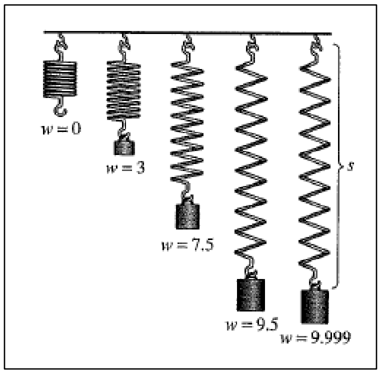
MA 15910 Lesson 9 Notes, Section 3.1 (2^{nd} half of text)
Limits

In everyday language, people refer to a speed limit, a wrestler's weight limit, the limit on one's endurance, or stretching a spring to its limit. These phrases all suggest that a limit is a bound, which on some occastions may not be reached but on other occasions may be reached or exceeded.

Consider a spring that will break or completely stretch out only if a weight of 10 pounds or more is attached. To determine how far the spring will stretch without breaking, you could attach increasingly heavier weights and measure the 'spring length' *s* for each weight *w*, as shown in the picture below. If the spring length approaches a value of *L*, then it is said that 'the limit of *s* as *w* approaches 10 is *L*.' A mathematical limit is much like the limit of the spring described. As a weight hanging from a spring approaches 10 pounds, the length of the stretch of the spring will approach a certain number called the 'limit'. Let us suppose that limit is 8 inches. (If any more weight than 10 pounds is put on the spring, it will break or stop stretching beyond 8 inches.) We could say 'the limit of the length of the stretched spring as the weight approaches 10 pounds is 8 inches. This would be written as $\lim_{w \to 10} (\text{spring}) = 8$ (the limit of the length of the spring as weight *w* approaches 10 pounds is 8 inches).



The general limit notation is $\lim_{x \to c} f(x) = L$, which is read 'the limit of $f(x)$ as x approaches c is L.'

Finding Limits

There are many different strategies used in calculus to find limits. One approach is to evaluate the function for numbers very close to *c*, slightly larger and/or slightly smaller than *c*. Examine these examples.

The first strategy for finding limits is **using a table** such as the next few examples.

1

Ex 1:

$\lim_{x\to 4} f(x)$ where $f(x) = 2x+3 \to \lim_{x\to 4}(2x+3)$ Select values of *x* slightly smaller or slightly larger than 4 and use the table of ordered pairs below.

x	3.9	3.99	3.999	4.001	4.01	4.1
f (x)	10.8	10.98	10.998	11.002	11.02	11.2

Approaching from the left $\rightarrow$ $\longrightarrow\longleftarrow$ $\leftarrow$ Approaching from the right

You can examine that the closer the *x* value is to 4, the function value is closer to 11. We say $\lim_{x\to 4}(2x+3) = 11$.

2

Coincidentally a 'direct substitution' of 4 into the function value 2*x* + 3 yielded the limit value of 11.

Sometimes the strategy of '**direct substitution'** works

$$\lim_{x\to 4}(2x+3) = 2(4)+3 = 11$$

The limit value is the function value (the *y* value). What *y*-value is approached as *x* approaches *a*? (*x*→*a*)

Ex 2:

$\lim_{x\to 3} g(x)$ where $g(x) = \frac{x^2-9}{x-3} \to \lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$ Select values of *x*, slightly smaller or slightly larger than 3 and use the table of ordered pairs below.

x	2.9	2.99	2.999	3.001	3.01	3.1
g (*x*)	5.9	5.99	5.999	6.001	6.01	6.1

Approaching from the left $\rightarrow$ $\leftarrow$ Approaching from the right

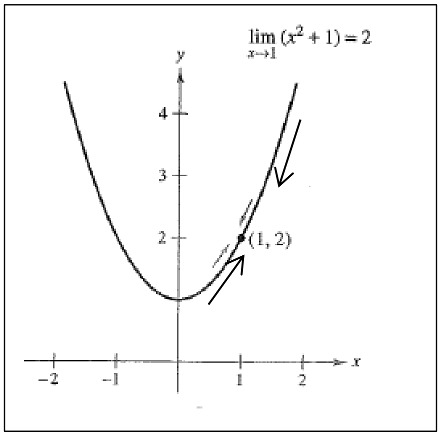
You can examine that the close the *x* value is to 3, the function value is closer to 6. We say

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = 6$$

Notice that this time a '**direct substitution'** would not work, because a zero denominator results in an undefined number. **Direct substitution yields 0/0, which is called an indeterminant form (limit may or may not exist).**

3

Sometimes a 'direct substitution' works, sometimes it does not. You can occasionally use tables such as above to determine limits. Sometimes **looking at a graph helps.** Look at the figure below right. This graph corresponds to the limit, $\lim_{x \to 1}(x^2 + 1)$.



Imagine 'crawling' toward the value $x = 1$ from both sides of the graph (<u>follow both arrows</u>). Either approaching from below left or above right; **as x approaches 1, the function value (y value) is going toward 2**.

A third strategy for finding a limit is to **<u>view a graph</u>**.
<u>Approach the x-value from the 'left' and the 'right'.</u>

$$\lim_{x \to 1}(x^2 + 1) = 2$$

Even if the point (1, 2) was an 'open' point, the limit would still be 2.

The arrows show that approaching the x value 1 **from either the left (values smaller than 1) or from the right (values larger than 1),** yield a function value (y value) of 2. We can easily see that $\lim_{x \to 1}(x^2 + 1) = 2$.

The limits shown below are called 'one-sided' limits.

The notation $\lim_{x \to a^-}(f(x))$ represents 'the limit of $f(x)$ as x approaches a from the left'.

The notation $\lim_{x \to a^+}(f(x))$ represents 'the limit of $f(x)$ as x approaches a from the right'.

If the limit of a function as x approaches a number a from both the left and the right is the same function value L, then the limit of the function as x approaches the number a is L. In other words, **<u>if the left sided limit equals the right sided limit, then that value is the limit value in general.</u>**

<u>Ex 3</u>: (a) Graph $f(x)=\frac{|x-2|}{x-2}$

Use the graph to find $\lim_{x\to 2} f(x)$.

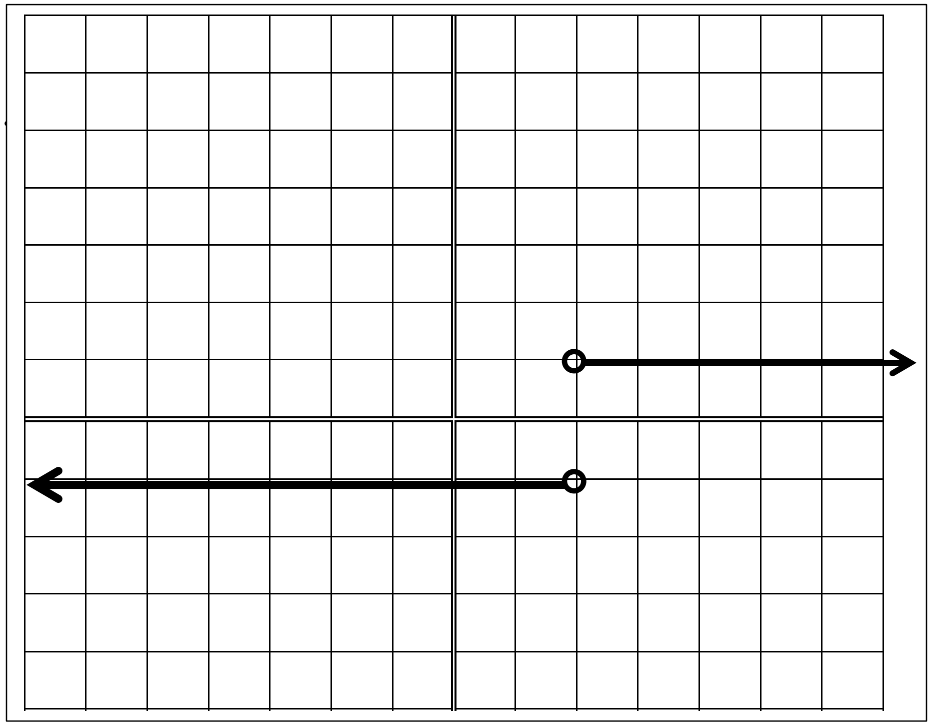
As you follow the graph toward x = 2 from the left or from the right, does it appear to be approaching a certain y value (function value) ?

From left

x	y
-2	-1
-1	-1
0	-1
1	-1
1.5	-1
2	undefined

From right

x	y
4	1
3	1
2 ½	1
2.1	1
2	undefined



(b) Let's also make a table.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	-1	-1	-1	1	1	1

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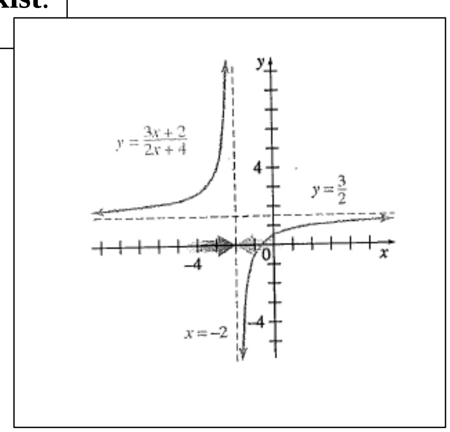
Using the table, does the $\lim_{x\to 2}(f(x))$ appear to be approaching the same function value from 'the left' and from 'the right'? We say $\lim_{x\to 2}(f(x))$ **does not exist**.

<u>Ex 4:</u> Here is another examples where a limit does not exist. $\lim_{x\to -2}\left(\frac{3x+2}{2x+4}\right)$ Examine the graph below. As x approaches -2 from the left, the graph goes toward ∞. Then x approaches -2 from the right, the graph goes toward -∞. Therefore $\lim_{x\to -2}\left(\frac{3x+2}{2x+4}\right)$ **does not exist**.



A table and graph shows the same results.

x	-2.1	-2.01	-2.001	-2	-1.999	-1.99	-1.9
$h(x)$	21.5	201.5	2001.5	undefined	-1998.5	-198.5	-18.5

As x approaches -2 from the left, the function value goes toward ∞. As x approaches -2 from the right, the function value goes toward -∞. **Therefore the limit does not exist.**

After examination of the examples above and other examples shown in the textbook, the following are three important conclusions.

1) Saying that the limit of $f(x)$ as x approaches a is L means that the function value gets very, very close to L as x gets closer and closer to a.
2) For a limit L to exist, you must allow x to approach a from **either side of *a***. For the limit to exist, the value found by approaching from either the left or the right must be the same.
3) The function does not have to be defined at a in order to have a limit as $x \to a$. In other words, a limit may exist even though the function value does not.

Techniques for Evaluating Limits:

1. **With a polynomial function (or many other functions), direct substitution sometimes can be used to find the limit.** See the examples below.

$a)$ $\lim\limits_{x\to(-2)}(x^2 - x) = (-2)^2 - (-2) = 4 + 2 = 6$

$b)$ $\lim\limits_{n\to 3}(2n-4) = 2(3) - 4 = 6 - 4 = 2$

$c)$ $\lim\limits_{a\to 5}\sqrt{2a+6} = \sqrt{2(5)+6} = \sqrt{16} = 4$

$d)$ $\lim\limits_{n\to 2}\left(\dfrac{2n-5}{n+1}\right) = \dfrac{2(2)-5}{2+1} = \dfrac{-1}{3}$ or $-\dfrac{1}{3}$

2. **With a rational function or rational expression where direct substitution yields 0/0, you can sometimes write an equivalent expression for the function by simplifying, then use 'direct substitution' in the equivalent expression.** (When direct substitution yields 0/0, it is an indeterminant form.)

$a)$ $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right) = \lim\limits_{x\to 3}\left(\dfrac{(x+3)(x-3)}{x-3}\right) = \lim\limits_{x\to 3}(x+3) = 3+3 = 6$

$b)$ $\lim\limits_{c\to 5}\left(\dfrac{c^2-4c-5}{c-5}\right) = \lim\limits_{c\to 5}\left(\dfrac{(c-5)(c+1)}{c-5}\right) = \lim\limits_{c\to 5}(c+1) = 5+1 = 6$

$c)$ $\lim\limits_{\Delta x\to 0}\left(\dfrac{3(x+\Delta x)-2-(3x-2)}{\Delta x}\right) = \lim\limits_{\Delta x\to 0}\left(\dfrac{3x+3(\Delta x)-2-3x+2}{\Delta x}\right)$

$= \lim\limits_{\Delta x\to 0}\left(\dfrac{3(\Delta x)}{\Delta x}\right) = \lim\limits_{\Delta x\to 0} 3 = 3$

Ex 5: Find each limit. If the limit does not exist, write 'does not exist'.

$a)$ $\lim\limits_{x\to 5}(2x+3)$

$b)$ $\lim\limits_{a\to(-3)}(2a^2 - 5a + 7) =$

$c)\quad \lim_{x\to 4}\sqrt{21+x}$

Compare *c* to this example

$\lim_{x\to -30}\sqrt{21+x} =$

Direct substitution yields $\sqrt{-9}$.

A graph verifies that the limit does not exist.

$d)\quad \lim_{x\to 2}\sqrt{x-5}$ (Hint: graph.)

$e)\quad \lim_{m\to 2}\left(\dfrac{\frac{1}{m}-\frac{1}{m+1}}{m}\right)$

<u>Ex 6</u>: Find each limit, if it exists. If not, write 'does not exist'. (<u>Notice: Direct substitution yields 0/0, an indeterminant form, in each.</u>)

$a)\quad \lim_{x\to -10}\left(\dfrac{x^2-100}{x+10}\right)$

$b)\quad \lim_{x\to 2}\left(\dfrac{x^2-4}{x^2+x-6}\right)$

$c)\quad \lim_{x\to 0}\left(\dfrac{1/2-1/(x+2)}{x}\right)$

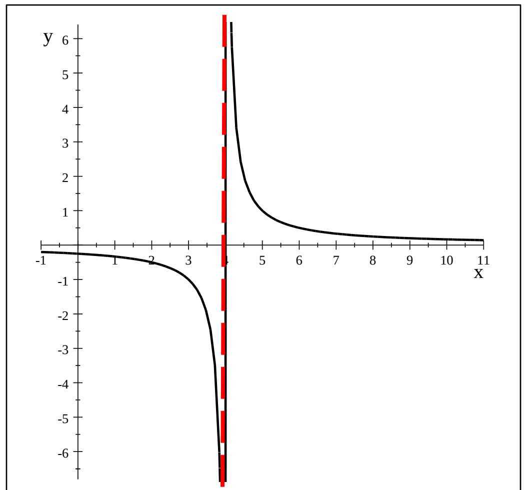
Ex 7:

Direct substitution in all of the following examples yields $\frac{0}{0}$, the indeterminant form.

Use a simplfying technique, if possible, then use substituion.

$a)\quad \lim_{x\to -5}\left(\frac{x^2-25}{x+5}\right)$

$b)\quad \lim_{x\to 4}\left(\frac{x-4}{x^2-8x+16}\right)$



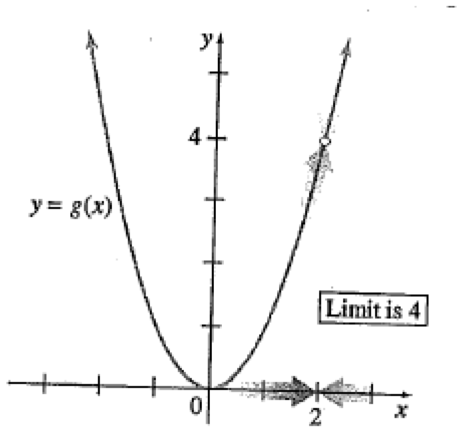
The graph of this function shows the right-sided limit is ∞ and the left-sided limit is -∞. The limit in general does not exist.

$c)\quad \lim_{t\to 2}\left(\frac{t^2+3t-10}{t^2-4}\right)$

$d)\quad \lim_{\Delta x\to 0}\left(\frac{4(x+\Delta x)+3-(4x+3)}{\Delta x}\right)$

$e)\quad \lim_{\Delta r\to 0}\left(\frac{(r+\Delta r)^2-2(r+\Delta r)-1-(r^2-2r-1)}{\Delta r}\right)$

f) This is a graph of $g(x)=\dfrac{x^3-2x^2}{x-2}$. We want to find $\lim_{x\to 2}(g(x))$. We know the function value $g(2)$ does not exist (there is a hole in the graph at $x = 2$), but the limit may still exist. We will look at the graph and approach x at 2 from both the left side and the right side. We can also use an algebraic approach by simplifying the rational expression and using a substitution.



$$\lim_{x\to 2} g(x)=\lim_{x\to 2}\left(\frac{x^3-2x^2}{x-2}\right)=\lim_{x\to 2}\left(\frac{x^2(x-2)}{x-2}\right)$$
$$=\lim_{x\to 2}(x^2)=2^2=4$$

g) Suppose the function *h* was defined as $h(x)=\begin{cases} x^2 & \text{if } x\neq 2 \\ 1 & \text{if } x=2 \end{cases}$. Its graph would be the same as the graph in *f* above except there would be the point (2, 1) as well (see graph below). The limit of $g(x)$ as x approaches 2 would again be 4, even though the function value when $x = 2$ is 1.

$$\boxed{\lim_{x\to 2} h(x)=4, \text{ although } h(2)=1}$$

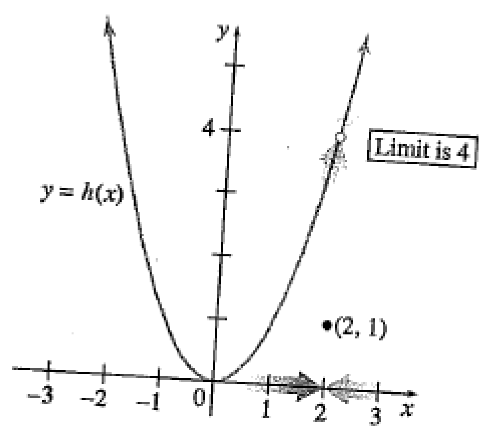


FIGURE 5