1. Laplace Operator and the Laplace Equation

Let U be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is said to be twice continuously differentiable if all of the second partial derivatives of f exist and continuous on U. The set of all real valued twice continuously differentiable functions on U is denoted by $C^2(U)$.

Let $f \in C^2(U)$. The Laplacian of f is defined to be

$$\Delta f = \sum_{i=1}^{n} f_{x_i x_i}.$$

Definition 1.1. A function $f \in C^2(U)$ is said to be harmonic if $\Delta f = 0$.

Let $g : (0, \infty) \to \mathbb{R}$ be a C^2-function. We define a function $f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ by

$$f(x, y) = g(r), \quad r = \sqrt{x^2 + y^2}.$$

Let us compute Δf. By chain rule,

$$f_x = g'(r)\frac{x}{r}, \quad f_y = g'(r)\frac{y}{r}$$

and

$$\begin{aligned} f_{xx} &= g''(r)\left(\frac{x}{r}\right)^2 + g'(r)\frac{r^2 - x^2}{r^3}, \\ f_{yy} &= g''(r)\left(\frac{y}{r}\right)^2 + g'(r)\frac{r^2 - y^2}{r^3}. \end{aligned}$$

We obtain that

$$\begin{aligned} \Delta f = f_{xx} + f_{yy} &= g''(r)\frac{x^2 + y^2}{r^2} + g'(r)\frac{2r^2 - (x^2 + y^2)}{r^3} \\ &= g''(r) + \frac{1}{r}g'(r). \end{aligned}$$

Hence $\Delta f = 0$ if and only if

$$g''(r) + \frac{1}{r}g'(r) = 0, \quad r > 0.$$

Let $h(r) = g'(r)$ for $r > 0$ We have $rh'(r) + h(r) = 0$, for $r > 0$ i.e. $(rh(t))' = 0$ for $r > 0$. By mean value theorem, $rh'(r) = c_1$ for $r > 0$. Therefore

$$g'(r) = h(r) = \frac{c_1}{r} \text{ for } r > 0.$$

By integrating $g'(r)$, we have

$$g(r) = \int_{r_0}^{r} \frac{c_1}{t} dt = c_1 \ln r + c_2$$

where $c_2 = -c_1 \ln r_0$ and $r_0 > 0$. This shows that

$$f(x, y) = \frac{c_1}{2} \log(x^2 + y^2) + c_2.$$

Let us try to solve the Laplace equation

$$\Delta f = 0 \text{ on } \mathbb{B}(\mathbf{0}, 1).$$

Proposition 1.1. Let $f \in C^2(\mathbb{B}(\mathbf{0},1)) \cap C(\mathbb{D}(\mathbf{0},1))$. We set

$$u(r,\theta) = f(r\cos\theta, r\sin\theta), \quad 0 < r < 1, \theta \in \mathbb{R}.$$

Then u is a periodic function in θ with period 2π.

$$\Delta f = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Assume that $u(r,\theta) = R(r)\Theta(\theta)$ is a nonzero solution to $\Delta f = 0$. Then

$$R''(r)\Theta(\theta) + \frac{R'(r)}{r}\Theta(\theta) + \frac{R(r)}{r^2}\Theta''(\theta) = 0.$$

Dividing the above equation by $R(r)\Theta(\theta)$ and multiplying the above by r^2, we obtain

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

The left hand side of the equation is a function of r which is independent of θ while the right hand side of the equation a function of θ which is independent of r. Thus both of the right hand side and the left hand side of the equation must be a constant. Set the constant to be λ We obtain two differential equation

$$\begin{array}{ccccc} r^2R''(r) & + & rR'(r) & -\lambda R(r) & = 0, \\ \Theta''(\theta) & + & \lambda\Theta(\theta) & & = 0. \end{array}$$

When $\lambda = 0$, $\Theta(\theta) = a\theta + b$ for some a, b. Since Θ is a period function, $a = 0$. Therefore $r^2R''(r) + rR'(r) = 0$. We have seen that $R(r) = c_1 \log r + c_2$. Since we assume that f is continuously differentiable at $\mathbf{0}$, it is continuous at $\mathbf{0}$. This implies that $c_1 = 0$.

When $\lambda \neq 0$, for each $n \geq 1$, the function $c_n(\theta) = \cos n\theta$ and $s_n(\theta) = \sin n\theta$ are both periodic of period 2π and satisfy $\Theta'' + n^2\Theta = 0$. In this case, we obtain that for each $n \geq 1$,

$$r^2R''(r) + rR'(r) - n^2R(r) = 0.$$

To solve this equation, we define a new function $\phi(t) = R(e^t)$, i.e. we make a change of variable $r = e^t$. Then we obtain that

$$\phi''(t) - n^2\phi(t) = 0.$$

By theory of O.D.E., we find that $\phi(t)$ is of the form

$$\phi(t) = C_1e^{nt} + C_2e^{-nt}.$$

This implies that

$$R(r) = C_1r^n + C_2r^{-n}.$$

Again, by the continuity of f, we find $C_2 = 0$. Set $u_n(r,\theta) = r^n\cos n\theta$ and $v_n(r,\theta) = r^n \sin n\theta$. Then $\{1, u_n(r,\theta), v_n(r,\theta)\}$ are all solutions to the Laplace equation. Fourier's idea: combine all these solutions into one. More precisely, we set

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n \sin n\theta). \tag{1.1}$$

How do we determine a_0, a_n, b_n? Observe that

$$u(1,\theta) = f(\cos\theta, \sin\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta). \tag{1.2}$$

Let us define $g(\theta) = f(\cos\theta, \sin\theta)$ for $\theta \in \mathbb{R}$. It follows from the definition that g is a periodic function of period 2π, in fact,

$$g(\theta + 2\pi) = f(\cos(\theta + 2\pi), \sin(\theta + 2\pi)) = f(\cos\theta, \sin\theta) = g(\theta)$$

for any $\theta \in \mathbb{R}$. We can also use the notation $f|_{S^1}$ to denote g. We call g the restriction of f to $S^1 = \partial\mathbb{B}(\mathbf{0}, 1)$. Hence if we know g, the restriction of f to the boundary of S^1, and express g as an infinite series of the form (1.2), then we can find (solve for) f. This leads to the following Dirichlet problem

$$\begin{cases} \Delta f &= 0 \quad \text{on } \mathbb{B}(\mathbf{0}, 1) \\ f|_{S^1} &= g \end{cases} \tag{1.3}$$

Hence if g has the following series representation

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \tag{1.4}$$

then f is given by (1.1).

Example 1.1. Solve (1.3) with the boundary condition

$$g(\theta) = 1 - \cos\theta + \sin\theta + 3\sin 3\theta.$$

The harmonic function is given by

$$f(r\cos\theta, \sin\theta) = 1 - r\cos\theta + r\sin\theta + 3r^3 \sin 3\theta.$$

We can also express f in terms of x, y as follows. By De Moivre's law, we know

$$(x + iy)^n = r^n(\cos n\theta + i\sin n\theta).$$

We set $C_n(x, y) = \mathrm{Re}(x + iy)^n$ and $S_n(x, y) = \mathrm{Im}(x + iy)^n$. The above function can be rewritten as

$$f(x, y) = 1 - C_1(x, y) + S_1(x, y) + 3S_3(x, y).$$

Since $C_1(x, y) = x$ and $S_1(x, y) = y$ and $S_3(x, y) = 3xy^2 - y^3$,

$$f(x, y) = 1 - x + y + 9x^2 y - 3y^3.$$

In general, if the boundary condition is given by (1.4), then the solution to (1.3) is given by

$$f(x, y) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n C_n(x, y) + b_n S_n(x, y)). \tag{1.5}$$

Later, we will show that any continuous periodic function of period 2π always possess such a series representation as (1.4) and discuss the convergence of (1.5). We need the notion of uniform convergence.

Example 1.2. Solve (1.3) with the boundary condition

$$g(\theta) = 1 + 2\cos\theta - 3\sin\theta + 4\sin^2\theta + \cos^2\theta + 5\sin\theta\cos\theta + \cos^3\theta + 6\sin^3\theta.$$