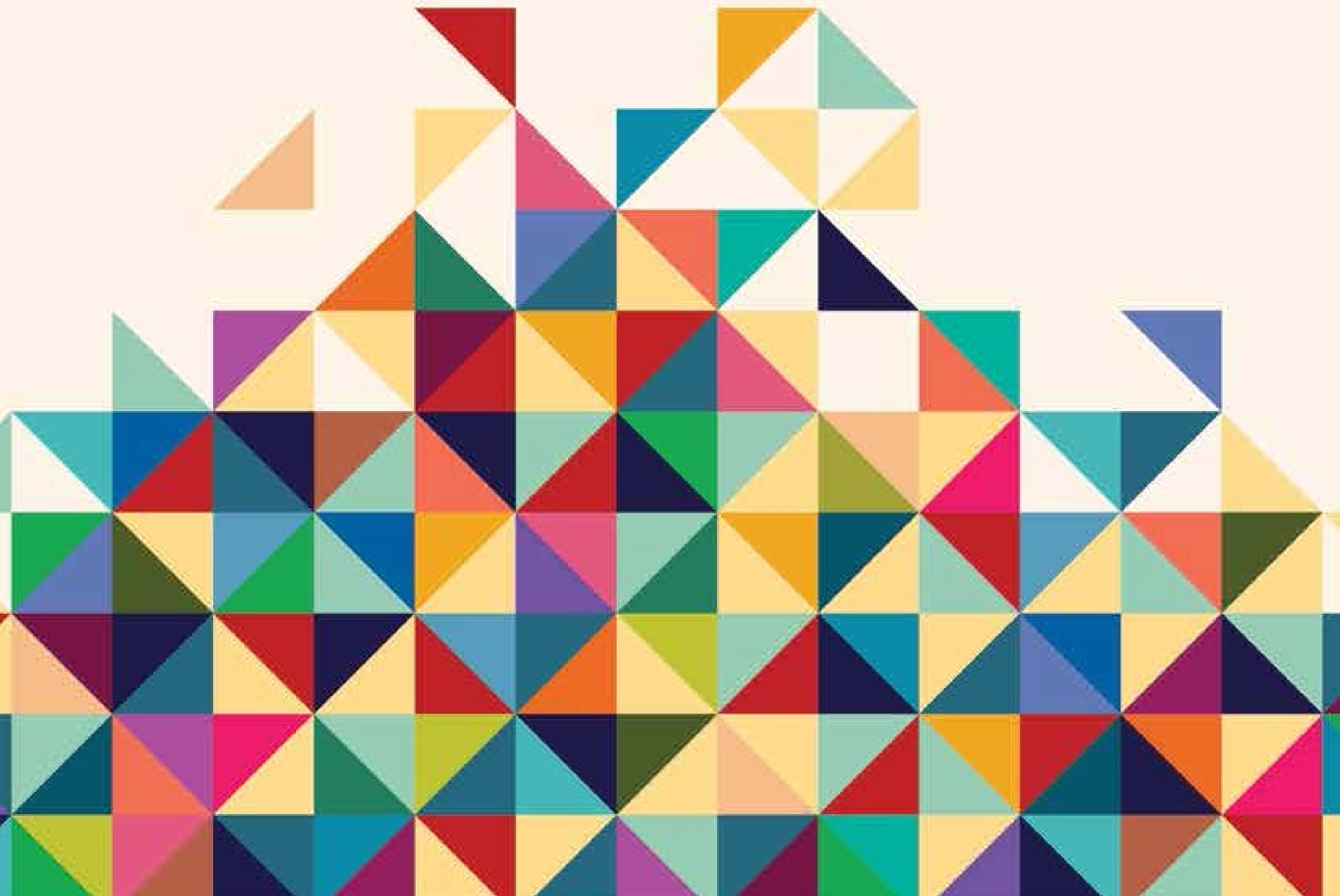
Separating Combinatorial Principles Using Nonstandard Models

The Stable Version of Ramsey's Theorem for Pairs Does Not Imply the General Version

by Yang Yue



The main objective of this research is to study the relative strength of combinatorial principles, in particular, the principles related to Ramsay's theorem. It turns out that the most interesting ones are those weaker than the Ramsey's theorem for pairs. The strength is measured by hierarchies from either recursion theory or reverse mathematics.

Let us recall the precise statement of Ramsey's Theorem: Any function f from n-element subsets of the set of natural numbers to natural number $k=\{0,1,\ldots, k-1\}$ has an infinite homogeneous set H, namely, f is constant on n-element subsets of the set H. One informal reading of the theorem says, if we think of f as a k-coloring of the n-element subsets of natural numbers, then there is an infinite set H, whose n-element subsets of have the same color. It is customary to think of this kind of ``for all $f$ there exists $H$...'' statements as ``our opponent posed a problem (e.g., coloring) and we must provide a solution (e.g., the infinite homogenous set).

The version above is denoted by RT^n_k. Our main focus is on RT^2_2 – Ramsey's Theorem for Pairs. We now give a proof of it. Let f be a coloring of pairs, say by red and blue. We first find an infinite subset C of natural numbers on which f is "stable", i.e. for all x, the limit $f(x,y)$ exists, when y tends to infinity and y is in C. We call such a set C cohesive for f. Next we consider the following two sets: $D^R=\{x$: x is an element of C and is "eventually red"$\}$ and $D^B=\{x$: x is an element of C and is "eventually blue"$\}$. One of them must be infinite, say it is D^R. Now it is fairly easy to obtain a solution from D^R.

We extract two combinatorial principles out of the proof: Let R be an infinite set and $R^s=\{t|(s,t)$ is in $R\}$. A set G is said to be R-cohesive if for all s, either G intersects R^s is finite or G intersects the complement of R^s is finite. The cohesive principle COH states that for every R, there is an infinite G that is R-cohesive. The other principle is called the stable Ramsey's Theorem for pairs, denoted by $\mathbf{SRT}^2_2$ which states that every stable coloring of pairs has a solution.

Theorem (Cholak, Jockusch and Slaman, 2001)

$$RT^2_2 = SRT^2_2 + COH.$$

Now we can make the aforementioned main objective more precise by asking the following concrete motivating questions: How complicated is the homogeneous set H? Is COH or SRT^2_2 as strong as RT^2_2? What are the logical consequences or strength of Ramsey's Theorem?

To answer these questions, we must determine if one principle P implies the other principle Q. It is usually more challenging to show that P does not imply Q. As we know from logic, one way to demonstrate that P does not imply Q is to "make P true and Q false". But given that these combinatorial principles are all true theorems from mathematics, how can you make it false? Thus we have to work in some weaker axiom system Γ and demonstrate that "Γ proves P but does not prove Q". Usually, we will have a hierarchy of systems $\Gamma_0<\Gamma_1<\ldots$, as our benchmarks and their relative strength has been established that Γ_i is strictly weaker than Γ_j for $i<j$. Therefore, to show that the P does not prove Q, it suffices to show that Γ_i proves P but Q proves Γ_j for some $j>i$. Notice that the last step requires that we prove axiom Γ_j from a theorem Q, which reverses the usual mathematical practice of proving theorems from axioms, that is where the name "reverse mathematics" comes.

We now introduce two most commonly used hierarchies of first- and second-order arithmetic. Recall that the language of first order Peano Arithmetic contains a constant symbol 0, three function symbols S, +, x, and a binary predicate <. Formulas over the language of arithmetic naturally form a hierarchy by the number of alternating blocks of quantifiers, which gives us the usual arithmetic hierarchy. Formulas with n alternating blocks of quantifiers with leading one existential (or universal) are called Σ^0_n and $\cap^0_n$ respectively. Furthermore, the Δ^0_n formulas are those having two equivalent forms, one Σ^0_n and $\cap^0_n$ 0n. Let $I\Sigma^0_n$0n denote the induction schema for Σ^0_n -formulas; and $B\Sigma^0_n$ denote the Bounding Principle for Σ^0_n-formulas. By a theorem of Kirby and Paris (1977)

$$\ldots \rightarrow I\Sigma^0_{n+1} \rightarrow B\Sigma^0_{n+1} \rightarrow I\Sigma^0_n \rightarrow \ldots$$

We have one benchmark in first order arithmetic. The other benchmark is by subsystems of second order arithmetic which is used in reverse mathematics. Here we only list three of those subsystems which are needed in the sequel: RCA_0 which contains Σ^0_1-induction and Δ^0_1-comprehension; WKL_0 which is RCA_0 plus every infinite binary tree has an infinite path; and ACA_0 which is RCA_0 plus arithmetical comprehension. Their relative strength is known:

$$RCA_0 < WKL_0 < ACA_0.$$

We also need the notion of models. A model M of second-order arithmetic consists of (M, 0, S, +, x, <, X) where (M, 0, S, +, x, <) is its first-order part and the set variables are interpreted as members of X. For example, if M is a model of RCA_0, then its second-order part X is closed under Turing reducibility and Turing join.

With the concept of hierarchies available, we can further rephrase the motivating questions: Suppose the coloring function f is recursive, what is the minimal syntactical complexity of a solution? Which system in Reverse Mathematics does Ramsey's Theorem correspond? E.g., does RT^2_2 imply ACA_0? What are the first-order consequences of Ramsey's Theorem? E.g., does RT^2_2 imply $I\Sigma^0_2$? Does SRT^2_2 imply RT^2_2? In other words, if X contains solutions for all stable colorings, how about for general colorings?

We now give a list of historical results. Some of the early studies are motivated by effective mathematics. We have modified their statements to suit our purposes.

Theorem (Jockusch, 1972) Over RCA_0,

$ACA_0 \rightarrow RT^3_2 \leftrightarrow RT^n_k$.

$ACA_0 \rightarrow RT^2_2$ and WKL_0 does not imply RT^2_2.

Theorem (Hirst, 1987) Over RCA_0,

$SRT^2_2 \rightarrow B\Sigma^0_2$.

This tells us a lower bound of SRT^2_2's first order strength.

Theorem (Seetapun and Slaman, 1995) Over RCA_0,

RT^2_2 does not imply ACA_0.

Seetapun's proof made clever use of trees, which leads to the Seetapun Conjecture: RT^2_2 implies WKL_0.

Theorem (Cholak, Jockusch and Slaman, 2001) Over RCA_0,

RT^2_2 does not imply $I\Sigma^0_3$.

After Cholak, Jockusch and Slaman's paper, the exact strength of RT^2_2 was studied extensively by practically every expert in the field and many failed attempts were made to solve it. However, the extensive study changed the whole field of reverse mathematics. For example, the usual big five subsystems are no longer the only benchmarks to use. In fact, around the RT^2_2, linear measurement is no longer sufficient; it is more like a "zoo" now. Hirschfeldt and Shore in their 2007 paper entitled Combinatorial principles weaker than Ramsey's theorem for pairs, made further progress on the exact strength of many important combinatorial principles weaker than RT^2_2. However, three major questions remain open: (1) Seetapun's Conjecture; (2) Over RCA_0 does RT^2_2 imply RT^2_2? (3) Does SRT^2_2 imply $I\Sigma^0_2$? If not, how about RT^2_2?

The first problem was solved by Jiayi Liu in 2011, when he showed that over RCA_0 RT^2_2 does not imply WKL_0. However, the solution for (2) and (3) remains elusive. The most natural approach is to show that stable colorings always have a low solution; or equivalently, every Δ^0_2-set contains or is disjoint from an infinite low set. However, Downey, Hirschfeldt, Lempp and Solomon in 2001 showed that there is a Δ^0_2-set D such that neither D nor the complement of D contains an infinite low subset, thus blocking the seemingly only promising approach.

It is Chitat Chong who suggested in 2005 that we should look at nonstandard models of fragments of arithmetic, because the theorem by Downey, Hirschfeldt, Lempp and Solomon was done on the standard model of arithmetic, whose proof involves infinite injury method thus requiring $I\Sigma^0_2$. Yet we know that in nonstandard models, things behave differently. For example, there is a model of $B\Sigma^0_2$ but not $I\Sigma^0_2$ in which every incomplete Δ^0_2-set is low. After almost 10 years' of work, this approach turns out to be fruitful:

Theorem (Chong, Slaman and Yang, 2014) Over RCA_0,

SRT^2_2 does not imply RT^2_2 and SRT^2_2 does not imply $I\Sigma^0_2$.

Theorem (Chong, Slaman and Yang, ta) Over RCA_0,
RT^2_2 does not imply $I\Sigma^0_2$.

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