ of the Haar measure μ on G/Γ is then a **finite** measure on $\mathbb{R}_{>0}$. This measure is invariant under the image of G under the map Δ_G. However, there are no finite measures on $\mathbb{R}_{>0}$ which are invariant under a non-trivial subgroup. Hence, the map Δ_G is identically 1 and G is unimodular. $\square$

For example, the group $G = \{g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0,\ b \in \mathbb{R}\}$ of Example 3 is not unimodular and hence does not contain lattices.

Quotients do not always have invariant Haar measures. As an example, consider $G = SL_2(\mathbb{R})$ and $H = B = \{g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0\ b \in \mathbb{R}\}$. Then, it is easy to see that G/B does not have a G-invariant measure.

2.3. **Haar measures on Lie Groups.** Suppose that G is a connected Lie group. Fix a metric at the tangent space at identity. By left translating this metric everywhere, we get a (left) G-invariant metric on G. Fix an orthonormal basis $w_1, w_2, \cdots, w_n$ of the cotangent space of G at identity. The left translate of w_i gives a left invariant differential one form ω_i for each i. Clearly, ω_i form an orthonormal basis of the cotangent space at every point of G.

Therefore, the wedge product $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ gives a nowhere vanishing top degree differential form, denoted ω on G and G is orientable.

Given a compactly supported smooth function f on G, the form $f\omega$ can be integrated on G and gives a positive linear functional on the dense space $\mathcal{C}_c^\infty(G)$, and by the Riesz representation theorem, this gives a Haar measure on G.

Thus, a Haar measure on a connected G can be easily constructed.

3. Lattices in $\mathbb{R}^n$

In this section, we consider the simple case of lattices on the real vector space $\mathbb{R}^n$. The results established in this section will be helpful in proving the Minkowski reduction of the next section.

We first note that $\mathbb{R}^n$ is the real span of the standard basis vectors $e_1, e_2, \cdot, e_n$. The *integral* span of $e_1, e_2, \cdots, e_n$ is the subgroup $\mathbb{Z}^n$ of n-tuples which have integral co-ordinates and is a discrete subgroup of $\mathbb{R}^n$. Clearly, $\mathbb{R}^n/\mathbb{Z}^n$ is the n-fold product of the circle group $\mathbb{R}/\mathbb{Z}$ with itself. We will now show that every lattice in $\mathbb{R}^n$ is the translate of $\mathbb{Z}^n$ by a non-singular linear transformation.

Proposition 3. *Suppose that $L \subset \mathbb{R}^n$ is a lattice. Then there exists a basis $v_1, v_2, \cdots v_n$ of $\mathbb{R}^n$ such that L is the integral linear span of $v_1, v_2, \cdots, v_n$.*

Proof. Suppose $L \subset \mathbb{R}^n$ is a lattice. Then the $\mathbb{R}$-linear span of L is vector subspace W, and the vector space $\mathbb{R}^n/W$ has finite volume, which means that $\mathbb{R}^n/W = \{0\}$, i.e. $W = \mathbb{R}^n$. Consequently, L contains a basis, call it $w_1, w_2, \cdots, w_n$ of $\mathbb{R}^n$. The *integral* linear span L' of the vectors $w_1, \cdots, w_n$ is contained in L and is already a lattice in $\mathbb{R}^n$. Therefore, L/L' is finite, and hence L is finitely generated and torsion-free.

Therefore, L is a free abelian group on k generators. Since L' has finite index in L and is free on n generators, it follows that $k = n$. The linear span of L is $\mathbb{R}^n$. Therefore, L is the integral linear span of a basis of $\mathbb{R}^n$.

□

Lemma 4. *Let $L \subset \mathbb{R}^n$ be a lattice in $v_1 \in L$ a non-zero vector of smallest norm. Let $v \in L \setminus \mathbb{Z}v_1$ be any vector and v' the projection of v to the orthogonal complement $v_1^\perp$. Then, we have the estimate*

$$|| \, v' \, || \geq \frac{\sqrt{3}}{2} \, || \, v_1 \, || \, .$$

Proof. Write $v = v' + \lambda v_1$ for some scalar λ. Definite the integer a by the property that $\mu = \lambda - a$ lies between $-\frac{1}{2}$ and $+\frac{1}{2}$. Then the projection of $w = v - av_1$ to the orthogonal complement $v_1^\perp$ is still v' and w lies in the lattice L.

The norm of $v - av_1$ is bounded from below by the norm of v_1 since v_1 has the smallest norm amongst nonzero elements of the lattice. Therefore, we get

$$|| v_1 ||^2 \leq || w ||^2 = || v' ||^2 + | \mu |^2 || v_1 ||^2 \leq || v' ||^2 + \frac{1}{4} || v_1 ||^2 .$$

Therefore,

$$|| v' ||^2 \geq \frac{3}{4} || v_1 ||^2 .$$

This gives the estimate of the Lemma. □

Lemma 5. *If L is a lattice, then there exists an element v_1, say, of smallest norm. Let $v_1^\perp$ denote the orthogonal complement to v_1 in $\mathbb{R}^n$ with respect to the standard inner product on $\mathbb{R}^n$. Then the projection of L into $v_1^\perp$ is a lattice.*

Proof. In view of the proposition, it is enough to show that the projection s discrete. By the preceding Lemma, if v is an element of the lattice that is not an integral multiple of v_1, then the norm of the projection of v to the orthogonal complement of v_1 is bounded away from zero. Therefore, the projection to $v_1^\perp$ is discrete. □

Lemma 6. *Given a lattice $L \subset \mathbb{R}^n$, there exists an integral basis $v_1, v_2, \cdots, v_n$ of L such that if v_i' denotes the orthogonal projection of the i-th basis vector v_i to the orthogonal complement of the span of the basis vectors $v_1, v_2, \cdots, v_{i-1}$, then*

$$| v_i' |^2 \geq \frac{3}{4} | v_{i-1} |^2 .$$

Proof. Given a lattice L in $\mathbb{R}^n$, fix a non-zero element $v_1 \in L$ of the smallest norm. If $v_1 = \lambda w$ is an integral multiple of another element $w \in L$, then the minimality of the norm of v_1 shows that λ has norm 1, whence $\lambda = \pm 1$.

Since the lattice L is isomorphic to $\mathbb{Z}^n$, we conclude that v_1 is part of an integral basis of L. From Lemma 4, we know that the projection of L to the orthogonal complement $v_1^\perp$ is a lattice L' and it has an integral basis $v_2', \cdots, v_n'$. Fix elements $v_2, \cdots, v_n \in L$ mapping onto $v_2', \cdots, v_n'$ and satisfying the inequalities of the Lemma (for $i \geq 3$) by the induction assumption (n replaced by $n-1$). Then it is clear that $v_1, v_2, \cdots, v_n$ form a basis of L: we have a short exact sequence $0 \to \mathbb{Z}v_1 \to L \to L' \to 0$ of free $\mathbb{Z}$ modules. Hence L is the integral span of $v_1, v_2, \cdots, v_n$. The inequality of the Lemma for $i = 2$ follows from Lemma 4. □

4. Minkowski Reduction and $SL_n(\mathbb{R})$

We view $GL_n(\mathbb{R})$ as ordered bases of $\mathbb{R}^n$ by identifying a matrix $g \in GL_n(\mathbb{R})$ with its columns $v_1, v_2, \cdots, v_n$. Given an element $h \in GL_n(\mathbb{R})$ and a basis $v_1, \cdots, v_n$, we get the new basis $h(v_1), \cdots, h(v_n)$. Under our identification, this new basis corresponds to the matrix hg. Thus, this identification of GL_n with bases of $\mathbb{R}^n$, respects the $SL_n(\mathbb{R})$-action on both sides.

4.1. **The Gram -Schmidt Process.** The following Lemma is a simple restatement of the Gram-Schmidt orthonormalisation process.

Lemma 7. *Every matrix g in $SL_n(\mathbb{R})$ may be written in the form*

$$g = kau,$$

where $k \in SO(n)$, a is a diagonal matrix with positive entries and u is an upper triangular matrix with 1's on the diagonal.

Proof. Let us view elements g $GL_n(\mathbb{R})$ as n linearly independent vectors $v_1, v_2, \cdots, v_n$ where $v_1, v_2, \cdots, v_n$ are the columns of the matrix g.

If u is an upper triangular matrix with 1's on the diagonal (such a matrix u has he property that $u - 1$ is nilpotent, and is called a *unipotent* matrix), then the right multiplication of g by u takes the vectors $v_1, v_2, \cdots, v_n$ into the vectors $v'_1, v'_2, \cdots, v'_n$ where for each $i \leq n$, the vector $v'_i = v_i + \sum_{j \leq i-1} u_{ij} v_j$ is the sum of v_i plus a linear combination of the previous members $v_1, v_2, \cdots, v_{i-1}$.

Recall that the Gram-Schmidt process takes a basis $v_1, v_2, \cdots, v_n$ into a linear combination of the form $v'_1, v'_2, \cdots, v'_n$ where v'_i is v_i plus a linear combination of $v_1, v_2, \cdots, v_{i-1}$, such that the vectors $v'_1, v'_2, \cdots, v'_n$ are orthogonal to each other. It follows from the preceding paragraph that given g there exists a unipotent upper triangular matrix u such that $g' = gu$ is a matrix whose columns are **orthogonal** vectors.

Given g' as above, and a diagonal matrix a with diagonal entries $a_1, a_2, \cdots, a_n$, the matrix $g'a$ has the columns $w'_1, w'_2, \cdots, w'_n$ where $w'_i = a_i v'_i$ for each i. Now the Gram- Schimdt process takes an orthogonal basis $(v'_i)_{1 \leq i \leq n}$ into a basis of the form $(w'_i)_{1 \leq i \leq n}$ where w'_i is a positive scalar multiple of w_i such that $w'_1, w'_2, \cdots, w'_n$ form an **orthonormal** basis. Consequently, there exists a diagonal matrix a whose diagonal entries are all positive, such that $g'a$ is a matrix whose columns form an orthonormal basis of $\mathbb{R}^n$. Moreover, $a_i = \frac{1}{|v'_i|}$ where v'_i

is the image of v_i to the orthogonal complement of $v_1, v_2, \cdots, v_{i-1}$ and $\mid v_i' \mid$ is its norm with respect to the standard inner product on $\mathbb{R}^n$.

By the last two paragraphs, given $g \in SL_n(\mathbb{R})$ we can find a unipotent upper triangular matrix u and a diagonal matrix a with positive diagonal entries such that the matrix gua has the property that its rows are of the form $w_1', w_2', \cdots, w_n'$ with w_i forming an orthonormal basis. In other words, $gua = k$ where $k \in O(n)$.

Taking absolute values of the determinants on both sides of the equation $k = gua$, we see that the determinant of a is one. Taking determinants on both sides of the equation $k = aug$ we then see that $k \in SO(n)$.

Thus, $g = ka^{-1}u^{-1}$ for some a and u as above. Writing u and a in place of u^{-1} and a^{-1} in place of u and a, we obtain the Lemma. □

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Lemma 8. *With the preceding notation, if $g = kau$ is the Iwasawa decomposition of a matrix g, then the i-th diagonal entry of a is $\mid v_i' \mid$.*

Proof. In the Gram Schmidt process, we multiplied g on the right by first a unipotent upper triangular matrix and then by a diagonal matrix a'. The entries of the diagonal matrix a' were proved to be the inverses of $\mid v_i' \mid$. In the Iwasawa decomposition of g, the diagonal matrix a is the inverse of a', and the lemma follows. □

We will now prove that $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$. We first compute the Haar measure with respect to the Iwasawa decomposition of $SL_n(\mathbb{R})$.

4.2. **Haar measure on $SL_n(\mathbb{R})$.**

Lemma 9. *If A and B are groups and $G = A \times B$ is their product, then the left Haar measure on G is $da \times db$ where da and db are the left Haar measures on A and B respectively.*

If Δ_A and Δ_B are the modular functions on A and B respectively, then the modular function Δ_G on $g = (a, b) \in G$ is given by

$$\Delta_G(a, b) = \Delta_A(a)\Delta_B(b).$$

The proof is routine and is omitted.

The group $GL_n(\mathbb{R})^+$ of non-singular $n \times n$-matrices with *positive* determinant is an open subset of the set $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ of $n \times n$-matrices. Let dX denote the Lebesgue measure on the vector space $M_n(\mathbb{R})$.

Lemma 10. *The Haar measure on $GL_n(\mathbb{R})^+$ is given by*

$$d\mu(g) = \frac{dg}{(detg)^n}.$$

Moreover, $GL_n(\mathbb{R})^+$ is unimodular.

Proof. Clearly, $d\mu$ is left and right invariant. □

Corollary 2. *The group $SL_n(\mathbb{R})$ is unimodular.*

Proof. The group $GL_n(\mathbb{R})^+ = SL_n(\mathbb{R}) \times \mathbb{R}^+$ is the direct product of $SL_n(\mathbb{R})$ and the multiplicative group $\mathbb{R}^+$ of positive real numbers. By the preceding lemma, $SL_n(\mathbb{R})$ is unimodular. □

Proposition 11. *If $G = K \times N \times A$ is the homeomorphism given by Lemma 7, then the Haar measure on $G = SL_n(\mathbb{R})$ is the product*

$$dk\ dn\ da.$$

where da and dn are the Haar measures on A and N respectively.

The decomposition of Lemma 7: $SL_n(\mathbb{R}) = SO(n) \times N \times A$, where A is the group of diagonal matrices in $SL_n(\mathbb{R})$ whose entries are positive and N is the group of upper triangular unipotent matrices. shows that there is a unique measure on $SL_n(\mathbb{R})$ which is left K invariant and right NA invariant: as a topological space, $G = K \times NA$ and the topological group $K \times NA$ has a unique measure invariant under K on the left and NA on the right.

By the corollary, this measure is the Haar measure on $SL_n(\mathbb{R})$. Therefore, if dg, $d(na)$ and dk denote the bi-invariant Haar measures on G, NA and N respectively, then

$$dg = dk\ d(na).$$

The group N is the commutator subgroup of NA, as can be easily seen. Therefore, $d(na) = dnda$ where A is the Haar measure on A and dn is the Haar measure on N. Consequently, we have

$$dg = dk\ dn\ da.$$

Definition 3. Given $t > 0$ and a constant $C > 0$, define the **Siegel Set** $\mathcal{S}_{t,C}$ as the set of elements g in $SL_n(\mathbb{R})$ which have the Iwasawa decomposition

$$g = kau,\ a = (a_1, a_2, \cdots, a_n)\ u = 1 + \sum_{i<j\leq n} u_{ij}E_{ij},$$

(where E_{ij} is the $n \times n$ matrix all of whose entries are zero except the ij-th entry, which is 1) such that

$$\frac{a_i}{a_{i+1}} < t,\ \mid u_{ij} \mid < C.$$

Lemma 12. *The volume of a Siegel set $\mathcal{S}_{t,C}$ in $SL_n(\mathbb{R})$ is finite.*

Proof. Write $g = kau = kva$ with $g \in \mathcal{S}_{t,C}$ and $v = aua^{-1}$. Then v is also an upper triangular unipotent matrix of the form

$$v = 1 + \sum_{i<j\leq n} \frac{a_i}{a_j} u_{ij}E_{ij},$$

The Haar measure in k, v, a co-ordinates is given by $dg = dk\ dv\ da$. It is easily seen that

$$dv = \prod_{i<j} dv_{ij} = \prod_{i<j} du_{ij}\frac{a_i}{a_j}.$$

Consequently, the volume of $\mathcal{S}_{t,C}$ with respect to the Haar measure $dk\ dv\ da$ is given by the integral

$$(\int_K dk)[\int_{\frac{a_i}{a_{i+1}}<t:\ i\leq n-1} \frac{da_1}{a_1}\frac{da_2}{a_2}\cdots\frac{da_{n-1}}{a_{n-1}}(\prod_{i<j}\frac{a_i}{a_j})]\prod_{i<j}\int_{u_{ij}<C} du_{ij}.$$

The integral over K is finite. The integral over u_{ij} is also finite since u_{ij} lie in the bounded interval $[-C, +C]$. To compute the integral over the variables $a_1, \cdots, a_n$ (with the condition that the product $a_1a_2\cdots a_n = 1$ since the diagonal element belongs to SL_n), we change variables from $a_1, \cdots, a_n$ to the variables

$$\alpha_1 = \frac{a_1}{a_2},\ \alpha_2 = \frac{a_2}{a_3}, \cdots,\ \alpha_{n-1} = \frac{a_{n-1}}{a_n}.$$

Then the integrand over a becomes

$$\prod_{\alpha_i<t:\ i\leq n-1} (d\alpha_1\alpha_1^{m_1})(d\alpha_2\alpha_2^{m_2})\cdots\ (d\alpha_{n-1}\alpha_{n-1}^{m_{n-1}}),$$

where m_i are **non-negative** integers. We have thus a product(over i) of an integral over the bounded interval $\alpha_i \in [0, t]$ of a monomial $\alpha_i^{m_i}$ for each i and this is finite. Therefore, the integral over a is also finite. □

4.3. **Finiteness of volume of the space of unimodular lattices.** We have viewed elements of $GL_n(\mathbb{R})$ as bases of the vector space $\mathbb{R}^n$ by identifying an element g with the basis of the rows $v_1, v_2, \cdots, v_n$ of the matrix g. Given an element$g \in GL_n(\mathbb{R})$, consider the lattice $L = L(g)$ in $\mathbb{R}^n$ given by the integral linear span of the basis $v_1, v_2, \cdots, v_n$ of its rows. We then get a map $\phi : GL_n(\mathbb{R}) \to \mathcal{L}$ of $GL_n(\mathbb{R})$ into the space of lattices in $\mathbb{R}^n$. On the latter there is an obvious GL_n action by sending a basis to its translate by the element of GL_n, and clearly, this map respects the action on both sides.

The map ϕ when restricted to $SL_n(\mathbb{R})$, takes an element g of $SL_n(\mathbb{R})$ to a unimodular lattice $L(g)$. Clearly the map ϕ is right $SL_n(\mathbb{Z})$ invariant. Hence we have the isomorphism

$$\phi : SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \to \mathcal{L}^0,$$

where $\mathcal{L}^0$ is the space of unimodular lattices in $\mathbb{R}^n$.

Theorem 13. *Given an element $g \in SL_n(\mathbb{R})$, there exists an element $\gamma \in SL_n(\mathbb{Z})$ such that the element $g\gamma$ has the Iwasawa decomposition*

$$g\gamma = kau,$$

where a is a diagonal matrix $a = (a_1, a_2, \cdots, a_n)$ with a_i positive, such that $\frac{a_i}{a_{i+1}} < \frac{2}{\sqrt{3}}$, and where u is an upper triangular matrix with 1's on the diagonal and whose entries above the diagonal are of the form $(u_{ij} : i < j \leq n)$ with $\mid u_{ij} \mid \leq \frac{1}{2}$.

Proof. We may view elements of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ as the space of lattices L in $\mathbb{R}^n$ such that the volume $\mathbb{R}^n/L$ is one (**unimodular lattices**), by sending an element of $SL_n(\mathbb{R})$ to the integral linear span L of its rows.

By Lemma 6, there exists a basis $v_1, v_2, \cdots, v_n$ of L such that for all $i \leq n-1$ we have $\mid v_{i+1} \mid^2 \geq \mid v_i \mid^2 \frac{3}{4}$.

We have thus proved (see Lemma 8) that given $g \in SL_n(\mathbb{R})$, there exists an element $\delta \in SL_n(\mathbb{Z})$ such that $g\delta$ has the Iwasawa decomposition $g = kav$ with $\frac{a_i}{a_{i+1}} < \frac{2}{\sqrt{3}}$ for each$i$.

An easy induction shows that given $v \in N(\mathbb{R})$ there exists an element $\theta \in N(\mathbb{Z})$ such that $v\theta = 1 + \sum_{i<j} u_{ij}E_{ij}$ with $\mid u_{ij} \mid \leq \frac{1}{2}$. The element $\gamma = \delta\theta$ is such that $g\gamma \in \mathcal{S}_{t,C}$ with $C = \frac{1}{2}$ and $t = \frac{2}{\sqrt{3}}$. □

Corollary 3. *$SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$. Moreover, the quotient $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is not compact.*

Proof. By the Minkowski reduction, given $g \in SL_n(\mathbb{R})$, there exists an element $\gamma \in SL_n(\mathbb{Z})$ such that $g\gamma$ lies in the Siegel set $\mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$. Thus, $SL_n(\mathbb{R}) = \mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}} SL_n(\mathbb{Z})$. The Siegel set has finite volume by Lemma 12. □

The following Theorem gives a criterion to check when a sequence in the quotient $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ tends to infinity (i.e. has no convergent subsequence). This is called the **Mahler Criterion**.

Theorem 14. *A sequence $g_m \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ does not have a convergent subsequence if and only if there exists a sequence $v_m \in \mathbb{Z}^n$ with $v_m \neq 0$ such that $g_m(v_m)$ tends to zero.*

Proof. By Minkowski Reduction, any element $g \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is represented by a matrix $g \in SL_n(\mathbb{R})$ whose Iwasawa decomposition is of the form $g = kan$, with n in a compact subset of the space of upper triangular unipotent matrices and with a diagonal whose diagonal entries are of the form $a_1, a_2, \cdots, a_n$), and which satisfy the inequalities

$$\frac{a_2}{a_1} \geq \frac{\sqrt{3}}{2},\ \frac{a_3}{a_2} \geq \frac{\sqrt{3}}{2},\ \cdots,\ \frac{a_n}{a_{n-1}} \geq \frac{\sqrt{3}}{2},$$

and such that the determinant

$$det(a) = \ a_1 a_2 \cdots a_n = \ 1.$$

We may write

$$1 = a_1 a_2 \cdots a_{n-1} a_n = a_1^n (\frac{a_2}{a_1})^{n-1} (\frac{a_3}{a_2})^{n-1} \cdots (\frac{a_n}{a_{n-1}})^{n-1}.$$

Since $\frac{a_{i+1}}{a_i} \geq t$, it follows that a_1 is bounded from above by a power of t, where $t = \frac{\sqrt{3}}{2}$. *If a_1 is bounded from below also by a positive constant*, then from these inequalities it follows that $a_2, \cdots, a_n$ are bounded both from above and from below. Consequently, the a_i lie in a compact subset of the space of positive real numbers. Therefore, $g = kan$ also lies in a compact set.

Therefore, a sequence $g_m \in SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ lies in a a compact set if and only if their Iwasawa representatives $g_m = k_m a_m n_m$ have the property that the first diagonal entry $(a_m)_1$ tends to zero. This means that $\mid g_m(e_1) \mid = \mid a_m(e_1) \mid = (a_m)_1$ tends to zero and this proves the Mahler Criterion. □

5. LATTICES IN OTHER GROUPS

5.1. Algebraic groups defined over $\mathbb{Q}$.

Definition 4. Suppose that $G \subset SL_n$ is a subgroup which is a set of zeroes of some polynomials in the matrix entries X_{ij}. Then the subgroup G is said to be an **algebraic subgroup** of SL_n.

If the set of polynomials in X_{ij} of which G is the set of zeroes have rational coefficients, we say that G is an algebraic subgroup **defined over** $\mathbb{Q}$. We similarly have the definition of an algebraic group defined over any field of characteristic zero.

We say that $f : G \to \mathbb{C}$ is an **algebraic function**, if f is a polynomial in the matrix entries X_{ij}.

We say that f is **defined over** $\mathbb{Q}$ if the polynomial which corresponds to f has rational coefficients.

More generally, if $G \subset SL_n$ and $H \subset SL_m$ are algebraic groups defined over $\mathbb{Q}$ and $f : G \to H$ is a homomorphism, we say that f is an algebraic map defined over $\mathbb{Q}$, if the matrix entries of the function $x \mapsto f(x)$ are algebraic functions on G defined over $\mathbb{Q}$.

We recall a result of Chevalley without proof.

Lemma 15. *(Chevalley) If $G \subset SL_n$ is a linear algebraic group defined over $\mathbb{Q}$ without any non-trivial characters $G \to GL_1$ defined over $\mathbb{Q}$, then there exists a representation $\rho : SL_n \to SL_m$ defined over $\mathbb{Q}$ and a non-zero vector $v \in \mathbb{Q}^m$ such that G is precisely the isotropy of v: G is the set of elements $x \in SL_n$ such that $\rho(x)v = v$.*

We define $G(\mathbb{Z}) = G \cap SL_n(\mathbb{Z})$. Consequently $G(\mathbb{Z})$ is a discrete subgroup of the topological group $G(\mathbb{R})$ which is a closed subgroup of $SL_n(\mathbb{R})$. we have a continuous map $G(\mathbb{R})/G(\mathbb{Z}) \to SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.

Lemma 16. *If G has no non-trivial algebraic characters defined over $\mathbb{Q}$, then the map $\phi : G(\mathbb{R})/G(\mathbb{Z}) \to SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is a proper map.*

Proof. We will first show that the map ϕ has closed image by using Chevalley's theorem (Lemma 15). Let v and ρ be as in Chevalley's Theorem. We may assume that $v \in \mathbb{Z}^m$. The map $x \mapsto \rho(x)v$ from $SL_n(\mathbb{R})$ into $\mathbb{R}^n$ is an algebraic map defined over $\mathbb{Q}$.

The inverse image of $\rho(SL_n(\mathbb{Z}))v$ under the continuous map $x \mapsto \rho(x)v$ is closed and is clearly, by Lemma 15, the product set $G(\mathbb{R})SL_n(\mathbb{Z})$. Hence, $G(\mathbb{R})SL_n(\mathbb{Z})$ is closed in $SL_n(\mathbb{R})$.By the definition of the quotient topology, this means that $\phi : G(\mathbb{R})/G(\mathbb{Z}) \to SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ has closed image.

The map

$$\phi : G(\mathbb{R})/G(\mathbb{Z}) \to G(\mathbb{R})SL_n(\mathbb{Z})/SL_n(\mathbb{Z}) \subset SL_n(\mathbb{R})/SL_n(\mathbb{Z}),$$

maps the quotient $G(\mathbb{R})/G(\mathbb{Z})$ into the closed set $G(\mathbb{R})SL_n(\mathbb{Z})/SL_n(\mathbb{Z})$. This map is clearly bijective and continuous. By a Baire category argument, the map can be shown to be an open map, and hence a homeomorphism. In particular, ϕ is a proper.

$\square$

5.2. **Borel Harish-Chandra Theorem and Arithmetic Groups.** For general algebraic groups defined over $\mathbb{Q}$ with no rational character, there is the following theorem of Borel and Harish-Chandra.

Theorem 17. *(Borel and Harish-Chandra) Suppose that $G \subset SL_n$ is a $\mathbb{Q}$-algebraic subgroup, such that every homomorphism $G \to GL(1)$ defined over $\mathbb{Q}$ is trivial. Then, $G(\mathbb{R})/G(\mathbb{Z})$ has finite volume.*

The Mahler condition enables us to test if $G(\mathbb{R})/G(\mathbb{Z})$ is compact. The following criterion for compactness is called the Godement Criterion.

Theorem 18. *(The Godement Criterion) If G is semi-simple, then $G(\mathbb{R})/G(\mathbb{Z})$ is non-compact if and only if there exist non-trivial unipotent elements in $G(\mathbb{Z})$.*

Suppose that G is a real semi-simple algebraic group without compact factors. Suppose that there exists a semi-simple $\mathbb{Q}$ algebraic group $\mathcal{G}$ and a morphism $\phi : \mathcal{G}(\mathbb{R}) \to G$ of real groups such that ϕ has compact kernel and finite Co-kernel. Then, by the Theorem of Borel and Harish-Chandra, the image of $\mathcal{G}(\mathbb{Z})$ under ϕ is a lattice in $G(\mathbb{R})$.

Definition 5. A lattice in G is said to be **arithmetic** if there exists $\mathcal{G}$ and ϕ as above such that Γ is commensurate with $\phi(\mathcal{G}(\mathbb{Z}))$.

Thus, the Borel-Harish-Chandra Theorem says that arithmetic subgroups of linear semi-simple Lie groups are lattices. For semi-simple groups of real rank at least two, there is a converse. To state it, we need first the notion of irreducibility of lattices and of rank of semi-simple groups.

Definition 6. Suppose G is a linear real semi-simple Lie Group without compact factors. A lattice $\Gamma \subset G$ is said to be **reducible**, if there exist closed normal semi-simple subgroups G_1 and G_2 of G such that $G = G_1G_2$ and lattices $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ such that the product group $\Gamma_1\Gamma_2 \subset \Gamma$ (and is of finite index in Γ). A lattice $\Gamma \subset G$ is an **irreducible lattice** if Γ is not reducible.

Definition 7. If $G \subset SL_n$ is an algebraic group defined over $\mathbb{R}$, then the **real rank of** G -denoted $\mathbb{R} - rank(G)$ is the maximum of the dimension of the intersection of the group of diagonals in SL_n with conjugates of G in $SL_n(\mathbb{R})$.

For example, the real rank of SL_n is $n - 1$.

In this connection, one has the famous **Arithmeticity Theorem** of Margulis.

Theorem 19. *(Margulis) If G is a real semi-simple algebraic group without compact factors and such that $\mathbb{R} - rank(G) \geq 2$, then every irreducible lattice in Γ is arithmetic.*

If $\mathbb{R} - rank(G)$ is one and G is not locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$, then the following result of Corlette and Gromov-Schoen says that again lattices are arithmetic.

Theorem 20. *If G is a real rank one simple Lie group locally isomorphic to $Sp(n, 1)$ or the real rank one form of F_4, then every lattice in G is arithmetic.*

Remark. It can be shown, using rigidity theorems, that if G is a non-compact simple Lie group not locally isomorphic to $SL_2(\mathbb{R})$, then any lattice in G can be conjugated to one whose elements have entries in an algebraic number field. The only general construction of lattices in semi-simple Lie groups is by arithmetic lattices, and the above Theorems tell us that for most groups, these are the only lattices.

5.3. $SL_2(\mathbb{R})$. By the Uniformisation Theorem, if X is a compact Riemann surface of genus $g \geq 2$, then its fundamental group may be identified to a co-compact torsion-free discrete subgroup of $SL_2(\mathbb{R})$. Conversely, a torsion-free co-compact discrete subgroup Γ of $SL_2(\mathbb{R})$ acts on the upper half plane $\mathfrak{h}$ properly discontinuously and freely and therefore, the quotient $X = \mathfrak{h}/\Gamma$ is a compact Riemann surface of genus ≥ 2.

Using this, it is not hard to show that there exists a continuous family of co-compact $\Gamma \subset SL_2(\mathbb{R})$ parametrised by an open set in $\mathbb{R}^{6g-6}$, all these Γ are isomorphic as abstract groups.

6. The example of $SO(n,1)$

6.1. **Action of $SO(n,1)$ on the unit ball in $\mathbb{R}^n$.** Define $SO(n,1)$ as the subgroup of $SL_{n+1}(\mathbb{R})$ which preserves the quadratic form

$$x_1^2+\cdots x_n^2-x_{n+1}^2.$$

In particular, $SO(n,1)$ preserves the "light cone" $C=\{x\in\mathbb{R}^{n+1}: x_1^2+\cdots+x_n^2-x_{n+1}^2<0\}$.

Since $SO(n,1)$ preserves the light cone and commutes with scalar multiplication, $SO(n,1)$ preserves the lines through the origin in $\mathbb{R}^{n+1}$ which lie in the cone C. This set of lines may be identified with the unit ball B_n in $\mathbb{R}^n$ via the map

$$(x_1,x_2,\cdots,x_n,x_{n+1})\mapsto(\frac{x_1}{x_{n+1}},\cdots,\frac{x_n}{x_{n+1}}),$$

The isotropy subgroup of $SO(n,1)$ at the origin in $\mathbb{R}^n$ is given by the group $O(n)=O(n+1)\cap SO(n,1)$.

6.2. **Action by certain subgroups of $SO(n,1)$.** Consider the subgroup $H=\{h=\begin{pmatrix} coshx & 0 & sinhx\\ 0 & 1_{n-1} & 0\\ sinhx & 0 & coshx\end{pmatrix}: x\in\mathbb{R}\}$ of $SO(n,1)$. This subgroup takes the origin into the open unit interval lying in the unit ball, such that the $x_2,\cdots x_n$ co-ordinates of its points vanish. The x_1 coordinate is $tanhx$ which lies between -1 and 1.

Moreover, any point in the unit ball, after a rotation by $On)$ may be brought into the unit interval. This proves that the group generated by $O(n)$ and H acts transitively on the unit ball. Moreover, this also proves that $SO(n,1)$ is generated by H and $O(n)$.

Lemma 21. *The volume form on the unit ball in $\mathbb{R}^n$ given by*

$$dv=\frac{dx_1dx_2\cdots dx_n}{(1-x_1^2-x_2^2-\cdots-x_n^2)^{(n+1)/2}}$$

is invariant under $SO(n,1)$.

Proof. The form dv depends only on the radius $\sqrt{(x_1^2+\cdots+x_n^2)}$. Hence dv is clearly invariant under $O(n)$. We need only check that H leaves dv invariant. □

The group $SO(n,1)$ acts transitively on the unit ball in $\mathbb{R}^n$ and the isotropy at the origin is $O(n)$. Consequently, if m is an $O(n)$-invariant metric on the tangent space at the origin, then there is an $SO(n,1)$ -invariant metric on B_n which coincides with the $O(n)$-invariant metric at the origin.

The invariant metric (up to positive scalar multiples) is

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)^2}.$$

This is the hyperbolic metric on the unit ball in $\mathbb{R}^n$, and, up to connected components, $SO(n,1)$ is the group of isometries of the unit ball with respect to this hyperbolic metric. Quotients of the unit ball B_n by torsion-free discrete subgroups Γ of $SO(n,1)$ thus give hyperbolic n-manifolds.

Denote by dg a Haar measure on $SO(n,1)$. Fix a discrete subgroup Let Γ of $SO(n,1)$.

Proposition 22. *(1) The discrete subgroup Γ operates without fixed points on B_n if and only if Γ is torsion-free.*

(2) If Γ is torsion-free then the quotient $B_n/Gamma$ is compact if and only if $\Gamma\backslash SO(n,1)$ is compact.

(3) If Γ is torsion-free then the quotient $\Gamma\backslash B_n$ has finite volume with respect to the hyperbolic metric if and only if $\Gamma\backslash G$ has finite measure with respect to dg.

6.3. **Equivalence of quadratic forms and Orthogonal Groups.** Suppose that K is a field of characteristic zero and V an k dimensional vector space over the field K. Let f and h be two non-degenerate quadratic forms on V with values in K. One says that h is equivalent to f if there exists a non-singular linear transformation $g \in GL(V)$ such that for all vectors $v \in V$, we have

$$h(v) = f(g(v)).$$

By fixing a basis of V over K, we may view f and h as non-singular symmetric matrices f and h of order k, and the foregoing equivalence is the equation

$$h =^t gfg,$$

where ${}^tg$ is the transpose of the matrix g.

If $SO(f)$ denotes the group of linear transformations in $GL(V)$ which preserve f, then it is clear that $SO(h) = gSO(f)g^{-1}$. Hence $SO(f)$ and $SO(h)$ are isomorphic.

If h is any non-degenerate quadratic form on V, then there exists a diagonal quadratic form

$$\lambda_1 x_1^2 + \cdots \lambda_d x_d^2,$$

which is equivalent to f, with $\lambda_i \in K \setminus \{0\}$.

6.4. **Example of real quadratic forms.** Take $K = \mathbb{R}$ and $k = n + 1$. Suppose that $\lambda_1, \cdots \lambda_n$ are n positive numbers and that λ_{n+1} is a negative real number. Consider the quadratic form

$$h = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots \lambda_n x_n^2 + \lambda_{n+1}^2 x_{n+1}^2.$$

There exists a non-singular matrix g such that $h =^t gfg$. Indeed, g can be taken to be the diagonal matrix with diagonal entries

$$(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}, \sqrt{-\lambda_{n+1}}).$$

6.5. **Rational Quadratic Forms.** Let $K = \mathbb{Q}$ and $k = n + 1$. If $\lambda_1, \cdots \lambda_n$ are positive rational numbers and λ_{n+1} is a negative rational number then the quadratic form

$$\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 + \lambda_{n+1} x_{n+1}^2,$$

is a nondegenerate quadratic form. By replacing f by an integral scalar multiple, we may assume that λ_i are integers.

In particular, if $V = \mathbb{R}^{n+1}$ and $L = \mathbb{Z}^{n+1}$, then for integral vectors $v \in \mathbb{Z}^{n+1}$, we have $f(v) \in \mathbb{Z}$.

The group $SO(f)(\mathbb{R})$ is a closed subgroup of $SL_{n+1}(\mathbb{R})$. Hence the intersection $SO(f)(\mathbb{R}) \cap SL_{n+1}(\mathbb{Z})$ is a discrete subgroup of $SO(n, 1)$.

The group $G = SO(f) = \{x \in SL_k :^t xfx = f\}$ is the set of zeroes of the collection Σ of the k^2-matrix entries of the difference matrix ${}^t gfg - f$. These consist of polynomials of degree two.

Proposition 23. *The map $G(\mathbb{R})/G(\mathbb{Z}) \to SL_k(\mathbb{R})/SL_k(\mathbb{Z})$ is proper with the image being closed.*

We repeat the proof of Theorem 18 in the case of $G = SO(f)$; the proof uses the fact that $SO(f)$ is the isotropy of a vector (namely the quadratic form f) in a suitable representation of SL_k (namely the space of quadratic forms on $V = \mathbb{R}^k$). Recall that the general case is similar, and we use the Theorem of Chevalley that a $\mathbb{Q}$ algebraic group without rational characters is the isotropy of a vector in a Q-representation V of SL_k.

Proposition 24. *The map*

$$\phi : SO(f)(\mathbb{R})/SO(f)(\mathbb{Z}) \to SL_{n+1}(\mathbb{R})/SL_{n+1}(\mathbb{Z})$$

is a proper map. Moreover, $SO(f)(\mathbb{R})/SO(f)(\mathbb{Z})$ is non-compact if and only if the quadratic form f represents a rational zero.

Proof. The image of the map ϕ is closed since it is the image of the closed saturated set $SO(f)(\mathbb{R})SL_{n+1}(\mathbb{Z})$; the latter is closed in $SL_{n+1}(\mathbb{R})$ since it is the inverse image of a subset of a discrete space, namely the **integral** points of the orbit of $SL_{n+1}(\mathbb{R})$ through the non-degenerate quadratic form f. Note that the space of quadratic forms is a module for the action $g * \psi \mapsto (v \mapsto \phi(g(v)))_{v \in \mathbb{R}^{n+1}}$.

Now a Baire category argument proves that ϕ maps the quotient $SO(f)(\mathbb{R})/SO(f)(\mathbb{Z})$ homeomorphically onto its image. By the preceding paragraph, the image is closed. This proves the properness of the map ϕ.

The quotient $SO(f)(\mathbb{R})/SO(f)\mathbb{Z})$ is non-compact if and only if there exists a sequence g_m in it tending to infinity. Since the map ϕ is proper, it follows that g_m tends to infinity in the quotient $SL_{n+1}(\mathbb{R})/SL_{n+1}(\mathbb{Z})$. By the Mahler criterion, this means that there is a sequence of integral vectors v_m such that $g_m(v_m) \to 0$. Evaluating f on this sequence, we get $f(g_m(v_m) \to 0$. Since g_m lie in the orthogonal group of f, they preserve the quadratic form f and hence $f(g_m(v_m)) = f(v_m) \to 0$. But v_m are integral vectors and f has integral coefficients hence e$f(v_m)$ is a sequence of integers which tends to zero, hence $f(v_m) = 0$ for large m. This proves that f represents a zero.

□

7. Arithmetic Groups in $SO(n,1)$

7.1. **non-uniform lattices in $SO(n,1)$.** We first give a general construction of non-co-compact lattices in $SO(n,1)$. Let $\lambda_1, \cdots \lambda_{n+1}$ be positive rational numbers and f the quadratic form

$$f(x) = \lambda_1 x_1^2 + \cdots \lambda_n x_n^2 - \lambda_{n+1} x_{n+1}^2.$$

The subgroup of $SL_{n+1}(\mathbb{Q})$ which preserves this quadratic form f is a rational subgroup $G = SO(f)$ of $SL_{n+1}(\mathbb{R})$ whose real points form a real algebraic group $G(\mathbb{R})$ isomorphic to $SO(n,1)$. The intersection $SO(f)(\mathbb{Z}) = G(\mathbb{Z}) \overset{defn}{=} G \cap SL_{n+1}(\mathbb{Z})$ of $G = SO(f)$ with $SL_{n+1}(\mathbb{Z})$ is a discrete subgroup of $SO(n,1)$.

Theorem 25. *The group $SO(f)(\mathbb{Z})$ is a lattice in $SO(n,1)$ (If $n \geq 4$, then $SO(f)(\mathbb{Z})$ is a nonuniform lattice).*

Proof. There is a result in number theory, called the Hasse principle for quadratic forms which implies the following result.

Theorem 26. *If f is a quadratic form in k variables with $k \geq 5$ and which represents a real zero, then it represents a rational zero.*

In view of this, our integral quadratic form f , since is of the form $x_1^2 + \cdots x_n^2 - x_{n+1}^2$ over the reals, represents a real zero, hence by the foregoing consequence of the Hasse principle, represents an integral zero. By the proposition, this means that the quotient $SO(f)(\mathbb{R})/SO(f)(\mathbb{Z})$ is non-compact. $\square$

Therefore, in order to get compact hyperbolic manifolds with $n \geq 4$,integral quadratic forms do not suffice. We will therefore have to consider quadratic forms over other number fields.

7.2. **Restriction of Scalars.** Suppose that K is a finite extension of $\mathbb{Q}$ (K is called a number field). Suppose that $\mathcal{G} \subset SL_r$ is a subgroup of SL_r which is the set of zeroes of a collection Σ of polynomials in the r^2-matrix variables, whose coefficients lie in K. Then $\mathcal{G}$ is called a K-algebraic subgroup of SL_r.

We may view K is a d dimensional vector space over $\mathbb{Q}$ and hence view K^k as a $rd = k$ dimensional vector space over $\mathbb{Q}$. The group SL_r over K is then a subgroup of SL_k over $\mathbb{Q}$, which is the subgroup which commutes with the action of K which acts as scalar multiplication on $K^r = \mathbb{Q}^{rd}$ via $\mathbb{Q}$ linear endomorphisms. This, SL_r over K is a $\mathbb{Q}$-algebraic subgroup of $SL_k = SL_{rd}$ over $\mathbb{Q}$.

The elements of Σ are polynomials in the matrix entries of $M_r(K)$. The r^2 variables which are matrix entries may be thought of as $r^2d^2 = k^2$ variables in the matrix entries of $M_r(K) \subset M_{rd}(\mathbb{Q})$. The coefficients of elements of Σ are in K and may be thought of as elements of a d dimensional vector space of $\mathbb{Q}$. Thus each equation $\phi = 0$ with $\phi \in \Sigma$ may be thought of as d equations involving polynomials in k^2 variables with $\mathbb{Q}$-coefficients. We may thus view $\mathcal{G}$ as an algebraic subgroup G of SL_{rd} defined over $\mathbb{Q}$. We denote $G = R_{K/\mathbb{Q}}(\mathcal{G})$, call G the Weil restriction of scalars of $\mathcal{G}$.

7.3. **Properties of Restriction of Scalars.** If E is an extension of $\mathbb{Q}$ and $G = R_{K/\mathbb{Q}}(SL_r)$ then $G(E)$ is by definition, the elements of $SL(E \otimes K^r)$. which commute with the action of K (and also with E). Therefore, these are just matrices in $SL(E \otimes K^r)$ which commute with $E \otimes K$. In other words, this group is just $SL_r(E \otimes K)$.

Using this observation, it is clear that if $\mathcal{G}$ is a K-subgroup of SL_r and $G = R_{K/\mathbb{Q}}(\mathcal{G})$ is the group obtained from $\mathcal{G}$ by restricting scalars from K to $\mathbb{Q}$, then $G(E) = \mathcal{G}(E \otimes K)$.

In particular, $G(\mathbb{R}) = \mathcal{G}(\mathbb{R} \otimes K)$. Now, if $K = \mathbb{Q}[X]/(P(X))$ is a quotient of the polynomial ring in X modulo the ideal (f) generated by an irreducible monic polynomial $P \in \mathbb{Q}X$, then $\mathbb{R} \otimes K = \mathbb{R}[X]/(P(X))$ is isomorphic as an algebra over $\mathbb{R}$ to the product $\mathbb{R}$-algebra $\mathbb{C}^{r_2} \times \mathbb{R}^{r_1}$, where $2r_2$ is the number of complex roots of P and r_1 is the number of real roots of P. Note that $d = 2r_2 + r_1$.

Consequently, $G(\mathbb{R}) = \mathcal{G}(\mathbb{C})^{r_2} \times \prod_v \mathcal{G}(K_v)$ where, for each real root v of P, denote by K_v the real embedding $\mathbb{R}[X]/(X - v) \simeq \mathbb{R}$ of the number field K.

In particular, if the number field K has the property that the irreducible polynomial P above has only real roots, then the quadratic form f is of the index (p_v, q_v) for each v above. We choose f so that

$$G(\mathbb{R}) = SO(n, 1) \times SO(n + 1)^{d-1}.$$

7.4. **Some Uniform Lattices in $SO(n, 1)$.** In this section, we will construct some **compact** hyperbolic manifolds of dimension n for every integer $n \geq 1$. This can be done, if we construct co-compact arithmetic lattices in $SO(n, 1)$; as we have seen, the unit groups of suitable quadratic forms with rational coefficients yield co-compact lattices only if $n \leq 4$. Therefore, we need to construct co-compact arithmetic lattices slightly differently.

We will therefore consider unit groups $SO(f)$ of quadratic forms over other number fields K. If the number field is totally real and f is suitably chosen, then we will see that these indeed give arithmetic co-compact lattices. To see this, we need to view $SO(f)$ as an algebraic group over $\mathbb{Q}$; this will be achieved by the Weil restriction of scalars construction.

Suppose now that K is a finite extension of $\mathbb{Q}$ (K is then called a number field) of degree d. Then, $K = \mathbb{Q}[X]/(g(X))$ is the quotient of the ring of polynomials in the variable X with $\mathbb{Q}$-coefficients, modulo an ideal generated by an irreducible polynomial $g \in \mathbb{Q}[X]$. The degree of g is d. Suppose that the polynomial g has the additional property that all its roots are real. Then K is said to be totally real (any embedding of the field K into the field $\mathbb{C}$ of complex numbers lies in the real field $\mathbb{R}$).

Let f be a non-degenerate quadratic form in $n+1$ variables over K. suppose that in $d-1$ distinct embeddings of K in $\mathbb{C}$ (the image of K will be in $\mathbb{R}$), the quadratic form f has index $(n+1, 0)$ and in the remaining real embedding, f has index $(n, 1)$. Then, $SO(f)(O_K)$ is a discrete subgroup of $SO(f)(K \otimes \mathbb{R}) = SO(n,1) \times SO(n+1)^{d-1}$.

Theorem 27. *If $d \geq 2$, then the projection of $SO(f)(O_K)$ to $SO(n,1)$ is a co-compact lattice in $SO(n,1)$.*

Proof. The embedding

$$SO(f)(\mathbb{R} \otimes K)/SO(f)(O_K) \subset SL_{(n+1)d}(\mathbb{R})/SL_{(n+1)d}(\mathbb{Z})$$

is a proper map. If the left hand side is not compact, there exists a sequence g_m of elements of $SO(f)(\mathbb{R} \otimes K)$ and $v_m \in O_K^r$ of nonzero vectors such that $g_m(v_m)$ tends to zero.

Applying f to $g_m(v_m)$ we get $f(v_m) = f(g_m(v_m))$ tends to zero. But $f(v_m)$ is a sequence of elements of O_K viewed as a discrete subgroup of R^d, whence $f(v_m) = 0$. Therefore, f represents a zero in K^r. But in one of the real embeddings K_v of K, f is positive definite, hence f does not represent a zero in K_v^r, contradicting the assumption. Therefore, if $d \geq 2$, then the quotient in the theorem is compact and $SO(f)(O_K)$ is a co-compact lattice in $SO(n,1)$.

□

Theorem 28. *If n is even and $n \geq 4$, then all arithmetic lattices in $SO(n,1)$ arise in this way. That is, given an arithmetic lattice in $SO(n,1)$, there exists a totally real number field K, a quadratic form f over K such that $SO(f)(K \otimes \mathbb{R}) = SO(n,1) \times SO(n+1)^{d-1}$ such that Γ is commensurable to $SO(f)(O_K)$.*

7.5. **More Arithmetic Lattices in** $SO(n,1)$**; Unitary Groups over skew-Hermitian forms.** Suppose that K is a totally real number field and D a quaternionic central division algebra over K. Then $D \otimes \mathbb{C} = M_2(\mathbb{C})$ is a matrix algebra for any embedding θ of K in $\mathbb{C}$. IF $x \in D \subset M_2(\mathbb{C})$ then the trace of this matrix denoted $tr(x)$ is actually an element of K viewed as a subset of $\mathbb{C}$ via the embedding θ.

Let $x \mapsto \iota(x)$ denote the involution $\iota(x) = tr(x) - x$. An involution on D reverses multiplication and is a vector space automorphism of order two. All involutions on D are conjugate by a elements of D(by the Skolem-Noether Theorem). We fix an involution $*$ on D the dimension of whose fixed points is three (in the matrix realisation of D, the involution transpose is one such example).

A map $h : D^m \times D^m \to D$ denoted $h(x,y)$ for $x \in D^m$ and $y \in D^m$ is said to be $*$-Hermitian if D is linear in the first variable x and $*$ linear in the second variable y and $h(y,x) = *(h(x,y))$ for all $x, y \in D^m$.

Define the unitary group

$$SU(h) = \{g \in SL(V)_D : h(g(x), g(y)) = h(x,y) \forall x, y \in V.\}$$

If $\overline{K}$ denotes an algebraic closure of K, then under the assumptions on D and h, we have that $SU(h)(\overline{K}) = SO(2m, \overline{K})$.

Suppose that h is a Hermitian form in m variables over the quaternionic division algebra D such that in all the real embeddings of K except one, $SU(h)(\mathbb{R})$ becomes isomorphic to $SO(2m)$ and in the remaining one it becomes $SO(2m-1,1)$.

Theorem 29. *The group $SU(h)O_K$ is a co-compact lattice in $SO(2m-1,1)$.Moreover, all arithmetic lattices in $SO(n,1)$ arise in either this way or are unit groups of quadratic forms over totally real number fields.*

7.6. **Non-Arithmetic Lattices in** $SO(n,1)$**.** The construction by Gromov and Pjatetskii-Shapiro proves that there exist non-arithmetic lattices in $SO(n,1)$ for every integer $n \geq 3$.

8. Lattices in Arbitrary Connected Groups

If G is a connected group, let R be its maximal connected solvable (closed) subgroup. Then, it is patent that the quotient group G/R is semi-simple. The group R is called the *radical* of G.

We have then the following theorem of Auslander.

Theorem 30. *(*Auslander*) If $\Gamma \subset G$ is a lattice, then its projection to the quotient G/R is also a lattice. Moreover, the intersection $R \cap \Gamma$ is a lattice in the radical R.*

Thus lattices in arbitrary connected Lie groups can be "understood" in terms of lattices in connected solvable groups and lattices in semi-simple Lie groups. As we said before, lattices in higher rank groups are arithmetic, and thus only lattices in $SO(n,1)$ and perhaps $SU(n,1)$ need be classified.

It is a deep theorem of Mostow that lattices in solvable Lie groups R are arithmetic.

9. Exercises

9.1. **Haar measures.** If $G = SL_2(\mathbb{R})$ and $B = \{g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0,\ b \in \mathbb{R}\}$, then show that G/B is compact and that there is no G-invariant regular Borel measure on the quotient G/B.

Prove the same for $G = SL_n(\mathbb{R})$ and B the group of upper triangular matrices B_n in $SL_n(\mathbb{R})$, for any $n \geq 2$. Find the left invariant Haar measure on B_n.

If $G = SL_n(\mathbb{R})$ and H is the group of upper triangular matrices whose diagonal entries are all 1, then show that on G/H there is indeed a left G-invariant regular Borel measure.

9.2. **lattices in $\mathbb{R}^n$.** If $\mathbb{R}^2$ is viewed as the complex plane $\mathbb{C}$, define two lattices L, L' in $\mathbb{C}$ to be equivalent, if there exists a scalar $\lambda \in \mathbb{C}$ such that $L' = \lambda L$ (i.e. elements of L' are of the form λv for some $v \in L$).

Let $\mathfrak{h}$ denote the upper half plane in $\mathbb{C}$, consisting of complex numbers whose imaginary parts are positive. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in $SL_2(\mathbb{R})$ and $\tau \in \mathfrak{h}$, then show that $g(\tau) = \frac{a\tau+b}{c\tau+d}$ lies in $\mathfrak{h}$. Show that this defines an action of $SL_2(\mathbb{R})$ on $\mathfrak{h}$.

Then show that every lattice in $\mathbb{C}$ is equivalent to one of the form $\mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathfrak{h}$. Further, show that the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ is equivalent to the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau'$ if and only if there exists $\gamma \in SL_2(\mathbb{Z})$ such that $\tau' = \gamma(\tau)$ where the action of $SL_2(\mathbb{Z})$ on the upper half plane is as in the preceding paragraph.

9.3. **Minkowski Reduction.** Fix a diagonal element g in $SL_2(\mathbb{R})$ of the form $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $0 < a < 1$. If u is an upper triangular unipotent element in $SL_2(\mathbb{R})$, show that the sequence $g^m u g^{-m}$ tends to the identity matrix as $m \in \mathbb{Z}$ $(m > 0)$ tends to infinity. Similarly show that if u is a *lower* triangular matrix with 1's on the diagonal, then $g^{-m} u g^m$ tends to the identity.

If H is a Hilbert space and $\rho : G = SL_2(\mathbb{R}) \to U(H)$ is a homomorphism into the group of unitary transformation of H such that the map $G \times H \to H$ given by $(g, v) \mapsto \rho(g)(v)$ is continuous, then we call ρ (or H), a representation of G. If $v \in H$ is fixed by the element g as in the preceding paragraph, show that all of $SL_2(\mathbb{R})$ fixes the vector v.

Hint: consider the function $f(x) = x \mapsto \mid \rho(x)v - v \mid^2$ on $SL_2(\mathbb{R})$ where $\mid w \mid$ is the norm of the vector $w \in H$ (the norm defined with respect to the $SL_2(\mathbb{R})$- invariant inner product). Show that $f(gx) = f(x) = f(xg)$ for all x and use the preceding exercise.

Prove that if $E \subset SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ is a Borel set invariant under the element g as above, then either E or its complement has Haar measure zero. One says then that the element g acts ergodically on the quotient $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$

Hint: Use the fact that $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ has finite volume, and apply the foregoing exercise on Hilbert spaces to a suitable vector in $L^2(SL_2(\mathbb{R})/SL_2(\mathbb{Z}))$.

Similar arguments are involved in the proof of the following result, called the **Howe-Moore Ergodicity Theorem**.

Theorem 31. *If G is a non-compact simple real Lie group and Γ is a lattice, then every non-compact subgroup H of G acts ergodically on the quotient G/Γ.*

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